BOUNDS ON (t, r) BROADCAST DOMINATION OF n-DIMENSIONAL GRIDS

TOM SHLOMI

ABSTRACT. In this paper, we look at a variant of graph domination known as (t, r) broadcast domination, first defined in [1], and we describe some upper and lower bounds on the density of a (t, r) dominating pattern of an infinite grid, as well as methods of computing them. When $r \geq 2$, we describe a family of counterexamples to a generalization of Vizing's Conjecture to (t, r) broadcast domination.

CONTENTS

1. INTRODUCTION

Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set* D of G is any subset of V for which every vertex in G is either contained in D or is adjacent to a vertex in D. The *domination number* $\gamma(G)$ is defined to be the size of the smallest possible dominating set of G.

There have been over 2000 papers written on domination theory and 80 different domination related parameters defined on graphs in the past 50 years, and we refer the interested reader to the survey text written by Haynes, Hedetniemi, and Slater for a comprehensive introduction to other domination parameters [5]. In this paper, we focus primarily on (t, r) broadcast domination, which is a generalization of domination theory and distance domination theory defined by Blessing, Insko, Johnson, and Mauretour in 2015 [1, 4]. Following their conventions, a (t, r) *broadcast domination set* D is defined as follows: Fix a desired transmission strength $t \in \mathbb{Z}$ and desired reception strength $r \in \mathbb{Z}$, with $t \geq r > 0$. Define the *transmission neighborhood* $N_t(v)$ of a vertex v in V to be the set of points a distance less than or equal to t from v. The define the reception

---

$r(v)$ at each vertex $v \in V$ to be

$$r(v) := \sum_{d \in D \cap N_t(v)} t - dis(d, v)$$

where $dis(v_1, v_2)$ is the distance between two vertices of a graph. A set $D \subseteq V$ is a (t, r) dominating set if every vertex has a reception of at least r, i.e. for all vertices v, $r(v) \geq r$. The (t, r) *broadcast domination number* $\gamma_{t,r}(G)$ is defined to be the smallest cardinality of any (t, r) broadcast dominating set for G. Section 1.1 has an example where $t = 3$ and $r = 2$.

In their paper, Blessing, Insko, Johnson, and Mauretour established exact numbers for the (t, r) broadcast domination number of small grids, as well as bounds for larger ones[1]. Drews, Harris, and Randolph expanded this work by computing (t, r) broadcast domination patterns of infinite grids and finding their densities [3].

Perhaps the most famous conjecture in domination theory is one posed by Vizing in 1968 which states that the domination number of a Cartesian product of graphs is at least as large as the product of their domination numbers.

Conjecture 1 (Vizing [6]). *Let G and H be finite graphs, and let $G \square H$ be their Cartesian product. Then*

$$\gamma(G \square H) \geq \gamma(G)\gamma(H).$$

While this conjecture remains unresolved, there have been many special cases of the conjecture proven in the past 50 years as detailed in the survey article by Brešar, Dorbec, Goddard, Hartnell, Henning, Klavžar, and Rall [2]. Since (t, r) broadcast domination generalizes domination and distance domination, it seems natural to ask whether an analog of Vizing's conjecture holds for (t, r) broadcast domination.

In Section 2 of this paper we show that this is not the case when $r > 1$ by providing a family of counterexamples. We conjecture that it holds for $r = 1$, which is a strengthened form of Vizing's Conjecture.

In Section 3, we work on the problem of calculating how many vertices there are a certain distance away from a given point in an n-dimensional hypercubic lattice. We give both combinatorial formulas and generating functions, and explore a connection with the Delannoy numbers. The (m, n)th Delannoy number counts the number of lattice paths to an integer point (m, n) with only the steps $(1, 0)$, $(0, 1)$, and $(1, 1)$. This connection was previously found by Zaitsev in 2017 [7]. In Theorem 9 we find a surprising bijection between points in $\mathbb{Z}^n$ that are within distance d of the origin and points in $\mathbb{Z}^d$ that are within distance n of the origin.

In Section 4, we derive a lower bound on the minimum density of a (t, r) broadcast dominating set of an infinite grid of any dimension. From that, we derive a lower bound on (t, r) broadcast domination number of a finite grid.

We conclude by describing an algorithm which generates upper bounds on the density (t, r) dominating set of a 2 or 3 dimensional infinite grid. The algorithm is similar to the one described in a recent paper by Drews, Harris, and Randolph [3]. However, our algorithm works in more dimensions and we show it is more computationally efficient.

Regular graph domination has numerous applications in computer science and other fields. While they are not explored in this paper, there are many potential applications

for (t, r) broadcast domination as well. For example, it could be useful for municipal planning. In a city, every house needs to be within a certain radius of a fire station. However, it is possible that having two fire stations just outside that radius is acceptable, since it increases the chance that a fire truck from one of them could make it on time. This scenario could be approximately modeled using the city as a graph and the fire stations being (t, r) broadcasts.

