Combinatorial Proof of the Minimal Excludant Theorem

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Abstract

The maximal excludant of a partition λ, $\operatorname{mex}(\lambda)$, is defined to be the least gap of λ. For each positive integer n, the function $\sigma \operatorname{mex}(n)$ is defined to be the sum of the least gaps in all partitions of n. Recently, Andrews and Newman obtained a new combinatorial interpretations for $\sigma \operatorname{mex}(n)$. They used generating functions to show that $\sigma \operatorname{mex}(n)$ equals the number of partitions of n into distinct parts using two colors. In this paper, we provide a purely combinatorial proof of this result and new properties of the function $\sigma \operatorname{mex}(n)$.

Keywords: Minimal excludant, MEX, least gap in partition, partitions, 2-color partitions

MSC 2010: 11A63, 11P81, 05A19

1 Introduction

In 2015, Fraenkel and Peled have defined the minimal excludant or mex-function on a set S of positive integers as the least positive integer not in S and applied it to the combinatorial game theory [8].

Recently, Andrews and Newman [4] considered the mex-function applied to integer partitions. They defined the maximal excludant of a partition λ, $\operatorname{mex}(\lambda)$, to be the smallest positive integer that is not a part of λ. In addition, for each positive integer n, they defined

$$\sigma\,\mathrm{mex}(n) := \sum_{\lambda\in\mathcal{P}(n)} \mathrm{mex}(\lambda),$$

where $\mathcal{P}(n)$ is the set of all partitions of n. Elsewhere in the literature, the minimal excludant of a partition λ is referred to as the least gap or smallest gap of λ. In [5], $\mathrm{mex}(\lambda)$ is denoted by $g_1(\lambda)$ and $\sigma\,\mathrm{mex}(n)$ is denoted by $S_1(n)$.

Let $\mathcal{D}_2(n)$ be the set of partitions of n into distinct parts using two colors and let $D_2(n) = |\mathcal{D}_2(n)|$. For ease of notation, we denote the colors of the parts of partitions in $\mathcal{D}_2(n)$ by 0 and 1. In [4], the authors give two proofs of the following Theorem.

Theorem 1.1. *Given an integer $n \geqslant 0$, we have*

$$\sigma\,\mathrm{mex}(n) = D_2(n).$$

In section 2 we provide a bijective proof of Theorem 1.1. In the proof, we make use of the fact that

$$\sigma\,\mathrm{mex}(n) = \sum_{k\geqslant 0} p(n - k(k+1)/2), \tag{1}$$

where, as usual, $p(n)$ denotes the number of partitions of n. A combinatorial proof of (1) is given in [5, Theorem 1.1] . The same argument is also described in the second proof of [4, Theorem 1.1]. We note the result proven in [5] is a the generalization of (1) to the sum of r-gaps in all partitions of n. The r-gap of a partition λ is the least positive integer that does not appear r times as a part of λ.

In [2], Andrews and Merca considered a restricted mex-function and defined $M_k(n)$ to be the number of partitions of n in which k is the least positive integer that is not a part and there are more parts $> k$ than there are parts $< k$.

When $k = 1$, $M_1(0) = 0$ and, if $n > 0$, $M_1(n)$ is the number of partitions of n that do not contain 1 as a part. Thus, if $n > 0$, we have $M_1(n) = p(n)-p(n-1)$. Then, from (1), we obtain

$$\sigma\,\mathrm{mex}(n) - \sigma\,\mathrm{mex}(n-1) - \delta(n) = \sum_{j=0}^{\infty} M_1\big(n - j(j+1)/2\big), \tag{2}$$

where δ is the characteristic function of the set of triangular numbers, i.e.,

$$\delta(n) = \begin{cases} 1, & \text{if } n = m(m+1)/2,\ m \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}$$

In section 3, we prove the following generalization of (2).

