Groups, graphs, and hypergraphs: average sizes of kernels of generic matrices with support constraints

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We develop a theory of average sizes of kernels of generic matrices with support constraints defined in terms of graphs and hypergraphs. We apply this theory to study unipotent groups associated with graphs. In particular, we establish strong uniformity results pertaining to zeta functions enumerating conjugacy classes of these groups. We deduce that the numbers of conjugacy classes of $\mathbf{F}_q$-points of the groups under consideration depend polynomially on q. Our approach combines group theory, graph theory, toric geometry, and p-adic integration.

Our uniformity results are in line with a conjecture of Higman on the numbers of conjugacy classes of unitriangular matrix groups. Our findings are, however, in stark contrast to related results by Belkale and Brosnan on the numbers of generic symmetric matrices of given rank associated with graphs.

Contents

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1 Introduction

In this article, we study enumerative questions related to spaces of matrices defined via support constraints. Our work is motivated by and has immediate applications to the study of (conjugacy) class numbers of finite p-groups. We will naturally touch three subjects: rank distributions in spaces of matrices, class numbers of unipotent groups, and zeta functions of groups. We begin by summarising key facts from each of these fields.

1.1 Counting matrices of given rank

Polynomiality. The study of rank distributions in combinatorially defined spaces of matrices has a long history and draws on contributions from several fields of mathematical research. The numbers of arbitrary [42] $n \times m$ matrices or of antisymmetric [16, Theorem 3], symmetric [49, Theorem 2], or traceless [7, 15] $n \times n$ matrices of a given rank over a finite field $\mathbf{F}_q$ are each given by an explicitly known polynomial in q; we assume that q odd in the (anti-)symmetric cases. Lewis et al. [44] and Klein et al. [39] obtained further polynomiality results for rank distributions in spaces of general, symmetric, and antisymmetric matrices obtained by insisting that entries in suitable positions be zero.

Wilderness. The study of rank distributions naturally involves algebro-geometric methods. Thanks to these, much is known about ideals of minors associated with generic, symmetric, and antisymmetric matrices [14, 71].

In drastic contrast to the polynomiality results above, Belkale and Brosnan [6, Theorem 0.5] demonstrated that enumerating matrices of a given rank is a “wild” problem, even for spaces of combinatorial origin. More precisely, given $n \geqslant 1$ and a set S, consider the space $\mathrm{Sym}_n(\mathbf{F}_q; S)$ of symmetric $n \times n$ matrices $[a_{ij}]$ over $\mathbf{F}_q$ with $a_{ij} = 0$ whenever $(i, j) \notin S$. Belkale and Brosnan showed that, in a precise technical sense, enumerating invertible matrices in $\mathrm{Sym}_n(\mathbf{F}_q; S)$ is as difficult as counting $\mathbf{F}_q$-points on arbitrary $\mathbf{Z}$-defined varieties. (To the authors’ knowledge, it is unknown whether the same conclusion holds for spaces of arbitrary or antisymmetric matrices with suitably constrained supports.) Belkale and Brosnan used their result to refute a conjecture of Kontsevich on the polynomiality of the numbers of $\mathbf{F}_q$-points of specific hypersurfaces associated with graphs.

Halasi and Pálfy [34] obtained results of a similar flavour on the numbers of matrices over finite fields that satisfy prescribed “rank restraints”. In this context, too, polynomiality results (requiring fairly restrictive combinatorial assumptions) mark the exception from the rule of “wild” variation of the relevant numbers with the prime power q.

1.2 Class numbers of unipotent groups

Class numbers. Let $\mathrm{k}(G)$ denote number of conjugacy classes (“class number”) of a finite group G. Let $\mathrm{U}_n(R)$ be the group of upper unitriangular $n \times n$ matrices over a ring R. In an influential paper [36], Higman asked whether $\mathrm{k}(\mathrm{U}_n(\mathbf{F}_q))$ is always given by a polynomial in q. This question has been answered affirmatively for $n \leqslant 13$ by Vera-López and Arregi [70] and for $n \leqslant 16$ by Pak and Soffer [53].

Beyond Higman's conjecture. We are interested in problems in the spirit of Higman's question for other types of unipotent groups. Let $\mathbf{G} \leqslant \mathrm{U}_n$ be a subgroup scheme—we may think of $\mathbf{G}$ as a subgroup of $\mathrm{U}_n(\mathbf{C})$ defined by the vanishing of polynomials with integer coefficients. What can be said about the class numbers $\mathrm{k}(\mathbf{G}(\mathbf{F}_q))$ as a function of q? In particular, when does $\mathrm{k}(\mathbf{G}(\mathbf{F}_q))$ depend polynomially on q?

Questions like these have been asked and answered, to varying degrees of generality, for numerous group schemes realising e.g. (Sylow subgroups of) Chevalley groups or relatively free p-groups of exponent p, by numerous authors, including the above-mentioned and Evseev, Goodwin, Isaacs, Le, Lehrer, Magaard, Mozgovoy, Röhrle, and Robinson, to name but a few; see, for instance, [29, 32, 51, 52] and the references therein.

The aforementioned problems are closely related to the enumeration of matrices of given rank in $\mathbf{Z}$-defined spaces of matrices over $\mathbf{F}_q$. We note, for example, that the work in [34] on matrices with given rank constraints was motivated by a study of class numbers of pattern groups, viz. certain combinatorially defined subgroups of $\mathrm{U}_n(\mathbf{F}_q)$. This connection also occurs in previous work [52, 57, 60] of both authors. Moreover, it turns out that if we are willing to exclude small exceptional characteristics, the study of $\mathrm{k}(\mathbf{G}(\mathbf{F}_q))$ for group schemes $\mathbf{G} \leqslant \mathrm{U}_n$ as above essentially reduces to those of class 2. (For a proof, combine [60, Proposition 6.4] and [60, Lemma 7.1].)

Alternating bilinear maps. As a variation of the classical Baer correspondence [2], we may construct a (unipotent) group scheme $\mathbf{G}_\diamond$ of class at most 2 from each alternating bilinear map $\diamond\colon \mathbf{Z}^n \times \mathbf{Z}^n \to M$, where M is a free $\mathbf{Z}$-module of finite rank; see §2.4 for details. We call $\mathbf{G}_\diamond$ the **Baer group scheme** associated with $\diamond$. Commutators in $\mathbf{G}_\diamond$ are given by $\diamond$. For example, if $\diamond\colon \mathbf{Z}^2 \times \mathbf{Z}^2 \to \mathbf{Z}$ is the standard symplectic form, then the associated Baer group scheme is U_3, the Heisenberg group scheme.

Rather than consider the maps $\diamond$, we may equivalently use antisymmetric matrices. Let $\mathfrak{so}_n(\mathbf{Z}) \subset \mathrm{M}_n(\mathbf{Z})$ denote the module of antisymmetric $n \times n$ matrices over $\mathbf{Z}$. Then every $\mathbf{Z}$-module homomorphism $M \xrightarrow{\theta} \mathfrak{so}_n(\mathbf{Z})$ defines an alternating bilinear map

$$[\theta]\colon \mathbf{Z}^n \times \mathbf{Z}^n \to M^* := \mathrm{Hom}(M, \mathbf{Z})$$

such that $x[\theta]y$ is the functional $a \mapsto x(a\theta)y^\top$. In particular, for a submodule $M \subset \mathfrak{so}_n(\mathbf{Z})$, we obtain a Baer group scheme $\mathbf{G}_M := \mathbf{G}_{[\iota]}$, where ι is the inclusion $M \hookrightarrow \mathfrak{so}_n(\mathbf{Z})$.

We note that using essentially a variation of the above construction, finite p-groups associated with spaces of antisymmetric matrices over $\mathbf{F}_p$ have found applications relating graph- and group-theoretic problems in recent work of Bei et al. [5] and Li and Qiao [45].

Average sizes of kernels. Again, up to excluding small characteristics, as a function of q, the study of the class numbers $\mathrm{k}(\mathbf{G}_M(\mathbf{F}_q))$ for modules $M \subset \mathfrak{so}_n(\mathbf{Z})$ turns out to be essentially equivalent to the study of $\mathrm{k}(\mathbf{G}(\mathbf{F}_q))$ for *arbitrary* unipotent group schemes $\mathbf{G}$. By focusing on the Baer group schemes of the form $\mathbf{G}_M$, we may easily relate the study of class numbers to that of enumerating antisymmetric matrices of given rank as in §1.1. Namely, if A is a finite ring and if $\bar{M} \subset \mathfrak{so}_n(A)$ denotes the submodule generated by the image of M, then

$$\mathrm{k}(\mathbf{G}_M(A)) = |A|^m \cdot \frac{1}{|\bar{M}|} \sum_{a \in \bar{M}} |\mathrm{Ker}(\bar{a})|,$$

where m is the rank of M as a $\mathbf{Z}$-module; cf. Proposition 2.5. That is, up to a harmless factor, the class number of $\mathbf{G}_M(A)$ is the average size of the kernels of the elements of $\bar{M}$ acting on A^n. Thus, if $A = \mathbf{F}_q$ is a finite field, then we may express $\mathrm{k}(\mathbf{G}_M(\mathbf{F}_q))$ in terms of the numbers of $\mathbf{F}_q$-points of the $\mathbf{Z}$-defined rank loci in M; see [57, §2.1] for details.

1.3 Zeta functions

We have seen that the study of the class numbers $\mathrm{k}(\mathbf{G}(\mathbf{F}_q))$ for group schemes $\mathbf{G} \leqslant \mathrm{U}_n$ is intimately related to the study of average sizes of kernels of matrices over $\mathbf{F}_q$. *Class counting* and *ask zeta functions* provide convenient tools for generalising this connection to much more general finite rings, including those of form $\mathbf{Z}/p^k\mathbf{Z}$ (p prime) and $\mathbf{F}_q[z]/(z^k)$.

Class counting zeta functions. Let R be the ring of integers of a local or a global field. Let $\mathbf{G}$ be a group scheme of finite type over R. The **class counting zeta function** of $\mathbf{G}$ is the Dirichlet series

$$\zeta^{\mathrm{cc}}_{\mathbf{G}}(s) := \sum_{0 \neq I \triangleleft R} \mathrm{k}(\mathbf{G}(R/I)) \cdot |R/I|^{-s}.$$

Class counting zeta functions were introduced by du Sautoy [23] for p-adic linear groups. They were further studied by Berman et al. [8] for Chevalley groups and by the first author [57, 60] and Lins [46–48] for unipotent groups; other names for these functions in the literature are "conjugacy class zeta functions" and "class number zeta functions". The use of zeta functions as a tool in group theory was pioneered by Grunewald et al. [33].

Euler products and variation of the place. As we will now explain, the study of class counting zeta functions in characteristic zero immediately reduces to a local analysis. Let K be a number field with ring of integers $\mathcal{O}$. Let $\mathcal{V}_K$ be the set of non-Archimedean places of K. For $v \in \mathcal{V}_K$, let $\mathcal{O}_v$ denote the valuation ring of the v-adic completion of K. Let $\mathfrak{K}_v$ denote the residue field of $\mathcal{O}_v$ and let $q_v = |\mathfrak{K}_v|$. Let $\mathbf{G}$ be a group scheme of finite type over $\mathcal{O}$. Then the Chinese remainder theorem yields an Euler product factorisation

$$\zeta^{\mathrm{cc}}_{\mathbf{G}}(s) = \prod_{v \in \mathcal{V}_K} \zeta^{\mathrm{cc}}_{\mathbf{G} \otimes \mathcal{O}_v}(s). \tag{1.1}$$

For a general $\mathbf{G}$, it is unknown how the Euler factors $\zeta^{\mathrm{cc}}_{\mathbf{G} \otimes \mathcal{O}_v}(s)$ vary with the place v. However, if $\mathbf{G}$ is a Chevalley group [8] or unipotent [57], then $\zeta^{\mathrm{cc}}_{\mathbf{G} \otimes \mathcal{O}_v}(s)$ can, for almost all $v \in \mathcal{V}_K$, be expressed in terms of the numbers of $\mathfrak{K}_v$-points of certain $\mathcal{O}$-defined varieties and rational functions in q_v and q_v^{-s}. In both cases, it is an open problem to prove meaningful theorems on the class of varieties "required" to describe $\zeta^{\mathrm{cc}}_{\mathbf{G} \otimes \mathcal{O}_v}(s)$.

Uniformity. Among the ways that the Euler factors of a class counting zeta function as above might depend on the place, the tamest conceivable case has played a central role in the literature. Namely, we say that the group scheme $\mathbf{G}$ over the ring of integers $\mathcal{O}$ of a number field K has **uniform** class counting zeta functions if there exists a rational function $W(X,T) \in \mathbf{Q}(X,T)$ such that $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}_v}(s) = W(q_v, q_v^{-s})$ for all $v \in \mathcal{V}_K$. For example, if ζ_K denotes the Dedekind zeta function of K, then

$$\zeta^{\mathrm{cc}}_{\mathrm{U}_3\otimes\mathcal{O}}(s) = \frac{\zeta_K(s-1)\zeta_K(s-2)}{\zeta_K(s)} = \prod_{v\in\mathcal{V}_K} W(q_v, q_v^{-s})$$

where $W(X,T) = \frac{1-T}{(1-XT)(1-X^2T)}$; see [8, §8.2] and [57, §9.3]. A natural variation of Higman's question asks if the class counting zeta function of each U_n is uniform. While the above notion of uniformity is natural in view of the Euler product (1.1), both stronger and weaker concepts are frequently of interest.

We wish to add a further direction by allowing local base extensions. Namely, we say that $\mathbf{G}$ as above has **strongly uniform** class counting zeta functions if there exists $W(X,T) \in \mathbf{Q}(X,T)$ such that for all compact discrete valuation rings (DVRs) $\mathfrak{O}$ endowed with an $\mathcal{O}$-algebra structure, we have $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathfrak{O}}(s) = W(q, q^{-s})$, where q denotes the residue field size of $\mathfrak{O}$. (Note, in particular, that we do not insist that $\mathfrak{O}$ has characteristic zero.) Again, U_3 is an example of a group scheme with strongly uniform class counting zeta functions; this can be verified directly or deduced from much more general results below.

While it is relatively easy to produce examples of group schemes with non-uniform class counting zeta functions (see [57, §7]), as in the study of class numbers over $\mathbf{F}_q$ in §1.2, it remains unknown just how erratically Euler factors of class counting zeta functions may vary with the place.

Ask zeta functions: analytic form. In the same way that the class counting zeta function $\zeta^{\mathrm{cc}}_{\mathbf{G}}(s)$ (and its Euler factors) of a group scheme $\mathbf{G}$ generalises the collection of class numbers $\mathrm{k}(\mathbf{G}(\mathbf{F}_p))$ as p ranges over the primes, we may similarly define a Dirichlet series which generalises the average sizes of kernels that appeared in §1.2.

Let R be a commutative ring. Consider an R-module homomorphism $M \xrightarrow{\theta} \mathrm{M}_{n\times m}(R)$, where M is finitely generated. If R is finite, then the average size of the kernel associated with θ is the rational number

$$\mathrm{ask}(\theta) := \frac{1}{|M|}\sum_{a\in M} |\mathrm{Ker}(a\theta)|.$$

For each R-algebra S, we obtain a map $M\otimes S \xrightarrow{\theta^S} \mathrm{M}_{n\times m}(S)$. Suppose that R is the ring of integers of a local or global field. The **(analytic) ask zeta function** [57, 60] of θ is

$$\zeta^{\mathsf{ask}}_{\theta}(s) := \sum_{0\neq I\triangleleft R} \mathrm{ask}(\theta^{R/I}) \cdot |R/I|^{-s}.$$

Class counting and ask zeta functions. The following by-product of [60] (to be proved in §2.4) asserts that ask zeta functions associated with modules of antisymmetric matrices essentially coincide with class counting zeta functions of associated group schemes.

Proposition 1.1. *Let $M \overset{\iota}{\hookrightarrow} \mathfrak{so}_n(\mathbf{Z})$ be the inclusion of a submodule of $\mathbf{Z}$-rank m. Define a unipotent group scheme $\mathbf{G}_M$ as in §1.2. Let R be ring of integers of a local or global field of arbitrary characteristic. Then*

$$\zeta^{\mathsf{cc}}_{\mathbf{G}_M \otimes R}(s) = \zeta^{\mathsf{ask}}_{\iota^R}(s-m).$$

Beyond antisymmetric matrices, by [57, §8], ask zeta functions associated with general $\mathbf{Z}$-module homomorphisms $M \to \mathrm{M}_{n\times m}(\mathbf{Z})$ are of natural group-theoretic interest: they enumerate linear orbits of suitable groups (although perhaps not conjugacy classes).

Ask zeta functions: algebraic form. In the local case, it will be convenient to switch freely between the above ask zeta functions and the following algebraic counterpart. Let $\mathfrak{O}$ be a compact DVR and let $M \xrightarrow{\theta} \mathrm{M}_{n\times m}(\mathfrak{O})$ be an $\mathfrak{O}$-linear map, where M is finitely generated. Let $\mathfrak{P}$ be the maximal ideal of $\mathfrak{O}$. Then

$$\mathsf{Z}^{\mathsf{ask}}_{\theta}(T) := \sum_{k=0}^{\infty} \mathrm{ask}(\theta^{\mathfrak{O}/\mathfrak{P}^k})T^k \in \mathbf{Q}[\![T]\!]$$

is the **(algebraic) ask zeta function** of θ. The Dirichlet series $\zeta^{\mathsf{ask}}_{\theta}(s)$ and ordinary generating function $\mathsf{Z}^{\mathsf{ask}}_{\theta}(T)$ determine each other in the sense that

$$\zeta^{\mathsf{ask}}_{\theta}(s) = \mathsf{Z}^{\mathsf{ask}}_{\theta}(q^{-s}),$$

where $q = |\mathfrak{O}/\mathfrak{P}|$ denotes the residue field size of $\mathfrak{O}$. For this reason, we shall call each of these functions "the" ask zeta function of θ.

1.4 Groups, graphs, and hypergraphs

In this section, we introduce (somewhat informally) the real protagonists of the present article: graphical groups and adjacency and incidence representations of graphs and hypergraphs, respectively; a more complete and rigorous account will be given in §3.

Throughout, graphs are finite without parallel edges but they may contain loops; graphs without loops are **simple**.

Graphical groups and negative adjacency representations. Let Γ be a simple graph; for simplicity, we assume that $1, \dots, n$ are the vertices of Γ.

The following construction of a module of antisymmetric matrices derived from Γ was used by Tutte [68]. Let e_{ij} denote the $n \times n$ matrix with entry 1 in position (i,j) and zeros elsewhere. Let $M^-(\Gamma) \subset \mathfrak{so}_n(\mathbf{Z})$ be the submodule generated by all matrices $e_{ij} - e_{ji}$, where (i,j) runs over pairs of adjacent vertices. In the spirit of §1.1, $M^-(\Gamma)$ is the largest module of antisymmetric $n \times n$ matrices over $\mathbf{Z}$ such that the support of each matrix in $M^-(\Gamma)$ is contained in the set of pairs (i,j) with i adjacent to j in Γ.

Let $\mathbf{G}_\Gamma := \mathbf{G}_{M^-(\Gamma)}$ be the group scheme associated with $M^-(\Gamma)$ as in §1.2. We call $\mathbf{G}_\Gamma$ the **graphical group scheme** associated with Γ; see §3.4 for details. We refer to the groups of points $\mathbf{G}_\Gamma(R)$ over rings R as the **graphical groups** associated with Γ over R. For example, it is easy to see that if P_n denotes the path on n vertices, then $\mathbf{G}_{\mathrm{P}_n}(\mathbf{Z})$ is the largest nilpotent quotient of class at most 2 of $\mathrm{U}_{n+1}(\mathbf{Z})$. More generally, the groups $\mathbf{G}_\Gamma(\mathbf{Z})$ are precisely the class-2 quotients of right-angled Artin groups; see Remark 3.8.

Among the central objects of interest in the present article are the class counting zeta functions of graphical group schemes. Based on what we described above, the study of these class counting zeta functions becomes a part of the study of ask zeta functions. Namely, define the **negative adjacency representation** γ_- of Γ to be the inclusion $M^-(\Gamma) \hookrightarrow \mathfrak{so}_n(\mathbf{Z})$. By Proposition 1.1, the ask zeta function of γ_- essentially coincides with the class counting zeta function $\zeta^{\mathsf{cc}}_{\mathbf{G}_\Gamma}(s)$ of the graphical group scheme $\mathbf{G}_\Gamma$.

Positive adjacency representations. As the adjective "negative" indicates, the functions just defined admit "positive" analogues. Suppose that Γ is a graph as before except that we now allow Γ to contain loops. Let $M^+(\Gamma)$ be the submodule of the module $\mathrm{Sym}_n(\mathbf{Z})$ of symmetric $n \times n$ matrices over $\mathbf{Z}$ generated by the matrices $e_{ij} + e_{ji}$ for different adjacent vertices i and j and all e_{ii} for loops i. We define the **positive adjacency representation** γ_+ of Γ to be the inclusion $M^+(\Gamma) \hookrightarrow \mathrm{Sym}_n(\mathbf{Z})$.

Even though the ask zeta functions associated with the maps γ_+ lack an obvious group-theoretic interpretation (akin to our interpretation of $\zeta^{\mathsf{ask}}_{\gamma_-}(s)$ in terms of the class counting zeta function of $\mathbf{G}_\Gamma$), they are of natural interest in light of the results due to Belkale and Brosnan [6] mentioned in §1.1. Using our present terminology, Belkale and Brosnan showed that, as Γ varies over all finite graphs (with loops permitted), the number of invertible matrices in the image of $M^+(\Gamma) \otimes \mathbf{F}_q$ in $\mathrm{Sym}_n(\mathbf{F}_q)$ is "arbitrarily wild" as a function of q. It is therefore natural to ask whether this wildness survives taking the average both over $\mathbf{F}_q$ and, similarly, on the level of suitable ask zeta functions.

Hypergraphs and incidence representations. As we saw, graphs (with loops permitted) provide a combinatorial formalism for discussing modules of antisymmetric or symmetric matrices with support contained in a given set of positions. In the same spirit, we may use hypergraphs to encode modules of arbitrary rectangular matrices with constrained support. Here, a hypergraph H on the vertex set $\{1, \ldots, n\}$ consists of symbols $e_1, \ldots, e_m$ called hyperedges and, for each $j = 1, \ldots, m$, a support set $|e_i|$ which is an arbitrary subset of $\{1, \ldots, n\}$. Define $M(\mathsf{H}) \subset \mathrm{M}_{n\times m}(\mathbf{Z})$ to be the module of all matrices $[a_{ij}]$ with $a_{ij} = 0$ whenever the vertex i and hyperedge e_j are not incident (i.e. whenever $i \notin |e_j|$). We refer to the inclusion $M(\mathsf{H}) \overset{\eta}{\hookrightarrow} \mathrm{M}_{n\times m}(\mathbf{Z})$ as the **incidence representation** of H.

The ask zeta functions $\zeta^{\mathsf{ask}}_{\eta}(s)$ associated with hypergraphs are of interest in view of work of Lewis et al. [44], Klein et al. [39], and others on rank distributions in spaces of matrices defined in terms of support constraints. In addition, over the course of the present article, we will encounter group-theoretic incentives for studying these functions.

1.5 Results I: strong uniformity

Our first main result establishes that whatever wild geometry can be found in the rank loci of the modules $M^{\pm}(\Gamma) \subset \mathrm{M}_n(\mathbf{Z})$ and $M(\mathsf{H}) \subset \mathrm{M}_{n\times m}(\mathbf{Z})$ from §1.4 disappears on average in the sense that it is invisible on the level of ask zeta functions. As before, q denotes the residue field size of a compact DVR $\mathfrak{O}$.

Theorem A (Strong uniformity)**.**

(i) *Let* H *be a hypergraph with incidence representation* η *over* $\mathbf{Z}$*. Then there exists* $W_{\mathsf{H}}(X,T) \in \mathbf{Q}(X,T)$ *such that, for each compact* DVR $\mathfrak{O}$*,*

$$\mathsf{Z}^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(T) = W_{\mathsf{H}}(q,T).$$

(ii) *Let* Γ *be a simple graph with negative adjacency representation* γ_- *over* $\mathbf{Z}$*. Then there exists a rational function* $W^-_{\Gamma}(X,T) \in \mathbf{Q}(X,T)$ *such that, for each compact* DVR $\mathfrak{O}$*,*

$$\mathsf{Z}^{\mathsf{ask}}_{\gamma_-^{\mathfrak{O}}}(T) = W^-_{\Gamma}(q,T).$$

(iii) *Let* Γ *be a (not necessarily simple) graph with positive adjacency representation* γ_+ *over* $\mathbf{Z}$*. Then there exists a rational function* $W^+_{\Gamma}(X,T) \in \mathbf{Q}(X,T)$ *such that, for each compact* DVR $\mathfrak{O}$ *of odd residue characteristic,*

$$\mathsf{Z}^{\mathsf{ask}}_{\gamma_+^{\mathfrak{O}}}(T) = W^+_{\Gamma}(q,T).$$

By [57, Theorem 1.4], each of the generating functions $\mathsf{Z}^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(T)$, $\mathsf{Z}^{\mathsf{ask}}_{\gamma_-^{\mathfrak{O}}}(T)$, and $\mathsf{Z}^{\mathsf{ask}}_{\gamma_+^{\mathfrak{O}}}(T)$ in Theorem A is rational in T provided that $\mathfrak{O}$ has characteristic zero. What is remarkable is that these functions are in fact rational in both T and q without any restrictions on $\mathfrak{O}$. This is not a general phenomenon for ask zeta functions; see [57, §7].

The dichotomy between "tame" (i.e. strongly uniform) and "wild" behaviour is a recurring theme in the study of zeta functions associated with various group-theoretic counting problems. Uniformity results (akin to our Theorem A) have been obtained in various situations; see e.g. [33, Theorem 2], [66, Theorem B] and [17, Theorem 1.2].

By minor abuse of notation, we refer to the rational functions $W_{\mathsf{H}}(X,T)$ and $W^{\pm}_{\Gamma}(X,T)$ in Theorem A as the ask zeta functions associated with H and Γ, respectively. These rational functions are, to the best of our knowledge, new invariants of graphs and hypergraphs which, as we will see, reflect interesting structural features of the latter.

An immediate consequence of Theorem A is that upon taking the average, the arbitrarily wild numbers of invertible matrices over $\mathbf{F}_q$ provided by Belkale and Brosnan cancel.

Corollary 1.2. *Let* $n \geqslant 1$ *and a set* S *be given. Define* $\mathrm{Sym}_{n,r}(\mathbf{F}_q; S)$ *to be the set of matrices of rank* r *in* $\mathrm{Sym}_n(\mathbf{F}_q; S)$ *(see §1.1). Then there exists a polynomial* $f_{n,S}(X) \in \mathbf{Q}[X]$ *such that for each odd prime power* q*,*

$$\sum_{r=0}^{n} |\mathrm{Sym}_{n,r}(\mathbf{F}_q; S)|\, q^{n-r} = f_{n,S}(q). \tag{1.2}$$

Proof. Let d be the $\mathbf{F}_q$-dimension of $\mathrm{Sym}_n(\mathbf{F}_q; S)$ and note that d does not depend on q. Let Γ be the (not necessarily simple) graph with vertices $1, \dots, n$ and such that two vertices i and j of Γ are adjacent if and only if $(i, j), (j, i) \in S$. Let $f_{n,S}(X) \in \mathbf{Q}(X)$ be the coefficient of T of the rational power series $W_\Gamma^+(X, X^d T)$ in T from Theorem A(iii). By the definition of ask zeta functions in §1.3, (1.2) is satisfied for all odd prime powers q. It is a simple exercise to show that since $f_{n,S}(q)$ is an integer for infinitely many q, the rational function $f_{n,S}(X)$ is in fact a polynomial, as claimed. ⧫

In the same way, parts (i)–(ii) of Theorem A imply analogous results for spaces of general $n \times m$ and antisymmetric $n \times n$ matrices with supports constrained by sets.

Proposition 1.1 and Theorem A(ii) imply the following group-theoretic result (see §3.4).

Corollary B (Class counting zeta functions of graphical group schemes)**.**
Let Γ be a simple graph with m edges. Then, for each compact DVR $\mathfrak{O}$ *(of arbitrary characteristic) and with residue field size q,*

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathfrak{O}}(s) = W_\Gamma^-(q, q^{m-s}).$$

In particular, graphical group schemes have strongly uniform class counting zeta functions.

As a very special case, we obtain the following consequence in the spirit of Higman's question on the class numbers $\mathrm{k}(\mathrm{U}_n(\mathbf{F}_q))$ for graphical groups over $\mathbf{F}_q$.

Corollary 1.3. *Let Γ be a simple graph. Then there exists a polynomial $f_\Gamma(X) \in \mathbf{Q}[X]$ such that, for all prime powers q, we have $\mathrm{k}(\mathbf{G}_\Gamma(\mathbf{F}_q)) = f_\Gamma(q)$.*

Proof. We may take $f_\Gamma(X) \in \mathbf{Q}(X)$ to be the coefficient of T of the rational power series $W_\Gamma^-(X, X^m T)$ in T. As in the proof of Corollary 1.2, $f_\Gamma(X)$ is a polynomial. ⧫

Ingredients of the proof of Theorem A. While our proof of Theorem A(i) can be recast in terms of existing machinery from the theory of zeta functions ("monomial p-adic integrals" as in §1.6), parts (ii)–(iii) involve the development of several new tools that are likely to have further applications beyond the present article. These include (a) a new type of zeta function associated with modules over polynomial rings (see §2) and, more generally, over toric rings (see §4), (b) a notion of "torically combinatorial" modules (see §4.4) which provides an algebraic explanation of uniformity, and (c) a novel blend of graph theory and toric geometry in §6.

We note that the first author previously used toric geometry in the study of zeta functions of groups and related structures; see [54, 55]. However, in that work, it turned out to be extremely challenging to characterise those groups or algebras that are amenable to toric methods. In contrast, in the present setting, every graph provides an example of such a group (scheme) via Theorem A(ii) and Corollary B.

Beyond uniformity. Apart from being surprising in light of what is known about rank distributions and matrices with restricted support, Theorem A also raises intriguing follow-up questions. Which general features do the rational functions $W^{\pm}_{\Gamma}(X,T)$ and $W_{\mathsf{H}}(X,T)$ possess? How do they depend on the graph Γ and hypergraph H, respectively? Do they afford a meaningful combinatorial interpretation? Can they be computed?

Our proof of Theorem A is constructive and will thus provide an affirmative answer to the last of these questions. Regarding the first question, general results on ask zeta functions from [57] have consequences such as the following:

Corollary 1.4 (Functional equations)**.** *Let $W(X,T)$ be one of the rational functions $W_{\mathsf{H}}(X,T)$ or $W^{\pm}_{\Gamma}(X,T)$ associated with a hypergraph or graph on n vertices. Then:*

$$W(X^{-1},T^{-1}) = -X^nT\,W(X,T).$$

Proof. Combine [57, Theorem 4.18] and [59, §4]. ♦

Corollary 1.5 (Reduced zeta functions)**.** *Let the notation be as in Corollary 1.4. Then $W(1,T) = 1/(1-T)$.*

Proof. Apply [57, §4.6]. ♦

Theorem A and general results on zeta functions of algebraic structures (cf. e.g. [57, Theorem 4.10]) imply that each of the rational functions in Theorem A can be written in the form $f(X,T)/g(X,T)$, where $f(X,T) \in \mathbf{Q}[X^{\pm 1},T]$ and $g(X,T)$ is a product of factors of the form $1 - X^aT^b$ for $a,b \in \mathbf{Z}$ with $b \geqslant 0$. As we will see, we can often be much more precise here. In particular, our next main results will cast light on the rational functions $W_{\mathsf{H}}(X,T)$ for arbitrary hypergraphs and on the rational functions $W^{-}_{\Gamma}(X,T)$ (and hence associated class counting zeta functions) for certain graphs, namely the so-called *cographs.*

1.6 Results II: weak orders and explicit formulae for hypergraphs

While constructive, the intricate recursive nature of our proof of Theorem A(ii)–(iii) provides few indications as to how the rational functions obtained depend on the graph in question. In contrast, in the case of hypergraphs we make the uniformity statement in Theorem A(i) fully explicit, as our next main result shows.

Up to isomorphism, a hypergraph H as in §1.4 is completely determined by a vertex set V and, for each subset $I \subset V$, a "hyperedge multiplicity" μ_I which counts how many hyperedges of H have support I. We can explicitly describe $W_{\mathsf{H}}(X,T)$ in terms of these multiplicities. Let $\widehat{\mathrm{WO}}(V)$ denote the poset of flags of subsets of V, i.e. (essentially) the poset of weak orders on V; see Definition 5.3.

Theorem C (Ask zeta functions of hypergraphs and weak orders)**.** *Let H be a hypergraph with vertex set V and given by a family $\boldsymbol{\mu} = (\mu_I)_{I\subset V} \in \mathbf{N}_0^{\mathcal{P}(V)}$ of hyperedge multiplicities. Then*

$$W_{\mathsf{H}}(X,T) = \sum_{y\in\widehat{\mathrm{WO}}(V)} (1-X^{-1})^{|\mathrm{sup}(y)|} \prod_{J\in y} \frac{X^{|J|-\sum_{I\cap J\neq\varnothing}\mu_I}T}{1-X^{|J|-\sum_{I\cap J\neq\varnothing}\mu_I}T}. \tag{1.3}$$

The number of summands in (1.3) grows rather quickly. Indeed, let $n = |V|$. As explained in Remark 5.4, $|\widehat{\mathrm{WO}}(V)| = 4f_n$, where f_n is the nth Fubini (or ordered Bell) number, enumerating weak orders on V. In particular,

$$f_n \sim \frac{n!}{2(\log 2)^{n+1}} \tag{1.4}$$

grows super-exponentially as a function of n; see [3] or [72, §5.2]. The value of Theorem C lies not primarily in providing an algorithm for computing $W_{\mathsf{H}}(X,T)$ but in the rich combinatorial structure of these functions that it reveals.

We note that the right-hand side of (1.3) is similar but not identical to the "weak order zeta functions" of Carnevale et al. [17, §1.2].

Consequences. We exhibit three main applications of Theorem C. First, it imposes severe restrictions on the denominators of the functions $W_{\mathsf{H}}(X,T)$. This turns out to have remarkable consequences for analytic properties of ask zeta functions associated with hypergraphs; see Theorem 5.26. Secondly, for specific families of hypergraphs of special interest here, we will obtain more manageable versions of Theorem C; see §§5.1.1, 5.2.1, and 5.3.1. Finally, Theorem C will allow us to capture the effects of several natural operations for hypergraphs on the level of the rational functions $W_{\mathsf{H}}(X,T)$; see §5.4. This will prove to be particularly valuable when combined with the results in §1.7.

Ingredients of the proof of Theorem C. Let $\mathfrak{O}$ be a compact DVR. Beginning with the integral formalism for ask zeta functions from [57], our proof of Theorem C is based on a formula of the same type as (1.3) for multivariate monomial integrals such as

$$\mathcal{Z}(\boldsymbol{s}) := \int\limits_{\mathfrak{O}^n \times \mathfrak{O}} \prod_{I=\{i_1,i_2,\dots\}\subset\{1,\dots,n\}} \|x_{i_1}, x_{i_2}, \dots; y\|^{s_I} \,\mathrm{d}\mu(x,y), \tag{1.5}$$

where $\boldsymbol{s} = (s_I)_{I\subset\{1,\dots,n\}}$ is a family of complex variables, $\|\cdot\|$ denotes the (suitably normalised) maximum norm, and μ denotes the additive Haar measure on $\mathfrak{O}^{n+1}$ with $\mu(\mathfrak{O}^{n+1}) = 1$; see Theorem 5.5.

Weak orders on a set encode the possible rankings of its elements that allow for ties. Given non-zero $x_1, \dots, x_n, y \in \mathfrak{O}$, their valuations give rise to such a ranking via the usual order. In this way, weak orders naturally arise in the study of the integrals (1.5).

1.7 Results III: cographs and their models

Most of what we will learn about ask zeta functions associated with hypergraphs rests upon explicit formula such as (1.3). As indicated above, the starting point of these formulae is an expression for the local ask zeta functions (i.e. those over compact DVRs) associated with a hypergraph by means of a monomial integral as in (1.5). We have no reason to expect that such an approach will succeed for adjacency representations of graphs. (Example 7.5 will show that the integrals in (1.5) cannot suffice.) This explains why our proof of parts (ii)–(iii) of Theorem A is vastly more involved than that of part (i).

Our next main result exhibits a miraculous connection between the rational functions $W^-_\Gamma(X,T)$ associated with certain simple graphs and the rational functions $W_{\mathsf{H}}(X,T)$ associated with hypergraphs in Theorem C.

Cographs. The class of graphs known as **cographs** admits numerous equivalent characterisations; see §7.1. For instance, it is the smallest class of graphs which contains an isolated vertex and which is closed under both disjoint unions (denoted by $\oplus$) and "joins" (denoted by $\vee$) of graphs; here, the **join** of two graphs Γ_1 and Γ_2 is obtained from their disjoint union by inserting edges connecting each vertex of Γ_1 to each vertex of Γ_2. Equivalently, cographs are precisely those graphs that do not contain a path on four vertices as an induced subgraph.

Theorem D (Cograph Modelling Theorem)**.** *Let Γ be a cograph. Then there exists an explicit hypergraph H on the same vertex set as Γ such that*

$$W^-_\Gamma(X,T) = W_{\mathsf{H}}(X,T).$$

Informally, we think of the hypergraph H in Theorem D as a "model" of Γ in the sense that, through the techniques that we developed here, the former allows us to determine and study the rational function $W^-_\Gamma(X,T)$ much more easily than by the using methods underpinning Theorem A(ii). In particular, for a cograph Γ, Theorem D allows to express $W^-_\Gamma(X,T)$ via Theorem C. We will construct a particular hypergraph H as in Theorem D for each cograph Γ; we refer to this hypergraph as "the" model of Γ in the following.