1.1. **Examples.** Figure 1 illustrates examples of distinct sets of $(3, 2)$ broadcasts in a 5×5 grid. Black vertices have a reception of at least 3 (since they are a distance 0 away from a broadcast), red vertices have a reception of at least 2 (since they have a distance 1 from a broadcast), pink vertices have a reception of at least 2 (since they have a distance 2 from at least two broadcasts), gray vertices have a reception of 1 (since they have a distance 2 from exactly 1 broadcast), and white vertices have a reception of 0 since they have a distance of at least 3 from all broadcasts. For a grid to be $(3, 2)$ dominated by its black vertices, there must be no gray or white vertices. Thus, Figure 1a is fully dominated while Figure 1b is not.
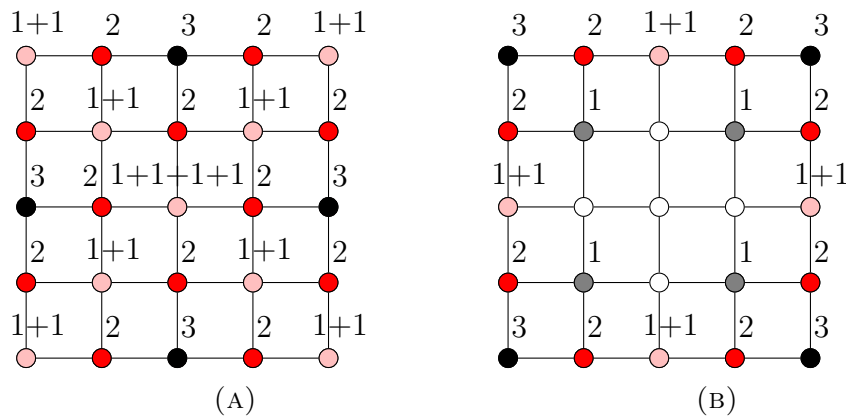

FIGURE 1. A (3,2) broadcast dominating set of $G_{5,5}$ and a (3,2) non-dominating broadcast set of $G_{5,5}$

2. THE (t, r) ANALOG OF VIZING'S CONJECTURE

Since (t, r) broadcast domination theory generalizes domination theory and distance domination theory, it is reasonable to ask whether Vizing's conjecture applies more generally to (t, r) broadcast domination.

Question 1. For all graphs G and H, let $\gamma_{t,r}(G)$ be the (t, r) broadcast domination number of G. Is it true that

$$\gamma_{t,r}(G \square H) \geq \gamma_{t,r}(G)\gamma_{t,r}(H)$$

for all $t, r \in \mathbb{Z}$ such that $t \geq r \geq 1$?

The main result of this section shows that this is not the case when $r > 1$. Before constructing a family of counterexamples to the (t, r) broadcast domination analog of

Vizing's conjecture, we first prove a lemma describing the (t, r) broadcast domination numbers of cyclic graphs.

Lemma 1. *Let C_n denote the cycle graph with n nodes. Then the (t, r) broadcast domination number of $C_{2(t-r+1)}$ is 2.*

Proof. Let $v = 2(t - r + 1)$. Label the vertices of C_v as $(0, 1, 2, \ldots, v - 1)$. Suppose without loss of generality that a broadcast of strength t is placed at vertex (0). Then vertex $(\frac{v}{2})$ has reception $r - 1$, because it is distance $t - r + 1$ from vertex (0), and so it is not yet dominated. Since vertex $(\frac{v}{2})$ is the only non-dominated vertex in C_v if a broadcast is placed at vertex (0), every vertex can be dominated if a broadcast is added at $(\frac{v}{2})$. Hence $\gamma_{t,r}(C_{2(t-r+1)}) = 2$. $\square$

The following result establishes a family of counterexamples to the inequality in Question 1.

Theorem 2. *Let C_v be a cycle graph with $v = 2(t - r + 1)$. Then the (t, r) broadcast domination number of $C_v \square C_v$ is $\gamma_{t,r}(C_v \square C_v) = 2$.*

Proof. Let $G = C_v \square C_v$. Place a broadcast at $(0, 0)$ and at $(\frac{v}{2}, \frac{v}{2})$. Since (0) and $(\frac{v}{2})$ have the maximum possible distance between them in C_v and since distance in G is the sum of the distance between the two pairs of coordinates, the vertices $(0, 0)$ and $(\frac{v}{2}, \frac{v}{2})$ must have the maximum possible distance in G. As a result, for any vertex w in G, the sum of the distance from w to $(0, 0)$ and from w to $(\frac{v}{2}, \frac{v}{2})$ is v. The reception at w is $t - d((0, 0), w) + t - d((\frac{v}{2}, \frac{v}{2}), w) = 2t - v = 2r - 2$. Since we assume $r \geq 2$, the reception at w is greater than r, so $\gamma_{t,r}(G) = 2$. $\square$