Theorem 1.2. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^j \sigma\,\mathrm{mex}\big(n-j(3j-1)/2\big)-\delta(n)\right)$$
$$=\sum_{j=0}^{\infty} M_k\big(n-j(j+1)/2\big).$$

As a corollary of this theorem we obtain the following infinite family of linear inequalities involving $\sigma\,\mathrm{mex}$.

Corollary 1.3. *Let k be a positive integer. Given an integer $n \geqslant 0$, we have*

$$(-1)^{k-1}\left(\sum_{j=-(k-1)}^{k}(-1)^j \sigma\,\mathrm{mex}\big(n-j(3j-1)/2\big)-\delta(n)\right)\geqslant 0,$$

with strict inequality if $n \geqslant k(3k+1)/2$.

In section 3 we also give a combinatorial interpretation for

$$\sum_{j=0}^{\infty} M_k\big(n-j(j+1)/2\big)$$

in terms of the number of partitions into distinct parts using three colors and satisfying certain conditions.

In sections 4 and 5, we introduce connections to certain subsets of overpartitions and partitions with distinct parts, respectively.

2 Combinatorial Proof of Theorem 1.1

To prove the theorem, we adapt Sylvester's bijective proof of Jacobi's triple product identity [10] which was later rediscovered by Wright [12]. For the interested reader, it is probably easier to follow Wright's short article for the description of the bijection.

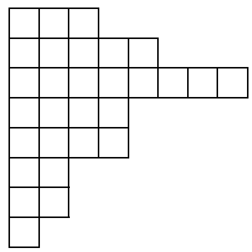
Given a partition λ in $\mathcal{D}_2(n)$, let $\lambda^{(j)}$, $j=0,1$, be the (uncolored) partition whose parts are the parts of color j in λ. Then, $\lambda^{(1)}$ and $\lambda^{(2)}$ are partitions into distinct parts.

Example 2.1. If $\lambda = 4_1+3_0+3_1+2_0+1_0 \in \mathcal{D}_2(13)$, then $\lambda^{(0)} = 3+2+1$ and $\lambda^{(1)} = 4+3$.

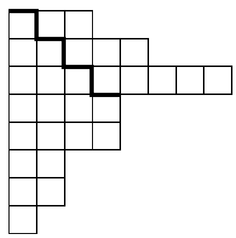
Denote by $\eta(k)$ the staircase partition $\eta(k) = k+(k-1)+(k-2)+\cdots+3+2+1$. If $k=0$ we define $\eta(k)=\emptyset$. For any partition λ we denote by $\ell(\lambda)$ the number of parts in λ.

Definition 1. Given diagram of left justified rows of boxes (not necessarily the Ferrers diagram of a partition), the *staircase profile* of the diagram is a zig-zag line starting in the upper left corner of the diagram with a right step and continuing in alternating down and right steps until the end of a row of the diagram is reached.

Example 2.2. The staircase profile of the diagram



is



Given a Ferrers diagram (with boxes of unit length) of a partition λ into distinct parts, the *shifted Ferrers diagram* of λ is the diagram in which row i is shifted $i-1$ units to the right.

We create a map

$$\varphi : \bigcup_{k\geqslant 0} \mathcal{P}(n-k(k+1)/2) \to \mathcal{D}_2(n)$$

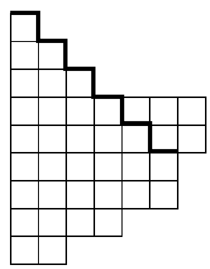
as follows. Start with $\lambda \in \mathcal{P}(n-k(k+1)/2)$ for some $k \geqslant 0$. Append a diagram with rows of lengths $1, 2, \ldots k$ (i.e., the Ferrers diagram of $\eta(k)$ rotated by $180°$) the top of Ferrers diagram of λ. We obtain a diagram with n boxes. Draw the staircase profile of the new diagram. Let α be the partition whose parts are the length of the columns to the left of the staircase profile and let β be the partition whose parts are the length of the rows to the right of the staircase profile. Then α and β are partitions with distinct parts. Moreover, $k \leqslant \ell(\beta) - \ell(a) \leqslant k+1$. Color the parts of α with color $k \pmod 2$ and the parts of β with color $k+1$ (mod 2). Then $\varphi(\lambda)$ is defined as the 2-color partition of n whose parts are the colored parts of α and β.