Our construction reveals a number of specific properties of models. For instance, models always have fewer hyperedges than vertices. Moreover, the sum over the entries of an incidence matrix of a model is always even (this will follow from Remark 7.25), just as for graphs. These conditions further illustrate the level of generality of Theorem C.

We note that the special case of Theorem D obtained by taking Γ to be a complete graph Γ on n vertices and H to be a hypergraph on n vertices with $n-1$ hyperedges, the support of each is the set of all vertices, was (implicitly) proved in [57, Proposition 5.11].

Ingredients of the proof of Theorem D. In the same way that our proof of Theorem A goes beyond merely establishing uniformity of zeta functions by elucidating the structure of certain modules, the cograph modelling theorem is based on more than a mere coincidence of rational functions. Instead, it is a consequence of a structural counterpart (Theorem 7.1) of Theorem D which establishes that for each cograph Γ, there exists an (explicit) hypergraph H such that the "negative adjacency module" of Γ and the "incidence module" of H, while generally non-isomorphic, are "torically isomorphic" (up to a well-understood direct summand). Our proof of this fact involves once again a blend of graph theory and toric geometry. We note that we have found no evidence that would point towards a modelling theorem for the rational functions $W^+_\Gamma(X,T)$ associated with an interesting class of (not necessarily simple) graphs Γ.

Group-theoretic applications. By a **cographical group (scheme)**, we mean a graphical group (scheme) (see §1.4) arising from a cograph. By combining Corollary B, Theorem C, and Theorem D, we obtain an explicit formula for (local) class counting zeta functions of cographical group schemes in terms of the associated modelling hypergraphs.

In particular, many of our results on ask zeta functions of hypergraphs (e.g. explicit formulae and information on analytic properties) have immediate applications to the class counting zeta functions of the associated cographical group schemes. These are recorded in §8. For instance, as a substantial generalisation of several previously known formulae, we explicitly determine the (local) class counting zeta functions of the cographical group schemes associated with the following classes of cographical groups over $\mathbf{Z}$:

(i) The class of finite direct products of finitely generated free class-2-nilpotent groups.

(ii) The class of class-2-nilpotent free products of free abelian groups of finite rank.

(iii) The smallest class of groups which contains $\mathbf{Z}$ and which is closed under both direct products with $\mathbf{Z}$ and class-2-nilpotent free products with $\mathbf{Z}$.

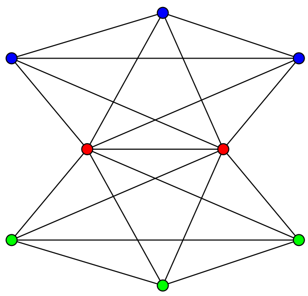
The class counting zeta functions of the cographical group schemes associated with free class-2-nilpotent groups and class-2-nilpotent free products of two free abelian groups have been previously determined by Lins [47, Corollary 1.5].

As we noted above, right-angled Artin groups are close relatives of our graphical groups. Right-angled Artin groups associated with cographs have e.g. been studied in [38, 63].

1.8 A recurring example

We illustrate Theorems A, C, and D by means of a simple yet instructive example that we will repeatedly revisit throughout this paper.

Example 1.6. Let Γ be the following simple graph:



Using the constructive arguments underpinning Theorem A(ii)–(iii), we may explicitly compute the rational functions $W_\Gamma^{\pm}(X,T)$ (see §9):

$$W_\Gamma^-(X,T) = \frac{1 + X^{-6}T - 2X^{-4}T - 2X^{-3}T + X^{-1}T + X^{-7}T^2}{(1-T)^2(1-XT)} \quad \text{and} \tag{1.6}$$

$$W_\Gamma^+(X,T) = F(X,T)/((1-X^{-7}T^2)(1-X^{-5}T^2)(1-X^{-5}T)(1-X^{-4}T)$$
$$(1-X^{-3}T^2)(1-X^{-2}T)^5(1-T^{-2}X^2)(1-T)^2), \tag{1.7}$$

where the (unwieldy) numerator $F(X,T)$ of $W_\Gamma^+(X,T)$ is recorded in Table 5 on p. 103.

Alternatively, the first of these rational functions can be found using Theorems C–D. Indeed, Γ is a cograph for the subgraph induced by all vertices excluding those two depicted on the central horizontal edge is a disjoint union of two complete graphs on three vertices each. As the aforementioned central vertices are connected to all other vertices, it follows that Γ is a cograph; in fact, we have just shown that Γ is isomorphic to $(\mathrm{K}_3 \oplus \mathrm{K}_3) \vee \mathrm{K}_2$, where K_n denotes the complete graph with vertices $1,\dots,n$.

Let H be a hypergraph on 8 vertices with 7 hyperedges and incidence matrix

$$\begin{bmatrix} 1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&1&1&0&0&0\\ 1&1&0&0&1&1&0\\ 1&1&0&0&1&1&0\\ 1&1&0&0&1&1&0 \end{bmatrix} \in \mathrm{M}_{8\times 7}(\mathbf{Z}). \tag{1.8}$$

Write $[n] = \{1,\dots,n\}$. Using the notion of hyperedge multiplicities from §1.6, up to isomorphism, H is thus given by the family $\boldsymbol{\mu} = (\mu_I)_{I\subset[8]}$ with

$$\mu_{[8]} = \mu_{[5]} = \mu_{\{1,2,6,7,8\}} = 2, \quad \mu_{[2]} = 1,$$

and $\mu_I = 0$ for all remaining subsets $I \subset [8]$. Then the explicit form of Theorem D (see §7) shows that $W_\Gamma^-(X,T) = W_{\mathsf{H}}(X,T)$; see Example 7.28. In particular, the formula (1.6) for $W_\Gamma^-(X,T)$ is, in principle, given by Theorem C, as a sum indexed by the poset $\widehat{\mathrm{WO}}([8])$. Rather than handle a sum over the 2,183,340 elements of this poset directly, it is far more convenient to apply some of the tools for recursively computing ask zeta functions associated with hypergraphs that we will develop in §5. For details of this short computation of $W_{\mathsf{H}}(X,T)$, see Example 5.25.

1.9 Results IV and open problems

We collect consequences of our main results from above—to be proved in §8.1—that are both closely related to topics of interest in asymptotic and finite group theory and that seem likely to provide promising avenues for fruitful further research.

Non-negativity. Let Γ be a simple graph with m edges. Let $W_\Gamma^-(X,T)$ be as in Theorem A(ii) and expand

$$W_\Gamma^-(X,X^mT) = \sum_{k=0}^{\infty} f_{\Gamma,k}(X)T^k$$

for $f_{\Gamma,k}(X) \in \mathbf{Q}(X)$. By Corollary B, $f_{\Gamma,k}(q)$ is the class number of the graphical group $\mathbf{G}_\Gamma(\mathfrak{O}/\mathfrak{P}^k)$ for each compact DVR $\mathfrak{O}$ with maximal ideal $\mathfrak{P}$ and residue field size q.

In particular, $f_{\Gamma,1}(X)$ is precisely the polynomial (!) that we denoted by $f_\Gamma(X)$ in Corollary 1.3. Our proof of the latter result implies that, in fact, each $f_{\Gamma,k}(X)$ is a polynomial in X. Inspired by Lehrer's conjecture [43] on character degrees and similar results on class numbers of the groups $\mathrm{U}_n(\mathbf{F}_q)$ by Vera-López et al. (see, for instance, [69]), both refinements of Higman's conjecture from §1.2, we obtain the following.

Theorem E. *Let Γ be a cograph. Then the coefficients of each $f_{\Gamma,k}(X)$ as a polynomial in $X-1$ are non-negative.*

Question 1.7. For which simple graphs Γ does the conclusion of Theorem E hold?

Analytic properties. The most fundamental analytic invariant of a (non-negative) Dirichlet series is its abscissa of convergence which encodes the precise degree of polynomial growth of the series's partial sums. In a seminal paper [24], du Sautoy and Grunewald showed that subgroup zeta functions associated with nilpotent groups have rational abscissae of convergence. The same turns out to be true for class counting zeta functions of arbitrary Baer group schemes. (For a proof, combine Proposition 2.5 below and [57, Theorem 4.20].) For cographical group schemes, we can do much better.

Theorem F. *Let Γ be a cograph with n vertices and m edges. There exists a positive integer $\alpha(\Gamma) \leqslant n+m+1$ such that if $\mathcal{O}_K$ is the ring of integers of an arbitrary number field K, then the abscissa of convergence of $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathcal{O}_K}(s)$ is equal to $\alpha(\Gamma)$. Moreover, if $\mathfrak{O}$ is a compact* DVR*, then the real part of each pole of $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathfrak{O}}(s)$ is a positive integer.*

By [57, Theorem 4.20], for an arbitrary simple graph Γ, there is a (unique) positive rational number $\alpha(\Gamma)$ with properties as in Theorem F. However, it is not clear if $\alpha(\Gamma)$ is always an integer. The positivity of local poles in Theorem F is related to [57, Question 9.4]. The integrality of local poles in Theorem F does not carry over to arbitrary graphs. For instance, for the graph Γ in Example 7.5, the function $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathfrak{O}}(s)$ has a pole at $3/2$.

Question 1.8. Let Γ be a simple graph.

(i) Is $\alpha(\Gamma)$ always an integer?

(ii) Are the real parts of the poles of $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathfrak{O}}(s)$ for compact DVRs $\mathfrak{O}$ always half-integers?

(iii) Is there are a meaningful combinatorial formula (in the spirit of Theorem C) for the functions $W^{\pm}_\Gamma(X,T)$ which is valid for *all* graphs on a given vertex set?

(iv) What do the numbers $\alpha(\Gamma)$ and the poles of class counting zeta functions of graphical group schemes tell us about a graph? How are they related to other graph-theoretic invariants?

1.10 Outline

Section 2. In §2, we collect basic facts about ask zeta functions including, in particular, the crucial duality operations from [60]. Along the way, in §2.4, we formally define

Baer group schemes and relate their class counting zeta functions to ask zeta functions attached to alternating bilinear maps. Apart from reviewing background material, we also develop a “cokernel formalism” (see §2.5) for expressing ask zeta functions in terms of p-adic integrals. In §2.6, we use this to interpret ask zeta functions as special cases of a more general class of zeta functions attached to modules over polynomial rings.

Section 3. After reviewing basic constructions and terminology pertaining to graphs and hypergraphs in §3.1 we define, in §§3.2–3.3, the adjacency and incidence representations informally described in §1.4. We further define adjacency and incidence *modules* and relate their zeta functions in the sense of §2.6 to the ask zeta functions associated with adjacency and incidence representations. In §3.4, we formally define graphical groups and group schemes and relate class counting zeta functions of the latter to ask zeta functions of adjacency representations.

Section 4. Toric geometry enters the scene in §4. We begin by collecting basic facts from convex geometry in §4.1 and on toric rings and schemes in §4.2. In §4.3, we further enlarge the class of zeta functions introduced in §2.6 (which, as we saw, includes ask zeta functions) by attaching zeta functions to modules over toric rings. In §4.4, we prove Theorem A(i) and introduce the key concept of “torically combinatorial” modules that will also form the basis of our proof of Theorem A(i)–(ii).

Section 5. §5 is devoted to a detailed analysis of the rational functions $W_{\mathsf{H}}(X,T)$ attached to hypergraphs H via Theorem A(i). In §5.1, we prove (a slightly more general version of) Theorem C. The remainder of §5 then focuses on two main themes. First, for several classes of hypergraphs of interest, we provide more manageable forms of Theorem A(i). These classes are the “staircase hypergraphs” in §5.1.1, disjoint unions of “block hypergraphs” in §5.2.1, and the “reflections” of the latter family in §5.3.1. Secondly, as we will explore and exploit throughout §5.2–5.4, the general formula provided by Theorem C behaves very well with respect to natural operations on hypergraphs. Finally we deduce, in §5.5, consequences for analytic properties of ask zeta functions of hypergraphs.

Later on, our results from §5 will find group-theoretic applications in §8 via the Cograph Modelling Theorem (Theorem D, proved in §7). In particular, the hypergraph operations alluded to above will translate to natural group-theoretic operations.

Section 6. In §6, we prove Theorem A(ii)–(iii) and also Corollary B. Our proof considers positive and negative adjacency representations of graphs simultaneously by means of a common generalisation, the “weighted signed multigraphs” (WSMs) introduced in §6.1. Multigraphs are more general than graphs in that they allow parallel edges. Each WSM gives rise to an adjacency module (over a suitable toric ring) which generalises the positive and negative adjacency modules of graphs from §3. In §6.2, we describe a number of “surgical procedures” for WSMs that do not affect the associated adjacency modules. Even when the original multigraph was a graph, these procedures may introduce parallel edges—this justifies introducing the concept of WSMs. After some technical preparations in §6.3, we use these procedures in §6.4 to give an inductive proof of Theorem A(ii)–(iii).

Section 7. The Cograph Modelling Theorem (Theorem D) is the subject of §7. We first recall basic facts about cographs in §7.1. In §7.2, we then explain how Theorem D follows from a structural comparison result (Theorem 7.1) relating adjacency modules of cographs and incidence modules of hypergraphs. Extending upon ideas underpinning the proof of Theorem A(ii)–(iii), the remainder of §7 is then devoted to proving Theorem 7.1. An overview of the ingredients featuring in our proof and of our overall strategy is given in §7.3. This is followed by an implementation of this strategy in §§7.4–7.7.

Section 8. In this section we combine Theorem A, Corollary B, Theorem C, Theorem D, and our further analysis of the rational functions $W_{\mathsf{H}}(X,T)$ associated with hypergraphs from §5 to deduce structural properties and produce explicit formulae for class counting zeta functions associated with "cographical group schemes", i.e. graphical group schemes arising from cographs. We consider, in particular, the cographical group schemes associated with the families of nilpotent groups listed in the final part of §1.7 and also relate our results to work of Lins [46, 47] on bivariate conjugacy class zeta functions.

Section 9–10. Based on our constructive proof of Theorem A and computational techniques developed by the first author, in §9, we provide further examples of the rational functions $W_\Gamma^{\pm}(X,T)$ associated with graphs Γ on few vertices. Many of these examples are not covered by Theorems C–D. Motivated by such computational evidence, in §10, we pose and discus a number of questions for further research beyond those already mentioned in §1.9.

1.11 Notation

Sets. The symbol "$\subset$" signifies not necessarily strict inclusion. We write $\sqcup$ for the disjoint union (= coproduct) of sets. Throughout, V denotes a finite set, typically of vertices of a graph or hypergraph and of cardinality n. The power set of V is denoted by $\mathcal{P}(V)$. We write $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. The complement of I within some ambient set V is denoted by $I^{\mathrm{c}} := V \setminus I$. We write $[n] = \{1, 2, \dots, n\}$ and $[n]_0 = [n] \cup \{0\}$.

Rings and modules. Rings are assumed to be associative, commutative, and unital. Let R be a ring. The unit group of R is denoted by $R^\times$. The dual of an R-module M is $M^* = \mathrm{Hom}(M, R)$. An R-algebra consists of a ring S together with a ring map $R \to S$.

For a set V, let $RV = \bigoplus\limits_{v \in V} Rv$ be the free module on V; we extend this notation to subsets of R in the evident way and e.g. write $\mathbf{R}_{\geqslant 0} V = \left\{ \sum\limits_{v \in V} \lambda_v v : \forall v \in V . \lambda_v \in \mathbf{R}_{\geqslant 0} \right\} \subset \mathbf{R}V$, where $\mathbf{R}_{\geqslant 0} = \{x \in \mathbf{R} : x \geqslant 0\}$. For $x \in RV$, we use the suggestive notation $x =$

$\sum_{v\in V} x_v v = (x_v)_{v\in V}$. Often, $X = (X_v)_{v\in V}$ denotes a family of algebraically independent elements over R.

We let $\mathrm{M}_{n\times m}(R)$ (resp. $\mathrm{M}_n(R)$) denote the module of all $n \times m$ (resp. $n \times n$) matrices over R. The transpose of a matrix a is denoted by $a^\top$.

Discrete valuation rings. Throughout this article, $\mathfrak{O}$ denotes a discrete valuation ring (DVR) with maximal ideal $\mathfrak{P}$. We write $\mathfrak{O}_k = \mathfrak{O}/\mathfrak{P}^k$ and $(\,\cdot\,)_k = (\,\cdot\,) \otimes \mathfrak{O}_k$. Let $q = |\mathfrak{O}/\mathfrak{P}|$ denote the size of the residue field of $\mathfrak{O}$. We write $(\underline{k})$ for the number $1 - q^{-k}$.

Let $\nu\colon \mathfrak{O} \to \mathbf{N}_0 \cup \{\infty\}$ denote the (surjective) normalised valuation on $\mathfrak{O}$ and let $|\cdot|$ be the absolute value $|a| = q^{-\nu(a)}$ on $\mathfrak{O}$. For a non-empty collection C of elements of $\mathfrak{O}$, write $\|C\| = \max\{|a| : a \in C\}$. For a free $\mathfrak{O}$-module M of finite rank, μ_M denotes the additive Haar measure on M with $\mu(M) = 1$.

For a non-zero $\mathfrak{O}$-module M, we write $M^\times = M \setminus \mathfrak{P}M$; we let $\{0\}^\times = \{0\}$.

Miscellaneous. Maps usually act on the right. In particular, we regard an $n \times m$ matrix over a ring R as a linear map $R^n \to R^m$.

The $n \times n$ identity matrix is denoted by 1_n. In contrast, $\mathbf{1}_{n\times m}$ (and $\mathbf{1}_n = \mathbf{1}_{n\times n}$) denotes the respective all-one matrix. The all-one vector of length n is denoted $\mathbf{1}^{(n)}$. The free nilpotent group of class at most c on d generators is denoted by $F_{c,d}$.

We write δ_{ij} for the usual "Kronecker delta"; more generally, for a Boolean value P, we let $\delta_P = 1$ if P is true and $\delta_P = 0$ otherwise.

Further notation

Notation	comment	reference
$\zeta^{\mathrm{cc}}_{\mathbf{G}}(s)$	class counting zeta function	§1.3
$W_{\mathsf{H}}(X,T)$, $W^{\pm}_{\Gamma}(X,T)$	ask zeta functions of (hyper)graphs	Thm A
$\theta^S, \eta^{\mathfrak{O}}, \gamma^R_{\pm}, \dots$	base change of module representations	§2.1
θ°, $\theta^\bullet$	Knuth duals	§2.1
$\mathsf{A}^{U,V,W}_\theta(Z)$, $\mathsf{C}^{U,V,W}_\theta(X)$	matrices of a module representations	§2.2
$\mathrm{ask}(\theta)$, $\zeta^{\mathsf{ask}}_\theta(s)$, $\mathsf{Z}^{\mathsf{ask}}_\theta(T)$	average size of kernel, ask zeta functions	§2.3
$\zeta_M(s)$	zeta function associated with a module	§§2.6, 4.3
$\mathrm{V}(\mathsf{H})$, $\mathrm{E}(\mathsf{H})$	vertex and (hyper)edge set of a (hyper)graph	§3.1
$v \sim v'$, $v \sim e$	adjacency and incidence relation	§3.1
$\mathsf{H}(\boldsymbol{\mu})$, $\mathsf{H}(V \mid \sum \mu_I I)$	hypergraph with given hyperedge multiplicities	Def. 3.1
$\mathsf{H}_1 \oplus \mathsf{H}_1$, $\mathsf{H}_1 \circledast \mathsf{H}_2$	disjoint resp. complete union	§3.1
inc, Inc, η	incidence modules and representations	§3.2
adj, Adj, $\gamma_\pm$	adjacency modules and representations	§§3.3, 7.5
$G_1 \circledast G_2$	free class-2-nilpotent product of groups	(3.11)
$\mathbf{G}_\Gamma$	graphical group scheme	§3.4
$\lvert\mathcal{F}\rvert$	support of a fan	§4.1
$\mathcal{F}_1 \wedge \mathcal{F}_2$	coarsest common refinement of fans	§4.1
$\leqslant_\sigma$	preorder defined by a cone	§4.1

R_σ	toric ring	§4.2
$\sigma(\mathfrak{O})$	"rational points" of a cone over a DVR	§4.1
$\mathrm{gp}\,(x)$ (resp. $\mathrm{gp}_0\,(x)$)	$\frac{x}{1-x}$ (resp. $\frac{1}{1-x}$)	§5.1
$\widehat{\mathrm{WO}}(V)$, $\widetilde{\mathrm{WO}}(V)$	weak orders	Def. 5.3
$F(T) \star G(T)$	Hadamard product	§5.2
$\mathsf{H}^{\mathbf{1}}, \mathsf{H}_{\mathbf{1}}, \mathsf{H}^{\mathbf{0}}, \mathsf{H}_{\mathbf{0}}$	insert all-one or all-zero row or column	Def. 5.22
$\mathbf{\Gamma}$	weighted signed multigraph (WSM)	§6.1
$\mathsf{H}(\mathcal{S})$, $\mathbf{\Gamma}(\mathcal{S})$	hypergraph and WSM defined by a scaffold	§7.5
$\mathrm{Kite}(k)$	kite graph	§8.4

2 Ask zeta functions and modules over polynomial rings

In this section, we recall background material on module representations and associated ask zeta functions from [57, 60]. We also relate the latter functions to class counting zeta functions associated with Baer group schemes. Finally, we develop a "cokernel formalism" for ask zeta functions which allows us to view the latter as special cases of a more general class of functions attached to modules over polynomial rings.

Throughout, let R be a ring.

2.1 Module representations

In §1.3, we attached ask zeta functions to module homomorphisms $M \to \mathrm{M}_{n\times m}(R)$. Rather than focusing on such parameterisations of modules of matrices, we will use the following coordinate-free approach from [60, §2]; as shown in [60], disposing of coordinates elucidates duality phenomena.

By a **module representation** over R, we mean a homomorphism $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$, where A, B, and C are R-modules.

Base change. For a ring map $R \xrightarrow{\lambda} S$, an R-module M, and an S-module N, we let $M^\lambda = M \otimes_R S$ (resp. N_λ) denote the extension (resp. restriction) of scalars of M (resp. N) along λ; this is an S-module (resp. an R-module). When the reference to λ is clear, we simply write $M^S = M^\lambda$ and $N_R = N_\lambda$.

Let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ be a module representation over R. Given a ring map $R \xrightarrow{\lambda} S$, extension of scalars along λ yields a module representation

$$A^\lambda \xrightarrow{\theta^\lambda} \mathrm{Hom}(B^\lambda, C^\lambda)$$

over S. When the reference to λ is clear, we write $\theta^S = \theta^\lambda$.

Knuth duals. Given a module representation $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$, let θ° denote the module representation $B \to \mathrm{Hom}(A, C)$ with $a(b\theta^\circ) = b(a\theta)$ for $a \in A$ and $b \in B$.

Write $(\,\cdot\,)^* = \mathrm{Hom}(\,\cdot\,, R)$. Apart from θ°, the module representation θ also gives rise to a module representation $C^* \xrightarrow{\theta^\bullet} \mathrm{Hom}(B, A^*)$ defined by $a(b(\psi\theta^\bullet)) = (b(a\theta))\psi$ for $a \in A$, $b \in B$, and $\psi \in C^*$.

Note that $(\theta^\lambda)^\circ = (\theta^\circ)^\lambda$ for each ring map $R \xrightarrow{\lambda} S$. Moreover, if A and C are both finitely generated and projective, then we may identify $(\theta^\bullet)^\lambda = (\theta^\lambda)^\bullet$. For more on the operations $\theta \mapsto \theta^\circ$ and $\theta \mapsto \theta^\bullet$, see [60, §§4–5].

Direct sums, homotopy, and isotopy. Let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ and $A' \xrightarrow{\theta'} \mathrm{Hom}(B', C')$ be module representations over R. The **direct sum** of θ and θ' is the module representation

$$A \oplus A' \xrightarrow{\theta \oplus \theta'} \mathrm{Hom}(B \oplus B', C \oplus C'), \quad (a, a') \mapsto a\theta \oplus a'\theta'.$$

A **homotopy** $\theta \to \theta'$ is a triple of homomorphisms $(A \xrightarrow{\alpha} A', B \xrightarrow{\beta} B', C \xrightarrow{\gamma} C')$ such that the following diagram commutes for each $a \in A$:

$$\begin{array}{ccc} B & \xrightarrow{a\theta} & C \\ {\scriptstyle\beta}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ B' & \xrightarrow{(a\alpha)\theta'} & C'. \end{array}$$

Module representations over R together with homotopies as morphisms naturally form a category. An invertible homotopy is called an **isotopy**.

2.2 Matrices associated with module representations involving free modules

Up to isotopy, a module representation involving *free* modules of finite rank can be equivalently expressed in terms of a matrix of linear forms. In detail, let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ be a module representation over R. Suppose that each A, B, and C is free of finite rank. By choosing bases U, V, and W of A, B, and C, respectively, we may identify $A = RU$, $B = RV$, and $C = RW$.

Let $Z = (Z_u)_{u \in U}$ consist of algebraically independent variables over R. Define an $R[Z]$-linear map

$$\mathsf{A}_\theta^{U,V,W}(Z) := \Bigl(\sum_{u \in U} Z_u u\Bigr)\theta^{R[Z]} \in \mathrm{Hom}(R[Z]V,\, R[Z]W).$$

Informally, $\mathsf{A}_\theta^{U,V,W}(Z)$ is the image of a "generic element" of $A = RU$ under θ. The matrix of $\mathsf{A}_\theta^{U,V,W}(Z)$ with respect to the bases V and W of $R[Z]V$ and $R[Z]W$, respectively, is the matrix of linear forms associated with θ (and the chosen bases) from [60, §4.4].

As we will now explain, by specialising $\mathsf{A}_\theta^{U,V,W}(Z)$, we may recover θ (and θ^λ for each ring map $R \xrightarrow{\lambda} S$).

Lemma 2.1. *Let S be an R-algebra and let $z \in SU$. Let S_z denote S regarded as an $R[Z]$-algebra via $Z_u s = z_u s$ $(u \in U, s \in S)$.*

(i) $\mathsf{A}_\theta^{U,V,W}(z) := \mathsf{A}_\theta^{U,V,W}(Z) \otimes_{R[Z]} S_z \in \mathrm{Hom}(SV, SW)$ *coincides with* $z\theta^S$.

(ii) $\mathrm{Coker}(z\theta^S) \approx_S \mathrm{Coker}(\mathsf{A}_\theta^{U,V,W}(Z)) \otimes_{R[Z]} S_z$.

Proof. Part (i) is clear. Part (ii) follows from (i) and right exactness of tensor products. ♦

Remark 2.2. The isomorphism types of the cokernels in Lemma 2.1(ii) (over S and $R[Z]$, respectively) clearly only depend on the isotopy type (see §2.1) of θ.

Recall the definition of θ° from §2.1. In subsequent sections, we will analyse and compute certain zeta functions associated with θ by studying the sizes of $\mathrm{Coker}(x(\theta^\circ)^S)$ for suitable finite R-algebras S and $x \in SV$. For this purpose, it will be convenient to use explicit (finite) presentations of these cokernels. Let $X = (X_v)_{v\in V}$ consist of algebraically independent variables over R. In accordance with the notation in [57, §4.3.5], let

$$\mathsf{C}_\theta^{U,V,W}(X) := \mathsf{A}_{\theta^\circ}^{V,U,W}(X) = \left(\sum_{v\in V} X_v v\right)(\theta^\circ)^{R[X]} \in \mathrm{Hom}\left(R[X]U,\, R[X]W\right). \tag{2.1}$$

Informally, the image of $\mathsf{C}_\theta^{U,V,W}(X)$ is the "additive orbit" $xM = \{xa : a \in M\}$, where x is a "generic element" of RV and M denotes the image of θ. We can read off explicit generators for the image of $\mathsf{C}_\theta^{U,V,W}(X)$ and hence a presentation for the latter's cokernel.

Lemma 2.3. $\mathrm{Im}(\mathsf{C}_\theta^{U,V,W}(X)) = \left\langle \Big(\sum_{v\in V} X_v v\Big)(u\theta^{R[X]}) : u \in U \right\rangle.$ ♦

Note that we may identify $u\theta^{R[X]} = u\theta \otimes_R R[X]$ for $u \in U$.

2.3 Reminder: ask zeta functions

Let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ be a module representation over a ring R. If A and B are both finite as sets, then the **average size of the kernel** of A acting on B via θ is the number

$$\mathrm{ask}(\theta) := \frac{1}{|A|} \sum_{a\in A} |\mathrm{Ker}(a\theta)|.$$

Suppose that R admits only finitely many ideals of a given finite index. Further suppose that A, B, and C are finitely generated. The **(analytic) ask zeta function** associated with θ is the Dirichlet series

$$\zeta_\theta^{\mathsf{ask}}(s) := \sum_{I \triangleleft R} \mathrm{ask}(\theta^{R/I}) |R/I|^{-s},$$

where s is a complex variable and the summation extends over those ideals $I \triangleleft R$ with $|R/I| < \infty$. If R is the ring of integers of a global or local field, then $\zeta_\theta^{\mathsf{ask}}(s)$ defines an analytic function on a right half-plane $\{s \in \mathbf{C} : \mathrm{Re}(s) > \alpha\}$; cf. [57, §§3.2–3.3]. The infimum of all such real numbers $\alpha > 0$ is the **abscissa of convergence** α_θ of $\zeta_\theta^{\mathsf{ask}}(s)$.

Let $\mathfrak{O}$ be a compact DVR with maximal ideal $\mathfrak{P}$. We write $\mathfrak{O}_k = \mathfrak{O}/\mathfrak{P}^k$ and, more generally, $(\,\cdot\,)_k = (\,\cdot\,) \otimes_{\mathfrak{O}} \mathfrak{O}_k$. Let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ be a module representation over $\mathfrak{O}$. Suppose that each of A, B, and C is free of finite rank. We may identify $\theta_k = \theta \otimes_{\mathfrak{O}} \mathfrak{O}_k$ and $\theta^{\mathfrak{O}_k}$ (see §2.1). Consider the generating function

$$\mathsf{Z}_\theta^{\mathsf{ask}}(T) := \sum_{k=0}^{\infty} \mathrm{ask}(\theta_k) T^k.$$

By slight abuse of terminology, we also refer to $\mathsf{Z}_\theta^{\mathsf{ask}}(T)$ as the **(algebraic) ask zeta function** of θ. By [57, Theorem 1.4], if $\mathfrak{O}$ has characteristic zero, then $\mathsf{Z}_\theta^{\mathsf{ask}}(T) \in \mathbf{Q}(T)$.

Note that the analytic function $\zeta_\theta^{\mathsf{ask}}(s)$ and its algebraic counterpart $\mathsf{Z}_\theta^{\mathsf{ask}}(T)$ determine one another: $\zeta_\theta^{\mathsf{ask}}(s) = \mathsf{Z}_\theta^{\mathsf{ask}}(q^{-s})$, where q denotes the residue field size of $\mathfrak{O}$. As explained in [57, §8], ask zeta functions (of either type) arise naturally in the enumeration of orbits and conjugacy classes of unipotent groups. Most of the main results of this article (e.g. Theorems A,C–D) are stated in terms of the generating functions $\mathsf{Z}_\theta^{\mathsf{ask}}(T)$ while the analytic functions $\zeta_\theta^{\mathsf{ask}}(s)$ feature in our proofs by means of suitable p-adic integrals.

We will rely heavily on the fact that the duality operations $\theta \mapsto \theta^\circ$ and $\theta \mapsto \theta^\bullet$ (see §2.1) have the following tame effects on ask zeta functions.

Theorem 2.4 ([60, Corollary 5.6])**.** $\mathsf{Z}_\theta^{\mathsf{ask}}(T) = \mathsf{Z}_{\theta^\circ}^{\mathsf{ask}}\left(q^{\mathrm{rk}(B)-\mathrm{rk}(A)}T\right) = \mathsf{Z}_{\theta^\bullet}^{\mathsf{ask}}(T)$.

2.4 Application: class counting zeta functions of Baer group schemes

Consider an alternating bilinear map $\diamond\colon A \times A \to B$ of $\mathbf{Z}$-modules. Suppose that B is uniquely 2-divisible (i.e. a $\mathbf{Z}[1/2]$-module). The **Baer group** associated with $\diamond$ is the nilpotent group $G_\diamond$ of class at most 2 on the set $A \times B$ with multiplication

$$(a, b) * (a', b') = \left(a + a', b + b' + \frac{1}{2}(a \diamond a')\right);$$

this construction is part of the **Baer correspondence** [2]. We now describe a version of the operation $\diamond \rightsquigarrow G_\diamond$ for group schemes.

Let V and E be disjoint finite sets and let $\diamond\colon \mathbf{Z}V \times \mathbf{Z}V \to \mathbf{Z}E$ be an alternating bilinear map. We obtain a nilpotent Lie $\mathbf{Z}$-algebra ("Lie ring") $\mathfrak{g}_\diamond$ of class at most 2 with underlying $\mathbf{Z}$-module $\mathbf{Z}V \oplus \mathbf{Z}E$ and Lie bracket $[x + c, y + d] = x \diamond y$ for $x, y \in \mathbf{Z}V$ and $c, d \in \mathbf{Z}E$.

Let $\sqsubseteq$ be a total order on $V \cup E$. By [66, §2.4.1], there exists a (unipotent) group scheme $\mathbf{G}_\diamond = \mathbf{G}_{\diamond,\sqsubseteq}$ over $\mathbf{Z}$ such that for each ring R, we may identify $\mathbf{G}_\diamond(R) = RV \oplus RE$ as sets and such that group commutators in $\mathbf{G}_\diamond(R)$ coincide with Lie brackets in $\mathfrak{g}_\diamond \otimes_{\mathbf{Z}} R$. The multiplication $*$ on $\mathbf{G}_\diamond(R)$ is characterised by the following properties:

(i) For $v_1 \sqsubseteq \cdots \sqsubseteq v_k$ in V and $r_1, \ldots, r_k \in R$, $(r_1 v_1) * \cdots * (r_k v_k) = r_1 v_1 + \cdots + r_k v_k$.

(ii) For $v, w \in V$ with $v \sqsubseteq w$ and $r, s \in R$, we have $(sw) * (rv) = rv + sw - rs(v \diamond w)$.

(iii) RE is a central subgroup of $\mathbf{G}_\diamond(R)$ and $x * c = x + c$ for $x \in \mathbf{G}_\diamond(R)$ and $c \in RE$.

Up to isomorphism, $\mathbf{G}_\diamond$ does not depend on $\sqsubseteq$. We call $\mathbf{G}_\diamond$ the **Baer group scheme** associated with $\diamond$. For each ring R in which 2 is invertible, $\mathbf{G}_\diamond(R)$ is isomorphic to the Baer group attached to the alternating bilinear map $RV \times RV \to RE$ obtained from $\diamond$.

Proposition 2.5. *Let* $\diamond\colon \mathbf{Z}V \times \mathbf{Z}V \to \mathbf{Z}E$ *be an alternating bilinear map. Let* $\mathbf{Z}V \xrightarrow{\alpha} \mathrm{Hom}(\mathbf{Z}V, \mathbf{Z}E)$ *be the module representation with* $v(w\alpha) = v \diamond w$ *for* $v, w \in V$. *Let* $m = |E|$. *Let* R *be the ring of integers of a local or global field of arbitrary characteristic. Let* $\mathbf{G}_\diamond$ *be the Baer group scheme associated with* $\diamond$ *as above. Then* $\zeta^{\mathsf{cc}}_{\mathbf{G}_\diamond \otimes R}(s) = \zeta^{\mathsf{ask}}_{\alpha^R}(s - m)$.

Proof. For each finite ring A, commutators in $\mathbf{G}_\diamond(A)$ are given by Lie brackets in $\mathfrak{g}_\diamond \otimes_{\mathbf{Z}} A$. By the same reasoning as in [60, Lemma 7.1], we see that $\mathrm{k}(\mathbf{G}_\diamond(A)) = |A|^m \,\mathrm{ask}(\alpha^A)$. ♦

Proof of Proposition 1.1. Let $\diamond\colon \mathbf{Z}^n \times \mathbf{Z}^n \to M^*$ be the alternating bilinear map given by $a(x \diamond y) = xay^\top$ from §1.2. Let $\mathbf{Z}^n \xrightarrow{\alpha} \mathrm{Hom}(\mathbf{Z}^n, M^*)$ with $x(y\alpha) = x \diamond y$. By Proposition 2.5, $\zeta^{\mathsf{cc}}_{\mathbf{G}_M \otimes R}(s) = \zeta^{\mathsf{ask}}_{\alpha^R}(s - m)$. We claim that $\alpha^\bullet$ (see §2.1) is isotopic to $M \overset{\iota}{\hookrightarrow} \mathfrak{so}_n(\mathbf{Z})$. By [60, Proposition 4.8], this is equivalent to $\iota^\bullet$ being isotopic to α which is easily verified using the isomorphism $(\mathbf{Z}^n)^* \approx \mathbf{Z}^n$ given by matrix transposition. Using Theorem 2.4, we conclude that $\zeta^{\mathsf{ask}}_{\alpha^R}(s) = \zeta^{\mathsf{ask}}_{\iota^R}(s)$ provided that the field of fractions of R is a local field. Finally, using [60, Remark 5.5], the global case reduces to the local one. ♦

2.5 Cokernel integrals

From now on, let $\mathfrak{O}$ be a compact DVR. Let $A \xrightarrow{\theta} \mathrm{Hom}(B, C)$ be a module representation over $\mathfrak{O}$, where each of A, B, and C is free of finite rank. For $a \in A$ and $y \in \mathfrak{O}$, the map $B \xrightarrow{a\theta} C$ gives rise to an induced map $B \otimes_{\mathfrak{O}} \mathfrak{O}/y \xrightarrow{a\theta \otimes \mathfrak{O}/y = (a \otimes 1)\theta^{\mathfrak{O}/y}} C \otimes_{\mathfrak{O}} \mathfrak{O}/y$. Define

$$\mathbb{C}_\theta(a, y) := |\mathrm{Coker}(a\theta \otimes \mathfrak{O}/y)|.$$

The following is equivalent to [57, Theorem 4.5], but with kernels replaced by cokernels.