Combining the results from Lemma 1 and Theorem 2 shows that

$$\gamma_{t,r}(C_v \square C_v) = 2 < \gamma_{t,r}(C_v)\gamma_{t,r}(C_v) = 4$$

when $v = 2(t - r + 1)$, and hence the analog of Vizing's Conjecture for (t, r) broadcast domination posed in Question 1 is false when $r > 1$. We illustrate this counterexample below for the case $(t, r) = (3, 2)$.
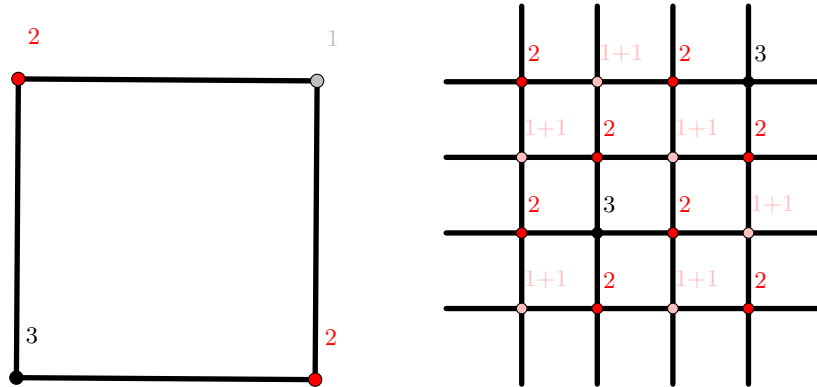

FIGURE 2. Under $(3, 2)$ broadcast domination, C_4 is not dominated by 1 broadcast, so $\gamma_{3,2}(C_4) \geq 2$, and $C_4 \square C_4$ is dominated by 2 broadcasts, so $\gamma_{3,2}(C_4 \square C_4) = 2$.

We have shown that the (t,r) broadcast domination analog of Vizing's conjecture does not hold when $r > 1$. However, we have not discovered any counterexamples that violate its bound by more that a factor of 2. In addition, none of our counterexamples apply for $(t,1)$ broadcast domination, which coincides with $(t-1)$-distance domination. As a result we state the following generalizations of Vizing's conjecture.

Conjecture 2. *For all graphs G and H, let $\gamma_{t,r}(G)$ be the (t,r) broadcast domination number of G. Then*

$$\gamma_{t,1}(G \square H) \geq \frac{1}{2} \gamma_{t,1}(G) \gamma_{t,1}(H)$$

Conjecture 3 (Generalizing Vizing's Conjecture)**.** *For all graphs G and H, let $\gamma_{t,1}(G)$ be the $(t,1)$ broadcast domination (or $(t-1)$-distance domination) number of G. Then*

$$\gamma_{t,1}(G \square H) \geq \gamma_{t,1}(G) \gamma_{t,1}(H).$$

Previous bounds on (t,r) domination of paths, combined with results later in this paper for grids, show that these are true for products of paths.

3. Balls in Lattices

Let $\mathbb{Z}^n$ denote an infinite n-dimensional $\mathbb{Z}$-lattice. Let $(x_1, \ldots, x_n)$ denote a point in $\mathbb{Z}^n$. Define $S_n(d)$ to be the set of vertices with graph distance d from the origin in $\mathbb{Z}^n$. Similarly, define $B_n(d)$ to be the set of vertices with graph distance less than or equal to d from the origin. This section focuses on counting the number of vertices in $S_n(d)$ and $B_n(d)$. We use these combinatorial formulas in deriving bounds on the density of dominating sets in Section 4 and combinatorial arguments interesting in their own right.

After deriving the formulas in Theorems 3 and 4 and the generating functions in Theorems 6 and 7, we searched Sloane's Online Encyclopedia of Integer Sequences (OEIS). We found $|S_n(d)|$ and $|B_n(d)|$ correspond to Sequences OEIS A265014 and OEIS A008288, both sequences have applications to cellular automata, and $|B_n(d)|$ correspond to the well-studied Delannoy numbers. This result was previously independently derived by Zaitsev in 2017. The (m,n)th Delannoy number counts the number of lattice paths from $(0,0)$ to (m,n) with the steps $(1,0)$, $(0,1)$, and $(1,1)$ [7].

