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A Recreational Application of Two Integer Sequences and the Generalized Repetitious Number Puzzle

Abstract

In this article, we give a particular recreational application of two integer sequences. These sequences are respectively the sequence A000533 and sequence A261544 in "The On-line Encyclopedia of Integer Sequences" (OEIS). The recreational application provides a direct extension to "The Repetitious Number" puzzle of Martin Gardner contained in The Second Scientific American Book of Mathematical Puzzles and Diversions published in 1961. We then generalize the repetitious number puzzle and give a puzzle similar to the *The Repetitious Number Puzzle* as an illustrative example of the *Generalized Repetitious Number Puzzle*. Finally, as a consequence of the generalization, we define a family of sequence in which the sequences A000533 and A261544 belong.

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1 Introduction

We begin this section by considering some brief introductory information on the OEIS. We then discuss the two integer sequences under study. Finally, we present the *Repetitious Number Puzzle* of Gardner that will serve as the "source" of the recreational application.

1.1 The OEIS

The OEIS is an on-line collection of over quarter-million number sequences initiated by Neil J.A. Sloane in early 1964 [1].

It mainly aims to "allow mathematicians or other scientists to find out if some sequence that turns up in their research has ever been seen before. If it has, they may find that the problem they're working on has already been solved, or partially solved, by someone else. Or they may find that the sequence showed up in some other situation, which may show them an unexpected relationship between their problem and something else. Another purpose is to have an easily accessible database of important, but difficult to compute, sequences." [2]

A particular example of important but difficult to compute sequence in the OEIS is the sequence of *Mersenne primes* [3]. The existence of which is equivalent to the existence of an even *perfect number* and the largest known *prime number* [4].

1.2 Two Integer Sequences in the OEIS

We now turn our attention to the two sequences in the OEIS. They are the sequences A000533 and A261544.

The sequence A000533 [5] in the OEIS is the sequence defined by

$$\begin{aligned} a(0) &= 1 \\ a(n) &= 10^n + 1, \quad n \geq 1. \end{aligned} \tag{1}$$

Its first 15 terms are:

1, 11, 101, 1001, 10001, 100001, 1000001, 10000001, 100000001, 1000000001, 10000000001, 100000000001, 1000000000001, 10000000000001, 100000000000001,....

We remark that "$a(1) = 11$ and $a(2) = 101$ are the only prime terms of the sequence up to $n = 100\,000$" [5], as verified by Daniel Arribas. Also, it is unknown whether there are other prime terms in the sequence.

The sequence A261544 [6] on the otherhand is the sequence defined by

$$b(n) = \sum_{k=0}^{n} 1000^k. \tag{2}$$

Its first 10 terms are:

1, 1001, 1001001, 1001001001, 1001001001001, 1001001001001001,
1001001001001001001, 1001001001001001001001,
1001001001001001001001001, 1001001001001001001001001001,....

It can be verified that unlike the first sequence, the terms of this sequence are all composite except for the zeroth term "1". A complete solution to this claim may be viewed in [7].

With the two sequences in the OEIS already introduced, we are now ready to consider the *Repetitious Number Puzzle*.

1.3 The Repetitious Number Puzzle

In his book "The Second Scientific American Book of Mathematical Puzzles and Diversions (1961)"[8], Martin Gardner presented the puzzle given below:

> " **The Repetitious Number**. An unusual parlor trick is performed as follows. Ask spectator A to jot down any three-digit number, and then to repeat the digits in the same order to make a six-digit number (e.g., 394 394). With your back turned so that you cannot see the number, ask A to pass the sheet of paper to spectator B, who is requested to divide the number by 7.
>
> Dont worry about the remainder, you tell him, because there won't be any. B is surprised to discover that you are right (e.g., 394 394 divided by 7 is 56 342). Without telling you the result, he passes it on to spectator C, who is told to divide it by 11. Once again you state that there will be no remainder, and this also proves correct (56 342 divided by 11 is 5 122).
>
> With your back still turned, and no knowledge whatever of the figures obtained by these computations, you direct a fourth spectator D, to divide the last result by 13. Again the division comes out even (5 122 divided by 13 is 394). This final result is written on a slip of paper which is folded and handed to you. Without opening it you pass it on to spectator A.

> Open this, you tell him, and you will find your original three-digit number.
>
> Prove that the trick cannot fail to work regardless of the digits chosen by the first spectator."

