Numerical Semigroups generated by Primes

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Abstract

Let $p_1 = 2, p_2 = 3, p_3 = 5, \dots$ be the consecutive prime numbers, S_n the numerical semigroup generated by the primes not less than p_n and u_n the largest irredundant generator of S_n. We will show, that

- $u_n \sim 3p_n$.

Similarly, for the largest integer f_n not contained in S_n, by computational evidence we suspect that

- f_n is an odd number for $n \geq 5$ and
- $f_n \sim 3p_n$; further
- $4p_n > f_{n+1}$ for $n \geq 1$.

If f_n is odd for large n, then $f_n \sim 3p_n$. In case $f_n \sim 3p_n$ every large even integer x is the sum of two primes. If $4p_n > f_{n+1}$ for $n \geq 1$, then the Goldbach conjecture holds true.

Further, Wilf's question in [12] has a positive answer for the semigroups S_n.

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Introduction

A *numerical semigroup* is an additively closed subset S of $\mathbb{N}$ with $0 \in S$ and only finitely many positive integers outside from S, the so-called *gaps* of S. The *genus* g of S is the number of its gaps. The set $E = S^* \setminus (S^* + S^*)$, where $S^* = S \setminus \{0\}$, is the (unique) minimal system of generators of S. Its elements are called the *atoms* of S; their number e is the *embedding dimension* of S. The *multiplicity* of S is the smallest element p of S^*.

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From now on we assume that $S \neq \mathbb{N}$. Then the greatest gap f is the *Frobenius number* of S. Since $(f+1)+\mathbb{N} \subseteq S^*$ we have $(p+f+1)+\mathbb{N} \subseteq p+S^*$, hence the atoms of S are contained in the interval $[p, p+f]$.

For our investigation of certain numerical semigroups S generated by prime numbers, the fractions

$$\frac{f}{p}, \ \frac{1+f}{p}, \ \frac{g}{1+f} \ \text{and} \ \frac{e-1}{e}$$

will play a role. For general S, what is known about these fractions?

First of all it is well known and easily seen that

$$\frac{1}{2} \leq \frac{g}{1+f} \leq \frac{p-1}{p},$$

and both bounds for $\frac{g}{1+f}$ are attained.

However, the following is still open:

Wilf's question ([12]): Is it (even) true that

$$\frac{g}{1+f} \leq \frac{e-1}{e} \tag{1}$$

for every numerical semigroup?

A partial answer is given by the following result of Eliahou:

[4, Corollary 6.5] If $\frac{1+f}{p} \leq 3$, then $\frac{g}{1+f} \leq \frac{e-1}{e}$.

In [13], Zhai has shown that $\frac{1+f}{p} \leq 3$ holds for almost all numerical semigroups of genus g (as g goes to infinity).

Therefore, for randomly chosen S, one has $\frac{g}{1+f} \leq \frac{e-1}{e}$ almost surely.

We shall consider the following semigroups: Let $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \ldots$ be the sequence of prime numbers in natural order and let S_n, for $n \geq 1$, be the numerical semigroup generated by all prime numbers not less than p_n; the multiplicity of S_n is p_n and we denote the aforementioned invariants of S_n by g_n, f_n, e_n and E_n. Since S_{n+1} is a subsemigroup of S_n it is clear that $f_n \leq f_{n+1}$ for all $n \geq 1$. The atoms of S_n are contained in the interval $[p_n, p_n + f_n]$; conversely, each odd integer from $S_n \cap [p_n, 3p_n[$ is an atom of S_n.

As a major result we will see that Wilf's question has a positive answer for S_n. Further g_n/p_n converges to $5/2$ for $n \to \infty$.

The prime number theorem suggests that there should be – like for the sequence (p_n) – some asymptotic behavior of (g_n), (f_n) and (e_n).