Conversely, start with $\mu \in \mathcal{D}_2(n)$. Let $\ell_j(\mu)$, $j = 0, 1$, be the number of parts of color j in μ.

(i) If $\ell_0(\mu) \geqslant \ell_1(\mu)$, let $r = \ell_0(\mu) - \ell_1(\mu)$. Let $k = r + \dfrac{(-1)^r - 1}{2}$. Remove the top k rows (i.e., the rotated Ferrers diagram of $\eta(k)$) from the conjugate of the shifted diagram of $\mu^{(0)}$ and join the remaining diagram with the shifted digram of $\mu^{(1)}$ so they align at the top. The obtained partition $\varphi^{-1}(\mu)$ belongs to $\mathcal{P}(n - k(k+1)/2)$.

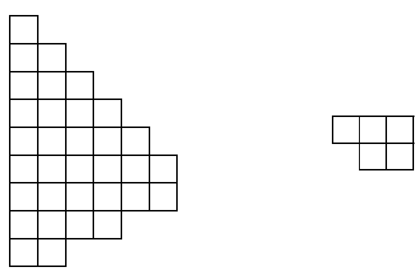
(ii) If $\ell_1(\mu) > \ell_0(\mu)$, let $r = \ell_1(\mu) - \ell_0(\mu)$. Let $k = r - \dfrac{(-1)^r + 1}{2}$. Remove the top k rows (i.e., the rotated Ferrers diagram of $\eta(k)$) from the conjugate of the shifted diagram of $\mu^{(1)}$ and join the remaining diagram with the shifted digram of $\mu^{(0)}$ so they align at the top. The obtained partition $\varphi^{-1}(\mu)$ belongs to $\mathcal{P}(n - k(k+1)/2)$.

Example 2.3. Let $n = 38, k = 3$, and let $\lambda = 7+7+6+6+4+2$ be a partition of $n - k(k+1)/2 = 32$. We add the rotated Ferrers diagram of $\eta(3)$ to the top of the Ferrers diagram of λ and draw the staircase profile.



Then $\alpha = 9 + 8 + 6 + 5 + 3 + 2$ and $\beta = 3 + 2$. Since k is odd, we have $\varphi(\lambda) = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0$.

Conversely, suppose $\mu = 9_1 + 8_1 + 6_1 + 5_1 + 3_1 + 3_0 + 2_1 + 2_0 \in \mathcal{D}(38)$. Then $\ell_0(\mu) = 2$ and $\ell_1(\mu) = 6$. We have $r = \ell_1(\mu) - \ell_0(\mu) = 4$ and $k = 3$. The diagrams of the conjugate of the shifted diagram of $\mu^{(1)}$ and the shifted diagram of $\mu^{(0)}$ are shown below.



Next, we remove the first 3 rows from the conjugate of the shifted diagram of $\mu^{(1)}$ and join the remaining diagram and the shifted digram of $\mu^{(0)}$ so they align at the top. We obtain $\varphi^{-1}(\mu) = 7 + 7 + 6 + 6 + 4 + 2 \in \mathcal{P}(32)$.

3 Proofs of Theorem 1.2

Analytic proof of Theorem 1.2. In [2], the authors considered Euler's pentagonal number theorem and proved the following truncated form:

$$\frac{(-1)^{k-1}}{(q;q)_\infty} \sum_{n=-(k-1)}^{k} (-1)^j q^{n(3n-1)/2} = (-1)^{k-1} + \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}, \quad (3)$$

where

$$(a;q)_n = \begin{cases} 1, & \text{if } n=0, \\ \prod_{k=0}^{n-1} (1-aq^k), & \text{otherwise,} \end{cases}$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