This puzzle was originally written by Yakov Perelman in his book "Figures for Fun Stories, Puzzles and Conundrums (1957)"[9].

In section 3, we present the solution of the puzzle and state some important questions necessary for its extension. For the mean time we discuss some important notations and mathematical concepts needed in understanding the solution of the puzzle and its extension.

2 Preliminaries

2.1 Some Terms and Notations

The following terms will be encountered in the succeeding sections of this article.

Definition 2.1. *Let $n = d_1d_2\ldots d_kd_1d_2\ldots d_k\ldots d_1d_2\ldots d_k$ be a positive repetitive integer. We say that the positive integer $g = d_1d_2\ldots d_k$ is a* **generator** *of n if g is a positive integer such that replicating g a finite number of times generates n.*

Definition 2.2. *Let $g = d_1d_2\ldots d_k$ be a generator of n. Then the* **length** *of g denoted by $l(g)$ is the number of digits in g.*

Definition 2.3. *Let $g = d_1d_2\ldots d_k$ be a generator of n. The* **replication number** *of g denoted by $r(g)$ is the number of replication performed in g in order to generate n.*

To fully understand the concepts being discussed, we consider some examples.

Example 2.4. *Consider the positive repetitive integer $n_1 = 394\ 394$ in the Repetitious Number Puzzle. The positive integer $g_1 = 394$ is a generator for n_1 with length $l(g_1) = 3$ and replication number $r(g_1) = 2$.*

Example 2.5. *The positive repetitive integer $n_2 = 111\ 111$ is generated by $g_2 = 1$ with length $l(g_2) = 1$ and replication number $r(g_2) = 6$. The integers 11, 111, and 111 111 are the other generators of n_2.*

Example 2.6. *The positive integer $n_3 = 223\ 344$ generates itself with length 6 and replication number 1.*

Remark 2.7. *We emphasize that a generator is not unique. Also, for any positive integer n, n is a generator of itself. Moreover, if n is not repetitive, then its generator is unique. Finally, if d generates n with replication number r, we write $n = d_r$.*

2.2 Essential Mathematical Concepts

For completeness, we recall some essential concepts in elementary Number Theory.

Definition 2.8. *Let a and b be two positive integers such that $a \leq b$. We say that a* **divides** *b written in symbol by $a \mid b$ if there is a positive integer c such that*

$$b = ac. \tag{3}$$

If there is no positive integer c that satisfies 3, then we say that a **does not divides** *b and this situation is denoted by $a \nmid b$. If $a \mid b$, we can also say the following:*

(i) b is a multiple of a, (ii) a is a divisor of b and (iii) a is a factor of b.

Example 2.9. *Let us consider the positive integer* 394 394. *Note that* 7 *divides* 394 394 *since*

$$394\,394 = 7 \times 56\,342.$$

However, 5 *does not divides* 394 394 *since we cannot find any positive integer c that can satisfy the equation*

$$394\,394 = 5 \times c.$$

The property of divisibility given below is important.

Lemma 2.10. *Let a, b and D be positive integers such that $D \leq a$ and $D \leq b$. If $D \mid a$ and $D \mid b$, then $D \mid (ax + by)$ for any positive integers x and y.*

The proof of Lemma 2.10 follows directly from the definition of divisibility and is standard in any elementary Number Theory textbooks. Letting $x = y = 1$ we arrive at the the corollary given below.

Corollary 2.11. *Let a, b and D be positive integers such that $D \leq a$ and $D \leq b$. If $D \mid a$ and $D \mid b$, then $D \mid (a + b)$.*

Remark 2.12. *The property of divisibility stated in Corollary 2.11 can be easily extended into a finite number of multiples. Given $D \mid a$ and $D \mid b$, by Corollary 2.11 we have $D \mid (a + b)$. If $D \mid c$ given $D \mid (a + b)$ applying Corollary 2.11 once more gives $D \mid ((a + b) + c)$ or $D \mid (a + b + c)$.*

If b is divided by a then either $a \mid b$ or $a \nmid b$. In both cases however, we may write b in terms of a; this is guaranteed by the next lemma.