Theorem 2.6. *For* $\mathrm{Re}(s) \gg 0$,

$$\zeta^{\mathsf{ask}}_\theta(s) = (1 - q^{-1})^{-1} \int\limits_{A \times \mathfrak{O}} |y|^{s - \mathrm{rk}(B) + \mathrm{rk}(C) - 1} \; \mathbb{C}_\theta(a, y) \;\, \mathrm{d}\mu_{A \times \mathfrak{O}}(a, y).$$

Proof. As in [57, Definition 4.4] (see [60, §3.5]), write $\mathbb{K}_\theta(a, y) = |\mathrm{Ker}(a\theta \otimes \mathfrak{O}/y)|$. The claim then follows immediately from [57, Theorem 4.5] (cf. [60, Theorem 3.5]) and the fact that by the first isomorphism theorem, $\mathbb{K}_\theta(a, y) = \mathbb{C}_\theta(a, y) \cdot |y|^{\mathrm{rk}(C) - \mathrm{rk}(B)}$ for $y \neq 0$. ♦

Remark 2.7. In the present article, we express ask zeta functions in terms of sizes of cokernels, rather than kernels; see Corollary 2.8. Cokernels turn out to be more convenient here since they commute with base change (both being colimits).

Explicit dual form of Theorem 2.6. We can make Theorem 2.6 more explicit by choosing bases. Let $A = \mathfrak{O}U$, $B = \mathfrak{O}V$, and $C = \mathfrak{O}W$, where U, V, and W are finite sets. As in §2.2, let $Z = (Z_u)_{u\in U}$ consist of algebraically independent elements. By Lemma 2.1,

$$\mathbb{C}_\theta(a, y) = |\mathrm{Coker}(\mathsf{A}_\theta^{U,V,W}(Z) \otimes_{\mathfrak{O}[Z]} (\mathfrak{O}/y)_a)|, \tag{2.2}$$

where $(\mathfrak{O}/y)_a$ denotes $\mathfrak{O}/y$ with the $\mathfrak{O}[Z]$-module structure $Z_u r = a_u r$ ($u \in U$, $r \in \mathfrak{O}/y$).

It will be convenient to express $\zeta_\theta^{\mathsf{ask}}(s)$ in terms of θ° via Theorem 2.4. Let $X = (X_v)_{v\in V}$ consist of algebraically independent variables over $\mathfrak{O}$. Recall from (2.1) the definition of the $\mathfrak{O}[X]$-homomorphism $\mathsf{C}_\theta^{U,V,W}(X)$.

Corollary 2.8. *For* $\mathrm{Re}(s) \gg 0$,

$$\zeta_\theta^{\mathsf{ask}}(s) = (1-q^{-1})^{-1} \int\limits_{\mathfrak{O}V\times\mathfrak{O}} |y|^{s-|V|+|W|-1}\, |\mathrm{Coker}(\mathsf{C}_\theta^{U,V,W}(X)) \otimes_{\mathfrak{O}[X]} (\mathfrak{O}/y)_x|\; \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x, y).$$

Proof. Combine Theorem 2.4, equation (2.2), and Theorem 2.6. ♦

2.6 Zeta functions associated with modules over polynomial rings

We will study ask zeta functions by considering a sequence of successive generalisations of the integrals featuring in Corollary 2.8. As our first step in this direction, we consider integrals obtained from Corollary 2.8 by allowing $\mathrm{Coker}(\mathsf{C}_\theta^{U,V,W}(X))$ to be a more general type of $\mathfrak{O}[X]$-module.

As before, we write $X = (X_v)_{v\in V}$, where the X_v are algebraically independent variables over $\mathfrak{O}$ (or whichever base ring we consider).

Let M be a finitely generated $\mathfrak{O}[X]$-module. The example of primary interest to us at this point is the case $M = \mathrm{Coker}(\mathsf{C}_\theta^{U,V,W}(X))$ (see (2.1)), where θ is a suitable module representation over $\mathfrak{O}$. As in §2.2, for $x \in \mathfrak{O}V$, let $\mathfrak{O}_x$ denote $\mathfrak{O}$ endowed with the $\mathfrak{O}[X]$-module structure $X_v r = x_v r$ ($v \in V$, $r \in \mathfrak{O}$). More generally, for an arbitrary $\mathfrak{O}$-module N, we let N_x denote the $\mathfrak{O}[X]$-module $N \otimes_{\mathfrak{O}} \mathfrak{O}_x$ (cf. Lemma 2.10).

Definition 2.9. Define a zeta function

$$\zeta_M(s) := \int\limits_{\mathfrak{O}V\times\mathfrak{O}} |y|^{s-1} \cdot |M_x \otimes_{\mathfrak{O}} \mathfrak{O}/y|\; \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x, y). \tag{2.3}$$

The following simple observation will be used repeatedly throughout this article.

Lemma 2.10. *Let R be a ring. Let S and S' be R-algebras. Let $\mathcal{X}' = \mathrm{Spec}(S')/R$ and let $x \in \mathcal{X}'(S)$. Let $S' \xrightarrow{\chi} S$ be the R-algebra map corresponding to χ. Let M be an S-module and let M' be an S'-module. Let $M'_x := (M')^\chi$ and $M_x := M_\chi$ (see §2.1); these are both (S, S')-bimodules. Then $M_x \otimes_{S'} M'$ and $M \otimes_S M'_x$ both carry naturally isomorphic (S, S')-bimodule structures.*

Proof. Apply [9, Ch. II, §3, no. 8] with $A = S$, $B = S'$, $E = M$, $F = S_\chi$, and $G = M'$. ♦

Remark 2.11. Note that for $x \in \mathfrak{O}V = \mathrm{Spec}(\mathfrak{O}[X])(\mathfrak{O})$, the definitions of the $\mathfrak{O}$-modules M_x and $(\mathfrak{O}/y)_x$ provided Lemma 2.10 coincide with those given above. In particular, Lemma 2.10 allows us to identify $M_x \otimes_{\mathfrak{O}} \mathfrak{O}/y = M \otimes_{\mathfrak{O}[X]} (\mathfrak{O}/y)_x$ in (2.3). (For a proof, relabel $M \leftrightarrow M'$ and take $S = \mathfrak{O}$, $S' = \mathfrak{O}[X]$, and $M = \mathfrak{O}/y$ in Lemma 2.10.)

Remark 2.12. If $\mathfrak{O}$ has characteristic zero, then standard arguments from p-adic integration show that $\zeta_M(s) \in \mathbf{Q}(q^{-s})$ and also establish "Denef-type formulae" in a global setting; cf. [57, §4.3.3].

The following is immediate from Corollary 2.8.

Corollary 2.13. *Let U, V, and W be finite sets and $X = (X_v)_{v\in V}$ as before. Let $\mathfrak{O}U \xrightarrow{\theta} \mathrm{Hom}(\mathfrak{O}V, \mathfrak{O}W)$ be a module representation over $\mathfrak{O}$. Let $M = \mathrm{Coker}(\mathsf{C}_\theta^{U,V,W}(X))$; see* (2.1). *Then*

$$\zeta_\theta^{\mathsf{ask}}(s) = (1-q^{-1})^{-1}\,\zeta_M(s - |V| + |W|). \qquad \blacklozenge$$

In particular, the zeta functions attached to modules over polynomial rings in Definition 2.9 generalise local ask zeta functions. We will further generalise the former functions by suitably replacing polynomial rings by toric rings; see Definition 4.4. As we will see, this greater generality will provide us with the means to study "toric properties" of ask zeta functions by purely combinatorial means.

3 Modules and module representations from (hyper)graphs

In this section, we begin by fixing our notation for various concepts related to graphs and hypergraphs. Formalising and generalising our discussion from §1.4, for each hypergraph H and (simple) graph Γ, we define the incidence representation η of H and the adjacency representations $\gamma_\pm$ of Γ, as well as corresponding incidence and adjacency modules $\mathrm{Inc}(\mathsf{H})$ and $\mathrm{Adj}(\Gamma; \pm 1)$. We also define graphical group schemes and express their class counting zeta functions in terms of ask zeta functions.

Throughout, let R be a ring.

3.1 Graphs, multigraphs, and hypergraphs

For a general reference on hypergraph theory, see e.g. [13].

Hypergraphs. A **hypergraph** is a triple $\mathsf{H} = (V, E, |\cdot|)$ consisting of a finite set V of **vertices**, a finite set E of **hyperedges**, and a **support function** $E \xrightarrow{|\cdot|} \mathcal{P}(V)$. We often tacitly assume that $V \cap E = \varnothing$. When confusion is unlikely, we often omit $|\cdot|$ (and occasionally even V and E) from our notation. We write $\mathrm{V}(\mathsf{H}) = V$, $\mathrm{E}(\mathsf{H}) = E$, and $|\cdot|_{\mathsf{H}} = |\cdot|$. Two hyperedges e and e' are **parallel** if $|e| = |e'|$. An **edge** is a hyperedge e with $\#|e| \in \{1, 2\}$. An edge e with $\#|e| = 1$ is a **loop**. The **reflection** of H is the hypergraph $\mathsf{H}^{\mathrm{c}} = (V, E, |\cdot|^{\mathrm{c}})$ with $|e|^{\mathrm{c}} = V \setminus |e|$. An **isomorphism** between hypergraphs H and H' consists of bijections $\mathrm{V}(\mathsf{H}) \xrightarrow{\phi} \mathrm{V}(\mathsf{H}')$ and $\mathrm{E}(\mathsf{H}) \xrightarrow{\psi} \mathrm{E}(\mathsf{H}')$ such that $|e|_{\mathsf{H}}\,\mathcal{P}(\phi) = |e\psi|_{\mathsf{H}'}$ for each $e \in \mathrm{E}(\mathsf{H})$, where $\mathcal{P}(\phi)$ is the direct image map $\mathcal{P}(\mathrm{V}(\mathsf{H})) \to \mathcal{P}(\mathrm{V}(\mathsf{H}'))$ induced by ϕ.

Incidence matrices. Let $\mathsf{H} = (V, E, |\cdot|)$ be a hypergraph. A vertex $v \in V$ and hyperedge $e \in E$ are **incident**, written $v \sim_{\mathsf{H}} e$ or simply $v \sim e$, if $v \in |e|$. Let $\mathbf{I}(\mathsf{H})$ denote the set of all pairs $(v, e) \in V \times E$ with $v \sim_{\mathsf{H}} e$. Write $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, where $n = |V|$ and $m = |E|$. The **incidence matrix** of H with respect to the given orderings of the elements of V and E is the $n \times m$ $(0,1)$-matrix $[a_{ij}]$ with $a_{ij} = 1$ if and only if $v_i \sim_{\mathsf{H}} e_i$.

Hypergraph operations. The **disjoint union** $\mathsf{H}_1 \oplus \mathsf{H}_2$ of hypergraphs $\mathsf{H}_i = (V_i, E_i, |\cdot|_i)$ $(i = 1, 2)$ is the hypergraph on the vertex set $V_1 \sqcup V_2$ (disjoint union) with hyperedge set $E_1 \sqcup E_2$ and support function $\|e_i\| = |e_i|_i$ for $e_i \in E_i$. If $A_i \in \mathrm{M}_{n_i \times m_i}(\mathbf{Z})$ is an incidence matrix of H_i, then the block-diagonal matrix

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \in \mathrm{M}_{(n_1+n_2)\times(m_1+m_2)}(\mathbf{Z})$$

is an incidence matrix of $\mathsf{H}_1 \oplus \mathsf{H}_2$.

The **complete union** $\mathsf{H}_1 \circledast \mathsf{H}_2$ is the hypergraph on the vertex set $V_1 \sqcup V_2$ with hyperedge set $E_1 \sqcup E_2$ and support function $\|e_i\| = |e_i|_i \sqcup V_j$ for $e_i \in E_i$ and $i + j = 3$. If $A_i \in \mathrm{M}_{n_i \times m_i}(\mathbf{Z})$ is an incidence matrix of H_i, then

$$\begin{bmatrix} A_1 & \mathbf{1}_{n_1 \times m_2} \\ \mathbf{1}_{n_2 \times m_1} & A_2 \end{bmatrix} \in \mathrm{M}_{(n_1+n_2)\times(m_1+m_2)}(\mathbf{Z})$$

is an incidence matrix of $\mathsf{H}_1 \circledast \mathsf{H}_2$; recall that $\mathbf{1}_{n\times m}$ denotes the $n \times m$ all-one matrix.

Both disjoint unions and complete unions naturally extend to families of more than two hypergraphs. These two operations are related by reflections of hypergraphs via the identity $(\mathsf{H}_1 \oplus \mathsf{H}_2)^{\mathrm{c}} = \mathsf{H}_1^{\mathrm{c}} \circledast \mathsf{H}_2^{\mathrm{c}}$.

(Multi-)graphs. A **multigraph** is a hypergraph $\Gamma = (V, E, |\cdot|)$ all of whose hyperedges are edges. A **graph** is a multigraph without parallel edges; note that we allow graphs to contain loops. Two vertices v and v' of a graph Γ are **adjacent** if there exists an edge $e \in E$ with $|e| = \{v, v'\}$; we write $v \sim_\Gamma v'$ or $v \sim_\Gamma v'$ in that case. (This notation is unambiguous whenever $V \cap E = \varnothing$ which we tacitly assume.) The set of **neighbours** of $v \in V$ in Γ is $\{w \in V : v \sim w\}$. A graph is **simple** if it contains no loops. The **join** $\Gamma_1 \vee \Gamma_2$ of two simple graphs Γ_1 and Γ_2 is the simple graph obtained from the disjoint union $\Gamma_1 \oplus \Gamma_2$ by adding an edge between each pair of vertices $(v_1, v_2) \in V_1 \times V_2$.

Parameterising hypergraphs. Up to isomorphism, a hypergraph $\mathsf{H} = (V, E, |\cdot|)$ determines and is determined by the cardinalities of the fibres $\mu_I := \#\{e \in E : |e| = I\} \in \mathbf{N}_0$ for $I \subset \mathrm{V}(\mathsf{H})$. More formally:

Definition 3.1. Let V be a finite set. Given a vector $\boldsymbol{\mu} = (\mu_I)_{I \subset V} \in \mathbf{N}_0^{\mathcal{P}(V)}$ of non-negative multiplicities, define a hypergraph $\mathsf{H}(\boldsymbol{\mu})$ with

$$\mathrm{V}(\mathsf{H}(\boldsymbol{\mu})) = V, \quad \mathrm{E}(\mathsf{H}(\boldsymbol{\mu})) = \{(I, j) : I \subset V, j \in [\mu_I]\}, \quad |(I, j)|_{\mathsf{H}(\boldsymbol{\mu})} = I.$$

In other words, $\mathsf{H}(\boldsymbol{\mu})$ contains precisely μ_I hyperedges with support I for each $I \subset V$. We often write $m = \sum_{I \subset V} \mu_I$ for the total number of hyperedges of $\mathsf{H}(\boldsymbol{\mu})$. Clearly, for each hypergraph H, there exists a unique vector $\boldsymbol{\mu}$ as in Definition 3.1 such that H and $\mathsf{H}(\boldsymbol{\mu})$ are isomorphic by means of an isomorphism fixing each vertex.

We will use the following shorthand notation for the hypergraphs $\mathsf{H}(\boldsymbol{\mu})$. For a finite set V, suppose that we are given numbers $\mu_I \in \mathbf{N}_0$ for some but perhaps not all subsets $I \subset V$. We may then extend the collection of these μ_I to a family $\boldsymbol{\mu}$ as in Definition 3.1 by setting $\mu_J = 0$ for the previously missing subsets $J \subset V$. We set

$$\mathsf{H}\Big(V \;\Big|\; \sum_I \mu_I I\Big) := \mathsf{H}(\boldsymbol{\mu});$$

to further simplify our notation, we often drop coefficients $\mu_I = 1$ and summands $\mu_I I$ with $\mu_I = 0$ from the left-hand side.

Important families of (hyper)graphs. The following hypergraphs will feature in several places throughout this article; most have vertex set $V = [n]$.

The **discrete (hyper)graph** on n vertices (often called an "empty graph" in the literature) is

$$\Delta_n := \mathsf{H}([n] \mid 0) := \mathsf{H}([n] \mid 0[n]). \tag{3.1}$$

The $n \times m$ **block hypergraph** is the hypergraph

$$\mathsf{BH}_{n,m} := \mathsf{H}([n] \mid m[n]). \tag{3.2}$$

We denote the reflection of $\mathsf{BH}_{n,m}$ by $\mathsf{PH}_{n,m}$; that is, $\mathsf{PH}_{n,m} = \mathsf{H}([n] \mid m\varnothing)$. More generally, given $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$ and $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbf{N}_0^r$, let

$$\mathsf{BH}_{\mathbf{n},\mathbf{m}} := \mathsf{BH}_{n_1,m_1} \oplus \cdots \oplus \mathsf{BH}_{n_r,m_r} \quad \text{and} \tag{3.3}$$

$$\mathsf{PH}_{\mathbf{n},\mathbf{m}} := \mathsf{BH}^{\mathrm{c}}_{\mathbf{n},\mathbf{m}} = \mathsf{PH}_{n_1,m_1} \circledast \cdots \circledast \mathsf{PH}_{n_r,m_r}\,. \tag{3.4}$$

The **complete graph** on n vertices is

$$\mathrm{K}_n := \mathsf{H}([n] \mid \sum_{1 \leqslant i < j \leqslant n} \{i,j\}). \tag{3.5}$$

The **star graph** on n vertices (with centre 1) is

$$\mathrm{Star}_n := \mathsf{H}([n] \mid \sum_{1 < i \leqslant n} \{1,i\}). \tag{3.6}$$

The **path graph** on n vertices is

$$\mathrm{P}_n := \mathsf{H}([n] \mid \sum_{1 \leqslant i < n} \{i,i+1\}). \tag{3.7}$$

The **cycle graph** on $n \geqslant 3$ vertices is

$$\mathrm{C}_n := \mathsf{H}([n] \mid \sum_{1\leqslant i<n} \{i, i+1\} + \{1, n\}). \tag{3.8}$$

The **staircase hypergraph** associated with $\boldsymbol{m} = (m_0, \dots, m_n) \in \mathbf{N}_0^{n+1}$ is

$$\Sigma\mathsf{H}_{\boldsymbol{m}} := \mathsf{H}\left([n] \mid \sum_{i=0}^{n} m_i[i]\right). \tag{3.9}$$

3.2 The incidence representation and module associated with a hypergraph

Let $\mathsf{H} = (V, E, |\cdot|)$ be a hypergraph. We construct a module representation η^R over R which we call the **incidence representation** of H over R.

Description of η in terms of hypergraph coordinates. For $(v, e) \in V \times E$, let $[ve]$ be the R-homomorphism $RV \to RE$ which satisfies $u[ve] = \delta_{uv} \cdot e$ $(u \in V)$. Recall that $\mathbf{I}(\mathsf{H}) = \{(v, e) \in V \times E : v \sim_{\mathsf{H}} e\}$. Define η^R to be the module representation

$$R\,\mathbf{I}(\mathsf{H}) \to \mathrm{Hom}(RV, RE), \quad (v, e) \mapsto [ve];$$

write $\eta = \eta^{\mathbf{Z}}$. We refer to η as *the* (absolute) incidence representation of H. Note that the notation η^R is unambiguous: if $R \to S$ is a ring map, then $(\eta^R)^S = \eta^S$.

Description of η in terms of matrices. Write $m = |E|$, $n = |V|$, $V = \{v_1, \dots, v_n\}$, and $E = \{e_1, \dots, e_m\}$. Let

$$M = \Big\{[a_{ij}] \in \mathrm{M}_{n\times m}(R) : a_{ij} = 0 \text{ whenever } v_i \notin |e_j|\Big\}.$$

Then η^R, as defined above, is isotopic (see §2.1) to the inclusion $M \hookrightarrow \mathrm{M}_{n\times m}(R)$.

Incidence modules. Let $X = (X_v)_{v\in V}$ consist of algebraically independent variables over R. Define

$$\mathsf{inc}(\mathsf{H}; R) := \Big\langle X_v e : v \sim_{\mathsf{H}} e \ (v \in V,\, e \in E)\Big\rangle \leqslant R[X]\,E.$$

The **incidence module** of H over R is

$$\mathrm{Inc}(\mathsf{H}; R) := \frac{R[X]\,E}{\mathsf{inc}(\mathsf{H}; R)};$$

the (absolute) incidence module of H is $\mathrm{Inc}(\mathsf{H}) := \mathrm{Inc}(\mathsf{H}; \mathbf{Z})$. Clearly,

$$\mathrm{Inc}(\mathsf{H}; R) \approx_{R[X]} \bigoplus_{e\in E} \frac{R[X]}{\langle X_v : v \in |e|\rangle}. \tag{3.10}$$

Lemma 3.2. $\mathrm{Inc}(\mathsf{H}; S) = \mathrm{Inc}(\mathsf{H}; R)^{S[X]}$ *for each ring map* $R \to S$.

Proof. Immediate from the right exactness of tensor products. ♦

The incidence module of H determines the ask zeta functions associated with η:

Proposition 3.3. *For each compact* DVR $\mathfrak{O}$*,*

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s) = (1-q^{-1})^{-1}\,\zeta_{\mathrm{Inc}(\mathsf{H};\mathfrak{O})}(s-|V|+|E|).$$

Proof. By Lemma 2.3, $\mathrm{Im}(\mathsf{C}^{\mathbf{I}(\mathsf{H}),V,E}_{\eta^{\mathfrak{O}}}(X)) = \mathsf{inc}(\mathsf{H};\mathfrak{O})$ (see (2.1)) whence $\mathrm{Inc}(\mathsf{H};\mathfrak{O}) = \mathrm{Coker}\Big(\mathsf{C}^{\mathbf{I}(\mathsf{H}),V,E}_{\eta^{\mathfrak{O}}}(X)\Big)$. The claim thus follows from Corollary 2.13. ♦

For any suitable ring R, we refer to $\zeta^{\mathsf{ask}}_{\eta^{R}}(s)$ as the **ask zeta function** of H over R. We can use the structure of $\mathrm{Inc}(\mathsf{H};R)$ in (3.10) to make Proposition 3.3 more explicit.

Proposition 3.4. *For each compact* DVR $\mathfrak{O}$*,*

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s) = (1-q^{-1})^{-1} \int\limits_{\mathfrak{O}V\times\mathfrak{O}} |y|^{s-|V|+|E|-1} \prod_{e\in E} \|x_e; y\|^{-1}\, \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y),$$

where $x_e := \{x_v : v \in |e|\}$.

Proof. For ideals $\mathfrak{a},\mathfrak{b} \triangleleft R$ of a ring R, there is a natural R-module isomorphism $R/\mathfrak{a}\otimes_R R/\mathfrak{b} \approx R/(\mathfrak{a}+\mathfrak{b})$; this follows e.g. from [10, Ch. I, §2, no. 8]. Equivalently, given a surjective ring map $R \xrightarrow{\lambda} S$, we have $R/\mathfrak{a}\otimes_R S \approx_S S/(\mathfrak{a}\lambda)$. Let $x \in \mathfrak{O}V$ and $0 \neq y \in \mathfrak{O}$. Using (3.10) and Remark 2.11, we then obtain $\mathfrak{O}$-module isomorphisms

$$\mathrm{Inc}(\mathsf{H};\mathfrak{O}) \otimes_{\mathfrak{O}[X]} (\mathfrak{O}/y)_x \approx \bigoplus_{e\in E} \frac{\mathfrak{O}[X]}{\langle X_v : v \in |e|\rangle} \otimes_{\mathfrak{O}[X]} (\mathfrak{O}/y)_x \approx \bigoplus_{e\in E} \frac{\mathfrak{O}}{\langle x_e; y\rangle}.$$

The common cardinality of these modules is therefore $\prod\limits_{e\in E} \|x_e; y\|^{-1}$. The claim now follows from Corollary 2.13. ♦

Remark 3.5. Suppose that $\mathsf{H} = \mathsf{BH}_{n,m}$ is the $n\times m$ block hypergraph; see (3.2). In terms of matrices, η^R parameterises all of $\mathrm{M}_{n\times m}(R)$. A formula for the ask zeta function of (the identity on) $\mathrm{M}_{n\times m}(\mathfrak{O})$ is given in [57, Proposition 1.5]; see Example 5.10(i). The integral in the proof of this proposition is exactly the corresponding special case of Proposition 3.4 here. Similarly, the determination of the ask zeta functions of modules of (strictly) upper triangular matrices over $\mathfrak{O}$ in [57, Proposition 5.15] proceeded by computing the integral in the corresponding special case of Proposition 3.4; see Example 5.10(ii). We thus recognise various ad hoc arguments from [57] as instances of the cokernel formalism here.

As we will see in §5, Proposition 3.4 can be used to produce an explicit formula for the ask zeta functions associated with *all* hypergraphs on a given vertex set.

3.3 Two adjacency representations and modules associated with a graph

Let $\Gamma = (V, E, |\cdot|)$ be a graph. We construct two module representations γ_+ and γ_- associated with Γ which we call the **adjacency representations** of Γ. The first of these module representations is defined for all graphs Γ and parameterises symmetric matrices with suitably constrained support. The second is defined whenever Γ is simple and parameterises antisymmetric matrices with support constrained by Γ. For our purposes, the antisymmetric case is usually more interesting.

Denote the exterior (resp. symmetric) square of a module M by $M \wedge M$ (resp. $M \odot M$).

Alternating case: construction of γ_-. Let Γ be simple. For an R-module M, let

$$(M \wedge M)^* \xrightarrow{\mathfrak{so}_M} \mathrm{Hom}(M, M^*)$$

be the module representation which sends $\psi \in (M \wedge M)^*$ to the map

$$M \to M^*, \quad m \mapsto (n \mapsto (m \wedge n)\psi).$$

If $M = R^n$, then the evident choices of bases furnish an isotopy between $\mathfrak{so}_M$ and the inclusion $\mathfrak{so}_n(R) \hookrightarrow \mathrm{M}_n(R)$, of alternating (= antisymmetric with zero diagonal) $n \times n$ matrices into $\mathrm{M}_n(R)$. We define the **negative adjacency representation** γ_-^R of Γ over R to be the composite

$$\left(\frac{RV \wedge RV}{\mathsf{N}(\Gamma, -1; R)}\right)^* \rightarrowtail (RV \wedge RV)^* \xrightarrow{\mathfrak{so}_{RV}} \mathrm{Hom}(RV, (RV)^*),$$

where $\mathsf{N}(\Gamma, -1; R)$ is the submodule of $RV \wedge RV$ generated by all $v \wedge w$ such that $v, w \in V$ are *non-adjacent* in Γ and the first map is the dual of the quotient map. For an explicit description of γ_-^R in terms of matrices, let $V = \{v_1, \dots, v_n\}$, where $n = |V|$. Let

$$M^- = \left\{[a_{ij}] \in \mathfrak{so}_n(R) : a_{ij} = 0 \text{ whenever } v_i \not\sim v_j\right\};$$

cf. the definition of $M^-(\Gamma)$ in §1.4. It is easy to see that γ_-^R is isotopic to the inclusion $M^- \hookrightarrow \mathrm{M}_n(R)$; a proof is implicitly given in the proof of Proposition 3.7 below.

We refer to $\gamma_- := \gamma_-^{\mathbf{Z}}$ as *the* (absolute) negative adjacency representation of Γ. As with incidence representations in §3.2, this notation is unambiguous: $(\gamma_-^R)^S = \gamma_-^S$ for each ring map $R \to S$.

Symmetric case: construction of γ_+. Let Γ be not necessarily simple. For an R-module M, let

$$(M \odot M)^* \xrightarrow{\mathrm{Sym}_M} \mathrm{Hom}(M, M^*)$$

be the module representation which sends $\psi \in (M \odot M)^*$ to the map

$$M \to M^*, \quad m \mapsto (n \mapsto (m \odot n)\psi).$$

Sym_{R^n} is isotopic to the inclusion $\mathrm{Sym}_n(R) \hookrightarrow \mathrm{M}_n(R)$ of symmetric matrices.

We define the **positive adjacency representation** γ_+^R of Γ over R to be the composite

$$\left(\frac{RV \odot RV}{\mathsf{N}(\Gamma, +1; R)}\right)^* \rightarrowtail (RV \odot RV)^* \xrightarrow{\mathrm{Sym}_{RV}} \mathrm{Hom}(RV, (RV)^*),$$

where $\mathsf{N}(\Gamma, +1; R)$ is the submodule of $RV \odot RV$ generated by all $v \odot w$ such that $v, w \in V$ are non-adjacent in Γ. In terms of matrices, let $V = \{v_1, \dots, v_n\}$, where $n = |V|$. Let

$$M^+ = \Big\{[a_{ij}] \in \mathrm{Sym}_n(R) : a_{ij} = 0 \text{ whenever } v_i \not\sim v_j\Big\};$$

cf. §1.1 and the definition of $M^+(\Gamma)$ in §1.4. Then γ_+^R is isotopic to the inclusion $M^+ \hookrightarrow \mathrm{M}_n(R)$. As above, we call $\gamma_+ := \gamma_+^{\mathbf{Z}}$ *the* (absolute) positive adjacency representation of Γ.

Adjacency modules. Let $X = (X_v)_{v \in V}$ as in §3.2. For $v, w \in V$, define

$$[v, w; \pm 1] := \begin{cases} X_v w \pm X_w v, & \text{if } v \neq w, \\ \pm X_v v, & \text{if } v = w, \end{cases}$$

an element of $\mathbf{Z}[X]\,V$, and

$$\mathsf{adj}(\Gamma, \pm 1; R) := \Big\langle [v, w; \pm 1] : v, w \in V,\, v \sim w \Big\rangle \leqslant R[X]\,V.$$

The **(positive resp. negative) adjacency module** of Γ over R is

$$\mathrm{Adj}(\Gamma, \pm 1; R) := \frac{R[X]\,V}{\mathsf{adj}(\Gamma, \pm 1; R)}.$$

Lemma 3.6. $\mathrm{Adj}(\Gamma, \pm 1; S) = \mathrm{Adj}(\Gamma, \pm 1; R)^{S[X]}$ *for each ring map* $R \to S$. ♦

We write $\mathrm{Adj}(\Gamma, \pm 1) := \mathrm{Adj}(\Gamma, \pm 1; \mathbf{Z})$.

Adjacency modules and ask zeta functions of graphs. In the same way that incidence modules of hypergraphs determine the ask zeta functions associated with incidence representations, adjacency modules of Γ are related to ask zeta functions derived from $\gamma_\pm$.

Proposition 3.7. *For each compact* DVR $\mathfrak{O}$,

$$\zeta^{\mathsf{ask}}_{\gamma_\pm^{\mathfrak{O}}}(s) = (1 - q^{-1})^{-1}\, \zeta_{\mathrm{Adj}(\Gamma, \pm 1; \mathfrak{O})}(s).$$

(Here, we assume that Γ *is simple in the negative case.)*

Proof. We only spell out the "negative case"; the positive one can be established along similar lines. Let $\sqsubseteq$ be an arbitrary total order on V. Let

$$\mathcal{A}(\Gamma, \sqsubseteq) := \{(v, w) \in V \times W : v \sim w \text{ and } v \sqsubseteq w\}.$$

Let $V^* = \{v^* : v \in V\}$ denote the dual basis associated with the basis V of RV. The images of the symbols $v \wedge w$ for $(v, w) \in \mathcal{A}(\Gamma, \sqsubseteq)$ form a basis of $\frac{RV \wedge RV}{\mathsf{N}(\Gamma, -1; R)}$; let

$\Phi := \{\phi_{vw} : (v,w) \in \mathcal{A}(\Gamma, \sqsubseteq)\}$ denote the associated dual basis of $\left(\frac{RV \wedge RV}{\mathsf{N}(\Gamma,-1;R)}\right)^*$, indexed in the natural way. Define a module representation $R\,\mathcal{A}(\Gamma,\sqsubseteq) \xrightarrow{\theta=\theta^R_{\sqsubseteq}} \mathrm{Hom}(RV, RV)$, where for $(v,w) \in \mathcal{A}(\Gamma,\sqsubseteq)$ and $u \in V$,

$$u((v,w)\theta) = \begin{cases} +w, & \text{if } u = v,\\ -v, & \text{if } u = w,\\ \ \ 0, & \text{otherwise.} \end{cases}$$

It follows that the diagram

$$\begin{array}{ccc} RV & \xrightarrow{(v,w)\theta} & RV \\ \Big\| & & \Big\downarrow \nu \\ RV & \xrightarrow{\phi_{vw}\gamma^R_{-}} & (RV)^* \end{array}$$

commutes, where $RV \xrightarrow{\nu} (RV)^*$ is the isomorphism $v\nu = v^*$ $(v \in V)$. Hence, γ^R_{-} and θ are isotopic (see §2.1). Lemma 2.3 shows that

$$\mathrm{Im}\Big(\mathsf{C}_{\theta}^{\mathcal{A}(\Gamma,\sqsubseteq),V,V}(X)\Big) = \Big\langle X_v w - X_w v : (v,w) \in \mathcal{A}(\Gamma,\sqsubseteq)\Big\rangle = \mathsf{adj}(\Gamma,-1;R)$$

whence

$$\mathrm{Coker}\Big(\mathsf{C}_{\gamma^R_{-}}^{\Phi,V,V^*} - (X)\Big) \approx_{R[X]} \mathrm{Coker}\Big(\mathsf{C}_{\theta}^{\mathcal{A}(\Gamma,\sqsubseteq),V,V}(X)\Big) = \mathrm{Adj}(\Gamma,-1;R).$$

The claim now follows from Corollary 2.13. ♦

3.4 Graphical groups and group schemes

Let $\Gamma = (V, E, |\cdot|)$ be a simple graph. Let $\sqsubseteq$ be an arbitrary total order on V. Define an alternating bilinear map

$$\diamond\colon \mathbf{Z}V \times \mathbf{Z}V \to \mathbf{Z}E$$

by letting, for $v, w \in V$ with $v \sqsubset w$,

$$v \diamond w := \begin{cases} e, & \text{if there exists } e \in E \text{ with } |e| = \{v,w\},\\ 0, & \text{otherwise.} \end{cases}$$

We leave it to the reader to verify that the isomorphism type of the Baer group scheme $\mathbf{G}_\diamond$ (see §2.4) associated with $\diamond$ only depends on Γ and not on the chosen total order $\sqsubseteq$. We call $\mathbf{G}_\Gamma := \mathbf{G}_\diamond$ the **graphical group scheme** associated with Γ. If Γ is a cograph (see §§1.7, 7.1), we talk about the **cographical group scheme** associated with Γ. By a (**co-**)**graphical group** (over R) we mean a group of rational points of a (co-)graphical group scheme, i.e. a group of the form $\mathbf{G}_\Gamma(R)$ for some ring R and (co)graph Γ.

Remark 3.8. The group $\mathbf{G}_\Gamma(\mathbf{Z})$ of $\mathbf{Z}$-points of $\mathbf{G}_\Gamma$ is a finitely generated torsion-free nilpotent group. It admits the presentation

$$\begin{aligned}\mathbf{G}_\Gamma(\mathbf{Z}) \approx \Big\langle V \sqcup E \;\Big|\; & [v,w] = e \text{ for } e \in E \text{ with } |e| = \{v,w\} \text{ and } v \sqsubset w,\\ & [v,w] = 1 \text{ for non-adjacent } v,w \in V, \text{ and}\\ & [v,e] = [e,f] = 1 \text{ for all } v \in V \text{ and } e,f \in E\Big\rangle.\end{aligned}$$

Equivalently, $\mathbf{G}_\Gamma(\mathbf{Z})$ is the maximal nilpotent quotient of nilpotency class at most 2 of the right-angled Artin group

$$\langle V \mid [v,w] = 1 \text{ for all non-adjacent } v,w \in V\rangle$$

associated with the *complement* of Γ; see e.g. [18]. It will prove advantageous for our graph-theoretic arguments in §§6–7 to work with Γ rather than with its complement.

The following variant of Proposition 1.1 (which was proved in §2.4) will be crucial in establishing Corollary B in §6.1.

Proposition 3.9. *Let* Γ *be a simple graph with* m *edges and let* γ_- *denote its negative adjacency representation over* $\mathbf{Z}$. *Let* R *be the ring of integers of a local or global field of arbitrary characteristic. Then* $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes R}(s) = \zeta^{\mathsf{ask}}_{\gamma_-^R}(s-m)$.

Proof. Let $\mathbf{Z}V \xrightarrow{\alpha} \mathrm{Hom}(\mathbf{Z}V,\mathbf{Z}E)$ be the module representation with $v(w\alpha) = v \diamond w$ for all $v,w \in V$. By Proposition 2.5, $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes R}(s) = \zeta^{\mathsf{ask}}_{\alpha^R}(s-m)$. A straightforward calculation as in the proof of Proposition 3.7 shows that the dual $\alpha^\bullet$ (see §2.1) of α is isotopic to γ_-. Using Theorem 2.4 and the final argument from the proof of Proposition 1.1 (see §2.4), we conclude that $\zeta^{\mathsf{ask}}_{\alpha^R}(s) = \zeta^{\mathsf{ask}}_{\gamma_-^R}(s)$ which completes the proof. ⧫

Disjoint unions and joins. Let Γ_1 and Γ_2 be simple graphs. Recall from §3.1 that $\Gamma_1 \oplus \Gamma_2$ and $\Gamma_1 \vee \Gamma_2$ denote the disjoint union and join of Γ_1 and Γ_2, respectively. Clearly, $\mathbf{G}_{\Gamma_1\oplus\Gamma_2}$ and $\mathbf{G}_{\Gamma_1} \times \mathbf{G}_{\Gamma_2}$ are isomorphic group schemes whence $\mathbf{G}_{\Gamma_1\oplus\Gamma_2}(R) \approx \mathbf{G}_{\Gamma_1}(R) \times \mathbf{G}_{\Gamma_2}(R)$ for each ring R. Denote the lower central series of a group G by $G = \boldsymbol{\gamma}_1(G) \geqslant \boldsymbol{\gamma}_2(G) \geqslant \cdots$. For groups G_1 and G_2, let

$$G_1 \circledast G_2 := (G_1 * G_2)/\boldsymbol{\gamma}_3(G_1 * G_2) \tag{3.11}$$

be their **free class-2-nilpotent product**, i.e. the maximal nilpotent quotient of class at most 2 of the free product $G_1 * G_2$. Note that $\mathbf{G}_{\Gamma_1\vee\Gamma_2}(R) \approx \mathbf{G}_{\Gamma_1}(R) \circledast \mathbf{G}_{\Gamma_2}(R)$ if $R = \mathbf{Z}$ or, more generally, $R = \mathbf{Z}/n\mathbf{Z}$ for $n \in \mathbf{Z}$.