Based on the list $f_1, f_2, \ldots, f_{2000}$ from [15], extensive calculations (cf. our table of values for $n \leq 10000$) gave evidence for the following three conjectures:

(C1) $f_n \sim 3p_n$, i. e. $\lim_{n\to\infty} \frac{f_n}{p_n} = 3$, as already observed by Kløve [7], see also the comments in [6, p. 56]; note that Kløve works with *distinct* primes, therefore his conjecture is formally stronger than ours, however see also [14, comment by user "Emil Jeřábek", Apr 4 '12].

By Proposition 1, we know that

$$3p_n - f_n \leq 6. \tag{2}$$

(C2) $f_{n+1} < 4p_n$ for all $n \geq 1$.

and

$3p_n < f_{n+1}$ for $n \geq 3$.

It is immediate from (2) that at least

$$3p_n \leq f_{n+1} \text{ for } n \geq 2.$$

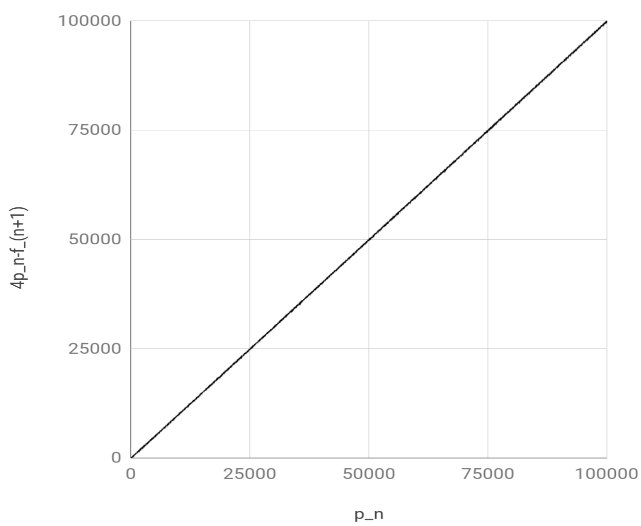

Figure 1: $4p_n - f_{n+1}$ vs p_n

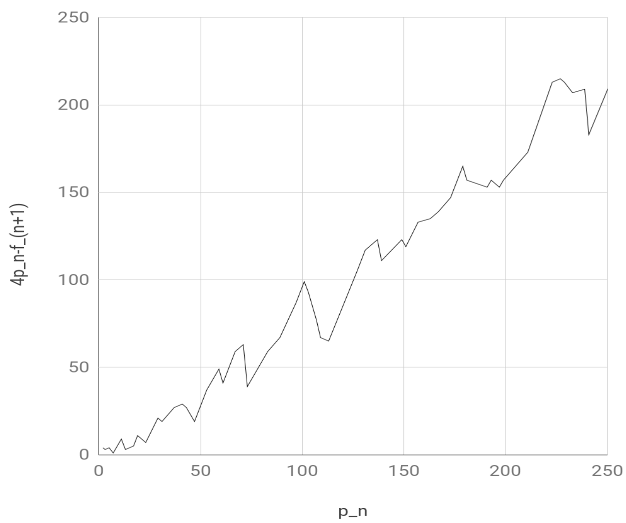

Figure 2: $4p_n - f_{n+1}$ vs p_n

As already noticed in [7] and in [14, answer by user "Woett", Apr 3 '12], both conjectures (C1) and (C2) are closely related to Goldbach's conjecture. As we will see in Proposition 4, (C1) is a consequence of conjecture

(C3) f_n is odd for $n \geq 5$.

Notice again, that a conjecture similar to (C3) was already formulated in [7], however for the (related) notion 'threshold of completeness' for the sequence of all prime numbers, in the sense of [6].

Figure 1 indicates, that $\lim_{n\to\infty} \frac{f_n}{p_n} = 3$ should be true.

As for (C2), by figure 1 and figure 2, evidently $4p_n - f_{n+1}$ should stay positive for all time.