Lemma 2.13. Division Algorithm. *Given integers a and b with $a > 0$, there are unique integers q and r satisfying*

$$b = qa + r, \qquad 0 \leq r < a.$$

Remark 2.14. *The integers q and r are respectively called the* **quotient** *and the* **remainder** *of b upon division by a. Note also that $a \mid b$ if and only if $r = 0$, and that $a \nmid b$ if and only if otherwise.*

Definition 2.15. *A positive integer $p > 1$ is said to be a* **prime number** *if its only positive divisors are* 1 *and p itself.*

Example 2.16. *The positive integers* $7, 11$ *and* 13 *are all prime numbers; since their only positive divisors are* 1 *and their selves.While the positive integer* $394\,394$ *is not a prime number; since from Example 2.9, we know that not only the positive integers* 1 *and* $394\,394$ *divides* $394\,394$ *but also the integer* 7.

Lemma 2.17. Fundamental Theorem of Arithmetic. *Every positive integer $n > 1$ is either a prime or a product of primes; this representation is unique, apart from the order in which the factors occur.*

Lemma 2.17 is a well known result in elementary Number Theory, its proof is included in most of elementary Number Theory textbooks. Consider [10] for instance.

Example 2.18. *Let us consider the positive integer* 1001. *From Lemma 2.17, either* 1001 *is a prime or a product of primes. The latter holds true since*

$$1001 = 7 \times 11 \times 13.$$

After a brief recall in some essential results in Elementary Number Theory, we are now ready to present the solution of the "Repetitious Number Puzzle".

2.3 Solution of the Repetitious Number Puzzle

The solution discussed in this subsection is due to the solution presented in [8].

Any three digit number takes the form $d_1d_2d_3$ where d_1, d_2 and d_3 are non negative integers with bounds

$$0 < d_1 \leq 9$$

$$0 \leq d_2 \leq 9$$

$$0 \leq d_3 \leq 9$$

Repeating the digits in the same order yields the six-digit integer $d_1d_2d_3d_1d_2d_3$. This integer can be factored into $1001 \times d_1d_2d_3$ as shown in the following computation

$$
\begin{array}{rcccccc}
 & & & 1 & 0 & 0 & 1 \\
\times & & & & d_1 & d_2 & d_3 \\
\hline
 & & & d_3 & 0 & 0 & d_3 \\
 & & d_2 & 0 & 0 & d_2 & \\
+ & d_1 & 0 & 0 & d_1 & & \\
\hline
 & d_1 & d_2 & d_3 & d_1 & d_2 & d_3.
\end{array}
$$

Thus, $d_1d_2d_3$ and 1001 divides $d_1d_2d_3d_1d_2d_3$ and that $d_1d_2d_3d_1d_2d_3 = 1001 \times d_1d_2d_3$. From Example 2.18, 1001 can be expressed as a product of primes $7, 11$ and 13. Hence $7, 11$ and 13 divides $d_1d_2d_3d_1d_2d_3$ and that

$$d_1d_2d_3d_1d_2d_3 = 7 \times 11 \times 13 \times d_1d_2d_3. \tag{4}$$

In lieu of Lemma 2.13, we have

$$d_1d_2d_3d_1d_2d_3 = 7 \times (11 \times 13 \times d_1d_2d_3) + 0.$$

So, dividing the integer $d_1d_2d_3d_1d_2d_3$ by 7 gives the integer $11 \times 13 \times d_1d_2d_3$ with remainder 0.

Next, we consider the integer $11 \times 13 \times d_1d_2d_3$. In lieu of Lemma 2.13, we have

$$11 \times 13 \times d_1d_2d_3 = 11 \times (13 \times d_1d_2d_3) + 0.$$

So, dividing the integer $11 \times 13 \times d_1d_2d_3$ by 11 gives the integer $13 \times d_1d_2d_3$ with remainder 0.

Finally, we consider the integer $13 \times d_1d_2d_3$. Note that this integer can be written as

$$13 \times d_1d_2d_3 = 13 \times (d_1d_2d_3) + 0.$$

So, dividing the integer $13 \times d_1d_2d_3$ by 13 gives the integer $d_1d_2d_3$ with remainder 0.

Hence, dividing the six-digit repetitive number $d_1d_2d_3d_1d_2d_3$ in succession by the integers $7, 11$ and 13 returns the repetitive number into its generator $d_1d_2d_3$. This solves the puzzle.