Multiplying both sides of (3) by

$$\frac{(q^2,q^2)_\infty}{(q,q^2)_\infty} = \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

we obtain

$$(-1)^{k-1} \left(\left(\sum_{n=0}^{\infty} \sigma \operatorname{mex}(n) q^n \right) \left(\sum_{n=-(k-1)}^{k} (-1)^j q^{n(3n-1)/2} \right) - \sum_{n=0}^{\infty} q^{n(n+1)/2} \right)$$
$$= \left(\sum_{n=0}^{\infty} q^{n(n+1)/2} \right) \left(\sum_{n=0}^{\infty} M_k(n) q^n \right),$$

where we have invoked the generating function for $\sigma \operatorname{mex}(n)$ [5, 4],

$$\sum_{n=0}^{\infty} \sigma \operatorname{mex}(n) q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty (q;q^2)_\infty}$$

and the generating function for $M_k(n)$ [2],

$$\sum_{n=0}^{\infty} M_k(n) q^n = \sum_{n=k}^{\infty} \frac{q^{\binom{k}{2}+(k+1)n}}{(q;q)_n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$$

The proof follows easily considering Cauchy's multiplication of two power series.

□

Combinatorial proof of Theorem 1.2. The statement of Theorem 1.2 is equivalent to identity (2) together with

$$\sigma\,\mathrm{mex}\left(n-\frac{k(3k+1)}{2}\right)-\sigma\,\mathrm{mex}\left(n-\frac{k(3k+5)}{2}-1\right)$$
$$=\sum_{j=0}^{\infty}\Big(M_k\big(n-j(j+1)/2\big)+M_{k+1}\big(n-j(j+1)/2\big)\Big). \tag{4}$$

Using (1), identity (4) becomes

$$\sum_{j=0}^{\infty}\left(p\left(n-\frac{j(j+1)}{2}-\frac{k(3k+1)}{2}\right)-p\left(n-\frac{j(j+1)}{2}-\frac{k(3k+5)}{2}-1\right)\right)$$
$$=\sum_{j=0}^{\infty}\Big(M_k\big(n-j(j+1)/2\big)+M_{k+1}\big(n-j(j+1)/2\big)\Big). \tag{5}$$

Identity (5) was proved combinatorially in [13]. Together with the combinatorial proof of (1), this gives a combinatorial proof of Theorem 1.2. □

Next, we give a combinatorial interpretation for $\sum_{j=0}^{\infty} M_k\big(n-j(j+1)/2\big)$. First, we introduce some notation. Given an integer r, let sign(r) denote the sign of r, i.e.

$$\mathrm{sign(r)}=\begin{cases}1 & \text{if } r\geqslant 0\\ -1 & \text{if } r<0.\end{cases}$$

For integers k,n such that $k\geqslant 1$ and $n\geqslant 0$, we denote by $D_3^{(k)}(n)$ the number of partitions μ of n into distinct parts using three colors and satisfying the following conditions:

(i) μ has exactly k parts of color 2 and, if $k>1$, twice the smallest part of color 2 is greater than largest part of color 2.

(ii) Let $r=\ell_0(\mu)-\ell_1(\mu)$ be the signed difference in the number of parts colored 0 and the number of parts colored 1 in μ. Let $j=|r|-\frac{1}{2}+\mathrm{sign(r)}\frac{(-1)^{\mathrm{r}}}{2}$. The largest part of color $j\,(\mathrm{mod}\ 2)$ must equal j more that the smallest part of color 2.

Then, we have the following proposition.

Proposition 3.1. *For integers k,n such that $k\geqslant 1$ and $n\geqslant 0$, we have*

$$\sum_{j=0}^{\infty} M_k\big(n-j(j+1)/2\big)=D_3^{(k)}(n). \tag{6}$$

Proof. Take a partition counted by $M_k\big(n-j(j+1)/2\big)$ and consider its Ferrers diagram. Remove the first k columns and color the length of each of these columns with color 2. To the remaining Ferrers diagram, add the rotated Ferrers diagram of a staircase $\eta(j)$ of height j and perform the transformation in the combinatorial proof of Theorem 1.1. It is now straight forward that this transformation is a bijection between the sets of partitions counted by the two sides of (6). ☐

Combining Theorems 1.1 and 1.2, and Proposition 3.1 we obtain the following corollary which, by the discussion above, has both analytic and combinatorial proofs.