Observations Numerical experiments suggest that similiar conjectures can be made if one restricts the generating sequence to prime numbers in a fixed arithmetic progression $a + kd$ for $(a, d) = 1$. In such a case the limit of $\frac{f_n}{p_n}$ would apparently be $d + 1$ (d even) or $2d + 1$ (d odd). (See figure 3.)

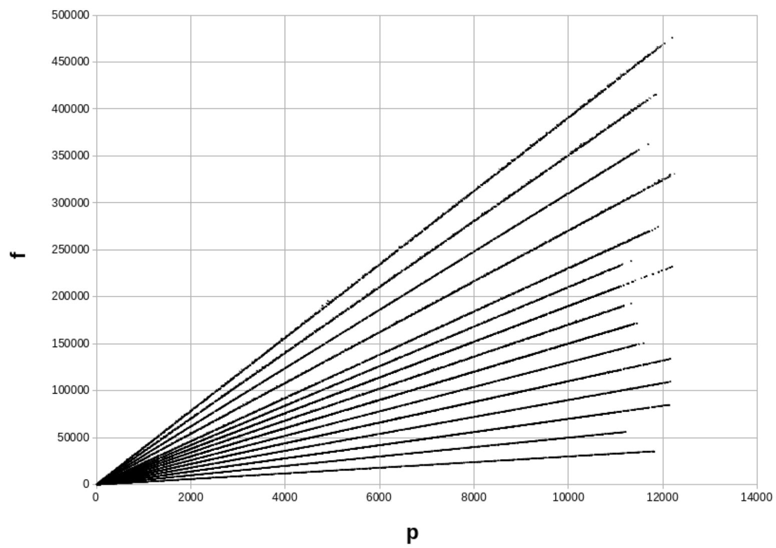

Figure 3: f vs. p for some series of semigroups as in the 'Observations'

The following version of Vinogradov's theorem is due to Matomäki, Maynard and Shao. It is fundamental for the considerations in this paper.

[8, Theorem 1.1] Let $\theta > \frac{11}{20}$. Every sufficiently large odd integer n can be written as the sum $n = q_1 + q_2 + q_3$ of three primes with the restriction

$$\left| q_i - \frac{n}{3} \right| \leq n^\theta \text{ for } i = 1, 2, 3.$$

Of course we could have used just as well one of the predecessors of this theorem, see the references in [8].

1 Variants of Goldbach's conjecture

For $x, y \in \mathbb{Q}$, $x \leq y$ we denote by $[x, y]$ the 'integral interval'

$$[x, y] := \{n \in \mathbb{Z} | x \leq n \leq y\},$$

accordingly we define $[x, y[$, $]x, y]$, $]x, y[$, $[x, \infty[$.

For $x \geq 2$ we define S_n^x to be the numerical semigroup generated by the primes in the interval $I_n^x := [p_n, x \cdot p_n[$ and f_n^x its Frobenius number.

A minor step towards a proof of conjecture (C1) is

Proposition 1

$$f_n \geq 3p_n - 6.$$

In particular for the zero sequence $r(n) := 6/p_n$ we have

$$\frac{f_n}{p_n} \geq 3 - r(n) \text{ for every } n \geq 1.$$

Proof For $n \geq 3$, obviously, the odd number $3p_n - 6$ is neither a prime nor the sum of primes greater than or equal to p_n, hence $3p_n - 6$ is not contained in S_n. □

Remark A final (major) step on the way to (C1) would be to find a zero sequence $l(n)$ such that

$$3 + l(n) \geq \frac{f_n}{p_n}.$$

Proposition 2 If (C1) is true then every sufficiently large even number x can be written as the sum $x = p + q$ of prime numbers p, q.
Addendum The prime number p can be chosen from the interval $]\frac{x}{4}, \frac{x}{2}]$.