In particular, we conclude that the class of cographical groups over $\mathbf{Z}$ is precisely the smallest class of torsion-free finitely generated groups which contains $\mathbf{Z}$ and which is closed under taking both direct and free class-2-nilpotent products.

4 Modules over toric rings and associated zeta functions

By Corollary 2.13, the functions $\zeta_M(s)$ attached to modules M over polynomial rings generalise ask zeta functions. In this section, we introduce a further generalisation of these functions by replacing polynomial rings by more general toric rings. This more general setting will provide us with a sufficient criterion (Proposition 4.8) for proving uniformity results such as Theorem A. Part (i) of the latter will be proved here while parts (ii)–(ii) will be proved in §6.

Throughout, as before, V is a finite set and R is a ring.

4.1 Cones and fans

We recall some standard notions from convex and toric geometry; see [21, Ch. 1–3]. Unless otherwise indicated, by a **cone** in $\mathbf{R}V$ we mean a closed, rational, and polyhedral cone—in other words, cones are finite intersections of $\mathbf{Z}$-defined linear half-spaces in $\mathbf{R}V$.

A **fan** in $\mathbf{R}V$ is a non-empty finite set $\mathcal{F}$ consisting of cones in $\mathbf{R}V$ such that

(i) every face of every cone in $\mathcal{F}$ belongs to $\mathcal{F}$ and

(ii) the intersection of any two cones in $\mathcal{F}$ is a common face of both.

The **support** of a fan $\mathcal{F}$ is $|\mathcal{F}| = \bigcup \mathcal{F}$. The fan $\mathcal{F}$ is **complete** if $|\mathcal{F}| = \mathbf{R}V$. Let $\mathcal{F}$ and $\mathcal{G}$ be two fans in $\mathbf{R}V$. We say that $\mathcal{G}$ **refines** $\mathcal{F}$ if every cone in $\mathcal{G}$ is contained in some cone in $\mathcal{F}$. The **coarsest common refinement** of $\mathcal{F}$ and $\mathcal{G}$ is the fan (!) $\mathcal{F} \wedge \mathcal{G} := \{\sigma \cap \tau : \sigma \in \mathcal{F}, \tau \in \mathcal{G}\}$; its support is $|\mathcal{F} \wedge \mathcal{G}| = |\mathcal{F}| \cap |\mathcal{G}|$.

Let $\cdot$ be the standard inner product $x \cdot y = \sum\limits_{v \in V} x_v y_v$ on $\mathbf{R}V$. If $\sigma \subset \mathbf{R}V$ is a cone, then so is its **dual** $\sigma^* = \{x \in \mathbf{R}V : \forall y \in \sigma \colon x \cdot y \geqslant 0\}$.

Let $\sigma \subset \mathbf{R}_{\geqslant 0}V$ be a cone. Recall that a **preorder** on a set is a reflexive and transitive relation. If all elements are comparable, then the preorder is **total**. We note that "total preorders" and "weak orders" (cf. §§1.6, 5.1) are equivalent concepts. We define a preorder $\leqslant_\sigma$ on $\mathbf{Z}V$ by letting $x \leqslant_\sigma y$ if and only if $y - x \in \sigma^*$.

Lemma 4.1. *For every fan $\mathcal{F}$ in $\mathbf{R}V$ and finite set $\Phi \subset \mathbf{Z}V$, there exists a refinement $\mathcal{F}'$ of $\mathcal{F}$ with $|\mathcal{F}'| = |\mathcal{F}|$ and such that $\leqslant_\sigma$ induces a total preorder on Φ for each $\sigma \in \mathcal{F}'$.*

Proof. We may assume that $\Phi \neq \varnothing$. For $x \in \mathbf{R}V$, let $x^{\pm} := \{y \in \mathbf{R}V : \pm x \cdot y \geqslant 0\}$ be the associated linear half-space and $x^{=} := x^{+} \cap x^{-} = x^{\perp}$. We obtain a complete fan $\mathcal{F}_x := \{x^{+}, x^{-}, x^{=}\}$ consisting of precisely three cones, except when $x = 0$ in which case $\mathcal{F}_x = \{\mathbf{R}V\}$. Clearly, the refinement $\mathcal{F}' := \mathcal{F} \wedge \bigwedge\limits_{x,y \in \Phi} \mathcal{F}_{x-y}$ has the desired property. ♦

4.2 Affine toric schemes and their rational points over DVRs

Toric rings and affine toric schemes. Let $\sigma \subset \mathbf{R}V$ be a cone. By Gordan's lemma (see [21, Proposition 1.2.17]), the additive monoid $\sigma^* \cap \mathbf{Z}V$ is finitely generated. Let $X = (X_v)_{v \in V}$ consist of algebraically independent variables over R. For $\alpha \in \mathbf{Z}V$, write

$X^\alpha = \prod_{v\in V} X_v^{\alpha_v}$. In the same way, we define x^α, where $x = \sum_{v\in V} x_v v$ and all the x_v are units (in some ambient ring). We let

$$R_\sigma := R[X^\alpha : \alpha \in \sigma^* \cap \mathbf{Z}V]$$

be the **toric ring** associated with σ and R. We let $\mathcal{X}_{\sigma,R} = \operatorname{Spec}(R_\sigma)$ be the associated **affine toric scheme** over R; we write $\mathcal{X}_\sigma := \mathcal{X}_{\sigma,\mathbf{Z}}$.

Rational points over DVRs. Let $\sigma \subset \mathbf{R}_{\geqslant 0}V$ be a cone and let $\mathfrak{O}$ be a DVR. Recall that ν denotes the normalised valuation on $\mathfrak{O}$. For $x = \sum_{v\in V} x_v v \in \mathfrak{O}V$ with $\prod_{v\in V} x_v \neq 0$, we write $\nu(x) := \sum_{v\in V} \nu(x_v) v \in \mathbf{Z}V$. Define

$$\sigma(\mathfrak{O}) := \Bigl\{ x \in \mathfrak{O}V : \prod_{v\in V} x_v \neq 0 \text{ and } \nu(x) \in \sigma \Bigr\}.$$

Alternatively, $\sigma(\mathfrak{O})$ admits the following dual description.

Lemma 4.2. $\sigma(\mathfrak{O}) = \{x \in \mathfrak{O}V : \prod_{v\in V} x_v \neq 0$ *and* $x^\alpha \in \mathfrak{O}$ *for each* $\alpha \in \sigma^* \cap \mathbf{Z}V\}$.

Proof. Let $x \in \mathfrak{O}V$ with $\prod_{v\in V} x_v \neq 0$. Then $x^\alpha \in \mathfrak{O}$ if and only if $\nu(x^\alpha) = \nu(x)\cdot\alpha \geqslant 0$. The latter condition holds for all $\alpha \in \sigma^* \cap \mathbf{Z}V$ if and only if $\nu(x) \in \sigma^{**} = \sigma$. ♦

Recall that $\mathcal{X}_\sigma = \operatorname{Spec}(\mathbf{Z}_\sigma)$. As before, write $X = (X_v)_{v\in V}$.

Lemma 4.3. *Let φ be the natural map $\mathcal{X}_\sigma(\mathfrak{O}) \to \mathfrak{O}V$ induced by the inclusion $\sigma \subset \mathbf{R}_{\geqslant 0}V$. Let $Z = \{x \in \mathfrak{O}V : \prod_{v\in V} x_v = 0\}$ and let Z' be the preimage of Z under φ. Then φ induces a bijection $\mathcal{X}_\sigma(\mathfrak{O}) \setminus Z' \to \sigma(\mathfrak{O})$.*

Proof. Let $\boldsymbol{x} \in \mathcal{X}_\sigma(\mathfrak{O}) \setminus Z'$. Then $\boldsymbol{x}$ corresponds to a ring map $\mathbf{Z}_\sigma \xrightarrow{\lambda} \mathfrak{O}$. Let $x := \boldsymbol{x}\varphi$ so that $x_v := X_v\lambda$. Since $\boldsymbol{x} \notin Z'$, we have $\prod_{v\in V} x_v \neq 0$. Let $\alpha \in \sigma^* \cap \mathbf{Z}V$ be arbitrary. Then there exists $\beta \in \mathbf{Z}_{\geqslant 0}V \subset \sigma^*$ with $\alpha + \beta \in \mathbf{Z}_{\geqslant 0}V$. We conclude that $x^{\alpha+\beta} = (X^{\alpha+\beta})\lambda = (X^\alpha)\lambda \cdot (X^\beta)\lambda = (X^\alpha)\lambda \cdot x^\beta$ and therefore $(X^\alpha)\lambda = x^\alpha \in \mathfrak{O} \setminus \{0\}$. By Lemma 4.2, $x = \boldsymbol{x}\varphi \in \sigma(\mathfrak{O})$. We have thus shown that λ (hence $\boldsymbol{x}$) is uniquely determined by x which implies that φ injectively maps $\mathcal{X}_\sigma(\mathfrak{O}) \setminus Z'$ onto a subset of $\sigma(\mathfrak{O})$. It remains to show that the latter subset is all of $\sigma(\mathfrak{O})$. Indeed, for each $y \in \sigma(\mathfrak{O})$, by Lemma 4.2, we obtain a ring map

$$\mathbf{Z}_\sigma \to \mathfrak{O}, \quad X^\alpha \mapsto y^\alpha$$

whose corresponding point $\boldsymbol{y} \in \mathcal{X}_\sigma(\mathfrak{O})$ does not belong to Z' and satisfies $\boldsymbol{y}\varphi = y$. ♦

We henceforth tacitly embed $\sigma(\mathfrak{O}) \subset \mathcal{X}_\sigma(\mathfrak{O})$ via Lemma 4.3.

4.3 Zeta functions associated with modules over toric rings

We now generalise the definition of the zeta functions ζ_M (see §2.6) attached to modules over polynomial rings to those over toric rings.

Let $\sigma \subset \mathbf{R}_{\geqslant 0}V$ be a cone. Recall that $\mathcal{X}_\sigma = \mathrm{Spec}(\mathbf{Z}_\sigma)$ is the affine toric scheme (over $\mathbf{Z}$) associated with σ. Let $\mathfrak{O}$ be a compact DVR and let M be a finitely generated $\mathfrak{O}_\sigma$-module. Generalising the definition of M_x in §2.6 (cf. Lemma 2.10), for each $x \in \mathcal{X}_\sigma(\mathfrak{O})$ $(= \mathcal{X}_{\sigma,\mathfrak{O}}(\mathfrak{O}))$, let M_x denote the $\mathfrak{O}$-module $M \otimes_{\mathfrak{O}_\sigma} \mathfrak{O}$, where the $\mathfrak{O}_\sigma$-module structure on $\mathfrak{O}$ is induced by the ring map $\mathfrak{O}_\sigma \to \mathfrak{O}$ corresponding to x. When $\sigma = \mathbf{R}_{\geqslant 0}V$, we recover the definition of M_x given in §2.6. Recall that we identify $\sigma(\mathfrak{O}) \subset \mathfrak{O}V$ with a subset of $\mathcal{X}_\sigma(\mathfrak{O})$ via Lemma 4.3.

Definition 4.4. Define a zeta function

$$\zeta_M(s) := \int\limits_{\sigma(\mathfrak{O})\times\mathfrak{O}} |y|^{s-1} \cdot |M_x \otimes \mathfrak{O}/y| \;\mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y).$$

Remark 4.5.

(i) If M is an $\mathfrak{O}[X]$-module ($= \mathfrak{O}_{\mathbf{R}_{\geqslant 0}V}$-module), then we recover Definition 2.9.

(ii) The function $\zeta_M(s)$ only depends on the isomorphism type of M as an $\mathfrak{O}_\sigma$-module.

(iii) Exactly as in Remark 2.11, we may identify $M_x \otimes_{\mathfrak{O}} \mathfrak{O}/y = M \otimes_{\mathfrak{O}_\sigma} (\mathfrak{O}/y)_x$.

Lemma 4.6. *Let $o \subset \mathbf{R}_{\geqslant 0}V$ be a cone. Let M_o be a finitely generated $\mathfrak{O}_o$-module. Let $\mathcal{F}$ be a fan with $|\mathcal{F}| = o$. For $\sigma \in \mathcal{F}$, let M_σ denote the $\mathfrak{O}_\sigma$-module $M_o \otimes_{\mathfrak{O}_o} \mathfrak{O}_\sigma$. Then*

$$\zeta_{M_o}(s) = \sum_{\varnothing \neq \Sigma \subset \mathcal{F}} (-1)^{|\Sigma|+1} \zeta_{M_\sigma}(s),$$

where we wrote $\sigma := \bigcap \Sigma$.

Proof. This follows by combining the inclusion-exclusion principle and the identification $(M_\sigma)_x = (M_o)_x$ for $\sigma \in \mathcal{F}$ and $x \in \sigma(\mathfrak{O}) \subset o(\mathfrak{O})$ ("transitivity of base change"). ♦

Global setting. We now provide a global setting for the functions ζ_M. Let R be a Noetherian ring, let $o \subset \mathbf{R}_{\geqslant 0}V$ be a cone, and let M_o be a finitely generated R_o-module. For each ring map $R \xrightarrow{\lambda} \mathfrak{O}$, let

$$\zeta_{M_o,\lambda}(s) := \zeta_{M_o \otimes_{R_o} \mathfrak{O}_o}(s),$$

where the ring map $R_o \xrightarrow{\lambda_o} \mathfrak{O}_o$ is induced by λ; when the reference to λ is clear, we also write $\zeta_{M_o,\mathfrak{O}}$ in place of $\zeta_{M_o,\lambda}$ in the following.

This global setting is compatible with Lemma 4.6 in the sense that for a cone $\sigma \subset o$, by transitivity of base change, we may identify $(M_o \otimes_{R_o} \mathfrak{O}_o) \otimes_{\mathfrak{O}_o} \mathfrak{O}_\sigma = (M_o \otimes_{R_o} R_\sigma) \otimes_{R_\sigma} \mathfrak{O}_\sigma$.

4.4 Combinatorial and torically combinatorial modules

Toric properties. Let R be a ring. Let $\mathcal{P}$ be a property of objects of types $\mathcal{A}, \mathcal{B}, \dots$ (e.g. modules) defined over all base R-algebras of the form R_σ for cones σ contained within some ambient cone in $\mathbf{R}V$. We assume that (i) $\mathcal{P}$ is invariant under isomorphisms, (ii) for each inclusion $\tau \subset \sigma$ of such cones, every R_σ-object A gives rise to an R_τ-object $A \otimes_{R_\sigma} R_\tau$, and (iii) this base change operation is transitive (up to isomorphism). We say that specific objects $A, B, \dots$ over a specific R-algebra R_σ have the property $\mathcal{P}$ **torically** if there exists a fan $\mathcal{F}$ of cones in $\mathbf{R}V$ with $|\mathcal{F}| = \sigma$ such that the R_τ-objects $A \otimes_{R_\sigma} R_\tau$, $B \otimes_{R_\sigma} R_\tau, \dots$ have property $\mathcal{P}$ for all $\tau \in \mathcal{F}$.

Transitivity of toric properties. Let the property $\mathcal{P}$ be as above. Further suppose that $\mathcal{P}$ is stable under shrinking cones in the sense that for each inclusion $\tau \subset \sigma$ of cones, whenever objects $A, B, \dots$ over R_σ have $\mathcal{P}$, then so do $A \otimes_{R_\sigma} R_\tau, B \otimes_{R_\sigma} R_\tau, \dots$.

Let $o \subset \mathbf{R}V$ be a cone and let $\mathcal{F}$ be a fan of cones in $\mathbf{R}V$ with support o. Let $A, B, \dots$ be objects over R_o and suppose that the R_σ-objects $A \otimes_{R_o} R_\sigma, B \otimes_{R_o} R_\sigma, \dots$ torically have property $\mathcal{P}$ for each $\sigma \in \mathcal{F}$. Then $A, B, \dots$ themselves torically have property $\mathcal{P}$ over R_o; for a proof, apply §6.3 (which is self-contained) below; cf. Corollary 6.16.

Combinatorial modules. Let $\sigma \subset \mathbf{R}V$ be a cone. Let R be a ring. By a **monomial ideal** I of R_σ, we mean an ideal generated by (finitely many) Laurent monomials X^α for $\alpha \in \sigma^* \cap \mathbf{Z}V$. We say that an R_σ-module is **combinatorial** if it is isomorphic to $R_\sigma/I_1 \oplus \dots \oplus R_\sigma/I_m$, where each I_j is a monomial ideal of R_σ.

Example 4.7 (Incidence modules are combinatorial)**.** Let H be a hypergraph with vertex set V. By (3.10), the incidence module $\mathrm{Inc}(\mathsf{H}; R)$ is a combinatorial $R[X]$-module, where $X = (X_v)_{v \in V}$.

Proposition 4.8 (Uniformity of zeta functions of torically combinatorial modules)**.** *Let $\sigma \subset \mathbf{R}_{\geqslant 0}V$ be a cone and let M be a torically combinatorial R_σ-module. Then there exists $W(X,T) \in \mathbf{Q}(X,T)$ such that $\zeta_{M,\lambda}(s) = W(q, q^{-s})$ for each compact* DVR $\mathfrak{O}$ *and ring map $R \xrightarrow{\lambda} \mathfrak{O}$.*

Proof. Fix $R \xrightarrow{\lambda} \mathfrak{O}$. First, let $A \subset \sigma^* \cap \mathbf{Z}V$ be a finite set. Let $I = \langle X^\alpha : \alpha \in A \rangle \triangleleft R_\sigma$ and $N = R_\sigma/I$. Let $x \in \sigma(\mathfrak{O})$ and $y \in \mathfrak{O} \setminus \{0\}$. The evident free presentation $R_\sigma A \to R_\sigma \to N \to 0$ yields, by base change, a presentation of the $\mathfrak{O}/y$-module

$$(N \otimes_{R_\sigma} \mathfrak{O}_\sigma)_x \otimes_{\mathfrak{O}} \mathfrak{O}/y \approx N \otimes_{R_\sigma} (\mathfrak{O}/y)_x \approx \mathfrak{O}/\langle x^\alpha \, (\alpha \in A); \, y \rangle =: N_{x,y};$$

cf. Proposition 3.4. In particular, $|N_{x,y}| = \|x^\alpha \, (\alpha \in A); \, y\|^{-1}$, independently of λ.

Next, by Lemma 4.6, after shrinking σ if necessary, we may assume that M is in fact combinatorial instead of merely torically combinatorial, say $M \approx R_\sigma/I_1 \oplus \dots \oplus R_\sigma/I_m$, where $I_j = \langle X^\alpha : \alpha \in A_j \rangle \triangleleft R_\sigma$ and each $A_j \subset \sigma^* \cap \mathbf{Z}V$ is finite. We thus obtain

$$\zeta_{M,\lambda}(s) = \int\limits_{\sigma(\mathfrak{O}) \times \mathfrak{O}} \frac{|y|^{s-1}}{\prod\limits_{j=1}^{m} \|x^\alpha \, (\alpha \in A_j); \, y\|} \, \mathrm{d}\mu_{\mathfrak{O}V \times \mathfrak{O}}(x,y).$$

The claimed uniformity result for p-adic integrals defined by such monomial expressions is well-known, see e.g. [54, Proposition 3.9]. ♦

Remark 4.9.

(i) If R admits *any* ring map to any compact DVR, then the rational function $W(X,T)$ in Proposition 4.8 is uniquely determined. Indeed, if $R \to \mathfrak{O}$ is such a ring map, where $\mathfrak{O}$ has residue field size q, then we obtain ring maps from R to a compact DVR with residue field size q^f for each $f \geqslant 1$. Uniqueness of $W(X,T)$ then essentially boils down to the fact that infinite subsets of $\mathbf{C}$ are Zariski dense.

(ii) The preceding condition is satisfied, in particular, if R is finitely generated over $\mathbf{Z}$. To see that, let $\mathfrak{m}$ be an arbitrary maximal ideal of R. By the Nullstellensatz for Jacobson rings (see e.g. [27, Theorem 4.19]), $R/\mathfrak{m}$ is then a finite field. We then e.g. obtain a ring map $R \to (R/\mathfrak{m})[\![z]\!]$.

Proof of Theorem A(i). Combine Example 4.7 and Proposition 4.8. ♦

In §6, we will show that negative adjacency modules of graphs are always torically combinatorial and that their positive counterparts are torically combinatorial over any ground ring in which 2 is invertible.

5 Ask zeta functions of hypergraphs

Let $\mathsf{H} = (V, E, |\cdot|)$ be a hypergraph. We write $n = |V|$ and and $m = |E|$. As explained in §3.2, this allows us to think of the incidence representation η of H in terms of a generic $n \times m$ matrix with support constrained by the hyperedge support function $|\cdot|$. In §5.1, we derive an explicit combinatorial formula for the rational function $W_{\mathsf{H}}(X,T)$ in Theorem A(i) and thus, for each compact DVR $\mathfrak{O}$, for the ask zeta function $\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s)$. We then consider two natural operations of hypergraphs: disjoint unions (see §5.2) and complete unions (see §5.3 and §3.1). As special cases, we derive explicit formulae for ask zeta functions of hypergraphs with pairwise disjoint (resp. codisjoint) hyperedge supports in §5.2.1 (resp. §5.3.1). In §5.4, we describe the effects of four further fundamental hypergraph operations on the rational functions $W_{\mathsf{H}}(X,T)$. In §5.5, we use our explicit formulae to deduce crucial analytic properties of local and global ask zeta functions associated with hypergraphs.

Throughout this section and beyond, we use the notation $(\underline{d}) = 1 - q^{-d}$.

5.1 An explicit formula for the ask zeta function of a hypergraph

The main result of this section, Corollary 5.6, provides an explicit formula for the rational function $W_{\mathsf{H}(\boldsymbol{\mu})}(X,T)$ (see Theorem A(i)) associated with an arbitrary hypergraph $\mathsf{H}(\boldsymbol{\mu})$ given by a vector $\boldsymbol{\mu}$ of hyperedge multiplicities; see Definition 3.1. This formula will, in particular, imply Theorem C.

Socles. For applications later on, it will prove advantageous to study the rational function $W_{\mathsf{H}(\boldsymbol{\mu})}(X,T)$ in (what appears to be) a slightly more general setup.

Definition 5.1. Given a d-element set D with $V \cap D = \varnothing$ and a vector $\boldsymbol{\mu}$ of hyperedge multiplicities as in Definition 3.1, define a hypergraph $\mathsf{H}(\boldsymbol{\mu}, D)$ with vertex set $D \sqcup V$ and vector of hyperedge multiplicities $(\nu_J)_{J\subset D\sqcup V}$ given by

$$\nu_J = \begin{cases} \mu_I, & \text{if } J = D \sqcup I \text{ for some } I \subset V, \\ 0, & \text{otherwise.} \end{cases}$$

Informally speaking, the hypergraph $\mathsf{H}(\boldsymbol{\mu}, D)$ arises from $\mathsf{H}(\boldsymbol{\mu})$ by inflating each hyperedge by the same fixed set ("socle") D. Thus, if $A \in \mathrm{M}_{n\times m}(\mathbf{Z})$ is an incidence matrix of $\mathsf{H}(\boldsymbol{\mu})$, then

$$\begin{bmatrix} \mathbf{1}_{d\times m} \\ A \end{bmatrix} \in \mathrm{M}_{(d+n)\times m}(\mathbf{Z})$$

is an incidence matrix of $\mathsf{H}(\boldsymbol{\mu}, D)$.

We now derive an explicit formula for $W_{\mathsf{H}(\boldsymbol{\mu},D)}(X,T)$; the shape of this formula will often allow us to reduce to the case $D = \varnothing$.

Setup and strategy. From now on, let $\mathfrak{O}$ be an arbitrary compact DVR with residue field cardinality q. Without loss of generality, suppose that $0 \notin V \sqcup D$ and write $D_0 := D \sqcup \{0\}$. Recall from §1.11 that for a non-trivial $\mathfrak{O}$-module M, we write $M^\times = M \setminus \mathfrak{P}M$ and $\{0\}^\times = \{0\}$. For $J \subset V$, define p-adic integrals

$$\mathsf{Z}_{J,D}(\boldsymbol{s}) := \mathsf{Z}_{J,D}\Big(s_0,\, (s_I)_{I\subset J}\Big) := \int\limits_{\mathfrak{O}J\times\mathfrak{O}D_0} |y_0|^{s_0} \prod_{I\subset J} \|x_I; y\|^{s_I} \, \mathrm{d}\mu_{\mathfrak{O}J\times\mathfrak{O}D_0}(x,y) \quad \text{and}$$

$$\mathcal{I}_{J,D}(\boldsymbol{s}) := \int\limits_{(\mathfrak{O}J)^\times\times\mathfrak{P}D_0} |y_0|^{s_0} \prod_{I\subset J} \|x_I; y\|^{s_I} \, \mathrm{d}\mu_{\mathfrak{O}J\times D_0}(x,y), \tag{5.1}$$

where $x_I := (x_i : i \in I) \in \mathfrak{O}I \subset \mathfrak{O}J$. Depending on context, we regard $\mathsf{Z}_{J,D}(\boldsymbol{s})$ and $\mathcal{I}_{J,D}(\boldsymbol{s})$ both as functions of the $1 + 2^{|J|}$ variables s_0 and $(s_I)_{I\subset J}$ and also as functions of the $1 + 2^{|V|}$ variables s_0 and $(s_I)_{I\subset V}$; in any case, s_0 and $s_\varnothing$ are different variables.

Let $\eta_{\boldsymbol{\mu},D}$ be the incidence representation of $\mathsf{H}(\boldsymbol{\mu}, D)$; see §3.2. We seek to determine $W_{\mathsf{H}(\boldsymbol{\mu},D)}(X,T)$ (and hence also $W_{\mathsf{H}(\boldsymbol{\mu})}(X,T)$) by expressing $\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}_{\boldsymbol{\mu},D}}(s)$ as a rational function in q and q^{-s}. By Proposition 3.4,

$$\begin{aligned} \zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}_{\boldsymbol{\mu},D}}(s) &= (1-q^{-1})^{-1} \int\limits_{\mathfrak{O}V\times\mathfrak{O}D_0} |y_0|^{s-(n+d)+m-1} \prod_{I\subset V} \|x_I; y\|^{-\mu_I} \, \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}D_0}(x,y) \\ &= (1-q^{-1})^{-1}\, \mathsf{Z}_{V,D}\Big(s - (n+d) + m - 1,\, (-\mu_I)_{I\subset V}\Big). \end{aligned} \tag{5.2}$$

This allows us to study $W_{\mathsf{H}}(X,T)$ by analysing the functions $\mathsf{Z}_{V,D}(\boldsymbol{s})$.

A recursive formula. Our first goal is to derive a recursive formula for $\mathsf{Z}_{V,D}(\boldsymbol{s})$; see Proposition 5.2. In the following, we write $t_I = q^{-s_I}$ (where $I \subset V$ or $I = 0$) and $t = q^{-s}$. We identify $\mathfrak{O}D_0 = \mathfrak{O} \times \mathfrak{O}D$ and decompose $\mathfrak{O}V \times \mathfrak{O}D_0 = \mathfrak{O}V \times \mathfrak{O} \times \mathfrak{O}D$ in the form

$$((\mathfrak{O}V)^\times \times \mathfrak{P} \times \mathfrak{P}D) \sqcup (\mathfrak{P}V \times \mathfrak{P} \times \mathfrak{P}D) \sqcup (\mathfrak{O}V \times \mathfrak{O}^\times \times \mathfrak{O}D) \sqcup (\mathfrak{O}V \times \mathfrak{P} \times (\mathfrak{O}D)^\times)\,. \tag{5.3}$$

Write $\mathrm{gp}\,(x) = x/(1-x)$ and $\mathrm{gp}_0\,(x) = 1/(1-x)$. Using a change of variables (cf. [37, Proposition 7.4.1]) and the well-known (and easily proved) identity

$$\int\limits_{\mathfrak{P}} |y_0|^{s_0}\, \mathrm{d}\mu_{\mathfrak{O}}(y_0) = (\underline{1})\,\mathrm{gp}\left(q^{-1}t_0\right), \tag{5.4}$$

we rewrite $\mathsf{Z}_{V,D}(\boldsymbol{s})$ as

$$\mathsf{Z}_{V,D}(\boldsymbol{s}) = \mathcal{I}_{V,D}(\boldsymbol{s}) + q^{-n-1-d}\, t_0 \left(\prod_{I \subset V} t_I\right) \mathsf{Z}_{V,D}(s) + (\underline{1}) \left(1 + (\underline{d})\mathrm{gp}\left(q^{-1}t_0\right)\right).$$

For $J \subset V$, let

$$\mathcal{Z}_{J,D}(\boldsymbol{s}) := \frac{1 - q^{-d-1}t_0}{1 - q^{-1}t_0} + (\underline{1})^{-1}\mathcal{I}_{J,D}(\boldsymbol{s}); \tag{5.5}$$

here, and in the following subsections, we often abbreviate $\mathcal{Z}_J(\boldsymbol{s}) := \mathcal{Z}_{J,\varnothing}(\boldsymbol{s})$. Thus,

$$\mathsf{Z}_{V,D}(\boldsymbol{s}) = \mathrm{gp}_0\Big(q^{-n-1-d}t_0 \prod_{I \subset V} t_I\Big)\, (\underline{1})\, \mathcal{Z}_{V,D}(\boldsymbol{s}) \tag{5.6}$$

and hence, by combining (5.2) and (5.6),

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}_{\boldsymbol{\mu},D}}(s) = \frac{1}{1-t}\, \mathcal{Z}_{V,D}\Big(s - (n+d) + m - 1, (-\mu_I)_{I \subset V}\Big). \tag{5.7}$$

The function $\mathcal{Z}_{V,D}(\boldsymbol{s})$ admits the following recursive expression.

Proposition 5.2.

$$\mathcal{Z}_{V,D}(\boldsymbol{s}) = \frac{1 - q^{-d-1}t_0}{1 - q^{-1}t_0} + \sum_{J \subsetneq V} (\underline{1})^{n-|J|}\ \mathrm{gp}\Big(q^{-d-1-|J|}t_0 \prod_{I \subset J} t_I\Big)\, \mathcal{Z}_{J,D}(\boldsymbol{s}). \tag{5.8}$$

Proof. We decompose the first factor of the domain of integration of $\mathcal{I}_{V,D}(\boldsymbol{s})$ defined in (5.1) according to precisely which entries of $x \in (\mathfrak{O}V)^\times$ are $\mathfrak{P}$-adic units; no such entry affects the integrand. By Fubini's theorem, we may then split off the relevant copies of $\mathfrak{O}^\times$, each of Haar measure $(\underline{1})$, and write

$$\mathcal{I}_{V,D}(\boldsymbol{s}) = \sum_{J \subsetneq V} (\underline{1})^{n-|J|} \underbrace{\int\limits_{\mathfrak{P}J \times \mathfrak{P}D_0} |y_0|^{s_0} \prod_{I \subset J} \|x_I; y\|^{s_I}\, \mathrm{d}\mu_{\mathfrak{O}J \times \mathfrak{O}D_0}(x,y)}_{=:\mathcal{I}^\circ_{J,D}(\boldsymbol{s})}\,. \tag{5.9}$$

A change of variables shows that

$$\mathcal{I}^{\circ}_{J,D}(\boldsymbol{s}) = q^{-d-1-|J|}\, t_0\Big(\prod_{I\subset J} t_I\Big) \int\limits_{\mathfrak{O}J\times\mathfrak{O}D_0} |y_0|^{s_0} \prod_{I\subset J} \|x_I; y\|^{s_I}\, \mathrm{d}\mu_{\mathfrak{O}J\times\mathfrak{O}D_0}(x,y).$$

Using a decomposition of $\mathfrak{O}J\times\mathfrak{O}D_0 = \mathfrak{O}J\times\mathfrak{O}\times\mathfrak{O}D$ analogous to (5.3), we obtain

$$\frac{\mathcal{I}^{\circ}_{J,D}(\boldsymbol{s})}{q^{-d-1-|J|}\, t_0\Big(\prod\limits_{I\subset J} t_I\Big)} = \mathcal{I}_{J,D}(\boldsymbol{s}) + \mathcal{I}^{\circ}_{J,D}(\boldsymbol{s}) + (\underline{1})\left(\frac{1-q^{-1-d}t_0}{1-q^{-1}t_0}\right)$$

and hence

$$\begin{aligned}\mathcal{I}^{\circ}_{J,D}(\boldsymbol{s}) &= \mathrm{gp}\left(q^{-d-1-|J|} t_0 \prod_{I\subset J} t_I\right)\left((\underline{1})\left(\frac{1-q^{-1-d}\,t_0}{1-q^{-1}\,t_0}\right) + \mathcal{I}_{J,D}(\boldsymbol{s})\right)\\ &= (\underline{1})\,\mathrm{gp}\left(q^{-d-1-|J|}\, t_0 \prod_{I\subset J} t_I\right) \mathcal{Z}_{J,D}(\boldsymbol{s}). \qquad (5.10)\end{aligned}$$

By combining (5.5) and (5.9)–(5.10), we finally obtain

$$\begin{aligned}\mathcal{Z}_{V,D}(\boldsymbol{s}) &= \frac{1-q^{-1-d}t_0}{1-q^{-1}t_0} + (\underline{1})^{-1}\mathcal{I}_{V,D}(\boldsymbol{s})\\ &= \frac{1-q^{-1-d}t_0}{1-q^{-1}t_0} + \sum_{J\subsetneq V} (\underline{1})^{n-|J|-1}\, \mathcal{I}^{\circ}_{J,D}(\boldsymbol{s})\\ &= \frac{1-q^{-1-d}t_0}{1-q^{-1}t_0} + \sum_{J\subsetneq V} (\underline{1})^{n-|J|}\ \mathrm{gp}\Big(q^{-d-1-|J|}\, t_0 \prod_{I\subset J} t_I\Big)\, \mathcal{Z}_{J,D}(\boldsymbol{s}). \qquad \blacklozenge\end{aligned}$$

An explicit formula in terms of weak orders. Our next goal is to translate the recursive formula in Proposition 5.2 into an explicit form given by a sum over a suitable combinatorial object.

Definition 5.3. Let $\widehat{\mathrm{WO}}(V)$ be the poset of flags of subsets of V. That is, $\widehat{\mathrm{WO}}(V)$ consists of elements of the form

$$y = \Big(I_1 \subsetneq I_2 \subsetneq \cdots \subsetneq I_\ell\Big),$$

where $\ell \geqslant 0$ and $I_i \subset V$ for $i = 1,\ldots,\ell$. Note that we allow both $I_1 = \varnothing$ and $I_\ell = V$ but do not require either condition to be satisfied. We define the **rank** of $y \in \widehat{\mathrm{WO}}(V)$ to be

$$\mathrm{rk}(y) = |\sup(y)| = \sup\{|I| : I \in y\} \in \mathbf{N}_0;$$

empty flags have rank 0. We denote by $\widetilde{\mathrm{WO}}(V)$ the subposet of $\widehat{\mathrm{WO}}(V)$ consisting of all flags of *non-empty* subsets of V only. We often write $\widehat{\mathrm{WO}}_n$ and $\widetilde{\mathrm{WO}}_n$ instead of $\widehat{\mathrm{WO}}([n])$ and $\widetilde{\mathrm{WO}}([n])$, respectively.

Remark 5.4.

(i) Clearly, $\mathrm{rk}(y) = 0$ if and only if y is either the empty flag or the singleton flag $(\varnothing)$. At the other extreme, $\mathrm{rk}(y) = |V| = n$ if and only if $V \in y$. The latter condition is satisfied for precisely half of the elements of $\widetilde{\mathrm{WO}}(V)$. The fact that $V \in y$ is permitted for elements y of $\widetilde{\mathrm{WO}}(V)$ marks the difference between the latter and the poset WO_n of weak orders of n objects; cf. e.g. [62, Section 2.3]. In particular, $\frac{1}{2}|\widehat{\mathrm{WO}}(V)| = |\widetilde{\mathrm{WO}}(V)| = 2|\mathrm{WO}_n| = 2f_n$, where f_n denotes the nth Fubini number as in §1.6; cf. [30], [19, p. 228], and (1.4).

(ii) The poset $\widetilde{\mathrm{WO}}(V)$ is isomorphic to $\mathrm{WO}_{\mathbf{1}^{(n)}}$ in [17, Section 3.1].

We obtain the following explicit formula for $\mathcal{Z}_{V,D}(\boldsymbol{s})$.

Theorem 5.5.

$$\mathcal{Z}_{V,D}\left(\boldsymbol{s}\right) = \frac{1 - q^{-d-1}t_0}{1 - q^{-1}t_0} \sum_{y \in \widehat{\mathrm{WO}}(V)} (\underline{1})^{\mathrm{rk}(y)} \prod_{J \in y} \mathrm{gp}\Big(q^{-1-n+|J|-d} t_0 \prod_{I \subset V \setminus J} t_I\Big).$$

Proof. Recursively apply Proposition 5.2 to the terms $\mathcal{Z}_{J,D}(\boldsymbol{s})$ on the right-hand side of (5.8). ♦

In particular, using (5.7), we obtain the following explicit formulae for the rational function $W_{\mathsf{H}(\boldsymbol{\mu},D)}(X,T)$ associated with the hypergraph $\mathsf{H}(\boldsymbol{\mu},D)$.