Theorem 3. [Theorem 1 [7]] When, $d \geq 1$, the number of vertices in $S_n(d)$ is given by

$$|S_n(d)| = \sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i} \binom{d-1}{n-i-1}.$$

Proof. We partition the set $S_n(d)$ into disjoint sets, indexed by how many coordinates are nonzero. First, we count the vertices for which all of the coordinates are greater than 0. This the same as the problem of distributing d identical items between n containers such that each container has at least one item. There are $\binom{d-1}{n-1}$ such vertices. If some of the coordinates are negative instead of positive, the number remains the same. There are 2^n ways of assigning each coordinate to be negative or positive. So there are $2^n \binom{d-1}{n-1}$ vertices within distance d away from the origin that have no coordinates that are 0.
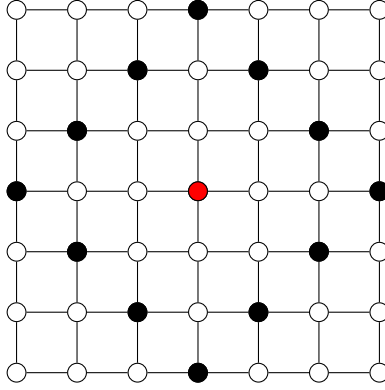
FIGURE 3. The red vertex is the origin and the black vertices are the elements of $S_2(3)$. The number of black vertices is $\sum_{i=0}^{1} \binom{2}{i} 2^{2-i} \binom{2}{1-i} = 12$.

n	$\lvert S_n(d) \rvert$
1	2
2	$4d$
3	$4d^2 + 2$
4	$\frac{8}{3}(d^3 + 2d)$
5	$\frac{2}{3}(2d^4 + 10d^2 + 3)$
6	$\frac{4}{15}(2d^5 + 20d^3 + 23d)$
7	$\frac{2}{45}(4d^6 + 70d^4 + 196d^2 + 45)$

Now assume i of the coordinates are 0. There is now d magnitude to divide between $n - i$ nonzero coordinates. This creates $\binom{d-1}{n-i-1}$ possibilities. Then, there are 2^{n-i} ways to make each of the nonzero coordinates positive or negative. Finally, there are $\binom{n}{i}$ ways to choose which of the coordinates should be 0. Multiplying these numbers, we find that there are $\binom{n}{i} 2^{n-i} \binom{d-1}{n-i-1}$ vertices in $S_n(d)$ that have exactly i coordinates that are 0. To get the total number of vertices in $S_n(d)$, we must sum over all the possible values of i. Since there must be some nonzero coordinate, $0 \le i \le n - 1$, we conclude that $\lvert S_n(d) \rvert = \sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i} \binom{d-1}{n-i-1}$. $\square$

In order to derive a generating function for $S_n(d)$ in Theorem 5, we first derive a generating function for $B_n(d)$. Let $B(x, y)$ be the multivariable generating function for $\lvert B_n(d) \rvert$, defined as

$$B(x, y) := \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \lvert B_i(j) \rvert x^i y^j.$$

Theorem 4. If $B(x, y)$ is the multivariable generating function for $\lvert B_n(d) \rvert$, then

$$B(x, y) = \frac{1}{1 - x - y - xy}.$$

Proof. To derive this, we find a recursive formula for $|B_n(d)|$. Partition $B_n(d)$ into three sets. The set of points a distance less than d away, the set of points a distance of exactly d away with an x_1 value of at least 0, and the set of points a distance of exactly d away from the origin with an x_1 value of less than 0. The first set is $B_n(d-1)$.

If we have a point a distance exactly d away from the origin in $\mathbb{Z}^n$ with coordinates $(a_1, \ldots, a_n)$, we know that

$$d = \sum_{i=1}^{n} |a_i| \Rightarrow |a_1| = d - \sum_{i=2}^{n} |a_i|.$$

Since all the points in the second set have an x_1 value of at least 0, this means that we can uniquely determine their x_1 values from their other coordinates. We then have that the second set is

$$\left\{ (a_1, \ldots, a_n) \,\middle|\, \sum_{i=2}^{n} |a_i| \leq d, a_1 = d - \sum_{i=2}^{n} |a_i| \right\}.$$

We also have that $B_{n-1}(d)$ is

$$\left\{ (a_2, \ldots, a_n) \in \mathbb{Z}^n \,\middle|\, \sum_{i=2}^{n} |a_i| \leq d \right\}.$$

Thus, there exists a bijection from $(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$ in the second set to $(a_2, a_3, \ldots, a_{n-1}, a_n)$ in $B_{n-1}(d)$.

Since all the points in the third set have an x_1 value of less than 0, this means that we can uniquely determine their x_1 values from their other coordinates. We then have that the third set is

$$\left\{ (a_1, \ldots, a_n) \in \mathbb{Z}^n \,\middle|\, \sum_{i=2}^{n} |a_i| < d, a_1 = -d + \sum_{i=2}^{n} |a_i| \right\}.$$

We also have that $B_{n-1}(d-1)$ is

$$\left\{ (a_2, \ldots, a_n) \,\middle|\, \sum_{i=2}^{n} |a_i| < d \right\}.$$

Thus, there exists a bijection from $(a_1, a_2, a_3, \ldots, a_{n-1})$ in the third set to $(a_2, a_3, \ldots, a_{n-1}, a_n)$ in $B_{n-1}(d-1)$.