Proof By the prime number theorem, we have $p_{n+1} \sim p_n$. (C1) implies

$$f_{n+1} \sim 3p_{n+1} \sim 3p_n,$$

i. e.

$$\lim_{n\to\infty} \frac{f_{n+1}}{p_n} = 3.$$

In particular, there exists $n_0 \geq 1$ such that $\frac{f_{n+1}}{p_n} < 4$ for all $n \geq n_0$.
It remains to show:

Lemma 1 If $n_0 \geq 1$ is such that $\frac{f_{n+1}}{p_n} < 4$ for all $n \geq n_0$ then every even number $x > 2$ with $x > f_{n_0}$ can be written as the sum

$$x = p + q \text{ with prime numbers } p \leq q \text{ and such that } \frac{x}{4} < p \leq \frac{x}{2}. \quad (1)$$

Proof By our hypothesis,

$$f_n \leq f_{n+1} < 4p_n < 4p_{n+1} \text{ for all } n \geq n_0$$

and hence, for $I_n := [1 + f_n, 4p_n[$ $(n \geq n_0)$,

$$[1 + f_{n_0}, \infty[= \bigcup_{n \geq n_0} I_n.$$

Therefore it suffices to prove (1) for all even numbers $x > 2$ from the interval I_n, for $n \geq n_0$.

By definition of f_n, every $x \in I_n$ can be written as the sum of primes $p \geq p_n$.

If in addition $x > 2$ is even, then, because of $f_n < x < 4p_n$, the number x is the sum of precisely two prime numbers $p \leq q$ with

$$p_n \leq p \leq q = x - p < 4p_n - p \leq 3p,$$

hence

$$\frac{x}{4} < p \leq \frac{x}{2}.$$

□

The special case $n_0 = 1$ of Lemma 1 gives

Proposition 3 If (C2) is true then every even number $x > 2$ can be written as the sum $x = p + q$ of prime numbers $p \leq q$ as described in the Addendum above. □

Proposition 4 If the Frobenius number f_n is odd for all large n, then $f_n \sim 3p_n$. In particular, conjecture (C3) implies conjecture (C1).

Proof From [8, Theorem 1.1] we get:

Lemma 2 Let $\varepsilon > 0$. For odd N large enough, there are prime numbers q_1, q_2, q_3 with

$$N = q_1 + q_2 + q_3$$

and such that

$$\frac{1}{3+\varepsilon} \cdot N < q_i < \frac{3+2\varepsilon}{9+3\varepsilon} \cdot N, \text{ i. e. } \left| q_i - \frac{N}{3} \right| < \frac{\varepsilon}{9+3\varepsilon} \cdot N \text{ for } i = 1, 2, 3.$$

Proof of Lemma 2 The claim follows immediately from [8, Theorem 1.1], since $\theta := \frac{3}{5} > \frac{11}{20}$ and, for large N, $N^{\frac{3}{5}} < \frac{\varepsilon}{9+3\varepsilon} \cdot N$. $\square_{\text{Lemma 2}}$

By our hypothesis, f_{n+1} is odd for large n. In Lemma 3 below we will show that, for each $\varepsilon > 0$, we have $f_{n+1} < (3 + \varepsilon)p_n$ for large n; then the claim of Proposition 4 follows from Proposition 1. $\square_{\text{Proposition 4}}$

Lemma 3 Let $\varepsilon > 0$. Then for large n, each odd integer $N \geq (3 + \varepsilon)p_n$ is contained in S_{n+1}. In particular, for large n

$$f_{n+1} < (3 + \varepsilon)p_n \text{ if } f_{n+1} \text{ is odd, and}$$

$$f_{n+1} < (3 + \varepsilon)p_n + p_{n+1} \text{ if } f_{n+1} \text{ is even,}$$

since then $f_{n+1} - p_{n+1}$ is odd and not in S_{n+1}.

Proof Since N is odd and large for large n, by Lemma 2 there exist prime numbers q_1, q_2, q_3 with

$$N = q_1 + q_2 + q_3$$

and such that

$$\frac{N}{3+\varepsilon} < q_i \text{ for } i = 1, 2, 3.$$

By assumption, $\frac{N}{3+\varepsilon} \geq p_n$, hence

$$q_i > p_n, \text{ i. e. } q_i \geq p_{n+1}$$

for the prime numbers q_i. This implies $N = q_1 + q_2 + q_3 \in S_{n+1}$. □

For a similar argument, see [14, answer by user "Anonymous", Apr 5'12].