Corollary 5.6.

$$\begin{aligned} &W_{\mathsf{H}(\boldsymbol{\mu},D)}(X,T) \\ &= \frac{1 - X^{n-m}T}{(1 - X^{d+n-m}T)(1-T)} \sum_{y \in \widehat{\mathrm{WO}}(V)} (1 - X^{-1})^{\mathrm{rk}(y)} \prod_{J \in y} \mathrm{gp}\left(X^{|J| - \sum_{I \cap J \neq \varnothing} \mu_I} T\right) \qquad (5.11) \\ &= \frac{1 - X^{n-m}T}{1 - X^{d+n-m}T} \sum_{y \in \widetilde{\mathrm{WO}}(V)} (1 - X^{-1})^{\mathrm{rk}(y)} \prod_{J \in y} \mathrm{gp}\left(X^{|J| - \sum_{I \cap J \neq \varnothing} \mu_I} T\right) \qquad (5.12) \\ &= \frac{1 - X^{n-m}T}{1 - X^{d+n-m}T} W_{\mathsf{H}(\boldsymbol{\mu})}(X,T). \end{aligned}$$

♦

Proof of Theorem C. Apply Corollary 5.6 with $D = \varnothing$. ♦

Remark 5.7.

(i) For $n = 0$, we recover the formula for the ask zeta function associated with the block hypergraph $\mathsf{BH}_{d,m}$ in (3.2); see Example 5.10(i).

(ii) The rational functions in (5.11) and (5.12) are reminiscent of the "generalised Igusa function" $I^{\mathrm{wo}}_{\mathbf{1}^{(n)}}(\boldsymbol{X})$ associated with the all-one-vector $\mathbf{1}^{(n)} = (1, \ldots, 1)$ in [17, Definition 3.5]. The curious factors $(1 - X^{-1})^{\mathrm{rk}(y)}$, however, set these two types of combinatorially defined functions apart.

Example 5.8. We write out the formulae for the functions $W_{\mathsf{H}(\boldsymbol{\mu})}(X,T) = W_{\mathsf{H}(\boldsymbol{\mu},\varnothing)}(X,T)$ given in (5.12) for $n \in \{2,3\}$. We identify $V = [n]$ and set, for $J \subset [n]$, $\mathrm{gp}_J := \mathrm{gp}_J(\boldsymbol{\mu}) := \mathrm{gp}\Big(X^{|J|-\sum_{I\cap J\neq\varnothing}\mu_I}T\Big)$. Write $\mathrm{gp}_i := \mathrm{gp}_{\{i\}}$ and similarly $\mathrm{gp}_{ij\dots} := \mathrm{gp}(\{i,j,\dots\})$.

(i) $(n = 2)$ The ranks of the six flags in $\widetilde{\mathrm{WO}}_2$ are given as follows.

$y \in \widetilde{\mathrm{WO}}_2$	$()$	$(\{1,2\})$	$(\{1\} \subsetneq \{1,2\})$	$(\{2\} \subsetneq \{1,2\})$	$(\{1\})$	$(\{2\})$
$\mathrm{rk}(y)$	0	2	2	2	1	1

Thus

$$W_{\mathsf{H}(\boldsymbol{\mu})}(X,T) = \\ \frac{1}{1-T}\Big(1 + (1-X^{-1})^2\mathrm{gp}_{12}\,(1+\mathrm{gp}_1+\mathrm{gp}_2) + (1-X^{-1})\,(\mathrm{gp}_1+\mathrm{gp}_2)\Big), \quad (5.13)$$

where the relevant substitutions are given by the numerical data

$$X^{|J|-\sum_{I\cap J\neq\varnothing}\mu_I}T = \begin{cases} X^{2-\mu_1-\mu_2-\mu_{12}}T, & \text{for } J=\{1,2\},\\ X^{1-\mu_2-\mu_{12}}T, & \text{for } J=\{1\},\\ X^{1-\mu_1-\mu_{12}}T, & \text{for } J=\{2\}.\end{cases}$$

(ii) $(n = 3)$ Here, $|\widetilde{\mathrm{WO}}_3| = 26$ and

$$\begin{aligned} W_{\mathsf{H}(\boldsymbol{\mu})}(X,T) = \frac{1}{1-T}\Big(&1 + (1-X^{-1})^3\mathrm{gp}_{123}\,(1+\mathrm{gp}_1+\mathrm{gp}_2+\mathrm{gp}_3\\ &+\mathrm{gp}_{12}\,(1+\mathrm{gp}_1+\mathrm{gp}_2)+\mathrm{gp}_{13}\,(1+\mathrm{gp}_1+\mathrm{gp}_3)+\mathrm{gp}_{23}\,(1+\mathrm{gp}_2+\mathrm{gp}_3))\\ &+(1-X^{-1})^2\,(\mathrm{gp}_{12}\,(1+\mathrm{gp}_1+\mathrm{gp}_2)+\mathrm{gp}_{13}\,(1+\mathrm{gp}_1+\mathrm{gp}_3)\\ &+\mathrm{gp}_{23}\,(1+\mathrm{gp}_2+\mathrm{gp}_3)) + (1-X^{-1})\,(\mathrm{gp}_1+\mathrm{gp}_2+\mathrm{gp}_3)\Big); \end{aligned}$$

we omit the lengthy substitutions.

It seems remarkable how slight the dependence of $W_{\mathsf{H}(\boldsymbol{\mu},D)}(X,T)$ on the "socle" D is. The final equality in Corollary 5.6 often allows us to reduce to the case $D = \varnothing$ or, equivalently, to assume that no vertex of our hypergraph is incident to every hyperedge.

5.1.1 A special case: staircase hypergraphs

Let $\boldsymbol{m} = (m_0,\dots,m_n) \in \mathbf{N}_0^{n+1}$ and write $m = m_0 + \dots + m_n$. Recall the definition of the staircase hypergraph $\Sigma\mathsf{H}_{\boldsymbol{m}}$ from (3.9). The upper block-triagonal "staircase matrix"

$$M_{\boldsymbol{m}} = \Big[\delta_{j>\sum_{\iota<i} m_\iota}\Big]_{\substack{i=1,\dots,n\\ j=1,\dots,m}} \in \mathrm{M}_{n\times m}(\mathbf{Z})$$

is the incidence matrix of $\Sigma\mathsf{H}_{\mathbf{m}}$ with respect to the natural order on $[n] = \mathrm{V}(\Sigma\mathsf{H}_{\boldsymbol{m}})$ and the lexicographic order on $\mathrm{E}(\Sigma\mathsf{H}_{\boldsymbol{m}})$ (where the first components are ordered by inclusion); see Example 8.18 for an illustration.

The rational function $W_{\Sigma\mathsf{H}_{\mathbf{m}}}(X,T)$ associated with $\Sigma\mathsf{H}_{\mathbf{m}}$ admits the following concise description.

Proposition 5.9.

$$W_{\Sigma\mathsf{H}_{\boldsymbol{m}}}(X,T) = \frac{1}{1-T}\prod_{j=0}^{n-1}\frac{1-X^{-1+n-j-\sum_{\iota>j} m_\iota}T}{1-X^{n-j-\sum_{\iota>j} m_\iota}T}. \tag{5.14}$$

Proof. Combine Proposition 3.4 and [57, Lemma 5.6]. ♦

Proposition 5.9 generalises several previously known results.

Example 5.10.

(i) If $m_0 = \ldots = m_{n-1} = 0$ and $m_n = m$, then $\Sigma\mathsf{H}_{\mathbf{m}} = \mathsf{BH}_{n,m}$. Proposition 5.9 yields, in accordance with [57, Proposition 1.5],

$$\begin{aligned} W_{\Sigma\mathsf{H}_{\boldsymbol{m}}}(X,T) = W_{\mathsf{BH}_{n,m}}(X,T) &= \frac{1}{1-T}\prod_{j=0}^{n-1}\frac{1-X^{-1+n-j-m}T}{1-X^{n-j-m}T} \\ &= \frac{1-X^{-m}T}{(1-T)(1-X^{n-m}T)}. \end{aligned}$$

(ii) If $m_0 = 0$ and $m_1 = \ldots = m_n = 1$, then $M_{\boldsymbol{m}} = [\delta_{i\leqslant j}] \in \mathrm{M}_n(\mathbf{Z})$. Proposition 5.9 yields, in accordance with [57, Proposition 5.13(ii)],

$$W_{\Sigma\mathsf{H}_{\boldsymbol{m}}}(X,T) = \frac{1}{1-T}\prod_{j=0}^{n-1}\frac{1-X^{-1+n-j-(n-j)}T}{1-X^{n-j-(n-j)}T} = \frac{(1-X^{-1}T)^n}{(1-T)^{n+1}}.$$

5.2 Ask zeta functions of disjoint unions of hypergraphs

In this section, we consider ask zeta functions associated with disjoint unions of hypergraphs. As our main result, in §5.2.1, we record an explicit formula for ask zeta functions attached to hypergraphs with pairwise disjoint (hyperedge) supports.

Hadamard products. Recall that the **Hadamard product** of two generating functions $F(T) = \sum_{k=0}^{\infty} a_k T^k$ and $G(T) = \sum_{k=0}^{\infty} b_k T^k$ with coefficients in some common field is

$$F(T) \star G(T) := \sum_{k=0}^{\infty} a_k b_k T^k.$$

If $F(T)$ and $G(T)$ are both rational, then so is $F(T)\star G(T)$; see [65, Proposition 4.2.5].

Disjoint unions. Let $\mathsf{H}_1, \dots, \mathsf{H}_r$ be hypergraphs with pairwise disjoint vertex sets $V_1, \dots, V_r$. Let $\mathsf{H} := \mathsf{H}_1 \oplus \dots \oplus \mathsf{H}_r$ be the disjoint union of $\mathsf{H}_1, \dots, \mathsf{H}_r$ as in §3.1.

Proposition 5.11. $W_{\mathsf{H}_1\oplus\dots\oplus\mathsf{H}_r}(X,T) = W_{\mathsf{H}_1}(X,T) \star \dots \star W_{\mathsf{H}_r}(X,T)$.

Proof. Let η_i and η be the incidence representations of H_i and H, respectively. We may identify $\eta = \eta_1 \oplus \dots \oplus \eta_r$ (see §2.1). Now apply [60, Lemma 3.1]; cf. [57, Corollary 3.6]. ♦

We conclude that the set of rational functions $W_{\mathsf{H}}(X,T)$ associated with hypergraphs (or, equivalently, the class of rational functions given by the right-hand side of (5.11)) is closed under taking Hadamard products.

It is natural to seek to exploit this closure property. Let $\mathsf{H}_i \approx \mathsf{H}(\boldsymbol{\mu}^{(i)})$ for a vector $\boldsymbol{\mu}^{(i)}$ of hyperedge multiplicities as in Definition 3.1. By combining Corollary 5.6 and Proposition 5.11, we obtain

$$W_{\mathsf{H}_1\oplus\dots\oplus\mathsf{H}_r}(X,T) = \mathop{\bigstar}_{i=1}^{r} \sum_{y_i \in \widehat{\mathrm{WO}}(V_i)} (1-X^{-1})^{\mathrm{rk}(y_i)} \prod_{J\in y_i} \mathrm{gp}\Big(X^{|J| - \sum_{I\cap J\neq\varnothing} \mu_I^{(i)}} T\Big). \qquad (5.15)$$

The right-hand side of (5.15) falls short of being truly explicit due to the rather mysterious nature of Hadamard products. On the other hand, Corollary 5.6 provides an explicit formula for $W_{\mathsf{H}}(X,T)$ in terms of the hyperedge multiplicity vector $\boldsymbol{\mu} \in \mathbf{N}_0^{\mathcal{P}(V)}$, where $V := V_1 \sqcup \dots \sqcup V_r$ and $\mu_I := \sum_{i=1}^r \delta_{I\subset V_i}\, \mu^{(i)}_{I\cap V_i}$ for $I \subset V$. This approach, however, takes no advantage of the fact that H is a disjoint union. As we will now see, it turns out that we can do much better at least when each H_i is a block hypergraph as in (3.2).

5.2.1 A special case: hypergraphs with disjoint supports

Let $\mathbf{n} = (n_1, \dots, n_r) \in \mathbf{N}^r$ and $\mathbf{m} = (m_1, \dots, m_r) \in \mathbf{N}^r$; write $n = n_1 + \dots + n_r$ and $m = m_1 + \dots + m_r$. Let $\mathsf{H} := \mathsf{BH}_{\mathbf{n},\mathbf{m}}$ be the disjoint union of the block hypergraphs $\mathsf{H}_i := \mathsf{BH}_{n_i,m_i}$; see (3.3). Let V_i be the set of vertices of H_i and $V = V_1 \sqcup \dots \sqcup V_r$ be that of H. Note that $\mathbf{1}_{n_i\times m_i} \in \mathrm{M}_{n_i\times m_i}(\mathbf{Z})$ is the (unique!) incidence matrix of H_i. It follows that

$$\mathrm{diag}\,(\mathbf{1}_{n_1\times m_1}, \dots, \mathbf{1}_{n_r\times m_r}) \in \mathrm{M}_{n\times m}(\mathbf{Z}) \qquad (5.16)$$

is an incidence matrix of $\mathsf{BH}_{\mathbf{n},\mathbf{m}}$. Note that, up to reordering of rows and columns where necessary, this is the general form of incidence matrices of hypergraphs with disjoint supports (i.e. whenever $\mu_I\mu_J \neq 0$, then $I = J$ or $I \cap J = \varnothing$) and which also satisfy $\bigcup_{\mu_I>0} I = V$. (We will see that the latter condition imposes no real restrictions, nor would allowing some $m_i = 0$ offer anything new; see Remark 5.24.)

In Corollary 5.6, we obtained an expression for $W_{\mathsf{H}}(X,T)$ as a sum over $\widehat{\mathrm{WO}}(V) \approx \widehat{\mathrm{WO}}_n$. Our main result (Corollary 5.14) of this section provides an expression for $W_{\mathsf{H}}(X,T)$ as a sum over $\widehat{\mathrm{WO}}_r$. Apart from better reflecting the structure of the hypergraph H, in light of the rapid growth of Fubini numbers in (1.4), our formula has a more favourable complexity if $r \ll n$; see also Remark 5.20.

Auxiliary functions. We consider the specialisation

$$\mathcal{Z}_{\mathbf{n}}^{\oplus}(\boldsymbol{s}) := \mathcal{Z}_{\mathbf{n}}^{\oplus}(s_0;\, s_{V_1}, \dots, s_{V_r}) := \mathcal{Z}_V\Big(s_0, \Big(\delta_{\exists i\in[r]: I=V_i}\, s_I\Big)_{I\subset V}\Big) \tag{5.17}$$

of the function $\mathcal{Z}_V(\boldsymbol{s}) = \mathcal{Z}_{V,\varnothing}(\boldsymbol{s})$ from (5.5). In other words, $\mathcal{Z}_{\mathbf{n}}^{\oplus}(\boldsymbol{s})$ is obtained from $\mathcal{Z}_V(\boldsymbol{s})$ by setting all variables s_I corresponding to subsets $I \subset V$ to zero, except for those subsets equal to one of the pairwise disjoint sets V_i. From (5.7) (with $D = \varnothing$), we obtain

$$\zeta^{\mathsf{ask}}_{\eta_1^{\mathfrak{O}}\oplus\cdots\oplus\eta_r^{\mathfrak{O}}}(s) = \frac{1}{1-t}\,\mathcal{Z}_{\mathbf{n}}^{\oplus}\left(s-n+m-1; -m_1, \dots, -m_r\right).$$

Given $J \subset [r]$, write $n_J = \sum_{j\in J} n_j$ and $\mathbf{n}_J = (n_j)_{j\in J}$. Generalising (5.17), we define

$$\mathcal{Z}_{\mathbf{n}_J}^{\oplus}(\boldsymbol{s}) := \mathcal{Z}_V\left(s_0, \Big(\delta_{\exists j\in J: I=V_j}\, s_I\Big)_{I\subset V}\right).$$

A recursive formula. We obtain the following recursive formula for $\mathcal{Z}_{\mathbf{n}}^{\oplus}(\boldsymbol{s})$; the proof is similar to that of Proposition 5.2 and hence omitted.

Proposition 5.12.

$$\mathcal{Z}_{\mathbf{n}}^{\oplus}(\boldsymbol{s}) = 1 + \sum_{J\subsetneq[r]} \Big(\prod_{k\notin J} \underline{(n_k)}\Big)\, \mathrm{gp}\left(q^{-1-n_J} t_0 \prod_{j\in J} t_{V_j}\right) \mathcal{Z}_{\mathbf{n}_J}^{\oplus}(\boldsymbol{s}). \qquad \blacklozenge$$

An explicit formula. Just as Proposition 5.2 implies Theorem 5.5, we obtain the following by unravelling the recursive formula in Proposition 5.12.

Theorem 5.13.

$$\mathcal{Z}_{\mathbf{n}}^{\oplus}(\boldsymbol{s}) = \sum_{y\in\widehat{\mathrm{WO}}_r} \Big(\prod_{i\in\sup(y)} \underline{(n_i)}\Big) \prod_{J\in y} \mathrm{gp}\left(q^{-1-n+n_J} t_0 \prod_{j\in[r]\setminus J} t_{V_j}\right). \qquad \blacklozenge$$

In particular, Theorem 5.13 allows us to produce the following explicit formula for the rational function $W_{\mathsf{BH}_{\mathbf{n},\mathbf{m}}}(X,T)$ associated with the disjoint union $\mathsf{BH}_{\mathbf{n},\mathbf{m}} = \bigoplus_{i=1}^{r} \mathsf{BH}_{n_i,m_i}$.

Corollary 5.14.

$$W_{\mathsf{BH}_{\mathbf{n},\mathbf{m}}}(X,T) = \sum_{y\in\widehat{\mathrm{WO}}_r} \Big(\prod_{i\in\sup(y)} (1-X^{-n_i})\Big) \prod_{J\in y} \mathrm{gp}\left(X^{\sum_{j\in J} n_j - m_j} T\right). \quad \blacklozenge \tag{5.18}$$

Example 5.15. For $r = 2$, formula (5.18) for the ask zeta function associated with the disjoint union of two block hypergraphs BH_{n_i,m_i} reads

$$\begin{aligned} &W_{\mathsf{BH}_{(n_1,n_2),(m_1,m_2)}}(X,T) = \\ &\quad \frac{1}{1-T}\Big(1 + (1-X^{-n_1})\mathrm{gp}\left(X^{n_1-m_1}T\right) + (1-X^{-n_2})\mathrm{gp}\left(X^{n_2-m_2}T\right) + \\ &(1-X^{-n_1})(1-X^{-n_2})\mathrm{gp}\left(X^{n_1+n_2-m_1-m_2}T\right)\left(1+\mathrm{gp}\left(X^{n_1-m_1}T\right)+\mathrm{gp}\left(X^{n_2-m_2}T\right)\right)\Big). \end{aligned} \tag{5.19}$$

It is instructive to compare (5.19) and the general formula (5.13) for $W_{\mathsf{H}}(X,T)$ in the special case that H is a hypergraph on two vertices.

5.3 Ask zeta functions of complete unions of hypergraphs

Let H_1 and H_2 be hypergraphs on disjoint sets V_1 and V_2 of vertices. Recall from §3.1 the definition of the complete union $\mathsf{H}_1 \circledast \mathsf{H}_2$ of H_1 and H_2, a hypergraph with vertex set $V_1 \sqcup V_2$. In the main result of this section, Corollary 5.17, we express $W_{\mathsf{H}_1\circledast\mathsf{H}_2}(X,T)$ in terms of $W_{\mathsf{H}_1}(X,T)$ and $W_{\mathsf{H}_2}(X,T)$. In §5.3.1, we also record an explicit formula for the rational function $W_{\mathsf{H}}(X,T)$ whenever H has pairwise codisjoint hyperedge supports; such hypergraphs are precisely the reflections (see §3.1) of those considered in §5.2.1.

Let H_i have n_i vertices and m_i hyperedges; write $n = n_1 + n_2$, and $m = m_1 + m_2$. Let H_1, H_2, and $\mathsf{H} := \mathsf{H}_1 \circledast \mathsf{H}_2$ be given by the multiplicity vectors $\boldsymbol{\mu}^{(1)}$, $\boldsymbol{\mu}^{(2)}$, and $\boldsymbol{\mu}$, respectively; cf. Definition 3.1. For $I \subset V := V_1 \sqcup V_2$, let $I_i := I \cap V_i$ so that

$$\mu_I = \delta_{I_2=V_2}\, \mu^{(1)}_{I_1} + \delta_{I_1=V_1}\, \mu^{(2)}_{I_2}.$$

An auxiliary function. Recall the definition of $\mathcal{Z}_V(\boldsymbol{s}) = \mathcal{Z}_{V,\varnothing}(\boldsymbol{s})$ from (5.5) and consider the specialisation

$$\mathcal{Z}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s}) := \mathcal{Z}^{\circledast}_{(V_1,V_2)}(s_0,\, \boldsymbol{s}^{(1)}, \boldsymbol{s}^{(2)}) := \mathcal{Z}_V\Big(s_0,\, \Big(\delta_{I_2=V_2}\, s^{(1)}_{I_1} + \delta_{I_1=V_1}\, s^{(2)}_{I_2}\Big)_{I\subset V}\Big).$$

In other words, $\mathcal{Z}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s})$ is obtained from $\mathcal{Z}_V(\boldsymbol{s})$ by setting all variables s_I for $I \subset V$ to zero, except for those I that *contain* one of the disjoint sets V_1 and V_2; note that the variable s_V is substituted by $s^{(1)}_{V_1} + s^{(2)}_{V_2}$. Thus, $\mathcal{Z}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s})$ is a function of $1 + 2^{n_1} + 2^{n_2}$ complex variables s_0, $\boldsymbol{s}^{(1)} = \Big(s^{(1)}_{I_1}\Big)_{I_1\subset V_1}$, and $\boldsymbol{s}^{(2)} = \Big(s^{(2)}_{I_2}\Big)_{I_2\subset V_2}$. In particular, s_0, $s^{(1)}_{\varnothing}$, and $s^{(2)}_{\varnothing}$ are three distinct variables.

Let η_1, η_2, and $\eta_1 \circledast \eta_2$ be the incidence representations of H_1, H_2, and $\mathsf{H}_1 \circledast \mathsf{H}_2$, respectively. From (5.7) (with $D = \varnothing$), we obtain

$$\zeta^{\mathsf{ask}}_{(\eta_1\circledast\eta_2)^{\mathfrak{O}}}(s) = \frac{1}{1-t}\, \mathcal{Z}^{\circledast}_{(V_1,V_2)}\Big(s - n + m - 1,\, \Big(-\mu^{(1)}_{I_1}\Big)_{I_1\subset V_1},\, \Big(-\mu^{(2)}_{I_2}\Big)_{I_2\subset V_2}\Big).$$

Recursive formulae. This identity allows us to relate the rational functions $W_{\mathsf{H}}(X,T)$, $W_{\mathsf{H}_1}(X,T)$, and $W_{\mathsf{H}_2}(X,T)$. We first express $\mathcal{Z}^{\circledast}_{(n_1,n_2)}(\boldsymbol{s})$ in terms of (translates of) the functions $\mathcal{Z}_{V_i}(s_0, \boldsymbol{s}^{(i)})$; cf. (5.5). Let $t_0 := q^{-s_0}$.

Proposition 5.16.

$$\begin{aligned}\mathcal{Z}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s}) = \Big(&q^{-n-1}t_0 - 1 + \mathcal{Z}_{V_1}(s_0 + n_2, \boldsymbol{s}^{(1)})(1 - q^{-n_2-1}t_0)\\ &+ \mathcal{Z}_{V_2}(s_0 + n_1, \boldsymbol{s}^{(2)})(1 - q^{-n_1-1}t_0)\Big)/(1 - q^{-1}t_0) \qquad (5.20)\end{aligned}$$

Proof. It suffices to analyse the function

$$\mathcal{I}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s}) := \int_{(\mathfrak{O}V)^\times\times\mathfrak{P}} |y|^{s_0} \left(\prod_{I_1\subset V_1} \|x^{(1)}_{I_1}, x^{(2)}, y\|^{s_{I_1}}\right)\left(\prod_{I_2\subset V_2} \|x^{(2)}_{I_2}, x^{(1)}, y\|^{s_{I_2}}\right) \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y),$$

where $x^{(i)}_{I_i} = (x^{(i)}_j : j \in I_i)$ and $x = (x^{(1)}, x^{(2)}) = (x^{(1)}_1, \dots, x^{(1)}_{n_1}, x^{(2)}_1, \dots, x^{(2)}_{n_2})$. Indeed, $\mathcal{Z}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s}) = 1 + (\underline{1})^{-1}\mathcal{I}^{\circledast}_{(V_1,V_2)}(\boldsymbol{s})$; see (5.5). We proceed by decomposing the domain of integration of this function. On the set $S\times\mathfrak{P}$ for

$$S := \left\{(x^{(1)}, x^{(2)}) \in (\mathfrak{O}V)^\times : x^{(1)} \not\equiv 0 \not\equiv x^{(2)} \pmod{\mathfrak{P}}\right\},$$

the integral is very simple. Indeed, $\mu((\mathfrak{O}V)^\times \setminus S) = (\underline{n_1})q^{-n_2} + (\underline{n_2})q^{-n_1}$ whence

$$(\underline{1})^{-1}\int_{S\times\mathfrak{P}} |y|^{s_0} \left(\prod_{I_1\subset V_1} \|x^{(1)}_{I_1}, x^{(2)}, y\|^{s_{I_1}}\right)\left(\prod_{I_2\subset V_2} \|x^{(2)}_{I_2}, x^{(1)}, y\|^{s_{I_2}}\right) \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y) = (1 - q^{-n} - (\underline{n_1})q^{-n_2} - (\underline{n_2})q^{-n_1})\,\mathrm{gp}\left(q^{-1}t_0\right)$$

by (5.4). It remains to deal with

$$\begin{aligned}
&\int_{(\mathfrak{O}V)^\times\setminus S\times\mathfrak{P}} |y|^{s_0} \left(\prod_{I_1\subset V_1} \|x^{(1)}_{I_1}, x^{(2)}, y\|^{s_{I_1}}\right)\left(\prod_{I_2\subset V_2} \|x^{(2)}_{I_2}, x^{(1)}, y\|^{s_{I_2}}\right) \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y)\\
&= \int_{\mathfrak{P}V_1\times(\mathfrak{O}V_2)^\times\times\mathfrak{P}} |y|^{s_0} \left(\prod_{I_2\subset V_2} \|x^{(2)}_{I_2}, x^{(1)}, y\|^{s_{I_2}}\right) \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y)\\
&+ \int_{(\mathfrak{O}V_1)^\times\times\mathfrak{P}V_1\times\mathfrak{P}} |y|^{s_0} \left(\prod_{I_1\subset V_1} \|x^{(1)}_{I_1}, x^{(2)}, y\|^{s_{I_1}}\right) \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y)\\
&= \mathcal{I}_{V_2,V_1}(s_0, \boldsymbol{s}^{(2)}) + \mathcal{I}_{V_1,V_2}(s_0, \boldsymbol{s}^{(1)});
\end{aligned}$$

cf. (5.1). For $i = 1$, by invoking (5.5) again and also Theorem 5.5, we obtain

$$\begin{aligned}
\mathcal{I}_{V_1,n_2}(s_0, \boldsymbol{s}^{(1)}) &= (\underline{1})\left(\mathcal{Z}_{V_1,V_2}(s_0, \boldsymbol{s}^{(1)}) - \frac{1 - q^{-n_2-1}t_0}{1 - q^{-1}t_0}\right)\\
&= (\underline{1})\,\frac{1 - q^{-n_2-1}t_0}{1 - q^{-1}t_0}\left(\mathcal{Z}_{V_1}(s_0 + n_2, \boldsymbol{s}^{(1)}) - 1\right);
\end{aligned}$$

the argument for $i = 2$ is analogous. ♦

We now obtain the following expression for the rational function $W_{\mathsf{H}_1\circledast\mathsf{H}_2}(X, T)$ associated with the complete union $\mathsf{H}_1 \circledast \mathsf{H}_2$ of the hypergraphs H_1 and H_2.

Corollary 5.17.

$$\begin{aligned}W_{\mathsf{H}_1 \circledast \mathsf{H}_2}(X,T) = (X^{-m}T - 1 &\\ &+ W_{\mathsf{H}_1}(X, X^{-m_2}T)(1-X^{-m_2}T)(1-X^{n_1-m}T)\\ &+ W_{\mathsf{H}_2}(X, X^{-m_1}T)(1-X^{-m_1}T)(1-X^{n_2-m}T))\\ &/((1-T)(1-X^{n-m}T)).\end{aligned} \tag{5.21}$$

In particular, if H_1 *is the block hypergraph* BH_{n_1,m_1}*, then*

$$W_{\mathsf{H}_1 \circledast \mathsf{H}_2}(X,T) = W_{\mathsf{H}_2}(X, X^{-m_1}T)\,\frac{(1-X^{-m_1}T)(1-X^{n_2-m}T)}{(1-T)(1-X^{n-m}T)}. \tag{5.22}$$

Proof. Write $t = q^{-s}$. As

$$\zeta^{\mathsf{ask}}_{(\eta_1 \circledast \eta_2)^{\mathfrak{O}}}(s) = W_{\mathsf{H}_1 \circledast \mathsf{H}_2}(q,t) = \frac{1}{1-t}\mathcal{Z}^{\circledast}_{(V_1,V_2)}\Big(s-n+m-1, (-\mu^{(1)}_{I_1})_{I_1 \subset V_1}, (-\mu^{(2)}_{I_2})_{I_2 \subset V_2}\Big),$$

we seek to describe the effect of replacing s_0 by $s-n+m-1$ and each $s^{(i)}_{I_i}$ by $-\mu^{(i)}_{I_i}$ in each of the two functions $\mathcal{Z}_{V_i}(s_0 + n_{3-i}, \boldsymbol{s}^{(i)})$ in (5.20). Since

$$\zeta^{\mathsf{ask}}_{\eta_i^{\mathfrak{O}}}(s) = W_{\mathsf{H}_i}(q,t) = \frac{1}{1-t}\mathcal{Z}_{V_i}\Big(s-n_i+m_i-1, \Big(-\mu^{(i)}_{I_i}\Big)_{I_i \subset V_i}\Big)$$

by (5.7), we obtain

$$\mathcal{Z}_{V_i}\Big(s-n+m-1+n_{3-i}, \Big(-\mu^{(i)}_{I_i}\Big)_{I_i \subset V_i}\Big) = W_{\mathsf{H}_i}(q, q^{-m_{3-i}}t)\,(1-q^{-m_{3-i}}t).$$

This establishes the first claim. The special case follows from a simple computation using $W_{\mathsf{BH}_{n_1,m_1}}(X,T) = (1-X^{-m_1}T)/((1-T)(1-X^{n_1-m_1}T))$; see Example 5.10(i). ♦

Given hypergraphs $\mathsf{H}_1, \dots, \mathsf{H}_r$, repeated application of Corollary 5.17 yields explicit formulae for $W_{\mathsf{H}_1 \oplus \cdots \oplus \mathsf{H}_r}(X,T)$ in terms of $W_{\mathsf{H}_1}(X,T), \dots, W_{\mathsf{H}_r}(X,T)$.

5.3.1 A special case: hypergraphs with codisjoint supports

Let $\mathbf{n}, \mathbf{m} \in \mathbf{N}^r$. The main result of this section, namely Corollary 5.19, provides an explicit formula for the rational function $W_{\mathsf{H}}(X,T)$ associated with $\mathsf{H} := \mathsf{PH}_{\mathbf{n},\mathbf{m}} = \mathsf{PH}_{n_1,m_1} \circledast \cdots \circledast \mathsf{PH}_{n_r,m_r}$, the reflection of $\mathsf{BH}_{\mathbf{n},\mathbf{m}}$; see (3.3)–(3.4). Note that

$$\mathbf{1}_{n\times m} - \operatorname{diag}(\mathbf{1}_{n_1\times m_1}, \dots, \mathbf{1}_{n_r \times m_r}) \in \mathrm{M}_{n\times m}(\mathbf{Z})$$

is an incidence matrix of H; up to reordering rows and columns, this is the general form of incidence matrices of hypergraphs with codisjoint supports (i.e. whenever $\mu_I\mu_J \neq 0$ for $I, J \subset V$, then $I = J$ or $I^{\mathrm{c}} \cap J^{\mathrm{c}} = \varnothing$) and which also satisfy $\bigcap_{\mu_I > 0} I = \varnothing$.

Let V_i be the set of vertices of PH_{n_i,m_i} and let $V = V_1 \sqcup \cdots \sqcup V_r$ be that of $\mathsf{H} = \mathsf{PH}_{\mathbf{n},\mathbf{m}}$.

An auxiliary function. Consider the specialisation

$$\mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}(\boldsymbol{s}) := \mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}\Big(s_0;\, s_{V_1^{\mathrm{c}}},\ldots,s_{V_r^{\mathrm{c}}}\Big) := \mathcal{Z}_V\Big(s_0, \Big(\delta_{\exists i\in[r]:I=V_i^{\mathrm{c}}}\, s_I\Big)_{I\subset V}\Big)$$

of $\mathcal{Z}_V(\boldsymbol{s})$. In other words, $\mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}(\boldsymbol{s})$ is obtained from $\mathcal{Z}_V(\boldsymbol{s})$ by setting all variables s_I to zero, except for those with $I = V_i^{\mathrm{c}}$ for some i.

Let $n = n_1 + \cdots + n_r$ and $m = m_1 + \cdots + m_r$. Let $\eta_{\mathbf{n},\mathbf{m}}^{\circledast}$ be the incidence representation of H. From (5.7) (with $D = \varnothing$), we obtain the identity

$$\zeta^{\mathsf{ask}}_{(\eta_{\mathbf{n},\mathbf{m}}^{\circledast})^{\mathfrak{O}}}(s) = \frac{1}{1-t}\, \mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}\Big(s-n+m-1;\, -m_1,\ldots,-m_r\Big). \tag{5.23}$$

As before, we write $t_I := q^{-s_I}$.

An explicit formula and its consequences

Theorem 5.18.

$$\mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}(\boldsymbol{s}) = \frac{1}{1-q^{-1}t_0}\left(1-q^{-n-1}t_0\left(1-\sum_{i=1}^{r}\frac{(\underline{n_i})\, q^{n_i}\Big(t_{V_i^{\mathrm{c}}}-1\Big)}{1-q^{-n-1+n_i}\, t_0\, t_{V_i^{\mathrm{c}}}}\right)\right).$$

Prior to proving Theorem 5.18, we record our main result here, namely the following immediate consequence of Theorem 5.18 and (5.23).

Corollary 5.19.

$$W_{\mathsf{PH}_{\mathbf{n},\mathbf{m}}}(X,T) = \frac{1}{(1-T)(1-X^{n-m}T)}\left(1-X^{-m}T\left(1-\sum_{i=1}^{r}\frac{(X^{n_i}-1)(X^{m_i}-1)}{1-X^{n_i+m_i-m}T}\right)\right).$$

♦

Note that the formulae in Corollaries 5.17 and 5.19 indeed coincide where they overlap.

Remark 5.20. Using “Big Theta Notation”, the estimate from [3] for the nth Fubini number f_n cited in (1.4) implies that $f_n = \Theta\Big(\frac{n!}{(\log 2)^{n+1}}\Big)$. We have thus produced explicit formulae of three (generally strictly decreasing) complexities:

(i) For a general hypergraph H on n vertices, Corollary 5.6 expresses $W_{\mathsf{H}}(X,T)$ as a sum of $|\widehat{\mathrm{WO}}_n| = \Theta\Big(\frac{n!}{(\log 2)^{n+1}}\Big)$ rational functions.

(ii) If $\mathsf{H} = \mathsf{BH}_{\mathbf{n},\mathbf{m}}$ for $\mathbf{n},\mathbf{m} \in \mathbf{N}^r$, then Corollary 5.14 expresses $W_{\mathsf{H}}(X,T)$ as a sum of $\Theta\Big(\frac{r!}{(\log 2)^{r+1}}\Big)$ rational functions.

(iii) Finally, if $\mathsf{H} = \mathsf{PH}_{\mathbf{n},\mathbf{m}}$ is the reflection (see §3.1) of $\mathsf{BH}_{\mathbf{n},\mathbf{m}}$, then Corollary 5.19 expresses $W_{\mathsf{H}}(X,T)$ as a sum of $\Theta(r)$ rational functions.

Writing $m = |\mathsf{E}(\mathsf{H})|$, the rational functions appearing in these sums are products of $\mathcal{O}(n)$ factors of the form $\pm X^a$, $\pm X^a T$, $1 - X^a$, and $(1 - X^a T)^{\pm 1}$ for $a \in \mathbf{Z}$ with $|a| = \mathcal{O}(n+m)$.

For another point view, write each of the above formulae over a common denominator. Then we saw that the denominators can (essentially) be written as products of $\mathcal{O}(2^n)$ factors of the form $1 - X^A T$ in case (i), products of $\mathcal{O}(2^r)$ such factors in case (ii), and as products of $\mathcal{O}(r)$ factors in case (iii). While cancellations may reduce the actual number of factors for any given hypergraph, experiments suggest that our bounds generally indicate the correct order of magnitude.

Proof of Theorem 5.18. The following observation will be helpful.