We now have the recursion

$$|B_n(d)| = |B_{n-1}(d)| + |B_n(d-1)| + |B_{n-1}(d-1)|.$$

Since $|B_0(0)| = 1$ and because of the recursion above, we get

$$B(x, y) = xB(x, y) + yB(x, y) + xyB(x, y) + 1 \Rightarrow B(x, y) = \frac{1}{1 - x - y - xy}. \qquad \square$$

From Theorem 4, we can derive the generating function for $|S_n(d)|$.

Theorem 5. Let $S(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |S_i(j)| x^i y^j$. Then

$$S(x,y) = \frac{1-y}{1-x-y-xy}.$$

Proof. We know that $|S_n(d)| = |B_n(d)| - |B_n(d-1)|$. From that, we get that

$$S(x,y) = B(x,y) - yB(x,y) = \frac{1-y}{1-x-y-xy}. \quad \square$$

We may want to look at how $B_n(d)$ and $S_n(d)$ vary with n or d fixed. To help with this, we derive single variable generating functions, which can be derived in the same manner. Consider $B_d(x) = \sum_{i=0}^{\infty} |B_i(d)| x^i$ and $B_n(y) = \sum_{i=0}^{\infty} |B_n(i)| y^i$.

Theorem 6.

$$B_d(x) = \frac{(1+x)^d}{(1-x)^{d+1}}.$$

$$B_n(y) = \frac{(1+y)^n}{(1-y)^{n+1}}.$$

Proof. By the recursion for $B_n(d)$, we have

$$B_d(x) = xB_d(x) + B_{d-1}(x) + xB_{d-1}(x).$$

We know that $B_0(x) = \frac{1}{1-x}$, we have

$$B_d(x) = \frac{1+x}{1-x} B_{d-1}(x)$$

$$\Rightarrow B_d(x) = \frac{(1+x)^d}{(1-x)^{d+1}}.$$

Since $B(x,y)$ is symmetric over x and y, we must also have have

$$B_n(y) = \frac{(1+y)^n}{(1-y)^{n+1}}. \quad \square$$

We can do similar derivations for $S_d(x)$ and $S_n(y)$.

Theorem 7.

$$S_d(x) = \frac{2x(1+x)^{d-1}}{(1+x)^{d+1}} \text{ for } x > 0$$

$$S_n(y) = \frac{(1+y)^n}{(1-y)^n}.$$

Proof. If $d = 0$, $S_d(x) = \frac{1}{1-x}$. Otherwise

$$S_d(x) = B_d(x) - B_{d-1}(x) = \frac{(1+x)^d}{(1-x)^{d+1}} - \frac{(1+x)^{d-1}}{(1-x)^d} = \frac{2x(1+x)^{d-1}}{(1+x)^{d+1}}.$$

Similarly

$$S_n(y) = B_n(y) - xB_n(y) = (1-y)\frac{(1+y)^n}{(1-y)^{n+1}} = \frac{(1+y)^n}{(1-y)^n}. \quad \square$$

From the generating functions, we can conclude that $|B_n(d)|$ is equal to the (m,n)th Delannoy number, agreeing with the previously found result in [7, Theorem 2]. The Delannoy numbers describe the number of paths from the origin to the integer point (x,y), where each point (i,j) can reach $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$.

The fact that an n-dimensional ball of radius d has the same number of points as a d-dimensional ball of radius n is surprising, though an automatic consequence of the symmetry of the generating function $B(x,y)$, and suggests a simple bijection between the two. To identify such a bijection, we first introduce some notation.

Definition 8. Let a *tuple-sequence* be a list of signed tuples $[\pm(n_1, d_1), \pm(n_2, d_2) \ldots]$, where the n_is and d_is are positive integers. Let the *first sum* of a tuple-sequence be defined to be the sum of the first terms of the tuples, while the *second sum* is the sum of the second terms. Let $T_{n,d}$ denote the set of tuple-sequences with a first sum less than or equal to n and second sum less than or equal to d.

Every tuple-sequence describes a unique two-colored lattice path starting at $(0,0)$, with steps that increase both coordinates. Thus the set $T_{n,d}$ corresponds to the set of lattice paths that end at a point (m,c) such that $m \leq n$ and $c \leq d$.

Theorem 9. There is a bijection between the sets $B_n(d)$, $B_d(n)$, $T_{n,d}$, and $T_{d,n}$.