Remarks
a) It is immediate from Lemma 3 that

$$\limsup_{n\to\infty} \frac{f_n}{p_n} \leq 4.$$

As a consequence, a proof of $\limsup_{n\to\infty} \frac{f_n}{p_n} \neq 4$ would imply the binary Goldbach conjecture for large x with the Addendum from above – see Lemma 1 and the proof of Proposition 2.

b) The estimate $\limsup_{n\to\infty} \frac{f_n}{p_n} \leq 4$ together with a sketch of proof was already formulated in [14, comment by user "François Brunault" (Apr 6 '12) to answer by user "Anonymous" (Apr 5 '12)]. Our proof is essentially an elaboration of this sketch.

c) Lemma 3 shows that

$$f_{n+1} < 5p_{n+1} \text{ for large } n.$$

Because of $p_{n+1} < 2p_n$ (Bertrand's postulate) this implies also that there exists a constant C with

$$f_{n+1} < Cp_n \text{ for all } n. \tag{2}$$

Conjecture (C2) says that in (2) one can actually take $C = 4$.

Notice that (2) already follows from [1, Lemma 1].

Problem Find an explicit pair (n_0, C_0) of numbers such that

$$f_{n+1} < C_0 \cdot p_n \text{ for every } n \geq n_0.$$

Next we shall study the asymptotic behavior of the set of atoms of S_n. Lemma 2 will imply

Corollary Let $\varepsilon > 0$. Then $S_n = S_n^{3+\varepsilon}$ for large n.

In particular, $E_n \subseteq [p_n, (3+\varepsilon)p_n[$ for large n, and $\log u_n \sim \log p_n$.
On the other hand, the primes in $[p_n, 3p_n[$ are atoms of S_n. hence for large n, $\pi(3p_n) \leq \pi(u_n) \leq \pi((3+\varepsilon)p_n)$. The prime number theorem yields

$$3n \leq \pi(u_n) \leq (3+\varepsilon)n \text{ for large n.}$$

Consequently we have the following

Theorem $\pi(u_n) \sim 3n$, $e_n \sim 2n$ and $u_n \sim 3p_n$.

Proof of the Corollary It suffices to prove the claim for arbitrarily small values of ε:

First we show that, if $\varepsilon < 3$, then

$$S_{n+1}^{3+\varepsilon} \subseteq S_n^{3+\varepsilon}$$

for large n. For this it suffices to show that every prime number p on the interval $[p_{n+1}, (3+\varepsilon)p_{n+1}[$ is in $S_n^{3+\varepsilon}$:

Firstly, $p \geq p_{n+1} > p_n$.

Now we distinguish two cases:

I $p < (3+\varepsilon)p_n$: Then $p \in I_n^{3+\varepsilon}$, hence $p \in S_n^{3+\varepsilon}$.

II $p \geq (3+\varepsilon)p_n$: For n large enough, by Lemma 2 there exist prime numbers q_1, q_2, q_3 with

$$p = q_1 + q_2 + q_3$$

and such that

$$p_n \overset{\text{II}}{\leq} \frac{p}{3+\varepsilon} < q_i < \frac{3+2\varepsilon}{9+3\varepsilon} p \text{ for } i = 1, 2, 3.$$

By Chebyshev, Bertrand's postulate $p_{n+1} < 2p_n$ holds. Therefore,

$$p \overset{\text{hypothesis}}{<} (3+\varepsilon)p_{n+1} < (6+2\varepsilon)p_n$$

and hence

$$q_i < \frac{3+2\varepsilon}{9+3\varepsilon} p < \frac{3+2\varepsilon}{9+3\varepsilon}(6+2\varepsilon)p_n < (3+\varepsilon)p_n,$$

if $\varepsilon < 3$. It follows that

$$q_i \in [p_n, (3+\varepsilon)p_n[\text{ for } i = 1, 2, 3 \text{ and hence}$$

$$p = q_1 + q_2 + q_3 \in S_n^{3+\varepsilon},$$

which proves the above claim.