Remark 2.19. *The order of dividing the integer $d_1d_2d_3d_1d_2d_3$ by the integers $7, 11$ and 13 do not matter in the puzzle. For $d_1d_2d_3d_1d_2d_3$ can be written as*

$$7 \times (11 \times 13 \times d_1d_2d_3),\ 7 \times (13 \times 11 \times d_1d_2d_3)\ 11 \times (7 \times 13 \times d_1d_2d_3),$$
$$11 \times (13 \times 7 \times d_1d_2d_3),\ 13 \times (7 \times 11 \times d_1d_2d_3) \text{ and } 13 \times (11 \times 7 \times d_1d_2d_3).$$

We now present our results in the next section.

3 Results

3.1 Recreational Application of the Sequence A000533

The goal of this subsection is to show that the k^{th} term $a(k)$ of the sequence A000533 divides the $2k-digit$ repetitive number n generated by g with $l(g)=k$. Hence, when a $k-digit$ number g is duplicated resulting to n, dividing n with the prime factors of $a(k)$ gives the original number g. This result is due to

Theorem 3.1. *Let $n=(d_1d_2\ldots d_k)_2$ be a repetitive number generated by $g=d_1d_2\ldots d_k$ of length k. Then there is a finite sequence of divisors D_i such that n upon division by all of D_i becomes g.*

Proof. Given a repetitive number $n=(d_1d_2\ldots d_k)_2$, we express it as a sum of two positive integers both divisible by $g=d_1d_2\ldots d_k$. In particular n can be expressed as the sum

$$\begin{array}{r} d_1\, d_2 \ldots d_k\ 0\ \ 0\ \ldots\ 0 \\ +\qquad\qquad d_1\, d_2 \ldots\, d_k \\ \hline d_1\, d_2 \ldots d_k\, d_1\, d_2 \ldots d_k. \end{array}$$

Note that since $g \mid g$ and $g \mid d_1d_2\ldots d_k\underbrace{00\ldots0}_{\text{k-zeros}}$, by Corollary 2.11 we have

$$g \mid (g+d_1d_2\ldots d_k\underbrace{00\ldots0}_{\text{k-zeros}}).$$

So, $g \mid n$.

After factoring out the common factor g in both summands we have

$$\begin{aligned} n &= g\times(1+1\underbrace{00\ldots0}_{\text{k-zeros}})]] \\ &= g\times a(k). \end{aligned}$$

By the Fundamental Theorem of Arithmetic (Lemma 2.17), $a(k)$ is either a prime or a product of primes. If $a(k)$ is prime, then the finite sequence of divisors to be divided to n to become g is $a(k)$ itself. If $a(k)$ is non-prime then the finite sequence of divisors to be divided to n to become g is the finite sequence whose terms are the prime divisors of $a(k)$. ☐

The proof of Theorem 3.1 gives us a method on solving a particular extension of the *repetitious number puzzle*.

Problem. Suppose that in the *repetitious number puzzle* spectator A was asked to write down any $k-$digit positive integer. To what sequence of numbers does the resulting $2k-$digit number be divided in order to return to the original $k-$digit number?

Solution. Let g be the $k-$digit positive integer and let n be the resulting $2k-$digit number. Note that n is a repetitive number generated by g of length k with replication number 2. That is, $n = g_2$ with $l(g) = k$. Using the result contained in the proof of Theorem 3.1, we must divide n by the prime divisors of $1\underbrace{00\ldots0}_{\text{k-1-zeros}}1$, the k^{th} term of the sequence A000533 in order to return to the original number g.

Example 3.2. *Suppose that spectator A wrote the number* $g = 451\ 220\ 125$. *Duplicating* g *gives the number* $n = 451\ 220\ 125\ 451\ 220\ 125$. *Dividing* n *by the numbers* $7, 11, 13, 19$ *and* $52\ 579$, *the prime divisors of the ninth term of the sequence A000533 which is* $1\ 000\ 000\ 001$ *(See Table 1. Generated using Wolfram Alpha), gives the original number* $g = 451\ 220\ 125$.

3.2 Recreational Application of the Sequence A261544

The goal of this subsection is to show that the $(r-1)^{st}$ term $b(r-1)$ of the sequence A261544 divides the $3r-digit$ repetitive number n generated by g with $l(g) = 3$. Hence, when a $3-digit$ number g is replicated r-times resulting to n, dividing n with the prime factors of $b(r-1)$ gives the original number g. This result is due to

Theorem 3.3. *Let* $n = (d_1d_2d_3)_r$ *be a repetitive number generated by* $g = d_1d_2d_3$ *of length* 3. *Then there is a finite sequence of divisors* D_i *such that* n *upon division by all of* D_i *becomes* g.