Proof. Let ϕ be a mapping between points in $\mathbb{Z}^n$ and tuple-sequences. Consider point p in $B_n(d)$ with coordinates $(x_1, \ldots, x_n)$ such that

$$\sum_{i=1}^{n} |x_i| \leq d.$$

The map ϕ sends p to the tuple-sequence t, such that each second term of a tuple of t corresponds to the magnitude of a nonzero value of p, each first term corresponds to the distance between that value and the previous nonzero value (or the beginning of the sequence, if there is no first value), and the sign of that tuple corresponds to the sign of that value, i.e.,

$$(\pm(n_1, d_1), \pm(n_2, d_2), \ldots)$$
$$= \phi(([n_1 - 1\ 0s], \pm d_1, [n_2 - 1\ 0s], \pm d_2, \ldots, d_k, 0, 0, 0, \ldots))$$

The first sum of t is the position of the last nonzero term of p. Since p has n terms, that must be less than or equal to n. The second sum of t is equal to the sum of the magnitudes of the values of p, which is the distance from the origin to p. Since p is within a distance d from the origin, the second sum of t must be less than or equal to d. Thus, t is in $T_{n,d}$. For similar reasons, for any tuple-sequence s in $T_{n,d}$, $\phi^{-1}(s)$ is in $B_n(d)$. Since ϕ is a one to one invertible function, it must be a bijection.

Let ψ be the bijection between $T_{n,d}$ and $T_{d,n}$ defined by reversing the order of all the signed tuples in them. Since the composition of bijections is a bijection, this we see that $\phi \circ \psi \circ \phi^{-1}$ defines a bijection between $B_n(d)$ and $B_d(n)$. $\qquad\square$

The following example demonstrates the bijections described above.

Example 10. In $B_4(3) \overset{\phi}{\leftrightarrow} T_{4,3} \overset{\psi}{\leftrightarrow} T_{3,4} \overset{\phi^{-1}}{\longleftrightarrow} B_3(4)$:

$$(2,0,-1,0) \leftrightarrow (+(1,2),-(2,1)) \leftrightarrow (+(2,1),-(1,2)) \leftrightarrow (0,1,-2)$$

$$(-1,0,1,-1) \leftrightarrow (-(1,1),+(2,1),-(1,1)) \leftrightarrow (-(1,1),+(1,2),-(1,1)) \leftrightarrow (-1,2,-1)$$

$$(0,0,0,0) \leftrightarrow () \leftrightarrow () \leftrightarrow (0,0,0)$$

In the first example, we consider the vertex with coordinates $(2,0,-1,0)$, which is inside $B_4(3)$. First, we convert it to a tuple-sequence. Since the first nonzero coordinate is 2 and is the first coordinate, we get the signed tuple $+(1,2)$. Since the second nonzero coordinate is -1, which is two coordinates later, we get the signed tuple $-(2,1)$. Then we apply ψ, which flips all the tuples. Finally, we convert them back into a vertex in $B_3(4)$. Since the first tuple is $+(2,1)$, we get the that the second coordinate is 1. Since the next tuple is $-(1,2)$, we know that the next coordinate is -2. All others are 0. Thus, we get to $(0,1,-2)$.

4. Lower Bound on Densities for Infinite Grids

In this section, we calculate lower bounds for the density of a (t,r) broadcast pattern which dominates an infinite or finite grid of any dimension. To begin, we note that since any reception provided to a vertex beyond r is always wasted, it is often useful to ignore it. The *unwasted reception* $r^*(d,v)$ provided to a vertex v from a broadcast d is defined to be $min(r(d,v),r)$. Define the coverage $c(d)$ of a broadcast d to be $\sum_{v \in N_t(d)} r^*(d,v)$. Given a transmission strength t, a desired reception r, and a graph G, we define $C_{t,r}(G)$ to be the maximum coverage provided by any vertex in G.

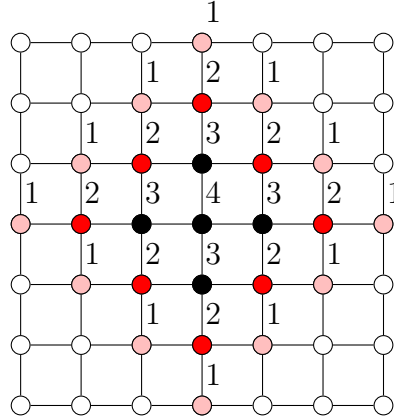FIGURE 4. A figure showing the $(4,3)$ coverage provided by the center vertex in $\mathbb{Z}^2$. The black vertices all receive at least 3 reception. Since $r = 3$, they receive exactly 3 unwasted reception, and they are dominated. Since there are 12 vertices receiving 1 reception, 8 receiving 2, and 4 receiving 3 unwasted reception, the coverage provided by the center is 40, so $C_{4,2}(\mathbb{Z}^2) = 40$.