Recursively, we get from $S_{n+1}^{3+\varepsilon} \subseteq S_n^{3+\varepsilon}$ that

$$p_k \in S_k^{3+\varepsilon} \subseteq S_n^{3+\varepsilon} \text{ for all } k \geq n.$$

Therefore,

$$S_n = S_n^{3+\varepsilon}.$$

□

By [4, Cor. 6.5], for arbitrary numerical semigroups S, Wilf's inequality $\frac{g}{1+f} \leq \frac{e-1}{e}$ holds, whenever $f < 3 \cdot p$. Further by [13], the latter is true for almost every numerical semigroup of genus g (as g goes to infinity).

In contrast, according to our table of values, for the semigroups S_n, the relation $f_n < 3 \cdot p_n$ seems to occur extremely seldom, but over and over again (see figure 4).

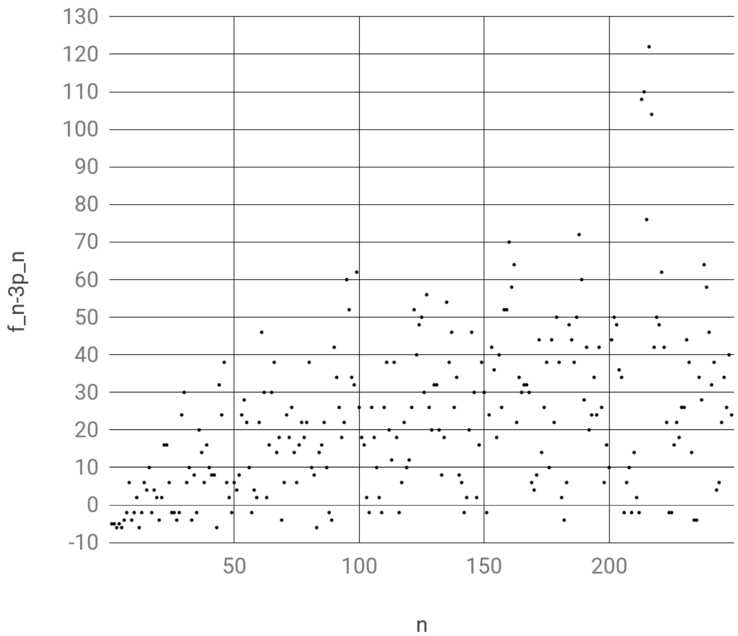

Figure 4: $f_n - 3p_n$ vs n

The following considerations are related to [14, answer by user "Aaron Meyerowitz", Apr 3 '12]:

Let $f_n < 3 \cdot p_n$. Then the odd number $3 \cdot p_n + 6$ is in S_n, but not a prime; hence $p_{n+1} \leq p_n + 6$.

1. If $p_{n+1} = p_n + 4$, since $3 \cdot p_n + 6 \in S_n$ is not a prime, $p_n + 6$ must be prime.

2. If $p_{n+1} = p_n + 6$, then the odd numbers $3p_n + 2$ and $3p_n + 4$ must be atoms in S_n, hence primes.

In any case:
Nota bene If $f_n < 3p_n$, then there is a twin prime pair within $[p_n, 3p_n + 4]$.

So we cannot expect to prove, that $f_n < 3p_n$ happens infinitely often, since this would prove the *twin prime conjecture*, that there are infinitely many twin prime pairs. Another consequence would be that

$$\liminf_{n\to\infty} \frac{f_n}{p_n} = 3,$$

since one always has that this limit inferior is ≥ 3, by Proposition 1.

The next section is attended to Wilf's question mentioned above.