Lemma 5.21. *For all $N \in \mathbf{N}$,*

$$\int_{\mathfrak{P}^N \times \mathfrak{P}} |y|^{a_0} \, \|x_1, \dots, x_{n-1}, y\|^{a_{N-1}} \, \mathrm{d}\mu_{\mathfrak{O}^N \times \mathfrak{O}}(x, y) \;=\; \frac{q^{-a_0 - a_{N-1} - (N+1)} (\underline{1}) \, (1 - q^{-a_0 - N})}{(1 - q^{-a_0 - 1})(1 - q^{-a_0 - a_{N-1} - N})}.$$

Proof. This is a straightforward corollary of [57, Lemma 5.8] which implies that

$$F_N(a_0, 0, \dots, 0, a_{N-1}, 0) := \int_{\mathfrak{O}^N \times \mathfrak{O}} |y|^{a_0} \, \|x_1, \dots, x_{N-1}, y\|^{a_{N-1}} \, \mathrm{d}\mu_{\mathfrak{O}^N \times \mathfrak{O}}(x, y) = \frac{(\underline{1})(1 - q^{-a_0 - N})}{(1 - q^{-a_0 - 1})(1 - q^{-a_0 - a_{N-1} - N})}. \quad \blacklozenge$$

Proof of Theorem 5.18. Consider

$$\mathcal{Z}^{\circledast, \mathrm{codis}}_{\boldsymbol{V}}(s_0;\, s_{V_1^{\mathrm{c}}}, \dots, s_{V_r^{\mathrm{c}}}) = 1 + (\underline{1})^{-1} \int_{(\mathfrak{O}V)^\times \times \mathfrak{P}} |y_0|^{s_0} \prod_{i=1}^{r} \|x_{V_i^{\mathrm{c}}}; y_0\|^{s_{V_i^{\mathrm{c}}}} \, \mathrm{d}\mu_{\mathfrak{O}V \times \mathfrak{O}}(x, y_0).$$

Note that the product in the integrand is trivial unless $x \in (\mathfrak{O}V)^\times$ has its $\mathfrak{P}$-adic units concentrated in exactly one of the sets V_i. We therefore split up the first factor of the domain of integration in the form $(\mathfrak{O}V)^\times = S \sqcup ((\mathfrak{O}V)^\times \setminus S)$, where

$$S := \Big\{ x \in (\mathfrak{O}V)^\times : \#\{ j \in [r] : \exists v \in V_j \colon x_v \in \mathfrak{O}^\times \} > 1 \Big\}.$$

Clearly $\mu((\mathfrak{O}V)^\times \setminus S) = \sum_{i=1}^{r} (\underline{n_i}) q^{n_i - n}$ whence

$$(\underline{1})^{-1} \int_{S \times \mathfrak{P}} |y_0|^{s_0} \prod_{i=1}^{r} \|x_{V_i^{\mathrm{c}}}; y\|^{s_{V_i^{\mathrm{c}}}} \, \mathrm{d}\mu_{\mathfrak{O}V \times \mathfrak{O}}(x, y) = \Big(1 - q^{-n} - \sum_{i=1}^{r} (\underline{n_i}) q^{n_i - n}\Big) \, \mathrm{gp}\left(q^{-1} t_0\right). \tag{5.24}$$

By applying Lemma 5.21 for each $j \in [r]$ (with $N = n - n_i + 1$, $a_0 = s_0$, $a_{N-1} = s_{V_i^{\mathrm{c}}}$), we obtain

$$\begin{aligned}
&(\underline{1})^{-1} \int_{(\mathfrak{O}V)^\times \setminus S \times \mathfrak{P}} |y|^{s_0} \prod_{i=1}^{r} \|x_{V_i^{\mathrm{c}}}; y\|^{s_{V_i^{\mathrm{c}}}} \, \mathrm{d}\mu_{\mathfrak{O}V \times \mathfrak{O}}(x, y) \\
&= (\underline{1})^{-1} \sum_{i=1}^{r} (\underline{n_i}) q \int_{\mathfrak{P}^{n-n_i+1} \times \mathfrak{P}} |y_0|^{s_0} \|x_{[n-n_i]}; y\|^{s_{V_i^{\mathrm{c}}}} \, \mathrm{d}\mu_{\mathfrak{O}^{n-n_i+1} \times \mathfrak{O}}(x, y) \\
&= \sum_{i=1}^{r} (\underline{n_i}) \frac{(1 - q^{-n-1+n_i} t_0)}{(1 - q^{-1} t_0)} \, \mathrm{gp}\left(q^{-n-1+n_i} t_0 t_{V_i^{\mathrm{c}}}\right). \qquad (5.25)
\end{aligned}$$

Combining (5.24) and (5.25) yields, after some trivial simplifications, that indeed

$$\begin{aligned}
\mathcal{Z}_{\boldsymbol{V}}^{\circledast,\mathrm{codis}}(s_0;\, s_{V_1^{\mathrm{c}}}, \ldots, s_{V_r^{\mathrm{c}}}) &= 1 + \left(1 - q^{-n} - \sum_{i=1}^{r} (\underline{n_i}) q^{n_i - n}\right) \mathrm{gp}\left(q^{-1} t_0\right) + \\
&\qquad \sum_{i=1}^{r} (\underline{n_i}) \frac{(1 - q^{-n-1+n_i} t_0)}{(1 - q^{-1} t_0)} \mathrm{gp}\left(q^{-n-1+n_i} t_0 t_{V_i^{\mathrm{c}}}\right) \\
&= \frac{1}{1 - q^{-1} t_0} \left(1 - q^{-n-1} t_0 \left(1 - \sum_{i=1}^{r} \frac{(\underline{n_i}) q^{n_i} \left(t_{V_i^{\mathrm{c}}} - 1\right)}{1 - q^{-n-1+n_i} t_0 \, t_{V_i^{\mathrm{c}}}}\right)\right).
\end{aligned}$$

♦

5.4 Four basic operations on hypergraphs

In this section, we study four fundamental operations for hypergraphs: insert either a row or a column of either all 0s or all 1s into any incidence matrix. In Proposition 5.23, we record the effects of these operations on associated ask zeta functions. For group-theoretic applications of these results, see §8.

Throughout, let $\boldsymbol{\mu} = (\mu_I)_{I \subset V}$ be the vector of hyperedge multiplicities of a hypergraph H on the vertex set V; see Definition 3.1 and the comments that follow it. Let $n = |V|$ and $m = |\mathrm{E}(\mathsf{H})| = \sum_{I \subset V} \mu_I$. Let $\bullet$ be a singleton set disjoint from V.

Definition 5.22. We define

(i) $\boldsymbol{\mu_1} = (\nu_J)_{J \subset V \sqcup \bullet}$ by $\nu_J = \begin{cases} \mu_I, & \text{if } J = I \sqcup \bullet \text{ for } I \subset V, \\ 0, & \text{otherwise,} \end{cases}$

(ii) $\boldsymbol{\mu_0} = (\nu_J)_{J \subset V \sqcup \bullet}$ by $\nu_J = \begin{cases} \mu_J, & \text{if } J \subset V, \\ 0, & \text{otherwise,} \end{cases}$

(iii) $\boldsymbol{\mu^1} = \left(\mu_I + \delta_{I=V}\right)_{I \subset V}$, and

(iv) $\boldsymbol{\mu^0} = \left(\mu_I + \delta_{I=\varnothing}\right)_{I \subset V}$.

In other words, beginning with an arbitrary incidence matrix of H, we obtain associated hypergraphs

$\mathsf{H}_{\mathbf{1}} := \mathsf{H}(\boldsymbol{\mu}_{\mathbf{1}})$ by inserting a **1**-row, $\mathsf{H}_{\mathbf{0}} := \mathsf{H}(\boldsymbol{\mu}_{\mathbf{0}})$ by inserting a **0**-row,
$\mathsf{H}^{\mathbf{1}} := \mathsf{H}(\boldsymbol{\mu}^{\mathbf{1}})$ by inserting a **1**-column, $\mathsf{H}^{\mathbf{0}} := \mathsf{H}(\boldsymbol{\mu}^{\mathbf{0}})$ by inserting a **0**-column.

Note that $\mathsf{H}(\boldsymbol{\mu}_{\mathbf{1}}) = \mathsf{H}(\boldsymbol{\mu}, \bullet)$ in the sense of Definition 5.1. We write $\boldsymbol{\mu}_{\mathbf{1}^{(0)}} = \boldsymbol{\mu}$ and, for $r \in \mathbf{N}$, $\boldsymbol{\mu}_{\mathbf{1}^{(r)}} = (\boldsymbol{\mu}_{\mathbf{1}^{(r-1)}})_{\mathbf{1}}$. Likewise, we write $\mathsf{H}^{\mathbf{1}^{(0)}} = \mathsf{H}$ and $\mathsf{H}^{\mathbf{1}^{(r)}} = \left(\mathsf{H}^{(\mathbf{1}^{(r-1)})}\right)$. We use analogous notation for the other three operations. All four operations turn out to have tame effects on the ask zeta functions associated with H.

Proposition 5.23.

$$W_{\mathsf{H}_{\mathbf{1}}}(X,T) = \frac{1 - X^{n-m}T}{1 - X^{1+n-m}T}\, W_{\mathsf{H}}(X,T), \tag{5.26}$$

$$W_{\mathsf{H}_{\mathbf{0}}}(X,T) = W_{\mathsf{H}}(X,XT), \tag{5.27}$$

$$W_{\mathsf{H}^{\mathbf{1}}}(X,T) = \frac{1 - X^{-1}T}{1-T} W_{\mathsf{H}}(X,X^{-1}T), \tag{5.28}$$

$$W_{\mathsf{H}^{\mathbf{0}}}(X,T) = W_{\mathsf{H}}(X,T). \tag{5.29}$$

Proof. The statement about $W_{\mathsf{H}_{\mathbf{0}}}(X,T)$ and $W_{\mathsf{H}^{\mathbf{0}}}(X,T)$ follow from [57, §3.4], the others by inspection of (5.11) (with $d = 1$ for $W_{\mathsf{H}_{\mathbf{1}}}(X,T)$). ♦

Remark 5.24. For the purpose of determining $W_{\mathsf{H}}(X,T)$ for a hypergraph $\mathsf{H} \approx \mathsf{H}(\boldsymbol{\mu})$, Proposition 5.23 allows us to assume that $\boldsymbol{\mu}$ satisfies $\mu_V = \mu_\varnothing = 0$, $\bigcap_{\mu_I>0} I = \varnothing$, and $\bigcup_{\mu_I>0} I = V$. In other words, we may assume that no incidence matrix of $\mathsf{H}(\boldsymbol{\mu})$ has rows or columns comprised exclusively of 0s or 1s. Conversely, by adding suitable rows or columns of **0**s, Proposition 5.23 also allows us to e.g. assume that incidence matrices of hypergraphs are squares; cf. [57, Cor. 3.7].

We may now continue the story that began in Example 1.6.

Example 5.25 (Example 1.6, part II)**.** Let H be the hypergraph on 8 vertices with incidence matrix (1.8) in Example 1.6. We are now in a position to compute the rational function $W_{\mathsf{H}}(X,T)$. Indeed, H is isomorphic to $(\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}} \circledast \mathsf{BH}_{2,2}$. Using equations (5.19) (with $n_1 = n_2 = 3$ and $m_1 = m_2 = 2$) and (5.29), we obtain

$$W_{(\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}}}(X,T) = \frac{1 + X^{-4}T - 2X^{-2}T - 2X^{-1}T + XT + X^{-3}T^2}{(1-T)(1-XT)(1-X^2T)}.$$

Therefore, by (5.22),

$$\begin{aligned} W_{\mathsf{H}}(X,T) &= W_{(\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}} \circledast \mathsf{BH}_{2,2}}(X,T) \\ &= W_{(\mathsf{H}^{(3)} \oplus \mathsf{H}^{(3)})^{\mathbf{0}}}(X, X^{-2}T)\, \frac{(1 - X^{-2}T)(1-X^{-1}T)}{(1-T)(1-XT)} \\ &= \frac{1 + X^{-6}T - 2X^{-4}T - 2X^{-3}T + X^{-1}T + X^{-7}T^2}{(1-T)^2(1-XT)}. \end{aligned} \tag{5.30}$$

Note that $W_{\mathsf{H}}(X,T)$ coincides with the formula for $W_{\Gamma}^{-}(X,T)$ given in (1.6). We will be able to explain this following our proof of the Cograph Modelling Theorem (Theorem D) in §7; see Example 7.28.

5.5 Analytic properties of ask zeta functions of hypergraphs

Let K be a number field with ring of integers $\mathcal{O} = \mathcal{O}_K$. Let $\zeta_K(s)$ be the Dedekind zeta function of K. As in §1.3, let $\mathcal{V}_K$ be the set of non-Archimedean places of K and, for $v \in \mathcal{V}$, let $\mathcal{O}_v$ be the valuation ring of the v-adic completion of K. Let q_v be the residue field size of $\mathcal{O}_v$.

Let η be the incidence representation (see §3.2) of a hypergraph $\mathsf{H} = \mathsf{H}(\boldsymbol{\mu})$ on a set V of cardinality $n \geqslant 1$, where $\boldsymbol{\mu} = (\mu_I)_{I \subset V} \in \mathbf{N}_0^{\mathcal{P}(V)}$ is a vector of hyperedge multiplicities. By [57, Proposition 3.4],

$$\zeta_{\eta^{\mathcal{O}}}^{\mathsf{ask}}(s) = \prod_{v \in \mathcal{V}_K} \zeta_{\eta^{\mathcal{O}_v}}^{\mathsf{ask}}(s) = \prod_{v \in \mathcal{V}_K} W_{\mathsf{H}}(q_v, q_v^{-s}).$$

The explicit formula for $W_{\mathsf{H}}(X,T)$ in Corollary 5.6 allows us to deduce the following.

Theorem 5.26. *Let $m' := \sum_{\varnothing \neq I \subset V} \mu_I$ be the number of non-empty hyperedges of* H.

(i) For each compact DVR $\mathfrak{O}$, *the real parts of the poles of $\zeta_{\eta^{\mathfrak{O}}}^{\mathsf{ask}}(s)$ are contained in*

$$\mathcal{P}_{\mathsf{H}} := \Big\{ |J| - \sum_{I \cap J \neq \varnothing} \mu_I : J \subset V \Big\} \subset \{1-m', 2-m', \dots, n-1, n\},$$

a set of integers (!) of cardinality at most $\min\{2^n, n+m'\}$.

(ii) The abscissa of convergence $\alpha(\mathsf{H})$ of $\zeta_{\eta^{\mathcal{O}_K}}^{\mathsf{ask}}(s)$ is a positive integer. It satisfies $\alpha(\mathsf{H}) \leqslant n+1$ and is independent of K.

Proof. First note that if $m' = 0$, then $\zeta_{\eta^{\mathfrak{O}}}^{\mathsf{ask}}(s) = 1/(1-q^{n-s})$ and $\zeta_{\eta^{\mathcal{O}_K}}^{\mathsf{ask}}(s) = \zeta_K(s-n)$; cf. [57, p. 577]. As $n \geqslant 1$, both claims then follow immediately. Henceforth, suppose that $m' > 0$ so that $|J| - \sum_{I \cap J \neq \varnothing} \mu_I \in \{1-m', \dots, n\}$ for each $J \subset V$. Since $\zeta_{\eta^{\mathfrak{O}}}^{\mathsf{ask}}(s) = W_{\mathsf{H}}(q, q^{-s})$, part (i) thus follows from Corollary 5.6 (with $D = \varnothing$).

For part (ii), we first paraphrase (5.12) in the form

$$W_{\mathsf{H}}(X,T) = \frac{1}{1-T}\left(1 + \sum_{i=1}^{N} f_i(X^{-1}) \prod_{j \in I_i} \mathrm{gp}\left(X^{A_{ij}} T\right)\right) \tag{5.31}$$

for some $N \in \mathbf{N}_0$, non-empty subsets $I_i \subset \mathbf{N}$, non-constant polynomials $f_i(Y) \in \mathbf{Z}[Y]$ with constant term $f_i(0) = 1$, and $A_{ij} \in \mathcal{P}_{\mathsf{H}}$. We may assume that $N > 0$ and write (5.31) over a common denominator

$$W_{\mathsf{H}}(X,T) = \frac{1 + \sum\limits_{k=1}^{\infty} \left(\sum\limits_{l=-\infty}^{\infty} a_{lk} X^l \right) T^k}{(1-T) \prod\limits_{i,j} (1 - X^{A_{ij}} T)}.$$

As a product of finitely many translates of $\zeta_K(s)$, the Euler product

$$\prod_{v\in\mathcal{V}_K} \left.\frac{1}{(1-T)\prod\limits_{i,j}(1-X^{A_{ij}}T)}\right|_{X=q_v,T=q_v^{-s}} = \zeta_K(s)\prod_{i,j}\zeta_K(s-A_{ij})$$

has abscissa of convergence $\alpha := \max\{1, A_{ij}+1 : i\in[N], j\in I_i\} \leqslant n+1$, where the estimate follows as in (i). Moreover, this product may be analytically continued to a meromorphic function on the whole complex plane. It thus suffices to show that the abscissa of convergence, α' say, of the Euler product

$$N_{\mathsf{H}}(s) := \prod_{v\in\mathcal{V}_K}\left.\left(1+\sum_{k=1}^{\infty}\left(\sum_{l=-\infty}^{\infty} a_{lk}X^l\right)T^k\right)\right|_{X=q_v,T=q_v^{-s}} \tag{5.32}$$

is strictly less than α. By [22, Lemma 5.4],

$$\alpha' \leqslant \max\left\{\frac{l+1}{k} : l\in\mathbf{Z}, k\in\mathbf{N}, a_{lk}\neq 0\right\} = \max\left\{\alpha'_1, \alpha'_{\geqslant 2}\right\},$$

where

$$\alpha'_1 = \sup\{l+1 : l\in\mathbf{Z}, a_{l1}\neq 0\} \quad\text{and}\quad \alpha'_{\geqslant 2} = \sup\left\{\frac{l+1}{k} : l\in\mathbf{Z}, k\in\mathbf{N}_{\geqslant 2}, a_{lk}\neq 0\right\}.$$

The coefficient of T in each Euler factor on the right-hand side of (5.32) is

$$\sum_{l=-\infty}^{\infty} a_{l1}X^l = \sum_{i,j}\Big(f_i(X^{-1})-1\Big)X^{A_{ij}}.$$

Hence, by the aforementioned properties of the polynomials $f_i(Y)$, we conclude that $\alpha'_1 < \alpha$. Next, for each subset $S \subset I_1\times\cdots\times I_N$ with $|S|\geqslant 2$,

$$\frac{1+\sum_{(i,j)\in S}A_{ij}}{|S|} < \frac{\sum_{(i,j)\in S}(1+A_{ij})}{|S|} \leqslant \max\{1+A_{ij} : (i,j)\in S\} \leqslant \alpha$$

whence $\alpha'_{\geqslant 2} < \alpha$. The independence of $\alpha(\mathsf{H})$ from K has been established, in greater generality, in [57, Theorem 4.20]. ♦

Remark 5.27.

(i) As we mentioned in Remark 5.20, for specific $\boldsymbol{\mu}$ the formula (5.12) may simplify due to cancellations, possibly leading to a much smaller set of real parts of poles than $\mathcal{P}_{\mathsf{H}}$. Based on experimental evidence, however, for suitably "generic" $\boldsymbol{\mu}$, we expect most of these at most 2^n candidate real poles in $\mathcal{P}_{\mathsf{H}}$ to survive cancellation.

(ii) Every integer from 1 up to (and including) $n+1$ arises as the abscissa of convergence of the ask zeta function of a hypergraph on n vertices. Indeed, $\alpha(\mathsf{BH}_{n,m}) = \max\{1, n-m+1\}$; see Example 5.10((i)) and cf. [57, Example 3.5].

(iii) The fact that both $\alpha(\mathsf{H})$ and all elements of $\mathcal{P}_{\mathsf{H}}$ are *integers* seems noteworthy. Indeed, the abscissae of convergence of Dirichlet generating functions arising from related counting problems in subgroup or representation growth tend to be *rational* but typically non-integral numbers; cf. [25, Theorem 1.3 and §6] and [56, Theorem A(ii)] for (non-)integrality results in the area of subgroup and submodule zeta functions and, for instance, [1, Theorem 1.2], [26, Corollary B], and [64, Theorem 4.22] in the context of representation zeta functions.

(iv) Example 1.6 shows that, in general, integrality statements such as those in Theorem 5.26 hold neither for the (Euler products of instances of the) functions $W_\Gamma^+(X,T)$ featuring in Theorem A(iii) nor for the functions $W_\Gamma^-(X,T)$ in Theorem A(ii), unless Γ is a cograph; cf. Theorem D and see Question 1.8.

For staircase hypergraphs (§5.1.1), we can considerably strengthen Theorem 5.26(ii). Indeed, inspection of (5.14) yields the following result.

Proposition 5.28. *Let* $\mathbf{m} = (m_0, \dots, m_n) \in \mathbf{N}_0^{n+1}$. *Let* $\sigma\eta_{\mathbf{m}}$ *denote the incidence representation of the staircase hypergraph* $\Sigma\mathsf{H}_{\mathbf{m}}$*; see* (3.9). *Then for each number field* K *with ring of integers* $\mathcal{O}_K$*, the abscissa of convergence of* $\zeta^{\mathsf{ask}}_{\sigma\eta_{\mathbf{m}}}(s)$ *is given by*

$$\alpha(\Sigma\mathsf{H}_{\mathbf{m}}) = \max\Big\{1,\, 1+n-j-\sum_{\iota>j} m_\iota : j=0,\dots,n-1\Big\}.$$

Moreover, the function $\zeta^{\mathsf{ask}}_{\sigma\eta_{\mathbf{m}}^{\mathcal{O}_K}}(s)$ *may be meromorphically continued to the whole of* $\mathbf{C}$. ♦

6 Uniformity for ask zeta functions of graphs

The main result of this section, Theorem 6.4, establishes that, subject to very mild assumptions, a simultaneous generalisation of the two types of adjacency modules from §3.3 is torically combinatorial (see §4.4). This will, in particular, provide a constructive proof of Theorem A(ii)–(iii).

In §6.1, we develop the general setup for the class of adjacency modules that appear in Theorem 6.4. In §6.2, we describe several graph-theoretic operations with tame effects on adjacency modules. In §6.3, we show that "torically torically combinatorial" and "torically combinatorial" are equivalent properties. Both §§6.2–6.3 are then employed in the proof of Theorem 6.4 in §6.4.

Throughout, let R be a ring.

6.1 Weighted signed multigraphs and their adjacency modules

Definition 6.1. A **weighted signed multigraph (WSM)** (over σ) is a quadruple

$$\mathbf{\Gamma} = (\Gamma, \sigma, \mathrm{wt}, \mathrm{sgn}),$$

where

(W1) $\Gamma = (V, E, |\cdot|)$ is a multigraph (see §3.1),

(W2) $\sigma \subset \mathbf{R}_{\geqslant 0}\, V$ is a cone,

(W3) wt is a function $E \to \mathbf{Z}V$ with $u + \mathrm{wt}(e) \in \sigma^*$ for all $e \in E$ and $u \in |e|$, and

(W4) sgn is a function $E \to \{\pm 1\}$.

Henceforth, let $\mathbf{\Gamma}$ be a WSM as above. Let $X = (X_v)_{v \in V}$ as before. For $u, v \in V$ and $\omega \in \mathbf{Z}V$, define

$$[u, v; \pm 1, \omega]_R := \begin{cases} X^{u+\omega} v \pm X^{v+\omega} u, & \text{if } u \neq v, \\ \pm X^{u+\omega} u, & \text{if } u = v, \end{cases}$$

an element of $R[X^{\pm 1}]\, V$; we usually drop the subscript R in the following. Note that if $u \neq v$, then $[v, u; \pm 1, \omega] = \pm [u, v; \pm 1, \omega]$. Further note that $[u, v; \pm 1, 0] = [u, v; \pm 1]$, where the right-hand side is defined as in §3.3.

By (W3) in Definition 6.1, for $e \in E$ and $u \in |e|$, we have $X^{u+\mathrm{wt}(e)} \in R_\sigma$; see §4.2 for a definition of R_σ. Let

$$\mathsf{adj}(\mathbf{\Gamma}; R) := \Big\langle [u, v; \mathrm{sgn}(e), \mathrm{wt}(e)]_R : e \in E \text{ with } |e| = \{u, v\} \Big\rangle \leqslant R_\sigma V.$$

The **adjacency module** of $\mathbf{\Gamma}$ over R is the R_σ-module

$$\mathrm{Adj}(\mathbf{\Gamma}; R) := \frac{R_\sigma\, V}{\mathsf{adj}(\mathbf{\Gamma}; R)}.$$

Remark 6.2.

(i) If Γ is a graph, then $\mathrm{Adj}(\Gamma, \mathbf{R}_{\geqslant 0} V, 0, \pm 1; R)$ coincides with the adjacency module $\mathrm{Adj}(\Gamma, \pm 1; R)$ of Γ over R as defined in §3.3. (Here, we assume that Γ is simple in the negative case.)

(ii) Of course, the signs of loops have no effect on adjacency modules. They are included for notational convenience only.

Lemma 6.3.

(i) $\mathrm{Adj}(\mathbf{\Gamma}; S) = \mathrm{Adj}(\mathbf{\Gamma}; R)^{S_\sigma}$ *for each ring map* $R \to S$.

(ii) Let $\mathbf{\Gamma}'$ *be the* WSM *obtained from* $\mathbf{\Gamma}$ *by replacing* σ *by a cone* $\tau \subset \sigma$. *Then* $\mathrm{Adj}(\mathbf{\Gamma}'; R) \approx_{R_\tau} \mathrm{Adj}(\mathbf{\Gamma}; R) \otimes_{R_\sigma} R_\tau$. ♦

The following result and its constructive proof constitute the main contribution of the present section; a proof will be given in §6.4.

Theorem 6.4. *Let* $\mathbf{\Gamma} = (\Gamma, \sigma, \mathrm{wt}, \mathrm{sgn})$ *be a weighted signed multigraph. Let* R *be a ring. Suppose that one of the following conditions is satisfied: (i)* $2 \in R^\times$. *(ii)* $2 = 0$ *in* R. *(iii)* $\mathrm{sgn} \equiv -1$ *(irrespective of* R*). Then* $\mathrm{Adj}(\mathbf{\Gamma}; R)$ *is torically combinatorial (see §4.4).*

Theorem 6.4 easily implies Theorem A:

Proof of Theorem A. Part (i) was already proved in §4.4. For (ii)–(iii), combine Proposition 3.7, Proposition 4.8, Remark 6.2(i), and Theorem 6.4 (with $R = \mathbf{Z}$ or $R = \mathbf{Z}[1/2]$). ⧫

Proof of Corollary B. Combine Theorem A and Proposition 3.9. ⧫

While the preceding two proofs only applied Theorem 6.4 in the special case that $\mathbf{\Gamma}$ arises as in Remark 6.2, our recursive proof of Theorem 6.4 heavily relies on the greater generality developed here.

A particularly easy special case of Theorem 6.4 deserves to be spelled out at this point. We say that a graph is **solitary** if each of its edges is a loop.

Proposition 6.5. *Let $\sigma \subset \mathbf{R}_{\geqslant 0}V$ be a cone. Let $\mathbf{\Gamma}$ be a* WSM *over σ (with underlying vertex set V) such that the underlying graph of $\mathbf{\Gamma}$ is solitary. Then $\mathrm{Adj}(\mathbf{\Gamma}; R)$ is a combinatorial R_σ-module.* ⧫

Informally, our proof of Theorem 6.4 given in §6.4 proceeds by induction on an invariant which measures to what extent a graph fails to be solitary; the base case of our induction will be provided by Proposition 6.5.

6.2 Multigraph surgery

As we will see in this subsection, subject to various assumptions, we may modify the edges (as well as their weights and signs) of weighted signed multigraphs without affecting the isomorphism type of the associated adjacency module. This will constitute the heart of our proof of Theorem 6.4 in §6.4.

Throughout let $\mathbf{\Gamma} = (\Gamma, \sigma, \mathrm{wt}, \mathrm{sgn})$ be a WSM, where $\Gamma = (V, E, |\cdot|)$. An **incident pair** of Γ is a pair ("formal product") $u.e$, where $e \in E$ and $u \in |e|$. Recall that $\leqslant_\sigma$ denotes the preorder on $\mathbf{Z}V$ such that $u \leqslant_\sigma v$ if and only if $v - u \in \sigma^*$. Given incident pairs $u.e$ and $w.f$ of Γ, we say that $u.e$ **dominates** $w.f$ (in $\mathbf{\Gamma}$) if $u + \mathrm{wt}(e) \leqslant_\sigma w + \mathrm{wt}(f)$.

The next three lemmas and their proofs (nearly) follow an identical pattern: subject to dominance conditions, suitable edges of Γ can be transplanted to produce a new WSM $\mathbf{\Gamma}'$ such that $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ for every ring R. The effects of the operations $\mathbf{\Gamma} \rightsquigarrow \mathbf{\Gamma}'$ on the underlying multigraphs are indicated in Figure 1. Note that these figures only depict those parts of the respective multigraphs that are relevant for the result in question.

Lemma 6.6 (Dominant loop vs non-loop)**.** *Let $u, v \in V$ be distinct. Let $\ell, h \in E$ with $|\ell| = \{u\}$ and $|h| = \{u, v\}$. Suppose that $u.\ell$ dominates $v.h$. Define a multigraph $\Gamma' := (V, E, \|\cdot\|)$, where*

$$\|e\| := \begin{cases} |e|, & \text{if } e \neq h, \\ \{v\}, & \text{if } e = h. \end{cases}$$

Define $\mathrm{wt}' \colon E \to \mathbf{Z}V$ *via*

$$\mathrm{wt}'(e) := \begin{cases} \mathrm{wt}(e), & \text{if } e \neq h, \\ u - v + \mathrm{wt}(h), & \text{if } e = h. \end{cases}$$

Then the following hold:

(i) $\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn})$ *is a* WSM.

(ii) $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ *for every ring* R*; in particular,* $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\mathbf{\Gamma}'; R)$.

Proof.

(i) We need to check that $x + \mathrm{wt}'(e) \in \sigma^*$ for all $e \in E$ and $x \in \|e\|$. For $e \neq h$, this clearly follows since $\mathbf{\Gamma}$ is a WSM. It also follows in the remaining case $e = h$ since $v + \mathrm{wt}'(h) = u + \mathrm{wt}(h) \in \sigma^*$, again since $\mathbf{\Gamma}$ is a WSM.

(ii) Let

$$I := \Big\langle [x, y; \mathrm{sgn}(e), \mathrm{wt}(e)] : e \in E \setminus \{h\} \text{ with } |e| = \{x, y\} \Big\rangle \subset \mathsf{adj}(\mathbf{\Gamma}; R) \cap \mathsf{adj}(\mathbf{\Gamma}'; R).$$

Further define

$$\begin{aligned} a &:= [u, v; \mathrm{sgn}(h), \mathrm{wt}(h)] \in \mathsf{adj}(\mathbf{\Gamma}; R), \\ a' &:= [v, v; \mathrm{sgn}(h), \mathrm{wt}'(h)] \in \mathsf{adj}(\mathbf{\Gamma}'; R), \text{ and} \\ b &:= [u, u; \mathrm{sgn}(\ell), \mathrm{wt}(\ell)] \in I \end{aligned}$$

and note that $\mathsf{adj}(\mathbf{\Gamma}; R) = \langle a \rangle + I$ and $\mathsf{adj}(\mathbf{\Gamma}'; R) = \langle a' \rangle + I$. Since $u.\ell$ dominates $v.h$, we have $t := X^{v + \mathrm{wt}(h) - u - \mathrm{wt}(\ell)} \in R_\sigma$. Therefore,

$$\mathrm{sgn}(h) a' = X^{u + \mathrm{wt}(h)} v = a - \mathrm{sgn}(h)\,\mathrm{sgn}(\ell) t b,$$

whence $a \equiv \pm a' \pmod{I}$. ♦

Lemma 6.7 (Dominant non-loop vs loop)**.** *Let* $u, v \in V$ *be distinct. Let* $\ell, h \in E$ *with* $|\ell| = \{v\}$ *and* $|h| = \{u, v\}$*. Suppose that* $u.h$ *dominates* $v.\ell$*. Define a multigraph* $\Gamma' := (V, E, \|\cdot\|)$*, where*

$$\|e\| := \begin{cases} |e|, & \text{if } e \neq \ell, \\ \{u\}, & \text{if } e = \ell. \end{cases}$$

Define $\mathrm{wt}' \colon E \to \mathbf{Z}V$ *via*

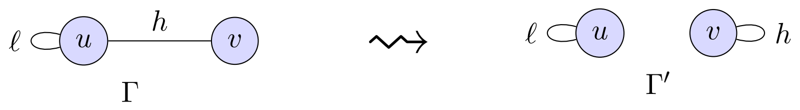
$$\mathrm{wt}'(e) := \begin{cases} \mathrm{wt}(e), & \text{if } e \neq \ell, \\ 2v - 2u + \mathrm{wt}(\ell), & \text{if } e = \ell \end{cases}$$

and $\mathrm{sgn}' \colon E \to \{\pm 1\}$ *via*

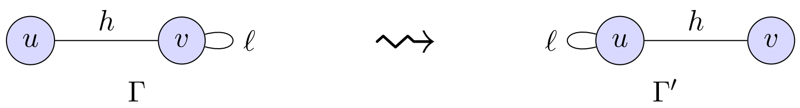
$$\mathrm{sgn}'(e) := \begin{cases} \mathrm{sgn}(e), & \text{if } e \neq \ell, \\ -\mathrm{sgn}(h)\,\mathrm{sgn}(\ell), & \text{if } e = \ell. \end{cases}$$

Then the following hold:

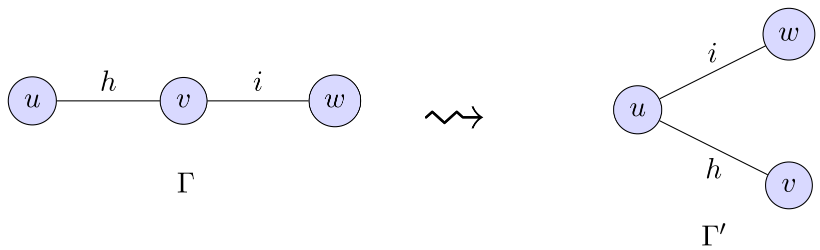
(i) $\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn}')$ *is a* WSM.



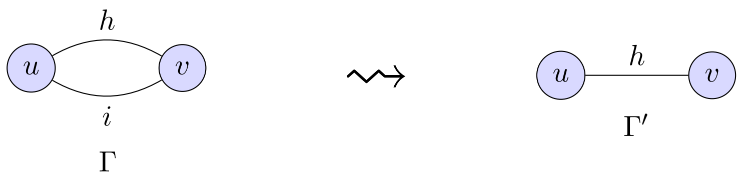
(a) Lemma 6.6: dominant loop vs non-loop



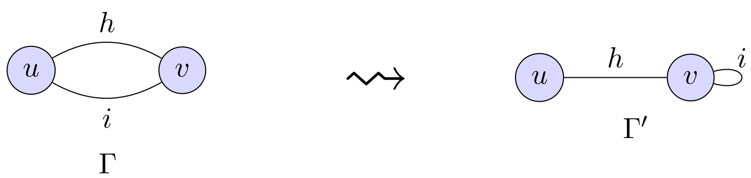
(b) Lemma 6.7: dominant non-loop vs loop



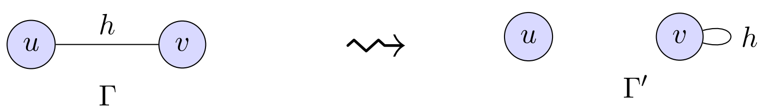
(c) Lemma 6.8: dominant non-loop vs non-loop



(d) Lemma 6.9: parallel edges I



(e) Lemma 6.10: parallel edges II



(f) Lemma 6.11: trimming spikes

Figure 1: Illustrations of Lemma 6.6–6.11

(ii) $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ *for every ring* R*; in particular,* $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\mathbf{\Gamma}'; R)$.

Proof.

(i) We need to check that $x + \mathrm{wt}'(e) \in \sigma^*$ for all $e \in E$ and $x \in \|e\|$. For $e \neq \ell$ and $x \in \|e\| = |e|$, we have $x + \mathrm{wt}'(e) = x + \mathrm{wt}(e) \in \sigma^*$ since $\mathbf{\Gamma}$ is a WSM. Moreover, $v + \mathrm{wt}(h) \in \sigma^*$ since Γ is a WSM and $v + \mathrm{wt}(\ell) - u - \mathrm{wt}(h) \in \sigma^*$ since $u.h$ dominates $v.\ell$. Hence, $u + \mathrm{wt}'(\ell) = 2v - u + \mathrm{wt}(\ell) \in \sigma^*$.

(ii) Let

$$I := \Big\langle [x, y; \mathrm{sgn}(e), \mathrm{wt}(e)] : e \in E \setminus \{\ell\} \text{ with } |e| = \{x, y\} \Big\rangle \subset \mathsf{adj}(\mathbf{\Gamma}; R) \cap \mathsf{adj}(\mathbf{\Gamma}'; R).$$

Further define

$$\begin{aligned} a &:= [u, v; \mathrm{sgn}(h), \mathrm{wt}(h)] \in I, \\ b &:= [v, v; \mathrm{sgn}(\ell), \mathrm{wt}(\ell)] \in \mathsf{adj}(\mathbf{\Gamma}; R), \text{ and} \\ b' &:= [u, u; \mathrm{sgn}'(\ell), \mathrm{wt}'(\ell)] \in \mathsf{adj}(\mathbf{\Gamma}'; R) \end{aligned}$$

and note that $\mathsf{adj}(\mathbf{\Gamma}; R) = \langle b \rangle + I$ and $\mathsf{adj}(\mathbf{\Gamma}'; R) = \langle b' \rangle + I$. Since $u.h$ dominates $v.\ell$, we have $t := X^{v + \mathrm{wt}(\ell) - u - \mathrm{wt}(h)} \in R_\sigma$. Therefore, $b - \mathrm{sgn}(\ell) t a = b'$, whence $b \equiv b' \pmod{I}$. ♦

Lemma 6.8 (Dominant non-loop vs non-loop)**.** *Let* $u, v, w \in V$ *be distinct. Let* $h \in E$ *with* $|h| = \{u, v\}$ *and* $i \in E$ *with* $|i| = \{v, w\}$*. Suppose that* $u.h$ *dominates* $w.i$*. Define a multigraph* $\Gamma' := (V, E, \|\cdot\|)$*, where*

$$\|e\| := \begin{cases} |e|, & \text{if } e \neq i, \\ \{u, w\}, & \text{if } e = i. \end{cases}$$

Define $\mathrm{wt}' \colon E \to \mathbf{Z}V$ *via*

$$\mathrm{wt}'(e) := \begin{cases} \mathrm{wt}(e), & \text{if } e \neq i, \\ v - u + \mathrm{wt}(i), & \text{if } e = i \end{cases}$$

and $\mathrm{sgn}' \colon E \to \{\pm 1\}$ *via*

$$\mathrm{sgn}'(e) := \begin{cases} \mathrm{sgn}(e), & \text{if } e \neq i, \\ -\mathrm{sgn}(h)\,\mathrm{sgn}(i), & \text{if } e = i. \end{cases}$$

Then the following hold:

(i) $\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn}')$ *is a* WSM.