Lemma 2.

$$C_{t,r}(\mathbb{Z}^n) = \sum_{d=t-r+1}^{t-1} (t-d)|S_n(d)| + r\sum_{d=1}^{t-r} |S_n(d)| + r.$$

Proof. Each broadcast provides r unwasted reception to itself, and it also provides r unwasted reception to each vertex a distance $t-r$ or less away. Thus it supplies a total unwasted reception to these vertices of

$$r + r\sum_{d=1}^{t-r} |S_n(d)|.$$

For each vertex a distance $t - r < d < t$ away, the broadcast provides $t - d$ reception, none of which is wasted. The total reception provided to these vertices is

$$\sum_{d=t-r+1}^{t-1} (t-d)|S_n(d)|.$$

Summing these, we get

$$C_{t,r}(G) = \sum_{d=t-r+1}^{t-1} (t-d)|S_n(d)| + r\sum_{d=1}^{t-r} |S_n(d)| + r$$

$$= \sum_{d=t-r+1}^{t-1} (t-d)\sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i} \binom{d-1}{n-i-1}$$

$$+ r\sum_{d=1}^{t-r}\sum_{i=0}^{n-1} \binom{n}{i} 2^{n-i} \binom{d-1}{n-i-1} + r$$

where the second equality comes from plugging in the cardinality $|S_n(d)|$ from Theorem 3. $\qquad\square$

n	$C_{t,r}$
1	$2tr - r^2$
2	$\frac{2r^3}{3} - 2r^2t + 2rt^2 + \frac{r}{3}$
3	$-\frac{r^4}{3} + \frac{4r^3t}{3} - 2r^2t^2 - \frac{2r^2}{3} + \frac{4rt^3}{3} + \frac{4rt}{3}$
4	$\frac{2r^5}{15} - \frac{2r^4t}{3} + \frac{4r^3t^2}{3} + \frac{2r^3}{3} - \frac{4r^2t^3}{3} - 2r^2t + \frac{2rt^4}{3} + 2rt^2 + \frac{r}{5}$

We are now ready to prove a lower bound on the (t, r) broadcast domination of a finite grid G.

Theorem 11. Let G be a finite n-dimensional grid graph. Then

$$\gamma_{t,r}(G) \geq \frac{r|V|}{C_{t,r}(\mathbb{Z}^n)}.$$

Proof. Since G is a subgraph of $\mathbb{Z}^n$, the maximum amount of coverage a vertex can provide is at most $C_{t,r}(\mathbb{Z}^n)$. The total reception needed to dominate the graph is equal to the product of r and the number of vertices in the graph. Since each vertex can provide at most $C_{t,r}(\mathbb{Z}^n)$ unwasted reception, we get

$$\gamma_{t,r}(G)C_{t,r}(\mathbb{Z}^n) \geq r|V|.$$

Dividing both sides by $C_{t,r}(\mathbb{Z}^n)$ gives a lower bound for $\gamma_{t,r}(G)$. □

5. Computer Implementation

To find more exact values for the density of (t, r) dominating patterns on the infinite grid, we built a program to test various values. This program uses tower notation, first defined in [3]. A tower set is denoted by $T(d, e)$, and is defined to be the set of vertices in $\mathbb{Z}^2$ with coordinates $(md+ne, n)$, with $m, n \in \mathbb{Z}$. By its definition $T(d, e)$ has density $\frac{1}{d}$. We have written a program which, for various values of (t, r), goes through various $T(d, e)$ and returns the one with the highest possible value of d. In the pseudocode below, $IsBroadcast(t, r, d, e)$ determines whether placing (t, r) broadcasts at $T(d, e)$ dominates the grid. MaxPotentialD(t, r) finds the highest possible value of d based on the formula from Section 4.

$$\text{MaxPotentialD}(t, r) = \left\lfloor \frac{1}{\gamma_{t,r}(G)} \right\rfloor \leq \frac{C_{t,r}(\mathbb{Z}^n)}{r|V|}.$$

Since the set of broadcasts is invariant under the transformations

$$(x, y) \to (x + md + n, y + n), m, n \in \mathbb{Z}$$

$$(x, y) \to (-x, y)$$

only the vertices from $(0, 0)$ to $(\lfloor d/2 \rfloor, 0)$ need to be checked. Each move up is equivalent to a shift by e in the range $(0, 0)$ to $(d-1, 0)$. Thus, the algorithm returns the reciprocal of the minimum density of a (t, r) broadcast dominating tower set.

Example 12. To better illustrate how the algorithm works we will walk through an example when $(t, r) = (4, 2)$. If $(t, r) = (4, 2)$ is plugged into the algorithm for a two dimensional grid, the following will occur:

For $(t, r) = (4, 2)$, the maximum number of vertices that could be dominated by a single broadcast is 19. The algorithm iterates over variables d and e, with d starting at 19 and going down to 1, and e starting at 1 and going up to $d - 1$. For $(t, r) = (4, 2)$, it starts at $(d, e) = (19, 0)$. The algorithm simulates an infinite grid with broadcasts places at points described by $T(d, e)$. This is the set of points (x, y) such that $x = ye \bmod d$. For example, $T(18, 5)$ has the points $(0, 0), (18, 0), (5, 1), (36, 0), (23, 1), (10, 2), \ldots$

Since $T(d, e)$ is symmetric around each broadcast, only the 18 vertices between the $(0, 0)$ and $(17, 0)$, inclusive, must be checked. We now add the receptions from only the broadcasts on the x-axis. The receptions now look like:

$$[4, 3, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 2, 3].$$

Since each row is shifted by 5 from the adjacent rows, to account for the other broadcasts we need to shift the reception pattern we got above by 5, decrease all nonzero terms by 1 since they are now further away, and add them to the receptions. So the reception received from the row above the x-axis is

$$[0, 0, 0, 1, 2, 3, 2, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0].$$

The table below shows the reception received from each row of broadcasts.