2 The question of Wilf for the semigroups $\mathbf{S_n}$

Proposition 5 For the semigroups S_n, Wilf's (proposed) inequality

$$\frac{g_n}{1+f_n} \leq \frac{e_n - 1}{e_n} \tag{1}$$

holds.

Proof For $n < 429$, have a look at our table of values. Now let $n \geq 429$. Instead of (1), we would rather prove the equivalent relation

$$e_n(1 + f_n - g_n) \geq 1 + f_n. \tag{2}$$

According to [4, Cor. 6.5] we may assume, that $3p_n < 1 + f_n$. Hence the primes in the interval $[p_n, 3p_n[$ are elements of S_n lying below $1 + f_n$, and in fact, they are atoms of S_n as well. This implies for the prime-counting function π

$$e_n(1 + f_n - g_n) \geq (\pi(3p_n) - n + 1)^2. \tag{3}$$

By Rosser and Schoenfeld [10, Theorem 2] we have

$$\pi(x) < \frac{x}{\log x - \frac{3}{2}} \quad \text{for } x > e^{\frac{3}{2}}, \text{ and} \tag{4}$$

$$\pi(x) > \frac{x}{\log x - \frac{1}{2}} \quad \text{for } x \geq 67. \tag{5}$$

Further $\lambda(x) := 3 \cdot \frac{\log x - \frac{3}{2}}{\log(3x) - \frac{1}{2}}$ is strictly increasing for $x > 1$, hence

$$2n < \pi(3p_n) < 3n \quad \text{for } n \geq 429. \tag{6}$$

Proof Since $\lambda(x)$ is strictly increasing, we get for $n \geq 429$, i.e. $p_n \geq 2971$

$$\pi(3p_n) \overset{(5)}{>} \frac{3p_n}{\log(3p_n) - \frac{1}{2}} \overset{(4)}{>} \pi(p_n) \cdot \lambda(p_n) \geq n \cdot \lambda(2971) > 2n, \text{ and}$$

$$\pi(3p_n) \overset{(4)}{<} \frac{3p_n}{\log p_n + \log 3 - \frac{3}{2}} < \frac{3p_n}{\log p_n - \frac{1}{2}} \overset{(5)}{<} 3n$$

□

In particular, by (3) and (6)

$$e_n(1 + f_n - g_n) \overset{(3)}{\geq} (\pi(3p_n) - n + 1)^2 \overset{(6)}{\geq} (n+2)^2.$$

It remains to prove

Lemma 4 If $n \geq 429$, then

$$f_n < n^2.$$

Proof Let $N \leq a_1 < \ldots < a_N$ be positive integers with $(a_1, \ldots, a_N) = 1$, $S = \langle a_1, \ldots, a_N \rangle$ the numerical semigroup generated by these numbers and f its Frobenius number. Then, by Selmer [11] we have the following theorem (see the book [9] of Ramírez Alfonsín). It is an improvement of a former result [5, Theorem 1] of Erdős and Graham.

[9, Theorem 3.1.11]

$$f \leq 2 \cdot a_N \left\lfloor \frac{a_1}{N} \right\rfloor - a_1. \tag{7}$$

We will apply this to the semigroup $S_n^3 \subseteq S_n$ generated by the primes

$$p_n = a_1 < p_{n+1} = a_2 < \ldots < p_{N+n-1} = a_N$$

in the interval $I_n^3 = [p_n, 3p_n[$, with Frobenius number f_n^3, hence

$$N = \pi(3p_n) - n + 1, a_N = p_{\pi(3p_n)} = \text{the largest prime in } I_n^3.$$

By [10, Theorem 3, Corollary, (3.12)] we have

$$p_n > n \log n \geq n \log 429 > 6n \overset{(6)}{>} N,$$

hence the above theorem can be applied.