Proof. Given a repetitive number $n = (d_1d_2d_3)_r$, we express it as a sum of r positive integers both divisible by $g = d_1d_2d_3$. In particular n can be expressed as the sum

$$n = d_1d_2d_3(0)_{3(r-1)} + d_1d_2d_3(0)_{3(r-2)} + \ldots + d_1d_2d_3(0)_{3(r-r)}.$$

Note that since $g \mid g$ and $g \mid d_1d_2d_3(0)_{3j}$, for $j = 1, 2, \ldots r-1$, by Corollary 2.11 we have

$$g \mid d_1d_2d_3(0)_{3(r-1)} + d_1d_2d_3(0)_{3(r-2)} + \ldots + d_1d_2d_3(0)_{3(r-r)}.$$

So, $g \mid n$.

Factoring out the common factor g in all of the summands we have

$$\begin{aligned} n &= g \times (1(0)_{3(r-1)} + 1(0)_{3(r-2)} + \ldots + 1(0)_{3(r-r)}) \\ &= g \times b(r-1). \end{aligned}$$

Table 1: **Prime factorization of the first 25 terms of the sequence A000533**

No. of Digits (k)	Rep. No. (r)	Terms of Sequence A000533	Prime Factorization
0	2	1	1
1	2	11	11
2	2	101	101
3	2	1001	$7 \cdot 11 \cdot 13$
4	2	10001	$73 \cdot 137$
5	2	100001	$11 \cdot 9091$
6	2	1000001	$101 \cdot 9901$
7	2	10000001	$11 \cdot 909091$
8	2	100000001	$17 \cdot 5882353$
9	2	1000000001	$7 \cdot 11 \cdot 13 \cdot 19 \cdot 52579$
10	2	10000000001	$101 \cdot 3541 \cdot 27961$
11	2	100000000001	$11 \cdot 23 \cdot 4093 \cdot 8779$
12	2	1000000000001	$73 \cdot 137 \cdot 99990001$
13	2	10000000000001	$11 \cdot 859 \cdot 1058313049$
14	2	100000000000001	$29 \cdot 101 \cdot 281 \cdot 121499449$
15	2	1000000000000001	$7 \cdot 11 \cdot 13 \cdot 211 \cdot 241 \cdot 2161 \cdot 9091$
16	2	10000000000000001	$353 \cdot 449 \cdot 641 \cdot 1409 \cdot 69857$
17	2	100000000000000001	$11 \cdot 103 \cdot 4013 \cdot 21993833369$
18	2	1000000000000000001	$101 \cdot 9901 \cdot 999999000001$
19	2	10000000000000000001	$11 \cdot 909090909090909091$
20	2	100000000000000000001	$73 \cdot 137 \cdot 1676321 \cdot 5964848081$
21	2	1000000000000000000001	$7 \cdot 11 \cdot 13 \cdot 127 \cdot 2689 \cdot 459691 \cdot 909091$
22	2	10000000000000000000001	$89 \cdot 101 \cdot 1052788969 \cdot 1056689261$
23	2	100000000000000000000001	$11 \cdot 47 \cdot 139 \cdot 2531 \cdot 549797184491917$
24	2	1000000000000000000000001	$17 \cdot 5882353 \cdot 9999999900000001$
25	2	10000000000000000000000001	$11 \cdot 251 \cdot 5051 \cdot 9091 \cdot 78875943472201$

By the Fundamental Theorem of Arithmetic (Lemma 2.17), $b(r-1)$ is either a prime or a product of primes. However, we know that (except for the zeroth term) the terms of the sequence A261544 are all composite. So the finite sequence of divisors to be divided to n to become g is the finite sequence whose terms are the prime divisors of $b(r-1)$. □

The proof of Theorem 3.2 gives us a method on solving another particular extension of the *repetitious number puzzle*.

Problem. Suppose that in the *repetitious number puzzle* spectator A was asked to write down any $3-$digit positive integer and replicate it $r-$times. To what sequence of numbers does the resulting $3r-$digit number be divided in order to return to the original $3-$digit number?