Since the sum of the receptions received from all rows is at least 2 at every vertex, $T(18, 5)$ dominates the grid under $(4, 2)$ broadcast domination. Since all tower sets

Algorithm 1: Optimal (t, r) Broadcast Dominating Pattern

---

    procedure MINDENSITY(t, r);

        $d = \text{MaxPotentialD}(t, r)$;

        while $d \geq 1$ **do**

         $e \leftarrow 1$;

         while $e < d$ **do**

           if *IsBroadcast(t, r, d, e)* **then**

             return d;

           end

           $e++$

         end

         $d--$

        end

    procedure ISBROADCAST(t, r, d, e);

        Array a = new Array[d];

        for $i = 0;\ i < t;\ i++$ **do**

         a[i]=a[-i]=t-i;

        end

        for $i; t - r + 1 \leq i \leq d/2$ **do**

         rcpt = a[i];

         for $j = 1;\ j < t;\ j++$ **do**

           if *a[(i+je) mod d] > j* **then**

             rcpt += a[(i+je) mod d]-j;

           end

           if *a[(i-je) mod d] > j* **then**

             rcpt += a[(i-je) mod d]-j;

           end

         end

         if $rcpt < r$ **then**

           return false;

         end

        end

        return true;

---

with lower densities have been shown to not dominate, this is the most efficient tower set for dominating the grid.

A similar algorithm is contained in [3]. Our algorithm works for 3-dimensional grids and runs about 415 times faster for $t, r \leq 10$. For instance, it takes our algorithm, when implemented in Cocalc, takes 1.39 seconds to calculate optimal broadcast densities for all (t, r) less than or equal to 10, while it takes the aforementioned algorithm 9 minutes and 36 seconds.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
2	0	0	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0
1	0	0	0	1	2	3	2	1	0	0	0	0	0	0	0	0	0	0
0	4	3	2	1	0	0	0	0	0	0	0	0	0	0	0	1	2	3
-2	0	0	0	0	0	0	0	1	2	1	0	0	0	0	0	0	0	0
-1	0	0	0	0	0	0	0	0	0	0	0	1	2	3	2	1	0	0
-3	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Sum	4	3	2	3	2	3	2	2	2	2	2	2	2	3	2	3	2	3

Table 1 contains the minimum densities of tower broadcasts for $(t, r) \leq (9, 9)$. A more complete set of data is found here, and the code used is found here. The actual density seems to always be close to the minimum density, and appears to limit to 0 as t grows. When $r = 1$, the density equals the minimum density.

	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$	$r = 9$
$t = 1$	1								
$t = 2$	5	3							
$t = 3$	13	8	5						
$t = 4$	25	18	13	10					
$t = 5$	41	32	25	18	14				
$t = 6$	61	50	41	34	26	22			
$t = 7$	85	72	61	50	42	36	29		
$t = 8$	113	98	85	74	62	54	43	39	
$t = 9$	145	128	113	98	86	76	65	58	49

TABLE 1. The reciprocal of the minimum density for small t and r.

6. Acknowledgements

References

[1] David Blessing, Katie Johnson, Christie Mauretour, and Erik Insko. On (t, r) broadcast domination numbers of grids. *Discrete Appl. Math.*, 187:19–40, 2015.

[2] Boštjan Brešar, Paul Dorbec, Wayne Goddard, Bert L. Hartnell, Michael A. Henning, Sandi Klavžar, and Douglas F. Rall. Vizing's conjecture: a survey and recent results. *J. Graph Theory*, 69(1):46–76, 2012.

[3] Benjamin F. Drews, Pamela E. Harris, and Timothy W. Randolph. Optimal (t, r) broadcasts on the infinite grid. *Discrete Appl. Math.*, 255:183–197, 2019.

[4] Michael Farina and Armando Grez. New upper bounds on the distance domination numbers of grids. *Rose-Hulman Undergrad. Math. J.*, 17(2):Art. 7, 133–145, 2016.

[5] Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. *Fundamentals of domination in graphs*, volume 208 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1998.

[6] V. G. Vizing. Some unsolved problems in graph theory. *Uspehi Mat. Nauk*, 23(6 (144)):117–134, 1968.

[7] Dmitry A. Zaitsev. A generalized neighborhood for cellular automata. *Theoret. Comput. Sci.*, 666:21–35, 2017.

FLORIDA GULF COAST UNIVERSITY
E-mail address: tshlomi1836@eagle.fgcu.edu