By (6) and (7), $p_{\pi(3p_n)} \overset{(6)}{<} p_{3n}$ and

$$f_n \leq f_n^3 \overset{(7)}{<} 2 \cdot p_{\pi(3p_n)} \cdot \frac{p_n}{\pi(3p_n) - n + 1} \overset{(6)}{<} 2 \cdot p_{3n} \cdot \frac{p_n}{n+2}.$$

From Rosser and Schoenfeld's result [10, Theorem 3, Corollary, (3.13)]

$$p_k < k(\log k + \log \log k) \text{ for } k \geq 6 \tag{8}$$

finally we shall conclude that $2 \cdot p_{3n} \cdot \frac{p_n}{n+2} < n^2$ for $n \geq 429$:

Elementary calculus yields

$$\lambda_2(x) := 6 \cdot (\log(3x) + \log\log(3x)) \cdot (\log x + \log \log x) < x + 2 \text{ for } x \geq 429, \tag{9}$$

hence

$$2 \cdot p_{3n} \cdot p_n \overset{(8)}{<} n^2 \cdot \lambda_2(n) \overset{(9)}{<} n^2 \cdot (n+2) \text{ for } n \geq 429.$$

□

See also P. Dusart's thèse [3] for more estimates like (4), (5) and (8).

Remark Looking at our table we see, that even

$$\pi(3p_n) > 2n \text{ for } n > 8 \text{ and } \pi(3p_n) < 3n \text{ for n} > 1$$

(which may be found elsewhere), and

$$f_n \leq n^2 \text{ for } n \neq 5.$$

At last we will see that, apparently, the quotient $\frac{g_n}{1+f_n}$ should converge to $\frac{5}{6}$ (whereas $\lim_{n\to\infty} \frac{e_n-1}{e_n} = 1$, since $e_n \sim 2n$ by our Theorem).

Proposition 6 The quotient $\frac{g_n}{p_n}$ converges and $\lim_{n\to\infty} \frac{g_n}{p_n} = \frac{5}{2}$. Hence under the assumption $\lim_{n\to\infty} \frac{p_n}{f_n} = \frac{1}{3}$ (C1) (which should be true by computational evidence) we have

$$\lim_{n\to\infty} \frac{g_n}{1+f_n} = \frac{5}{6}.$$

Proof For that, we consider the proportion $\alpha_k(n)$ of gaps of S_n among the integers in $[k \cdot p_n, (k+1) \cdot p_n]$, $(k, n \geq 1)$. Besides [8, Theorem 1.1], we shall need the following similar result about the representation of *even* numbers as the sum of two primes:

[2, Theorem 1, Corollary] Let $\varepsilon > 0$ and $A > 0$ be real constants. For $N > 0$ let $E(N)$ be the set of even numbers $2m \in [N, 2N]$, which cannot be written as the sum $2m = q_1 + q_2$ of primes q_1 and q_2 with the restriction

$$|q_j - m| \leq m^{\frac{5}{8}+\varepsilon} \text{ for } j = 1, 2.$$

Then there is a constant $D > 0$ such that $\#E(N) < D \cdot N/(\log N)^A$. □

From these two facts together with the prime number theorem, we conclude the following asymptotic behavior of the numbers $\alpha_k(n)$, as n goes to infinity:

$$\alpha_0(n) \to 1, \alpha_1(n) \to 1, \alpha_2(n) \to \frac{1}{2} \text{ and } \alpha_k(n) \to 0 \text{ for } k \geq 3.$$

Hence

$$\lim_{n\to\infty} \frac{g_n}{p_n} = 1 + 1 + \frac{1}{2} = \frac{5}{2}.$$

(Notice that for large n, by Lemma 3 we have $f_n < 5p_n$, hence $\alpha_k(n) = 0$ for $k \geq 5$.) □

Remark Let $f_{n,e}$ be the largest even gap of S_n. Ignoring the exceptional set $E(2p_n)$, the above theorem would imply that $f_{n,e} \sim 2p_n$, supported also by our computations. In this case, by Proposition 1 and Proposition 4, f_n is odd for large n and conjecture (C1) holds.

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