(ii) $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ *for every ring* R*; in particular,* $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\mathbf{\Gamma}'; R)$.

Proof.

(i) We need to check that $x + \mathrm{wt}'(e) \in \sigma^*$ for all $e \in E$ and $x \in \|e\|$. For $e \neq i$ and $x \in \|e\| = |e|$, $x + \mathrm{wt}'(e) = x + \mathrm{wt}(e) \in \sigma^*$. As $\mathbf{\Gamma}$ is a WSM, each of $v + \mathrm{wt}(h)$, $v + \mathrm{wt}(i)$, and $w + \mathrm{wt}(i)$ belongs to σ^*. Since $u.h$ dominates $w.i$, we conclude that $u + \mathrm{wt}'(i) = v + \mathrm{wt}(i) \in \sigma^*$ and

$$w + \mathrm{wt}'(i) = w + v - u + \mathrm{wt}(i) = (v + \mathrm{wt}(h)) + (w + \mathrm{wt}(i)) - (u + \mathrm{wt}(h)) \in \sigma^*.$$

(ii) Let

$$I := \Big\langle [x, y; \mathrm{sgn}(e), \mathrm{wt}(e)] : e \in E \setminus \{i\} \text{ with } |e| = \{x, y\} \Big\rangle \subset \mathsf{adj}(\mathbf{\Gamma}; R) \cap \mathsf{adj}(\mathbf{\Gamma}'; R).$$

Further define

$$\begin{aligned} a &:= [u, v; \mathrm{sgn}(h), \mathrm{wt}(h)] \in I, \\ b &:= [v, w; \mathrm{sgn}(i), \mathrm{wt}(i)] \in \mathsf{adj}(\mathbf{\Gamma}; R), \text{ and} \\ b' &:= [u, w; \mathrm{sgn}'(i), \mathrm{wt}'(i)] \in \mathsf{adj}(\mathbf{\Gamma}'; R) \end{aligned}$$

and note that $\mathsf{adj}(\mathbf{\Gamma}; R) = \langle b \rangle + I$ and $\mathsf{adj}(\mathbf{\Gamma}'; R) = \langle b' \rangle + I$.

Since $u.h$ dominates $w.i$, we have $t := X^{w + \mathrm{wt}(i) - u - \mathrm{wt}(h)} \in R_\sigma$. Therefore, $b - \mathrm{sgn}(i) t a = b'$, whence $b \equiv b' \pmod{I}$. ⧫

Lemma 6.9 (Parallel edges I)**.** *Let $h, i \in E$ be distinct with $|h| = |i|$. Suppose that* $\mathrm{wt}(h) \leqslant_\sigma \mathrm{wt}(i)$*. Define a multigraph* $\Gamma' := (V, E \setminus \{i\}, \|\cdot\|)$ *and a* WSM

$$\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn}'),$$

where $\|\cdot\|$, wt', *and* sgn' *are the restrictions of* $|\cdot|$, wt, *and* sgn *to* $E \setminus \{i\}$*, respectively. Suppose that one of the following conditions is satisfied: (i) h (hence i) is a loop. (ii)* $\mathrm{sgn}(h) = \mathrm{sgn}(i)$*. Then* $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ *for every ring R; in particular,* $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\mathbf{\Gamma}'; R)$.

Proof. Write $|h| = \{u, v\}$. Since $\mathrm{wt}(h) \leqslant_\sigma \mathrm{wt}(i)$, we have $t := X^{\mathrm{wt}(i) - \mathrm{wt}(h)} \in R_\sigma$. The claim follows since, in each one of the two cases listed,

$$t \cdot [u, v; \mathrm{sgn}(h), \mathrm{wt}(h)] = \pm [u, v; \mathrm{sgn}(i), \mathrm{wt}(i)].$$ ⧫

Lemma 6.10 (Parallel edges II)**.** *Let $u, v \in V$ be distinct. Let $h, i \in E$ with $|h| = |i| = \{u, v\}$,* $\mathrm{sgn}(h) = -\mathrm{sgn}(i)$*, and* $\mathrm{wt}(h) \leqslant_\sigma \mathrm{wt}(i)$*. Define a multigraph* $\Gamma' := (V, E, \|\cdot\|)$*, where*

$$\|e\| := \begin{cases} |e|, & \text{if } e \neq i, \\ \{v\}, & \text{if } e = i. \end{cases}$$

Define $\mathrm{wt}' \colon E \to \mathbf{Z}V$ *via*

$$\mathrm{wt}'(e) := \begin{cases} \mathrm{wt}(e), & \text{if } e \neq i, \\ u - v + \mathrm{wt}(i), & \text{if } e = i \end{cases}$$

Then the following hold:

(i) $\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn})$ *is a* WSM.

(ii) $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ *for every ring* R *in which 2 is invertible; in particular,* $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\mathbf{\Gamma}'; R)$ *for such rings* R.

Proof.

(i) For $e \neq i$ and $x \in \|e\| = |e|$, we have $x + \mathrm{wt}'(e) = x + \mathrm{wt}(e) \in \sigma^*$. As $\mathbf{\Gamma}$ is a WSM, $u + \mathrm{wt}(i)$ belongs to σ^* whence $v + \mathrm{wt}'(i) = u + \mathrm{wt}(i) \in \sigma^*$.

(ii) Let

$$I := \Big\langle [x, y; \mathrm{sgn}(e), \mathrm{wt}(e)] : e \in E \setminus \{i\} \text{ with } |e| = \{x, y\} \Big\rangle \subset \mathsf{adj}(\mathbf{\Gamma}; R) \cap \mathsf{adj}(\mathbf{\Gamma}'; R).$$

Further define

$$\begin{aligned} a &:= [u, v; \mathrm{sgn}(h), \mathrm{wt}(h)] \in I, \\ b &:= [u, v; \mathrm{sgn}(i), \mathrm{wt}(i)] \in \mathsf{adj}(\mathbf{\Gamma}; R), \text{ and} \\ b' &:= [v, v; \mathrm{sgn}(i), \mathrm{wt}'(i)] \in \mathsf{adj}(\mathbf{\Gamma}'; R) \end{aligned}$$

and note that $\mathsf{adj}(\mathbf{\Gamma}; R) = \langle b \rangle + I$ and $\mathsf{adj}(\mathbf{\Gamma}'; R) = \langle b' \rangle + I$.

Since $\mathrm{wt}(h) \leqslant_\sigma \mathrm{wt}(i)$, we have $t := X^{\mathrm{wt}(i) - \mathrm{wt}(h)} \in R_\sigma$. As $b + ta = \pm 2b'$, we conclude that $\mathsf{adj}(\mathbf{\Gamma}; R) = \mathsf{adj}(\mathbf{\Gamma}'; R)$ whenever 2 is invertible in R. ⧫

By a **spike** of $\mathbf{\Gamma}$, we mean a pair (u, v) of distinct vertices of Γ such that (i) u is the only neighbour of v, (ii) there is only one edge $e \in E$ with $|e| = \{u, v\}$, and (iii) $u \leqslant_\sigma v$.

Lemma 6.11 (Trimming spikes)**.** *Let* (u, v) *be a spike of* $\mathbf{\Gamma}$. *Let* $h \in E$ *be the unique edge with* $|h| = \{u, v\}$. *Define a multigraph* $\Gamma' := (V, E, \|\cdot\|)$, *where*

$$\|e\| := \begin{cases} |e|, & \text{if } e \neq h, \\ \{v\}, & \text{if } e = h. \end{cases}$$

Define $\mathrm{wt}' \colon E \to \mathbf{Z}V$ *via*

$$\mathrm{wt}'(e) := \begin{cases} \mathrm{wt}(e), & \text{if } e \neq h, \\ u - v + \mathrm{wt}(h), & \text{if } e = h. \end{cases}$$

(i) $\mathbf{\Gamma}' := (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn})$ *is a* WSM.

(ii) $\mathrm{Adj}(\mathbf{\Gamma}; R) \approx_{R_\sigma} \mathrm{Adj}(\mathbf{\Gamma}'; R)$ *for every ring* R. *(However, in contrast to the preceding lemmas,* $\mathsf{adj}(\mathbf{\Gamma}; R)$ *and* $\mathsf{adj}(\mathbf{\Gamma}'; R)$ *may differ.)*

Proof.

(i) For $e \in E \setminus \{h\}$ and $x \in \|e\| = |e|$, we have $x + \mathrm{wt}'(e) = x + \mathrm{wt}(e) \in \sigma^*$. Moreover, $v + \mathrm{wt}'(h) = u + \mathrm{wt}(h) \in \sigma^*$ since $\mathbf{\Gamma}$ is a WSM.

(ii) Since $u \leqslant_\sigma v$, we obtain an R_σ-module automorphism θ of $R_\sigma V$ given by

$$x\theta = \begin{cases} x, & \text{if } x \neq v, \\ v - \mathrm{sgn}(h) X^{v-u} u, & \text{if } x = v. \end{cases}$$

We now show that $\mathsf{adj}(\mathbf{\Gamma}; R)\theta = \mathsf{adj}(\mathbf{\Gamma}'; R)$; the claim then follows immediately.

Let $e \in E \setminus \{h\}$ with $|e| = \{x, y\}$. Since (u, v) is a spike but $e \neq h$, we have $v \notin |e|$. Hence, θ fixes $[x, y; \mathrm{sgn}(e), \mathrm{wt}(e)]$ $(= [x, y; \mathrm{sgn}(e), \mathrm{wt}'(e)])$. The claim follows since $[u, v; \mathrm{sgn}(h), \mathrm{wt}(h)]\,\theta = X^{u+\mathrm{wt}(h)} v = \pm[v, v; \mathrm{sgn}(h), \mathrm{wt}'(h)]$. ♦

Corollary 6.12. *Let the* WSM $\mathbf{\Gamma}' = (\Gamma', \sigma, \mathrm{wt}', \mathrm{sgn}')$ *be derived from* $\mathbf{\Gamma}$ *using any one of Lemmas 6.6–Lemma 6.9 or Lemma 6.11. If* $\mathrm{sgn} \equiv -1$*, then* $\mathrm{sgn}' \equiv -1$. ♦

Remark 6.13. Note that even if the underlying multigraph of a WSM admits no parallel edges, each of Lemmas 6.6–Lemma 6.8, Lemma 6.10, or Lemma 6.11 might introduce parallel edges.

6.3 Torically torically combinatorial modules are torically combinatorial

This section establishes the (intuitively evident) fact that a torically {torically combinatorial} module over a toric ring is itself torically combinatorial; see Corollary 6.16.

Lemma 6.14. *Let T be a non-empty finite set of cones in $\mathbf{R}V$. Then there exists a fan $\mathcal{F}'$ in $\mathbf{R}V$ such that the following conditions are satisfied:*

(i) For each $\tau \in T$, there exists $\Sigma \subset \mathcal{F}'$ with $\tau = \bigcup \Sigma$.

(ii) For each $\sigma \in \mathcal{F}'$, there exists $\tau \in T$ with $\sigma \subset \tau$.

(iii) $|\mathcal{F}'| = \bigcup T$.

Proof. For $x \in \mathbf{R}V$, define $x^{\pm}$ and $x^{=}$ as in the proof of Lemma 4.1. For each $\tau \in T$, there exists a non-empty finite set $H_\tau \subset \mathbf{Z}V$ such that $\tau = \bigcap\limits_{h \in H_\tau} h^+$. For $h \in H := \bigcup\limits_{\tau \in T} H_\tau$, define a complete fan $\mathcal{F}_h := \{h^+, h^-, h^=\}$. Let $\mathcal{F} := \bigwedge\limits_{h \in H} \mathcal{F}_h$ and $\mathcal{F}' := \{\sigma \in \mathcal{F} : \exists \tau \in T.\, \sigma \subset \tau\}$. Since $\mathcal{F}$ is a fan, so is $\mathcal{F}'$. We claim that $\mathcal{F}'$ has the desired properties, (ii) being satisfied by construction.

For (i), let $\tau \in T$. Recall that $\tau = \bigcap\limits_{h \in H_\tau} h^+$. Let $x \in \tau$ be arbitrary. For each $h \in H$, define $\sigma_x(h) \in \mathcal{F}_h$ via

$$\sigma_x(h) = \begin{cases} h^{\pm}, & \text{if } x \in h^{\pm} \setminus h^{=}, \\ h^{=}, & \text{if } x \in h^{=}; \end{cases}$$

in other words, $\sigma_x(h)$ is the unique cone in $\mathcal{F}_h$ which contains x in its relative interior. Let $\sigma_x = \bigcap\limits_{h \in H} \sigma_x(h) \in \mathcal{F}$ and note that $x \in \sigma_x$. Since $x \in \tau = \bigcap\limits_{h \in H_\tau} h^+$, for each $h \in H_\tau$, we have $\sigma_x(h) \in \{h^+, h^=\}$ and hence $\sigma_x(h) \subset h^+$. Thus,

$$\sigma_x = \bigcap_{h \in H} \sigma_x(h) \subset \bigcap_{h \in H_\tau} \sigma_x(h) \subset \bigcap_{h \in H_\tau} h^+ = \tau;$$

in particular, $\sigma_x \in \mathcal{F}'$. We may thus take Σ to be the finite (!) set $\{\sigma_x : x \in \tau\}$. Finally, by (i)–(ii), we have $\bigcup T \subset |\mathcal{F}'| \subset \bigcup T$. ⧫

In particular, we can construct "fans of fans" as follows.

Corollary 6.15. *Let $\mathcal{F}$ be a fan of cones in $\mathbf{R}V$. For each $\sigma \in \mathcal{F}$, let $\mathcal{F}_\sigma$ be a fan of cones in $\mathbf{R}V$ with $|\mathcal{F}_\sigma| = \sigma$. Then there exists a fan $\mathcal{F}''$ of cones in $\mathbf{R}V$ with the following properties:*

(i) $\mathcal{F}''$ refines $\mathcal{F}$.

(ii) $|\mathcal{F}''| = |\mathcal{F}|$.

(iii) For each $\sigma \in \mathcal{F}$ and $\sigma' \in \mathcal{F}_\sigma$, there exists $\Sigma'' \subset \mathcal{F}''$ with $\sigma' = \bigcup \Sigma''$.

(iv) For each $\sigma'' \in \mathcal{F}''$, there exist $\sigma \in \mathcal{F}$ and $\sigma' \in \mathcal{F}_\sigma$ with $\sigma'' \subset \sigma'$.

Proof. Let $\mathcal{F}'$ be as in Lemma 6.14 with $T := \bigcup\limits_{\sigma \in \mathcal{F}} \mathcal{F}_\sigma$. Let $\mathcal{F}'' := \mathcal{F} \wedge \mathcal{F}'$. The first property holds by definition and the second one since $|\mathcal{F}'| = \bigcup T = |\mathcal{F}|$. For (iii), let $\sigma \in \mathcal{F}$ and $\sigma' \in \mathcal{F}_\sigma \subset T$. By Lemma 6.14(i), there exists $\Sigma \subset \mathcal{F}'$ with $\sigma' = \bigcup \Sigma$. As $\sigma' \subset \sigma$, we have $\sigma' = \bigcup \Sigma'$, where $\Sigma' := \{\varrho \cap \sigma : \varrho \in \Sigma\} \subset \mathcal{F}''$. For (iv), every cone in $\mathcal{F}''$ is contained in a cone from $\mathcal{F}'$ and each cone in $\mathcal{F}'$ is contained in an element of T. ⧫

Corollary 6.16. *Let $o \subset \mathbf{R}_{\geqslant 0} V$ be a cone. Let M be an R_o-module. Suppose that $\mathcal{F}$ is a fan in $\mathbf{R}_{\geqslant 0} V$ with $|\mathcal{F}| = o$ such that $M \otimes_{R_o} R_\sigma$ is torically combinatorial over R_σ for each $\sigma \in \mathcal{F}$. Then M is torically combinatorial as an R_o-module.*

Proof. By assumption, for each $\sigma \in \mathcal{F}$, there exists a fan $\mathcal{F}_\sigma$ with support σ such that $(M \otimes_{R_o} R_\sigma) \otimes_{R_\sigma} R_\tau \approx_{R_\tau} M \otimes_{R_o} R_\tau$ is combinatorial over R_τ for each $\tau \in \mathcal{F}_\sigma$. Now apply the preceding corollary and note that a change of scalars of a combinatorial module along a natural ring map $R_\tau \hookrightarrow R_{\tau'}$ (coming from an inclusion $\tau' \subset \tau$ of cones) preserves the property of being combinatorial. ⧫

6.4 Proof of Theorem 6.4: "solitary induction"

Let $\mathbf{\Gamma} = (\Gamma, \sigma, \mathrm{wt}, \mathrm{sgn})$ be a WSM, where $\Gamma = (V, E, |\cdot|)$ and $\sigma \subset \mathbf{R}_{\geqslant 0} V$.

Define $s(\Gamma)$ (the "social degree" of Γ) to be the number of non-loops of Γ in E. Note that $s(\Gamma) = 0$ if and only if Γ is solitary (see §6.1). Let R be a ring. If $2 = 0$ in R, then $\mathrm{Adj}(\mathbf{\Gamma}; R)$ does not depend on $\mathrm{sgn}(\,\cdot\,)$ at all so we may assume that $\mathrm{sgn} \equiv -1$ in this case. To prove Theorem 6.4, it thus suffices to show that $\mathrm{Adj}(\mathbf{\Gamma}; R)$ is torically combinatorial whenever $\mathrm{sgn} \equiv -1$ or 2 is invertible in R. We proceed by induction on $s(\Gamma)$.

Base case. If $s(\Gamma) = 0$, then Γ is solitary and $\mathrm{Adj}(\mathbf{\Gamma}; R)$ is combinatorial (not merely *torically* combinatorial) by Proposition 6.5.

Henceforth, suppose that $s(\Gamma) > 0$.

General assumptions and reductions. We first carry out a number of general reductions; none of these increases $s(\Gamma)$.

The following operations amount to (i) constructing a fan $\mathcal{F}$ with support σ and (ii) considering the cases obtained by replacing σ by a cone in $\mathcal{F}$; this strategy is justified by Lemma 6.3 and Corollary 6.16.

Thus, by shrinking σ via Lemma 4.1 and using Lemma 6.9, we may assume that Γ has no parallel edges except possibly parallel non-loops with different signs. By shrinking σ yet further, we may also assume that for any two incident pairs of Γ, one of them dominates the other; see §6.2 for this notion. The last condition is clearly equivalent to the preorder $\leqslant_\sigma$ from §4.1 being *total* on elements $x + \mathrm{wt}(e) \in \mathbf{Z}V$ for $e \in E$ and $x \in |e|$.

Parallel non-loops with opposite signs. Suppose that Γ has parallel non-loops with opposite signs. In particular, $\mathrm{sgn} \not\equiv -1$ and we may assume that 2 is invertible in R. Let $u, v \in V$ be distinct and let $h, i \in E$ with $|h| = |i| = \{u, v\}$ but $\mathrm{sgn}(h) \neq \mathrm{sgn}(i)$. Our assumption on dominance of incidence pairs implies that $\mathrm{wt}(h) \leqslant_\sigma \mathrm{wt}(i)$ or $\mathrm{wt}(h) \geqslant_\sigma \mathrm{wt}(i)$. Without loss of generality, suppose that we are in the former case. Let $\mathbf{\Gamma}'$ be the WSM obtained from $\mathbf{\Gamma}$ using Lemma 6.10. By construction, $s(\Gamma') < s(\Gamma)$. Indeed, the non-loop i of Γ is a loop of Γ' and other edges coincide in the sense that they have the same support in each multigraph. By induction, $\mathrm{Adj}(\mathbf{\Gamma}'; R) = \mathrm{Adj}(\mathbf{\Gamma}; R)$ is torically combinatorial.

We may therefore assume that Γ has no parallel edges at all. Recall that we also assume that given any two incident pairs of Γ, one of them dominates the other. Since $s(\Gamma) > 0$, we may choose a connected component, Ξ say, of Γ with $s(\Xi) > 0$.

Using a dominant loop. Suppose that $\ell \in E$ is a loop at $u \in V$ in Ξ such that $u.\ell$ dominates each incident pair of Ξ. Since Ξ is not solitary but connected, it contains a non-loop $h \in E$ with $u \in |e|$. Let $\mathbf{\Gamma}'$ be the WSM obtained from $\mathbf{\Gamma}$ using Lemma 6.6. Since h is a loop of Γ' and all other edges are unchanged as above, $s(\Gamma') < s(\Gamma)$. Hence, $\mathrm{Adj}(\mathbf{\Gamma}'; R) = \mathrm{Adj}(\mathbf{\Gamma}; R)$ is torically combinatorial by induction.

Using a dominant non-loop. We may thus assume that $u.h$ is an incident pair of Ξ which dominates all incident pairs of Ξ and that h is not a loop. Write $|h| = \{u, v\}$ and note that $u \leqslant_\sigma v$ since $u.h$ dominates $v.h$.

For each edge $i \in E \setminus \{h\}$ with $v \in |i|$, we then obtain a WSM $\mathbf{\Gamma}'$ as in Lemma 6.7 or Lemma 6.8 with $v \not\sim_{\Gamma'} i$ and such that all other edges of Γ' have the same support in Γ and Γ'. We may repeatedly apply these lemmas to all such edges i, one after the other, to derive a WSM $\tilde{\mathbf{\Gamma}}$ with $\mathrm{Adj}(\mathbf{\Gamma}; R) = \mathrm{Adj}(\tilde{\mathbf{\Gamma}}; R)$ and such that the underlying graph $\tilde{\Gamma}$ of $\tilde{\mathbf{\Gamma}}$ satisfies $s(\tilde{\Gamma}) = s(\Gamma)$. By construction, (u, v) is then a spike of $\tilde{\mathbf{\Gamma}}$. By deriving $\tilde{\mathbf{\Gamma}}'$ from $\tilde{\mathbf{\Gamma}}$ via Lemma 6.11, we obtain $s(\tilde{\Gamma}') < s(\tilde{\Gamma}) = s(\Gamma)$ whence $\mathrm{Adj}(\tilde{\mathbf{\Gamma}}'; R) \approx_{R_\sigma} \mathrm{Adj}(\tilde{\mathbf{\Gamma}}; R) = \mathrm{Adj}(\mathbf{\Gamma}; R)$ is torically combinatorial by induction.

Restrictions on R. We only made use of the assumption that 2 be invertible in R when we considered parallel edges with opposite signs. If all edge signs of a WSM $\mathbf{\Gamma}$ are -1, then by Corollary 6.12, the same is true for all the graphs derived from $\mathbf{\Gamma}$ as part of our inductive proof above. Hence, no restrictions on R are needed in this case and this completes the proof of Theorem 6.4. ♦

Remark 6.17. Given a WSM $\mathbf{\Gamma} = (\Gamma, \sigma, \mathrm{wt}, \mathrm{sgn})$ as above, our inductive proof of Theorem 6.4 gives rise to a recursive algorithm for constructing a fan $\mathcal{F}$ with support σ and for each $\tau \in \mathcal{F}$ a WSM $\mathbf{\Gamma}_\tau$ with solitary underlying graph such that $\mathrm{Adj}(\mathbf{\Gamma}; R) \otimes_{R_\sigma} R_\tau \approx_{R_\tau} \mathrm{Adj}(\mathbf{\Gamma}_\tau; R)$ for each $\tau \in \mathcal{F}$ (and subject to the assumptions on R from above). Together with Proposition 4.8 and the techniques for computing monomial integrals from [54, 58], we thus obtain an algorithm for explicitly computing the rational functions in Theorem A. This algorithm turns out to be quite practical; see §9.

Remark 6.18. In the setting of Theorem A(iii), the arguments developed in this section do not apply to compact DVRs of characteristic 2 due to the factors ± 2 in the penultimate line of the proof of Lemma 6.10. Indeed, the conclusion of Theorem A(iii) does not generally hold for compact DVRs $\mathfrak{O}$ with residue characteristic 2. For example, using either the method from [57, §9.1] or the one developed here (see §9.1), we find that

$$\begin{aligned} W^{+}_{\mathrm{K}_3}(X,T) &= (T^2 + T + 1 - 3X^{-1}T^2 - 6X^{-1}T + 6X^{-2}T^2 + 3X^{-2}T - X^{-3}T^3 - X^{-3}T^2 \\ &\qquad - X^{-3}T)/(1-T)^4 \\ &= 1 + (\underbrace{5 - 6X^{-1} + 3X^{-2} - X^{-3}}_{=:g(X)})T + \mathcal{O}(T^2). \end{aligned}$$

On the other hand, a simple calculation shows that the average size of the kernel of a matrix of the form

$$\begin{bmatrix} 0 & x & y \\ x & 0 & z \\ y & z & 0 \end{bmatrix}$$

over $\mathbf{F}_{2^f}$ is given by $h(2^f)$, where $h(X) = 1 + X + X^{-2}$; note that $g(x) \neq h(x)$ for all real $x > 1$. In particular, for each compact DVR $\mathfrak{O}$ with residue field size $q = 2^f$, the function $W^{+}_{\mathrm{K}_3}(q, q^{-s})$ differs from the ask zeta function of the positive adjacency representation associated with K_3 over $\mathfrak{O}$.

7 Graph operations and ask zeta functions of cographs

In this section, we deduce the Cograph Modelling Theorem (Theorem D) from a structural result (Theorem 7.1) which relates incidence modules of hypergraphs and adjacency modules of cographs. After collecting some facts about cographs in §7.1, we formally state Theorem 7.1 in §7.2. We give an outline of the latter theorem's proof in §7.3 which is then fleshed out in §§7.4–§7.7.

Since we will focus exclusively on negative adjacency representations of simple graphs, in this section, we frequently omit references to the "negative" part. Throughout, R is a ring, V is a finite set, and $X = (X_v)_{v \in V}$ consists of algebraically independent variables over R. All graphs are assumed to be simple in this section.

7.1 Background on cographs

A **cograph** is a graph which belongs to the smallest class of graphs which contains isolated vertices and which is closed under both disjoint unions and joins of graphs. In this definition, "joins" can be replaced by "taking complements". Cographs have appeared in various contexts and under various names such as "complement reducible graphs" and "P_4-free graphs"; see [20]. They admit numerous equivalent characterisations; see [20, Theorem 2]. For instance, cographs are precisely those graphs all of whose connected induced subgraphs have diameter at most 2. Moreover, cographs are also precisely those graphs that do not contain a path on four vertices as an induced subgraph.

As explained in [20], each cograph can be represented by a **cotree**: a rooted tree whose internal vertices are labelled using one of the symbols $\oplus$ and $\vee$ (corresponding to disjoint unions and joins, respectively) and whose leaves correspond to the vertices of the cograph. This representation is unique up to isomorphism of rooted trees provided that (i) each internal vertex has at least two descendants and (ii) adjacent internal vertices are labelled differently.

7.2 Comparing adjacency and incidence modules

Generalising the definition of incidence modules $\mathrm{Inc}(\mathsf{H}; R)$ in §3.2, for a hypergraph $\mathsf{H} = (V, E, |\cdot|)$ and cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$, we let

$$\mathsf{inc}(\mathsf{H}, \sigma; R) := \Big\langle X_v e : v \sim_{\mathsf{H}} e \ (v \in V,\, e \in E)\Big\rangle \leqslant R_\sigma E$$

and we define the **incidence module** of H with respect to σ over R to be

$$\mathrm{Inc}(\mathsf{H}, \sigma; R) := \frac{R_\sigma E}{\mathsf{inc}(\mathsf{H}, \sigma; R)}.$$

Clearly,

$$\mathrm{Inc}(\mathsf{H}, \sigma; R) \approx_{R_\sigma} \mathrm{Inc}(\mathsf{H}; R) \otimes_R R_\sigma. \tag{7.1}$$

For a simple graph Γ with vertex set V and a cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$, we obtain a weighted signed multigraph (see §6.1) $\mathbf{\Gamma} := (\Gamma, \sigma, 0, -1)$. We set

$$\mathsf{adj}(\Gamma, \sigma; R) := \mathsf{adj}(\mathbf{\Gamma}; R) = \Big\langle X_v w - X_w v : v, w \in V,\, v \sim_\Gamma w\Big\rangle \leqslant R_\sigma V$$

and define the **adjacency module** of Γ with respect to σ over R to be

$$\mathrm{Adj}(\Gamma, \sigma; R) := \mathrm{Adj}(\mathbf{\Gamma}; R) = \frac{R_\sigma V}{\mathsf{adj}(\Gamma, \sigma; R)}.$$

To further simplify our notation, we let $\mathrm{Adj}(\Gamma; R) := \mathrm{Adj}(\Gamma, \mathbf{R}_{\geqslant 0}V; R)$; note that this notation is consistent with §3.3. Observe that

$$\mathrm{Adj}(\Gamma, \sigma; R) \approx_{R_\sigma} \mathrm{Adj}(\Gamma; R) \otimes_{R[X]} R_\sigma; \tag{7.2}$$

cf. Lemma 6.3. In the following, we often omit R from our notation in case $R = \mathbf{Z}$.

The following is the main result of the present section.

Theorem 7.1 (Cograph Modelling Theorem: structural form)**.** *Let Γ be a cograph. Let C be the set of connected components of Γ. Then there exists a hypergraph H with $\mathrm{V}(\Gamma) = \mathrm{V}(\mathsf{H})$, $|\mathrm{E}(\mathsf{H})| = |\mathrm{V}(\Gamma)| - |C|$, and such that $\mathrm{Adj}(\Gamma)$ and $\mathrm{Inc}(\mathsf{H}) \oplus \mathbf{Z}[X]C$ are torically isomorphic $\mathbf{Z}[X]$-modules, where $X = (X_v)_{v\in V}$.*

Our proof of this theorem in §7.6 below is based on a number of algebraic and graph-theoretic techniques developed in the following. Our proof is effective: given a cograph Γ, we can write down an explicit hypergraph H (a "model" of Γ in a sense to be formalised in §7.6) as in Theorem 7.1. The final piece towards a proof of Theorem D is the following comparison result for adjacency and incidence representations.

Lemma 7.2. *Let Γ be a graph and let H be a hypergraph, both with common vertex set V. Let $c \geqslant 0$ and suppose that $|\mathrm{E}(\mathsf{H})| = |V| - c$. Let Σ be a set of cones with $\bigcup \Sigma = \mathbf{R}_{\geqslant 0}V$ and such that $\mathrm{Adj}(\Gamma,\sigma) \approx_{\mathbf{Z}_\sigma} \mathrm{Inc}(\mathsf{H},\sigma) \oplus \mathbf{Z}_\sigma^c$ for each $\sigma \in \Sigma$. Then $W_\Gamma^-(X,T) = W_{\mathsf{H}}(X,T)$.*

Proof. Let γ $(= \gamma_-)$ and η be the adjacency and incidence representation of Γ and H over $\mathbf{Z}$, respectively; see §§3.2–3.3. As always, let $\mathfrak{O}$ be a compact DVR. First, for each cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$ and finitely generated $\mathfrak{O}_\sigma$-module M, we clearly have $\zeta_{M\oplus\mathfrak{O}_\sigma}(s) = \zeta_M(s-1)$; cf. [57, Corollary 3.7]. Using Lemma 6.14 and (7.1)–(7.2), we may assume that Σ is a fan of cones with support $\mathbf{R}_{\geqslant 0}V$. Now combine Proposition 3.3, Proposition 3.7, and Lemma 4.6 to obtain

$$\zeta^{\mathsf{ask}}_{\gamma^{\mathfrak{O}}}(s) = (1-q^{-1})^{-1}\zeta_{\mathrm{Adj}(\Gamma)\otimes\mathfrak{O}[X]}(s) = (1-q^{-1})^{-1}\zeta_{\mathrm{Inc}(\mathsf{H})\otimes\mathfrak{O}[X]}(s-c) = \zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s). \ \blacklozenge$$

We may now deduce the version of the Cograph Modelling Theorem from the introduction.

Proof of Theorem D. Combine Theorem 7.1 and Lemma 7.2 with $c = |C|$. ♦

Remark 7.3. Assuming the validity of Theorem 7.1, we actually proved a slightly stronger result than Theorem D. Namely, for each cograph Γ, there exists a hypergraph H on the same set of vertices with $|\mathrm{E}(\mathsf{H})| < |\mathrm{V}(\Gamma)|$ and $W_\Gamma^-(X,T) = W_{\mathsf{H}}(X,T)$. By repeated application of (5.29), we may assume that $|\mathrm{E}(\mathsf{H})| = |\mathrm{V}(\Gamma)| - 1$. The incidence matrices of H are then "near squares" in the sense that only one column is missing from a square.

We note that the hypothesis of Theorem D itself is not optimal:

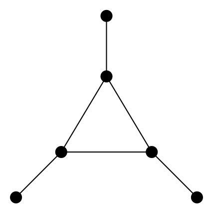
Example 7.4. The rational function $W^-_{\mathrm{P}_4}(X,T)$ associated with a path on four vertices coincides with $W_{\mathsf{H}}(X,T)$, where H is a hypergraph with incidence matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This can be verified by direct computations; see §9.2. We regard examples such as the above as evidence that the existence of a toric isomorphism in Theorem 7.1 is perhaps a more natural question to investigate than coincidence of rational functions.

Likewise, the conclusion of Theorem D does not hold for arbitrary graphs:

Example 7.5. Let Γ be the graph



By an explicit computation using §6 (see §9.1), we find that

$$W_\Gamma^-(X,T) = (-X^3T^4+5X^2T^4-6X^2T^3+4X^2T^2-6XT^4+14XT^3-15XT^2+4XT+T^5 - 5T^4+5T^3+5T^2-5T+1+4X^{-1}T^4-15X^{-1}T^3+14X^{-1}T^2-6X^{-1}T + 4X^{-2}T^3-6X^{-2}T^2+5X^{-2}T-X^{-3}T)/((1-T)(1-XT)^3(1-X^3T^2)), \tag{7.3}$$

where the numerator and denominator are both factored into irreducibles in $\mathbf{Q}(X)[T]$. In view of the quadratic irreducible factor $1-X^3T^2$ in (7.3), Theorem C shows that $W_\Gamma^-(X,T)$ is not of the form $W_\mathsf{H}(X,T)$ for *any* hypergraph H.

7.3 Informal overview of the proof of Theorem 7.1

Let Γ be a cograph with vertex set V. At the heart of our constructive proof of Theorem 7.1 lies the notion of a *scaffold* on V over a cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$; see Definition 7.7. Informally, scaffolds are forests (i.e. disjoint unions of trees) on V with the same connected components as Γ. These forests all come with *outgoing orientations* given by specifying a root in each of the forest's trees and letting all edges point away from their associated root. Crucially, these orientations are required to be compatible with the preorder $\leqslant_\sigma$ on $\mathbf{Z}V$ induced by the cone σ; see §4.1. By shrinking σ, we may further assume that the restriction of this preorder to V is total, i.e. a weak order. In addition to the above, the edges of a scaffold carry weights in the form of subsets of V. In this way, scaffolds give rise to hypergraphs and also to weighted signed multigraphs (WSMs; see §6.1) and adjacency modules.

We say that a scaffold *encloses* a (co)graph Γ over σ if the adjacency module of the WSM associated with the scaffold and the adjacency module of Γ (with respect to σ) coincide in a strong sense. In this case, we call the scaffold's hypergraph a *local model* of Γ over σ; see Definition 7.14(i).

A fundamental idea behind our proof of Theorem 7.1 is to approximate the graph Γ by scaffolds attached to various cones. These cones cover the positive orthant $\mathbf{R}_{\geqslant 0}V$. Crucially, all scaffolds realise the same (!) hypergraph H (up to suitable identifications) as local models of Γ. In this case, we call H a *global model* of Γ; see Definition 7.14(ii).

As cographs (save for singletons) arise as either disjoint unions or joins of smaller cographs, we are looking to recursively construct (global) models of disjoint unions and joins of cographs. The case of disjoint unions is comparatively simple: Proposition 7.23

establishes that if Γ_1 and Γ_2 are cographs with modelling hypergraphs H_1 and H_2, then the disjoint union $\mathsf{H}_1 \oplus \mathsf{H}_2$ is a model of $\Gamma_1 \oplus \Gamma_2$.

The case of joins of (co)graphs, which is settled in Theorem 7.24, is much more involved. In §7.7, we construct a model for the join of $\Gamma_1 \vee \Gamma_2$ of Γ_1 and Γ_2 by implementing the following strategy.

We fix a cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$ and scaffolds $\mathcal{S}_i(\sigma)$ enclosing Γ_i for $i = 1, 2$. In *Phase 1*, using a process governed by removing, one at a time, suitably chosen connecting edges between Γ_1 and Γ_2, we modify the disjoint union of the scaffolds $\mathcal{S}_i(\sigma)$ to obtain a scaffold $\mathcal{S}^{(N)}$ which "almost encloses" the join $\Gamma_1 \vee \Gamma_2$; more precisely, it encloses said join up to factoring out a particular submodule. It then remains to consider this "error term".