Solution. Let g be the $3-$digit positive integer and let n be the resulting $3r-$digit number. Note that n is a repetitive number generated by g of length 3 with replication number r. That is, $n = g_r$ with $l(g) = 3$. Using the result contained in the proof of Theorem 3.2, we must divide n by the prime divisors of $b(r-1)$, the $(r-1)^{st}$ term of the sequence A261544 in order to return to the original number g.

Example 3.4. *Suppose that spectator A wrote the number* $g = 721$. *Replicating* g *4-times gives the number* $n = 721\ 721\ 721\ 721$. *Dividing* n *by the numbers* $7, 11, 13, 101, 9091$ *the prime divisors of the third term of the sequence A261544 which is* $1\ 001\ 001\ 001$ *(See Table 2. Generated using Wolfram Alpha), gives the original number* $g = 721$.

3.3 Generalized Repetitious Number Puzzle

In this subsection, we generalize the *repetitious number puzzle* by allowing spectator A to write down any $j-$digit number and replicate it $r-$times to generate the integer $n = g_r$ with $l(g) = j$. The generalization is given in the next theorem.

Theorem 3.5. *Let* $n = (d_1 d_2 \ldots d_j)_r$ *be a repetitive number generated by* $g = d_1 d_2 \ldots d_j$ *of length* j. *Then the sequence of prime factors of the integer*

$$\left(1(0)_{k-1}\right)_{r-1}1$$

is a finite sequence such that n *upon division by all the sequence terms becomes* g.

Proof. Given a repetitive number $n = (d_1 d_2 \ldots d_j)_r$, we express it as a sum of r positive integers both divisible by $g = d_1 d_2 \ldots d_j$. In particular n can be expressed as the sum

$$n = d_1 d_2 \ldots d_j (0)_{j(r-1)} + d_1 d_2 \ldots d_j (0)_{j(r-2)} + \ldots + d_1 d_2 \ldots d_j (0)_{j(r-r)}.$$

Table 2: **Prime factorization of the first nine terms of the sequence A261544**

Number of Digits (k)	Number of Repetitions (r)	Terms of the Sequence A261544	Prime Factorization
3	1	1	1
3	2	1001	$7 \cdot 11 \cdot 13$
3	3	1001001	$3 \cdot 333667$
3	4	1001001001	$7 \cdot 11 \cdot 13 \cdot 101 \cdot 9091$
3	5	1001001001001	$31 \cdot 41 \cdot 271 \cdot 2906161$
3	6	1001001001001001	$3 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 52579 \cdot 333667$
3	7	1001001001001001001	$43 \cdot 239 \cdot 1933 \cdot 4649 \cdot 10838689$
3	8	1001001001001001001001	$7 \cdot 11 \cdot 13 \cdot 73 \cdot 101 \cdot 137 \cdot 9901 \cdot 99990001$
3	9	1001001001001001001001001	$33 \cdot 757 \cdot 333667 \cdot 440334654777631$

Note that since $g \mid g$ and $g \mid d_1d_2\ldots d_j(0)_{jk}$, for $k = 1, 2, \ldots r-1$, by Corollary 2.11 we have

$$g \mid d_1d_2\ldots d_j(0)_{j(r-1)} + d_1d_2\ldots d_j(0)_{j(r-2)} + \ldots + d_1d_2\ldots d_j(0)_{j(r-r)}.$$

So, $g \mid n$.

Factoring out the common factor g in all of the summands we have

$$\begin{aligned} n &= g \times (1(0)_{j(r-1)} + 1(0)_{j(r-2)} + \ldots + 1(0)_{j(r-r)}) \\ &= g \times \left(1(0)_{j-1}\right)_{r-1}1. \end{aligned}$$

By the Fundamental Theorem of Arithmetic (Lemma 2.17), $\left(1(0)_{j-1}\right)_{r-1}1$ is either a prime or a product of primes. If $\left(1(0)_{j-1}\right)_{r-1}1$ is prime, then the finite sequence of divisors to be divided to n to become g is $\left(1(0)_{j-1}\right)_{r-1}1$ itself. If $\left(1(0)_{j-1}\right)_{r-1}1$ is non-prime then the finite sequence of divisors to be divided to n to become g is the finite sequence whose terms are the prime divisors of $\left(1(0)_{j-1}\right)_{r-1}1$. □

Theorem 3.5 proves the validity of a Grade 7 teacher's clever way in verifying if his students correctly performed a sequence of division.