Corollary 3.2. *For integers k, n such that $k \geqslant 1$ and $n \geqslant 0$, we have*

$$(-1)^{\max(0,k-1)}\left(\sum_{j=-\max(0,k-1)}^{k}(-1)^j\sigma\operatorname{mex}\big(n-j(3j-1)/2\big)-\delta(n)\right)=D_3^{(k)}(n).$$

Note that, if $k=0$, the statement of the corollary reduces to Theorem 1.1.

4 Connections with overpartitions

Overpartitions are ordinary partitions with the added condition that the first appearance of any part may be overlined or not. There are eight overpartitions of 3:

$$3, \overline{3}, 2+1, \overline{2}+1, 2+\overline{1}, \overline{2}+\overline{1}, 1+1+1, \overline{1}+1+1.$$

In [3], the authors denoted by $\overline{M}_k(n)$ the number of overpartitions of n in which the first part larger than k appears at least $k+1$ times. For example, $\overline{M}_2(12)=16$, and the partitions in question are $4+4+4$, $\overline{4}+4+4$, $3+3+3+3$, $\overline{3}+3+3+3$, $3+3+3+2+1$, $3+3+3+\overline{2}+1$, $3+3+3+2+\overline{1}$, $3+3+3+\overline{2}+\overline{1}$, $\overline{3}+3+3+2+1$, $\overline{3}+3+3+\overline{2}+1$, $\overline{3}+3+3+2+\overline{1}$, $\overline{3}+3+3+\overline{2}+\overline{1}$, $3+3+3+1+1+1$, $3+3+3+\overline{1}+1+1$, $\overline{3}+3+3+1+1+1$, $\overline{3}+3+3+\overline{1}+1+1$.

We have the following identity.

Theorem 4.1. *For integers $k, n > 0$, we have*

$$\begin{aligned}&(-1)^k\left(\sigma\operatorname{mex}(n)+2\sum_{j=1}^{k}(-1)^j\sigma\operatorname{mex}(n-j^2)-\delta'(n)\right)\\&=\sum_{j=-\infty}^{\infty}(-1)^j\overline{M}_k\big(n-j(3j-1)\big),\end{aligned}$$

where

$$\delta'(n)=\begin{cases}(-1)^m, & \text{if } n=m(3m-1),\ m\in\mathbb{Z},\\ 0, & \text{otherwise.}\end{cases}$$

Proof. According to [3, Theorem 7], we have

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right) \tag{7}$$
$$= 1+2(-1)^k\frac{(-q;q)_k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(k+j+1)}(-q^{k+j+2};q)_\infty}{(1-q^{k+j+1})(q^{k+j+2};q)_\infty},$$

where

$$\sum_{n=0}^{\infty}\overline{M}_k(n)q^n = 2\frac{(-q;q)_k}{(q;q)_k}\sum_{j=0}^{\infty}\frac{q^{(k+1)(k+j+1)}(-q^{k+j+2};q)_\infty}{(1-q^{k+j+1})(q^{k+j+2};q)_\infty}.$$

Multiplying both sides of (7) by

$$(q^2,q^2)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)},$$

we obtain

$$(-1)^k\left(\left(\sum_{n=0}^{\infty}\sigma\,\mathrm{mex}(n)q^n\right)\left(1+2\sum_{j=1}^{k}(-1)^j q^{j^2}\right)-\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)}\right)$$
$$=\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)}\right)\left(\sum_{n=0}^{\infty}\overline{M}_k(n)q^n\right)$$

and the proof follows easily. □

Related to Theorem 4.1, we remark that there is a substantial amount of numerical evidence to conjecture the following inequality.