Example 3.6. A Division Involving Large Number Relay. *Sir Delta is grade 7 mathematics teacher in the Philippines. To test the proficiency of his students on performing division involving large numbers, he group his students in which each group has 10 members. He then instruct the first student which we name S1 to write down in a 1/4 sheet of paper any* $4-$*digit positive integer (say 2019) and replicate it* $8-$ *times to get a* $32-$*digit number (*20 192 019 201 920 192 019 201 920 192 019*). Then he asked S1 to give the paper containing the* $32-$*digit number to S2. S2 then was asked to divide the* $32-$*digit number by* 17 *and write down the answer*

(1 187 765 835 407 070 118 776 583 540 707)

in another 1/4 sheet of paper. After S2 was done writing the answer in a 1/4 sheet of paper, Sir Delta asked S2 to give the paper to S3.

Denote by A_n the answer of student n. Suppose that the process continues with the following given (See Table 3. Generated using Wolfram Alpha):

- *S3 performs* $A_2 \div 73$
- *S4 performs* $A_3 \div 137$
- *S5 performs* $A_4 \div 353$
- *S6 performs* $A_5 \div 449$

- *S7 performs* $A_6 \div 641$
- *S8 performs* $A_7 \div 1\ 409$
- *S9 performs* $A_8 \div 69\ 857$
- *S10 performs* $A_9 \div 5\ 882\ 353$.

Sir Delta then asked S10 to give his/her answer to him. Sir Delta then wants to determine if all the students performed their assigned division problem correctly or not. Prove that in order to determine if all the students performed their assinged division problem correctly or not, it is enough for Sir Delta to ask S1: "Is this your $4-$*digit number?"*

Example 3.7. *Given below are the correct answers for the assigned sequence of division in Example 3.6. (Generated using Wolfram Alpha)*

$$20\ 192\ 019\ 201\ 920\ 192\ 019\ 201\ 920\ 192\ 019 \div 17 = 1\ 187\ 765\ 835\ 407\ 070\ 118\ 776\ 583\ 540\ 707$$

$$1\ 187\ 765\ 835\ 407\ 070\ 118\ 776\ 583\ 540\ 707 \div 73 = 16\ 270\ 764\ 868\ 590\ 001\ 627\ 076\ 486\ 859$$

$$16\ 270\ 764\ 868\ 590\ 001\ 627\ 076\ 486\ 859 \div 137 = 118\ 764\ 707\ 070\ 000\ 011\ 876\ 470\ 707$$

$$118\ 764\ 707\ 070\ 000\ 011\ 876\ 470\ 707 \div 353 = 336\ 443\ 929\ 376\ 770\ 571\ 888\ 019$$

$$336\ 443\ 929\ 376\ 770\ 571\ 888\ 019 \div 449 = 749\ 318\ 328\ 233\ 342\ 030\ 931$$

$$749\ 318\ 328\ 233\ 342\ 030\ 931 \div 641 = 1\ 168\ 983\ 351\ 378\ 068\ 691$$

$$1\ 168\ 983\ 351\ 378\ 068\ 691 \div 1\ 409 = 829\ 654\ 614\ 178\ 899$$

$$829\ 654\ 614\ 178\ 899 \div 69\ 857 = 11\ 876\ 470\ 707$$

$$11\ 876\ 470\ 707 \div 5\ 882\ 353 = 2\ 019.$$

Table 3: **Some integers of the form $\left(1(0)_{j-1}\right)_{r-1}1$ with their corresponding prime divisors**

(j)	(r)	$\left(1(0)_{j-1}\right)_{r-1}1$	Prime Divisors
1	1	1	1
2	10	1010101010101010101	(41),(101),(271),(3541),(9091),(27961)
3	9	1001001001001001001001001	(3),(3),(757),(333667),(440334654777631)
4	8	10001000100010001000100010001	(17),(73),(137),(353),(449),(641),(1409),(69857),(5882353)
5	7	1000010000100001000010000100001	(71),(239),(4649),(123551),(102598800232111471)
6	6	1000001000001000001000001000001	(3),(19),(101),(9901),(52579),(333667),(999999000001)
7	5	10000001000000100000010000001	(41),(71),(271),(123551),(102598800232111471)
8	4	1000000010000000100000001	(17),(353),(449),(641),(1409),(69857),(5882353)
9	3	1000000001000000001	(3),(757),(440334654777631)
10	2	10000000001	(101),(3541),(27961)