Conjecture 1. *For $k,n>0$,*

$$\sum_{j=-\infty}^{\infty}(-1)^j\overline{M}_k\big(n-j(3j-1)\big)\geqslant 0,$$

with strict inequality if $n\geqslant (k+1)^2$.

It would be very appealing to have a combinatorial interpretation for the sum in this conjecture.

5 Connections with partitions into distinct parts

Following the notation for the number of partitions of n into distinct parts of two colors, we denote by $D_1(n)$ the number of partitions of n into distinct parts. We prove the following identity.

Theorem 5.1. *For any integer $n \geqslant 0$, we have*

$$\sum_{j=0}^{\infty}(-1)^{j(j+1)/2}\sigma\,\mathrm{mex}\big(n-j(j+1)/2\big)=\sum_{j=0}^{\infty}D_1\left(\frac{n-j(j+1)/2}{2}\right), \tag{8}$$

where $D_1(x)=0$ if x is not a positive integer.

Proof. Considering the classical theta identity [1, p. 23, eq. (2.2.13)]

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}=\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}, \tag{9}$$

we can write

$$\begin{aligned}\left(\sum_{n=0}^{\infty}\sigma\,\mathrm{mex}(n)q^n\right)\left(\sum_{n=0}^{\infty}(-q)^{n(n+1)/2}\right)&=\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q;q^2)_\infty}\cdot\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\\&=(-q^2;q^2)_\infty\cdot\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\\&=\left(\sum_{n=0}^{\infty}D_1(n)q^{2n}\right)\left(\sum_{n=0}^{\infty}q^{n(n+1)/2}\right)\end{aligned}$$

and the proof follows by equating the coefficients of q^n in this identity. □

To obtain a combinatorial interpretation for the sum on the right hand side of (8), let $D_2^*(n)$ be the number of partitions with distinct parts using two colors such that: (i) parts of color 0 form a gap-free partition (staircase) and (ii) only even parts can have color 1. Then, we have the following identity of Watson type [6].

Proposition 5.2. *For $n \geqslant 0$,*

$$\sum_{j=0}^{\infty}D_1\left(\frac{n-j(j+1)/2}{2}\right)=D_2^*(n).$$

Proof. To see this, let λ be a partition counted by $D_1\left(\frac{n-j(j+1)/2}{2}\right)$. Double the size of each part of λ to obtain a partition μ of $n-j(j+1)/2$ whose parts are even and distinct. Color the parts of μ with color 0 and add parts $1,2,\ldots,j$ in color 1 to obtain a partition counted by $D_2^*(n)$. This transformation is clearly reversible. □

In [3], the authors denoted by $MP_k(n)$ the number of partitions of n in which the first part larger than $2k-1$ is odd and appears exactly k times. All other odd parts appear at most once. For example, $MP_2(19)=10$, and the partitions in question are $9+9+1$, $9+5+5$, $8+5+5+1$, $7+7+3+2$, $7+7+2+2+1$, $7+5+5+2$, $6+5+5+3$, $6+5+5+2+1$, $5+5+3+2+2+2$, $5+5+2+2+2+2+1$.

We remark the following truncated form of Theorem 5.1.

Theorem 5.3. *For integers $n, k > 0$,*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}\sigma\operatorname{mex}\big(n-j(j+1)/2\big)-D_2^*(n)\right)$$
$$=\sum_{j=0}^{n} MP_k(j)D_2^*(n-j).$$

Proof. According to [3, Theorem 9], we have

$$\begin{aligned}\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}&\sum_{j=0}^{2k-1}(-q)^{j(j+1)/2} \\ &=1+(-1)^{k-1}\frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty},\end{aligned} \tag{10}$$

where

$$\sum_{n=0}^{\infty} MP_k(n)q^n=\frac{(-q;q^2)_k}{(q^2;q^2)_{k-1}}\sum_{j=0}^{\infty}\frac{q^{k(2j+2k+1)}(-q^{2j+2k+3};q^2)_\infty}{(q^{2k+2j+2};q^2)_\infty}.$$

The proof follows easily by multiplying both sides of (10) by

$$\frac{(q^2;q^2)_\infty}{(q;q)_\infty(q;q^2)_\infty}\cdot\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}.$$

□

A further interesting corollary of Theorem 5.3 relates to $\sigma\operatorname{mex}(n)$.

Corollary 5.4. *For integers $n, k > 0$,*

$$(-1)^{k-1}\left(\sum_{j=0}^{2k-1}(-1)^{j(j+1)/2}\sigma\operatorname{mex}\big(n-j(j+1)/2\big)-D_2^*(n)\right)\geqslant 0,$$

with strict inequality if $n \geqslant k(2k+1)$.

On the other hand, the reciprocal of the infinite product in (9) is the generating function for $\operatorname{pod}(n)$, the number of partitions of n in which odd parts are not repeated, i.e.,

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\operatorname{pod}(n)q^n. \tag{11}$$

The properties of the partition function $\operatorname{pod}(n)$ were studied in [9] by Hirschhorn and Sellers. We easily deduce the following convolution identity.

Corollary 5.5. *For $n \geqslant 0$,*

$$\sigma\,\mathrm{mex}(n) = \sum_{j=0}^{n} \mathrm{pod}(j) D_2^*(n-j).$$

Finally, we remark that combinatorial interpretations of

$$\sum_{j=0}^{n} MP_k(j) D_2^*(n-j)$$

would be very interesting.

6 Concluding remarks

The present work began with the search for a combinatorial proof of Theorem 1.1. We were further able to prove several truncated series formulas involving the function $\sigma\,\mathrm{mex}$. In [5] we worked with the generalization of this function: the sum, $S_r(n)$, of r-gaps in all partitions of n. There, we proved combinatorially that

$$S_r(n) = \sum_{j \geqslant 0} p(n - rj(j+1)/2). \tag{12}$$

We can employ a transformation similar to that in the combinatorial proof of Theorem 1.1 to prove its generalization to sums of r-gaps.

Let $\widetilde{D}_3^{(r)}(n)$ denote the number of partitions λ of n into distinct parts using three colors such that:

(i) The set of parts of color 2 is either empty of equal to $\{t(r-1) \mid 1 \leqslant t \leqslant j\}$ for some $j \geqslant 1$.

(ii) If $r > 0$, $\ell_{j \,(\mathrm{mod}\ 2)}(\lambda) - \ell_{j+1\,(\mathrm{mod}\ 2)}(\lambda) \in \{j, j+1\}$, where $j = 0$ if $\lambda^{(2)} = \emptyset$.

Theorem 6.1. *Let n, r be integers with $r > 0$ and $n \geqslant 0$. Then*

$$S_r(n) = \widetilde{D}_3^{(r)}(n).$$

Sketch of proof. Let μ be a partition of $n - rj(j+1)/2$ for some $j \geqslant 0$. To its Ferrers diagram add the rotated Ferrers diagram of the staircase $\eta(j)$ of length j. Perform the transformation in proof of Theorem 1.1 to obtain a partition ν in $\mathcal{D}_2(n-(r-1)j(j+1)/2)$ such that $\ell_{j\,(\mathrm{mod}\ 2)}(\nu) - \ell_{j+1\,(\mathrm{mod}\ 2)}(\nu) \in \{j, j+1\}$. To ν, add parts $(r-1), 2(r-1), \ldots, j(r-1)$ in color 2. Note that, if $j = 0$ or $r = 1$, there are no parts of color 2. Obtain a partition λ counted by $\widetilde{D}_3^{(r)}(n)$. Note that the conjugate of $\lambda^{(2)}$ is the fat staircase partition $(1^{r-1}, 2^{r-1}, \ldots, j^{r-1})$. This transformation is clearly reversible. □

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