3.4 The (j,r) Co-divisor Number and the (j,r) Co-divisor Sequences

We learned from the previous subsection that given a repetitive number $n = (d_1 d_2 \dots d_j)_r$ that is generated by $g = d_1 d_2 \dots d_j$ of length j, we have

$$n = g \times \left(1(0)_{j-1}\right)_{r-1} 1. \tag{5}$$

The number $\left(1(0)_{j-1}\right)_{r-1} 1$ due to its importance will be named and defined formally in the next definition.

Definition 3.8. *Let $j, r \in \mathbb{Z}^+$. The number $\left(1(0)_{j-1}\right)_{r-1} 1$ in Equation 5 is called the (j,r)* **co-divisor of** *g* **relative to** *n.*

Example 3.9. *Recall that in the Repetitious Number Puzzle we have*

$$394\,394 = 394 \times 1\,001.$$

Hence, the $(3,2)$ co-divisor of 394 *relative to* 394 394 *is* 1 001. *In general, given any positive integer g of length* 3 *when duplicated has the (j,r) co-divisor of* 1 001.

Remark 3.10. *To avoid redundancy, we drop the words "relative to n" in determining the (j,r) co-divisor of g. This is because the (j,r) co-divisor of an integer g is completely determined by the length of g which is j and the number of replications performed in g which is r.*

Example 3.11. *In Example 3.2, the $(9,2)$ co-divisor of* 451 220 125 *is* 1 000 000 001. *In general, any positive integer g of length* 9 *when duplicated has the (j,r) codivisor of* 1 000 000 001.

Example 3.12. *Let g be a* 3*-digit positive integer. The $(3,4)$ codivisor of g is* 1 001 001 001. *(See Example 3.4)*

Example 3.13. *Let g be a* 4*-digit positive integer. The $(4,8)$ codivisor of g is the number* 10 001 000 100 010 001 000 100 010 001. *(See Example 3.6)*

The concept of (j,r) co-divisor allows us to view the sequence A000533 and the sequence A261544 in the OEIS as a particular member of a family of sequence which we call (j,r) **co-divisor sequences**.

In particular, if we let $s(j,r) = \left(1(0)_{j-1}\right)_{r-1} 1$ we have

$$s(j,2) = a(j), j = 1,2,3,\dots$$

where $a(j)$ is the j^{th} term of the sequence A000533. We also have

$$s(3,r) = b(r-1), r = 1,2,3,\dots.$$

where $b(r-1)$ is the $(r-1)^{st}$ term of the sequence A261544.

We end this subsection and section by saying that further studies on the (j,r) co-divisor number and the (j,r) co-divisor sequences and their applications are recommended.

4 Conclusion

In this article, we discussed the *Repetitious Number Puzzle* and its solution. We established that the *Repetitious Number Puzzle* is equivalent to the problem:

Given a positive integer generator g of length j that is to be replicated $r-$times resulting to the integer n of length jr, by what prime numbers must n be divided such that upon dividing n by all of the prime numbers gives back g?

where the length of g is 3 and the number of replication r is 2.

We then provide a generalization to the puzzle by first taking $j \geq 3$. We showed that the solution to the puzzle when $j \geq 3$ is given by the prime divisors of $a(j)$ where $a(j)$ is the j^{th} term of the sequence A000533. Then fixing $j = 3$, we consider $r \geq 2$. In this case, we showed that the solution to the puzzle when $j = 3$ and $r \geq 2$ is given by the prime divisors of $b(r-1)$ where $b(r-1)$ is the $(r-1)^{st}$ term of the sequence A261544.

For the general case where $j \geq 3$ and $r \geq 2$, we showed that the solution to the puzzle is given by the prime divisors of the (j,r) co-divisor number $\left(1(0)_{j-1}\right)_{r-1}1$. The concept of (j,r) co-divisor number allowed the possibility to view the sequence A000533 and the sequence A261544 in the OEIS as a particular member of a family of sequence which we call (j,r) co-divisor sequences. Further studies on on the (j,r) co-divisor number and the (j,r) co-divisor sequences and their applications are then recommended.

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