The scaffold $\mathcal{S}^{(N)}$ differs from the disjoint union of the scaffolds $\mathcal{S}_1(\sigma)$ and $\mathcal{S}_2(\sigma)$ only in the weights borne by its edges. The disjoint union of two scaffolds is, in particular, a disjoint union of two forests. In order to obtain a scaffold enclosing the connected (!) graph $\Gamma_1 \vee \Gamma_2$, we grow, in *Phase 2* of our construction, a single oriented tree out of the two oriented forests comprising $\mathcal{S}^{(N)}$. In order to ensure that the resulting scaffold $\mathcal{S}^{(\infty)}$ has the desired property of giving rise to a local model of $\Gamma_1 \vee \Gamma_2$ over σ, we graft judiciously chosen (directed and weighted) edges between pairs of roots from both forests. A final analysis shows that the given procedure is sufficiently independent of the many choices made along the way and, crucially, the chosen cone σ. In particular, the hypergraph associated with $\mathcal{S}^{(\infty)}$ essentially only depends on the graphs Γ_1 and Γ_2 and the hypergraphs associated with the scaffolds $\mathcal{S}_1(\sigma)$ and $\mathcal{S}_2(\sigma)$. This allows us to combine global models of each of Γ_1 and Γ_2 into a global model of the join $\Gamma_1 \vee \Gamma_2$.

7.4 Outgoing orientations of forests

By an **orientation** of a graph $\Gamma = (V, E, |\cdot|)$, we mean a function $\mathrm{ori}\colon E \to V \times V$ which assigns an ordered pair $(u, v) = \mathrm{ori}(e)$ to each edge $e \in E$ with $|e| = \{u, v\}$. We call u and v the **source** and **target** of e, respectively. We use the notation $u \xrightarrow{e} v$ for an oriented edge e with $\mathrm{ori}(e) = (u, v)$.

The **indegree** (resp. **outdegree**) $\mathrm{indeg}(u)$ (resp. $\mathrm{outdeg}(v)$) of $u \in V$ with respect to an orientation is the number of edges with target (resp. source) u. An orientation of Γ is **outgoing** if each vertex has indegree at most one.

If T is a tree and u is a vertex of T, then the **rooted orientation** of T with root u has all edges pointing away from u. More formally, let e be any edge with $|e| = \{v, w\}$, where v precedes w on the unique simple path from u to w. We then define v to be the source of e. This orientation of T is clearly outgoing. Trees endowed with such orientations are often referred to as **arborescences** or **out-trees** in the literature. Outgoing and rooted orientations of trees are identical concepts:

Proposition 7.6 (Cf. [31, §3.5])**.** *Let* T *be a tree endowed with an outgoing orientation* ori*. Then* T *contains a vertex* u *such that* ori *is the rooted orientation of* T *with root* u*.*

Proof. Let n be the number of vertices of T. Then T contains precisely $n-1$ edges. Hence, the sum of the indegrees of all vertices is $n-1$. Since ori is an outgoing orientation, we conclude that a unique vertex u has indegree zero; see [35, Theorem 16.4]. Let $v_1, \dots, v_m$

be the distinct neighbours of u. Let $\mathsf{T}_1, \dots, \mathsf{T}_m$ be the different trees that constitute the forest obtained from T by deleting u; we assume that T_i contains v_i. Then each T_i inherits an outgoing orientation from T. Moreover, v_i is the unique vertex in T_i with indegree zero. By induction, the induced orientation of each T_i is therefore the rooted orientation with respect to v_i. The claim for T then follows immediately. ♦

In particular, each outgoing orientation of a forest Φ naturally induces a partial order $\prec$ on $\mathrm{V}(\Phi)$. In detail, vertices $u, v \in \mathrm{V}(\Phi)$ are comparable if and only if they belong to the same connected component, C say, and in that case, $u \prec v$ if and only if u precedes v on the unique simple path from the root of C to v. The $\prec$-minimal elements of $\mathrm{V}(\Phi)$ are exactly the roots of its connected components.

7.5 Scaffolds

Definition 7.7. A **scaffold** $\mathcal{S} = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|)$ on the vertex set V over a cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$ consists of a forest $\Phi = (V, E, |\cdot|)$ endowed with an outgoing orientation $\mathrm{ori}\colon E \to V \times V$ and a support function $\|\cdot\|\colon E \to \mathcal{P}(V)$ such that the following conditions are satisfied:

(S1) For each oriented edge $u \xrightarrow{e} v$ in Φ, we have $u \leqslant_\sigma v$ (see §4.1).

(S2) $\|e\| \neq \varnothing$ for each $e \in E$.

Given a scaffold $\mathcal{S}$ as in Definition 7.7, we obtain a hypergraph $\mathsf{H}(\mathcal{S}) := (V, E, \|\cdot\|)$; note that $\mathrm{E}(\mathsf{H}(\mathcal{S})) = \mathrm{E}(\Phi) = E$. Apart from the outgoing orientation, a scaffold consists of a forest Φ and a hypergraph $\mathsf{H}(\mathcal{S})$ related by a common set of (hyper)edges. A scaffold $\mathcal{S}$ as above also gives rise to a weighted signed multigraph (see Definition 6.1)

$$\mathbf{\Gamma}(\mathcal{S}) := (\Gamma(\mathcal{S}), \sigma, \mathrm{wt}_{\mathcal{S}}, -1)$$

constructed as follows:

(i) $\Gamma(\mathcal{S}) := (V, \mathrm{E}(\mathcal{S}), |\cdot|_{\mathcal{S}})$, where

$$\mathrm{E}(\mathcal{S}) := \{(e, w) : e \in \mathrm{E}(\Phi), w \in \|e\|\}$$

and $|(e, w)|_{\mathcal{S}} := |e|$ for each $(e, w) \in \mathrm{E}(\mathcal{S})$.

In other words, $\Gamma(\mathcal{S})$ is obtained from the forest Φ by replacing each edge e in Φ by a set of parallel edges with the same support as e, one for each element of $\|e\|$. By condition (S2), the forest Φ and multigraph $\Gamma(\mathcal{S})$ determine one another.

(ii) The weight of an edge (e, w) of $\Gamma(\mathcal{S})$ for an oriented (!) edge $u \xrightarrow{e} v$ of Φ and $w \in \|e\|$ is given by $\mathrm{wt}_{\mathcal{S}}(e, w) := w - u$.

Note that $u + \mathrm{wt}_{\mathcal{S}}(e, w) = w$ and $v + \mathrm{wt}_{\mathcal{S}}(e, w) = v + w - u$ both belong to σ^* (the latter since $u \leqslant_\sigma v$ by (S1)) so that condition (W3) in Definition 6.1 is satisfied.

For a scaffold $\mathcal{S}$, we write $\mathsf{adj}(\mathcal{S};R) := \mathsf{adj}(\mathbf{\Gamma}(\mathcal{S});R)$ and $\mathrm{Adj}(\mathcal{S};R) := \mathrm{Adj}(\mathbf{\Gamma}(\mathcal{S});R)$; as before, we often drop R when $R = \mathbf{Z}$. By definition, for each ring R,

$$\mathsf{adj}(\mathcal{S};R) = \Big\langle X^w v - X^{v+w-u} u : u \xrightarrow{e} v \text{ in } \Phi \text{ and } w \in \|e\| \Big\rangle \leqslant R_\sigma V. \tag{7.4}$$

Lemma 7.2 provides a sufficient condition for equality of ask zeta functions of adjacency and incidence representations. In order to use this lemma, we need to be able to establish "toric isomorphisms" between suitable adjacency and incidence modules. Scaffolds provide natural examples of such isomorphisms:

Proposition 7.8. *Let $\mathcal{S}$ be a scaffold as in Definition 7.7. Let C be the set of connected components of Φ. Then* $\mathrm{Adj}(\mathcal{S};R) \approx_{R_\sigma} \mathrm{Inc}(\mathsf{H}(\mathcal{S}),\sigma;R) \oplus R_\sigma C$.

Proof. By a *weak scaffold* on V, we mean a quadruple $\mathcal{S}$ as in Definition 7.7 except that Φ is allowed to be a forest on a *subset* $V_0 := \mathrm{V}(\Phi)$ of V. (This notion will not be used elsewhere.) The hypergraph $\mathsf{H}(\mathcal{S})$ associated with a weak scaffold has vertex set V_0. We define $\mathsf{adj}(\mathcal{S};R) \leqslant R_\sigma V_0$ as in (7.4) and $\mathrm{Adj}(\mathcal{S};R) := R_\sigma V_0/\mathsf{adj}(\mathcal{S};R)$.

We will establish Proposition 7.8 for weak scaffolds on V by induction on the number of edges of Φ; our proof uses the same idea as in Lemma 6.11. First, if Φ contains no edges, then $|C| = |V_0|$, $\mathrm{Adj}(\mathcal{S};R) = R_\sigma V_0 \approx_{R_\sigma} R_\sigma C$, and $\mathrm{Inc}(\mathsf{H}(\mathcal{S}),\sigma;R) = 0$.

Now suppose that Φ contains at least one edge. Consider any connected component T (a tree!) of Φ consisting of more than one vertex. Then T contains a vertex v with indegree 1 but outdegree 0 (i.e. a leaf distinct from the root). Let $u \xrightarrow{e} v$ be the unique oriented edge with target v in T (and in Φ). The change of coordinates on $R_\sigma V_0$ given by

$$x \mapsto \begin{cases} v + X^{v-u}u, & \text{if } x = v,\\ x, & \text{if } x \in V_0 \setminus \{v\}\end{cases}$$

maps the defining generator of $\mathsf{adj}(\mathcal{S};R)$ corresponding to e and $w \in \|e\|$ on the right-hand side of (7.4) to $X^w v$ while preserving generators arising from the other edges (as all these have trivial v-coordinate). Let $\mathcal{S}'$ be the weak scaffold on V obtained by deleting v and e from Φ; the underlying forest Φ' of $\mathcal{S}'$ satisfies $\mathrm{V}(\Phi') = V_0 \setminus \{v\}$. Note that

$$\mathrm{Adj}(\mathcal{S};R) \approx \mathrm{Adj}(\mathcal{S}';R) \oplus R_\sigma/\langle X^w : w \in \|e\|\rangle.$$

Since v is a leaf of T but not a root, deleting it did not increase the number of connected components; hence, Φ' has precisely $|C|$ connected components. Therefore, $\mathrm{Adj}(\mathcal{S}';R) \approx_{R_\sigma} \mathrm{Inc}(\mathsf{H}(\mathcal{S}');R) \oplus R_\sigma C$ by induction and the claim follows using (3.10). ♦

Proposition 7.8 allows us to compare adjacency and incidence modules associated with scaffolds. Our next step is to relate the latter to adjacency modules of general graphs.

Definition 7.9. Let Γ be a simple graph and $\mathcal{S} = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|)$ be a scaffold as in Definition 7.7, both on the same vertex set $V = \mathrm{V}(\Gamma) = \mathrm{V}(\Phi)$. We say that the scaffold $\mathcal{S}$ **encloses** Γ (over the underlying cone σ of $\mathcal{S}$) if the following conditions are satisfied:

(i) Γ and the underlying forest Φ of $\mathcal{S}$ have the same (vertex sets of) connected components.

(ii) $\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}(\mathcal{S})$.

Example 7.10 (Scaffolds and discrete graphs)**.** Recall that Δ_n is the discrete graph on the vertex set $V = \{1, \dots, n\}$; see (3.1). Trivially, for each cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$, there is a unique scaffold on V with underlying forest Δ_n and this scaffold encloses Δ_n.

Example 7.11 (Scaffolds and complete graphs)**.** Consider the complete graph K_n on n vertices; see (3.5). To avoid notational confusion, we also denote its vertices by $v_1, \dots, v_n$ (where $v_i = i$). Let

$$\sigma = \{x \in \mathbf{R}^n_{\geqslant 0} : x_1 \leqslant x_2, \dots, x_n\}.$$

Recall from (3.6) that Star_n denotes the star graph on $\{1, \dots, n\}$ with centre $v_1 = 1$. Let $\mathcal{S} = (\mathrm{Star}_n, \sigma, \mathrm{ori}, \|\cdot\|)$ be the scaffold on $\{1, \dots, n\}$, where (i) the edges of Star_n are oriented in the form $v_1 \to v_i$ and (ii) $\|e\| = \{v_1\}$ for each edge e of Star_n. Note that $\mathcal{S}$ is indeed a scaffold by our definition of σ. In other words, our orientation of Star_n is compatible with the preorder on vertices induced by σ. Figure 2 depicts the scaffold $\mathcal{S}$ and the graph K_n for $n = 3$; edges of the former are labelled by their supports in the associated hypergraph.

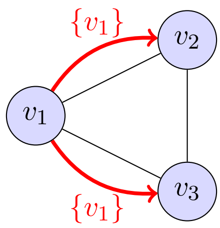


Figure 2: A scaffold enclosing K_3 over the cone $x_1 \leqslant x_2, x_3$

We claim that $\mathcal{S}$ encloses K_n over σ; our proof of this fact contains ideas that will feature in our proof of Theorem 7.1.

First note that we may identify

$$\mathbf{Z}_\sigma = \mathbf{Z}[X_1, \dots, X_n, X_1^{-1}X_2, \dots, X_1^{-1}X_n] = \mathbf{Z}[X_1, X_1^{-1}X_2, \dots, X_1^{-1}X_n].$$

Next,

$$\begin{aligned} \mathsf{adj}(\mathrm{K}_n, \sigma) &= \langle X_i v_j - X_j v_i : 1 \leqslant i < j \leqslant n \rangle \quad \text{and} \\ \mathsf{adj}(\mathcal{S}) &= \langle X_1 v_j - X_j v_1 : 2 \leqslant j \leqslant n \rangle . \end{aligned}$$

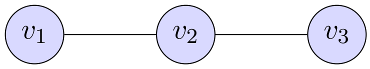
In particular, $\mathsf{adj}(\mathcal{S}) \subset \mathsf{adj}(\mathrm{K}_n, \sigma)$. To show the reverse inclusion, let $1 < i < j \leqslant n$.

Then, over $\mathbf{Z}_\sigma$ and modulo $\mathsf{adj}(\mathcal{S})$,

$$\begin{aligned} X_i v_j - X_j v_i &\equiv (X_i v_j - X_j v_i) + X_1^{-1} X_j (X_1 v_i - X_i v_1) \\ &= X_i v_j - X_1^{-1} X_i X_j v_1 \\ &= X_1^{-1} X_i \cdot (X_1 v_j - X_j v_1) \\ &\equiv 0 \pmod{\mathsf{adj}(\mathcal{S})}. \end{aligned}$$

Thus, $\mathsf{adj}(\mathcal{S}) = \mathsf{adj}(\mathrm{K}_n, \sigma)$ and $\mathcal{S}$ encloses K_n over σ.

Example 7.12 (Scaffolds and P_3)**.** Consider the path $\Gamma = \mathrm{P}_3$ on three vertices:



In the following, we identify $v_i = i$ as in Example 7.11. For $i = 1, 2, 3$, let

$$\sigma_i = \left\{ x \in \mathbf{R}^3_{\geqslant 0} : x_i \leqslant x_j \text{ for } j = 1, 2, 3 \right\}.$$

We construct scaffolds enclosing P_3 over σ_i for each $i = 1, 2, 3$. For $i = 2$, let $\mathcal{S}$ be the scaffold on $\{v_1, v_2, v_3\}$ with underlying forest P_3 oriented in the form $v_1 \leftarrow v_2 \rightarrow v_3$ and with supports $\|e\| = \{v_2\}$ for both edges e, as depicted in Figure 3.

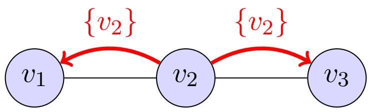


Figure 3: A scaffold enclosing P_3 over the cone $x_2 \leqslant x_1, x_3$

Arguments similar to those in Example 7.11 then show that $\mathcal{S}$ encloses Γ over σ_2. Next, by symmetry, the cases $i = 1$ and $i = 3$ are interchangeable; we only consider the former. Let $\mathcal{S}$ be the scaffold depicted in Figure 4. Since

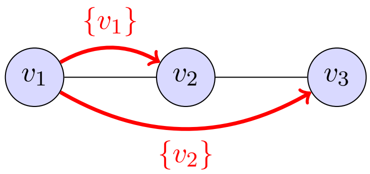


Figure 4: A scaffold enclosing P_3 over the cone $x_1 \leqslant x_2, x_3$

$$\begin{aligned} \mathsf{adj}(\mathrm{P}_3, \sigma) &= \langle \underbrace{X_1 v_2 - X_2 v_1}_{=: f_1}, \underbrace{X_2 v_3 - X_3 v_2}_{=: f_2} \rangle \\ &= \langle f_1, f_2 + X_1^{-1} X_3 f_1 \rangle \\ &= \left\langle f_1, X_2 \left(v_3 - X_1^{-1} X_3 v_1 \right) \right\rangle \\ &= \mathsf{adj}(\mathcal{S}), \end{aligned}$$

$\mathcal{S}$ encloses P_3 over σ_1.

Remark 7.13. Clearly, if $\mathcal{S}$ encloses Γ over σ and if $\sigma' \subset \sigma$ is a cone, then by shrinking the cone of $\mathcal{S}$, we obtain a scaffold $\mathcal{S}'$ which encloses Γ over σ'.

7.6 Models

Definition 7.14. Let Γ and H be a (simple) graph and hypergraph, respectively, both on the same vertex set V.

(i) We say that H is a **local model** of Γ over a cone $\sigma \subset \mathbf{R}_{\geqslant 0}V$ if there exists a scaffold $\mathcal{S}$ on V over σ which encloses Γ (see Definition 7.9) together with a bijection $\mathrm{E}(\mathsf{H}(\mathcal{S})) \xrightarrow{\phi} \mathrm{E}(\mathsf{H})$ such that $\|e\| = |e\phi|_{\mathsf{H}}$ for all $e \in \mathrm{E}(\mathsf{H}(\mathcal{S}))$.

(ii) We say that H is a **(global) model** of Γ is there exists a finite set Σ of cones with $\bigcup \Sigma = \mathbf{R}_{\geqslant 0}V$ such that H is a local model of Γ over each $\sigma \in \Sigma$.

In other words, H is a local model of Γ over σ if, up to relabelling of its hyperedges, H "is" the hypergraph $\mathsf{H}(\mathcal{S})$ (see §7.5) of some scaffold $\mathcal{S}$ enclosing Γ over σ. In the case of a global model, the particular scaffold and the relabelling of hyperedges may vary with the particular cone but the hypergraph remains fixed.

Remark 7.15. By Lemma 6.14 and Remark 7.13, we may equivalently require the set Σ in Definition 7.14(ii) to be a *fan* of cones.

Example 7.16. The discrete graph Δ_n is a model of itself.

Example 7.17. The block hypergraph $\mathsf{BH}_{n,n-1}$ (see (3.2)) is a model of the complete graph K_n. To see that, consider the cover $\mathbf{R}^n_{\geqslant 0} = \bigcup_{i=1}^{n} \sigma_i$, where

$$\sigma_i = \{x \in \mathbf{R}^n_{\geqslant 0} : x_i \leqslant x_j \text{ for } j = 1, \dots, n\};$$

here and in the following, we use the notation from Example 7.11. Let $\mathcal{S}_i$ be the scaffold over σ_i whose underlying graph is the star graph on $1, \dots, n$ with centre i, oriented edges of the form $i \to j$ for $i \neq j$, and all hyperedge supports of $\mathsf{H}(\mathcal{S}_i)$ equal to $\{i\}$. By Example 7.11, $\mathcal{S}_i$ encloses K_n over σ_i.

Let H_i be a hypergraph with vertices $1, \dots, n$ and $n-1$ hyperedges, each with support $\{i\}$. Then, up to relabelling of its hyperedges, $\mathsf{H}(\mathcal{S}_i)$ coincides with H_i which is therefore a local model of K_n over σ_i.

Next, by construction, X_i divides each X_j in $\mathbf{Z}_{\sigma_i}$. In particular, if we redefine all hyperedge supports of $\mathsf{H}(\mathcal{S}_i)$ to be $\{1, \dots, n\}$ instead of $\{i\}$, the resulting scaffold still encloses Γ over σ_i. Therefore, up to relabelling of its hyperedges, $\mathsf{H}(\mathcal{S}_i)$ coincides with $\mathsf{BH}_{n,n-1}$ for each $i = 1, \dots, n$. We conclude that $\mathsf{BH}_{n,n-1}$ is a global model of K_n.

Example 7.18 (A model of P_3)**.** We now construct a global model of P_3. We continue to use the notation from Example 7.12. For $i = 1, 2, 3$, let

$$\sigma_i = \{x \in \mathbf{R}^n_{\geqslant 0} : x_i \leqslant x_j \text{ for } j = 1, \dots, 3\}.$$

Let H_i be a hypergraph with vertices $1, 2, 3$ and incidence matrix A_i, where the rows are ordered naturally and A_i is given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

We showed in Example 7.12 that H_i is a local model of P_3 over σ_i for $i = 1, 2, 3$. Let H be a hypergraph with vertices $1, 2, 3$ and with incidence matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

By redefining hyperedge supports of scaffolds as in Example 7.17 and using that X_i divides each X_j in $\mathbf{Z}_{\sigma_i}$, we conclude that H is a global model of P_3.

Lemma 7.19. *Let H be a local model of Γ over some cone σ. Let c be the number of connected components of Γ. Then $|\mathrm{E}(\mathsf{H})| = |\mathrm{V}(\Gamma)| - c$.*

Proof. There exists a scaffold $\mathcal{S}$ which encloses Γ over σ. As the underlying forest, Φ say, of $\mathcal{S}$ and Γ have the same connected components and since a tree on n vertices contains $n - 1$ edges, the number of hyperedges of H (= number of edges of Φ) is as stated. ♦

Proposition 7.20. *Let H be a global model of Γ. Let C be the set of connected components of Γ. Then $\mathrm{Adj}(\Gamma)$ and $\mathrm{Inc}(\mathsf{H}) \oplus \mathbf{Z}C$ are torically isomorphic.*

Proof. Let Σ be a finite set of cones in $\mathbf{R}_{\geqslant 0}V$ with $\bigcup \Sigma = \mathbf{R}_{\geqslant 0}V$ and such that H is a local model of Γ over each $\sigma \in \Sigma$. By Remark 7.15, we may assume that Σ is a fan. Fix $\sigma \in \Sigma$. Then, up to relabelling of $\mathrm{E}(\mathsf{H})$, H is the hypergraph $\mathsf{H}(\mathcal{S})$ associated with a scaffold $\mathcal{S}$ which encloses Γ over σ. Hence, by Proposition 7.8, $\mathrm{Adj}(\Gamma, \sigma) = \mathrm{Adj}(\mathcal{S}) \approx_{\mathbf{Z}_\sigma} \mathrm{Inc}(\mathsf{H}, \sigma) \oplus \mathbf{Z}_\sigma C$. ♦

Corollary 7.21. *If H is a (global) model of Γ, then $W^-_\Gamma(X, T) = W_{\mathsf{H}}(X, T)$.*

Proof. Combine Lemma 7.2 and Proposition 7.20. ♦

Remark 7.22.

(i) Example 7.17 and Corollary 7.21 provide a new proof of the identity $\mathsf{Z}^{\mathsf{ask}}_{\mathfrak{so}_n(\mathfrak{O})}(T) = \mathsf{Z}^{\mathsf{ask}}_{\mathrm{M}_{n\times(n-1)}(\mathfrak{O})}(T)$ from [57, Proposition 5.11].

(ii) Corollary 7.21 shows that the graph in Example 7.5 does *not* admit a global model.

Our definition of models is specifically chosen to allow us to prove Proposition 7.20 and its consequence Corollary 7.21 as well as Proposition 7.23 and Theorem 7.24.

Proposition 7.23 (Models of disjoint unions of graphs)**.**
Let $\Gamma_i = (V_i, E_i, |\cdot|_i)$ be a graph for $i = 1, 2$. Let $\mathsf{H}_i = (V_i, H_i, \|\cdot\|_i)$ be a model of Γ_i. Then $\mathsf{H}_1 \oplus \mathsf{H}_2$ (see §3.1) is a model of $\Gamma_1 \oplus \Gamma_2$.

Proof. We may assume that $V_1 \cap V_2 = \varnothing$. For $i = 1,2$, there exist a collection of cones Σ_i with $\bigcup \Sigma_i = \mathbf{R}_{\geqslant 0} V_i$ and a collection of scaffolds $(\mathcal{S}_i(\sigma_i))_{\sigma_i \in \Sigma_i}$ on V_i such that each $\mathcal{S}_i(\sigma_i)$ encloses Γ_i over σ_i and such that each $\mathsf{H}(\mathcal{S}_i(\sigma_i))$ coincides with H_i up to relabelling of hyperedges. Let $V := V_1 \sqcup V_2$ and $\Sigma := \{\sigma_1 \times \sigma_2 : \sigma_i \in \Sigma_i;\ i = 1,2\}$ so that $\mathbf{R}_{\geqslant 0} V = \bigcup \Sigma$. For $\sigma_i \in \Sigma_i$ $(i = 1,2)$, let $\mathcal{S}(\sigma_1, \sigma_2)$ be the scaffold on V over $\sigma_1 \times \sigma_2$ whose underlying forest is the disjoint union of the underlying forests of $\mathcal{S}_1(\sigma_1)$ and $\mathcal{S}_2(\sigma_2)$, whose associated hypergraph is the disjoint union of $\mathsf{H}(\mathcal{S}_1(\sigma_1))$ and $\mathsf{H}(\mathcal{S}_2(\sigma_2))$, and whose outgoing orientation is induced by those of the scaffolds $\mathcal{S}_i(\sigma_i)$; note that conditions (S1)–(S2) are satisfied here so that $\mathcal{S}(\sigma_1, \sigma_2)$ is a scaffold. Next, note that $\mathsf{adj}(\Gamma_1 \oplus \Gamma_2, \sigma_1 \times \sigma_2)$ is generated by (the images of) $\mathsf{adj}(\Gamma_1, \sigma_1)$ and $\mathsf{adj}(\Gamma_2, \sigma_2)$. By construction, $\mathcal{S}(\sigma_1, \sigma_2)$ thus encloses $\Gamma_1 \oplus \Gamma_2$ over $\sigma_1 \times \sigma_2$ whence the claim follows. ♦

While formally similar to the preceding proposition, our next result requires considerably more work; a proof of the following theorem will be given in §7.7.

Theorem 7.24 (Models of joins of graphs)**.**
Let Γ_i be a non-empty graph for $i = 1,2$. Write $V_i = \mathrm{V}(\Gamma_i)$. Let $\mathsf{H}_i = (V_i, H_i, \|\cdot\|_i)$ be a model of Γ_i. Suppose that $V_1 \cap V_2 = \varnothing = H_1 \cap H_2$. Let c_i be the number of connected components of Γ_i. Let f_{ij} $(i = 1,2;\ j = 1,\dots,c_i - 1)$ and g be distinct symbols, none of which belongs to $H_1 \sqcup H_2$. Define a hypergraph $\mathsf{H} = (V, H, \|\cdot\|)$, where $V := V_1 \sqcup V_2$, $H := H_1 \sqcup H_2 \sqcup \{f_{ij} : i = 1,2; j = 1,\dots,c_i - 1\} \sqcup \{g\}$, and $H \xrightarrow{\|\cdot\|} \mathcal{P}(V)$ is defined by

$$\|h\| := \begin{cases} \|h\|_1 \sqcup V_2, & \text{if } h \in H_1, \\ V_1 \sqcup \|h\|_2, & \text{if } h \in H_2, \\ V_2, & \text{if } h = f_{1j} \text{ for } j = 1,\dots,c_1 - 1, \\ V_1, & \text{if } h = f_{2j} \text{ for } j = 1,\dots,c_2 - 1, \\ V_1 \sqcup V_2, & \text{if } h = g. \end{cases}$$

Then H is a model of the join $\Gamma_1 \vee \Gamma_2$ (see §3.1) of Γ_1 and Γ_2.

Remark 7.25.

(i) The hypergraph H in Theorem 7.24 may be expressed in terms of complete unions of hypergraphs as follows. For each $i \in \{1,2\}$, let $\mathsf{H}_i^{\square} = (\mathsf{H}_i)^{\mathbf{0}^{(c_i - 1)}}$; cf. Definition 5.22(iv). Informally speaking, H_i and $\mathsf{H}_i^{\square}$ coincide except for the multiplicity of the empty hyperedge; an incidence matrix of $\mathsf{H}_i^{\square}$ may be obtained from an incidence matrix of H_i by inserting $c_i - 1$ zero columns. One proves inductively that these are $|V_i| \times (|V_i| - 1)$-matrices, i.e. "near squares" (cf. Remark 7.3). Then $\mathsf{H} = (\mathsf{H}_1^{\square} \circledast \mathsf{H}_2^{\square})^{\mathbf{1}}$; cf. Definition 5.22(iii).

(ii) If $A_i \in \mathrm{M}_{n_i \times (n_i - c_i)}(\mathbf{Z})$ are incidence matrices of H_i, then the following (with $n = n_1 + n_2$) is an incidence matrix of H:

$$\begin{bmatrix} A_1 & \mathbf{1}_{n_1 \times (n_2 - c_2)} & \mathbf{0}_{n_1 \times (c_1 - 1)} & \mathbf{1}_{n_1 \times (c_2 - 1)} & \mathbf{1}_{n_1 \times 1} \\ \mathbf{1}_{n_2 \times (n_1 - c_1)} & A_2 & \mathbf{1}_{n_2 \times (c_1 - 1)} & \mathbf{0}_{n_2 \times (c_2 - 1)} & \mathbf{1}_{n_2 \times 1} \end{bmatrix} \in \mathrm{M}_{n \times (n-1)}(\mathbf{Z}).$$

Corollary 7.26. *Every cograph admits a model.*

Proof. Combine the description of cographs in terms of disjoint unions and joins in §7.1, Example 7.16 (for $n = 1$), Proposition 7.23, and Theorem 7.24. ♦

Proof of Theorem 7.1. Combine Corollary 7.26 and Proposition 7.20. ♦

Remark 7.27 (Canonical models)**.** Let Γ be a cograph represented by a cotree as in §7.1. The uniqueness of cotrees shows that the model, $\mathsf{H}(\Gamma)$ say, of Γ constructed in the proof of Corollary 7.26 is uniquely determined by Γ up to isomorphism of hypergraphs fixing all vertices of the set $\mathrm{V}(\Gamma) = \mathrm{V}(\mathsf{H}(\Gamma))$.

The hypergraphs of the form $\mathsf{H}(\Gamma)$ for cographs Γ are rather special. Recall that, by Corollary 7.21, $W_\Gamma^-(X,T) = W_{\mathsf{H}(\Gamma)}(X,T)$. Let $n = |\mathrm{V}(\Gamma)|$. By Lemma 7.19, $|\mathrm{E}(\mathsf{H}(\Gamma))| = n - c$, where c is the number of connected components of Γ. In particular, Remark 5.20 shows that $W_{\mathsf{H}(\Gamma)}(X,T)$ can be written over a denominator which is a product of fewer than $2n$ factors of the form $1 - X^A T$; for general hypergraphs on n vertices, we obtain an upper bound of 2^n such factors. For another restriction,

$$\sum_{e \in \mathrm{E}(\mathsf{H}(\Gamma))} \#\|e\|_{\mathsf{H}(\Gamma)} = 2|\mathrm{E}(\Gamma)|;$$

in particular, the number of non-zero entries in any incidence matrix of $\mathsf{H}(\Gamma)$ is even.

Example 7.28 (Example 1.6, part III)**.** We resume the story begun in Example 1.6. Let the graph $\Gamma \approx (\mathrm{K}_3 \oplus \mathrm{K}_3) \vee \mathrm{K}_2$ and hypergraph H be as defined there. As we observed in Example 5.25, $\mathsf{H} \approx (\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}} \circledast \mathsf{BH}_{2,2}$. By Example 7.17, the block hypergraph $\mathsf{BH}_{3,2}$ (resp. $\mathsf{BH}_{2,1}$) is a model of the complete graph K_3 (resp. K_2). Therefore, by Proposition 7.23, the disjoint union $\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2}$ is a model of $\mathrm{K}_3 \oplus \mathrm{K}_3$. By Theorem 7.24 (see also Remark 7.25(i)),

$$((\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}} \circledast \mathsf{BH}_{2,1})^{\mathbf{1}} \approx (\mathsf{BH}_{3,2} \oplus \mathsf{BH}_{3,2})^{\mathbf{0}} \circledast \mathsf{BH}_{2,2} \approx \mathsf{H}$$

is a model of Γ. By Corollary 7.21, $W_\Gamma^-(X,T) = W_{\mathsf{H}}(X,T)$ is thus given by (5.30).

7.6.1 Models and the addition of one vertex

Before we turn to the proof of Theorem 7.24 we record, for later use, the effects on models of taking disjoint unions and joins with a simple graph on a single vertex.

We denote, more precisely, by $\bullet = \mathrm{K}_1 = \Delta_1$ a fixed (simple) graph on one vertex and study the effects of the operations $\Gamma \rightsquigarrow \Gamma \oplus \bullet$ and $\Gamma \rightsquigarrow \Gamma \vee \bullet$. Given a hypergraph H, set $\mathsf{H}_{\mathbf{0}}^{\mathbf{0}} = (\mathsf{H}^{\mathbf{0}})_{\mathbf{0}} = (\mathsf{H}_{\mathbf{0}})^{\mathbf{0}}$ and likewise $\mathsf{H}_{\mathbf{1}}^{\mathbf{1}} = (\mathsf{H}^{\mathbf{1}})_{\mathbf{1}} = (\mathsf{H}_{\mathbf{1}})^{\mathbf{1}}$; cf. Definition 5.22.

Proposition 7.29. *Let* H *be a model of* Γ*.*

(i) $\mathsf{H}_{\mathbf{0}}^{\mathbf{0}}$ *is a model of* $\Gamma \oplus \bullet$*,*

(ii) *Let* c *be the number of connected components of* Γ*. Then* $(\mathsf{H}^{\mathbf{0}^{(c-1)}})_{\mathbf{1}}^{\mathbf{1}}$ *is a model of* $\Gamma \vee \bullet$*.*

Proof. This follows from Proposition 7.23 (for $\oplus$) and Theorem 7.24 (for $\vee$). ♦

Corollary 7.30.

(i) For each graph Γ*, we have* $W_{\Gamma\oplus\bullet}(X,T) = W_\Gamma(X,XT)$*,*

(ii) For each cograph Γ*, we have* $W_{\Gamma\vee\bullet}(X,T) = \frac{1-X^{-1}T}{1-XT}\, W_\Gamma(X,X^{-1}T)$*.*

Proof. The assertion in (i) follows from [57, §3.4]. For (ii), combine Proposition 7.29(ii) with (5.26) and (5.28). ♦

Remark 7.31. The question whether the assumption in (ii) that Γ be a cograph is unnecessary is generalised in Question 10.1.

7.7 Proof of Theorem 7.24

At this point, there is but one missing piece towards our proof of Theorem 7.1 (and Theorem D), namely Theorem 7.24, whose notation we now adopt. Write $\Gamma := \Gamma_1 \vee \Gamma_2$.

7.7.1 Phase 0: setup

Suitable collections of cones. Similar to the proof of Proposition 7.23, we obtain a collection Σ of cones with $\bigcup\Sigma = \mathbf{R}_{\geqslant 0}V$ and, for each $\sigma\in\Sigma$ and $i = 1,2$, a scaffold $\mathcal{S}_i = \mathcal{S}_i(\sigma)$ on V_i which encloses Γ_i over the image σ_i of σ under the projection $\mathbf{R}_{\geqslant 0}V \twoheadrightarrow \mathbf{R}_{\geqslant 0}V_i$ such that $\mathsf{H}(\mathcal{S}_i(\sigma))$ coincides with H_i up to relabelling of hyperedges. Using Lemma 4.1, Lemma 6.14, and Remark 7.13 to modify Σ if necessary, we may further assume that $\leqslant_\sigma$ (see §4.1) induces a total preorder on V for each $\sigma\in\Sigma$.

Note that in contrast to our proof of Proposition 7.23, we do not assume that each $\sigma\in\Sigma$ is of the form $\sigma = \sigma_1\times\sigma_2$. However, $\sigma\subset\sigma_1\times\sigma_2$ which allows us to identify $\mathbf{Z}_{\sigma_i}\subset\mathbf{Z}_{\sigma_1\times\sigma_2}\subset\mathbf{Z}_\sigma$ and also e.g. $\mathbf{Z}_{\sigma_i}V_i\subset\mathbf{Z}_\sigma V$.

A fixed cone. Henceforth, let $\sigma\in\Sigma$ be fixed but arbitrary. It suffices to construct a scaffold $\mathcal{S}$ which encloses Γ over σ and whose associated hypergraph $\mathsf{H}(\mathcal{S})$ coincides with H (as defined in the statement of Theorem 7.24) up to relabelling hyperedges.

Write $\mathcal{S}_i := \mathcal{S}_i(\sigma) = (\Phi_i,\sigma_i,\mathrm{ori}_i,\|\cdot\|_i)$ and $\Phi_i = (V_i,E_i,|\cdot|_i)$. We may assume that $E_1\cap E_2 = \varnothing$. For $u_i\in V_i$ and $v_j\in V_j$, we use the suggestive notation u_iv_j both for $(u_i,v_j)\in V_i\times V_j$ and, if $i=j$ and $u_i\sim v_i$ in Φ_i, for the oriented edge $u_i\to v_i$ of Φ_i.

Strategy. For $u,v\in V$, let $[uv] := X^uv - X^vu\in\mathbf{Z}_\sigma V$ (so $[uv] = [u,v;-1]$ as in §3.3). For $A\subset V_1\times V_2$, write $[A] := \{[v_1v_2] : v_1v_2\in A\}$. Let $\mathcal{S}^{(0)} = (\Phi^{(0)},\sigma,\mathrm{ori}^{(0)},\|\cdot\|^{(0)})$ be the "disjoint union" of $\mathcal{S}_1$ and $\mathcal{S}_2$ as constructed in the proof of Proposition 7.23. This is a scaffold enclosing the disjoint union $\Gamma_1\oplus\Gamma_2$ over σ. The underlying forest $\Phi := \Phi^{(0)} = \Phi_1\oplus\Phi_2$ satisfies $\mathrm{E}(\Phi^{(0)}) = E_1\sqcup E_2$ with the evident support function.

Let $M^{(0)} := V_1 \times V_2$. Recall that σ_i denotes the image of σ under the projection $\mathbf{R}_{\geqslant 0}V \twoheadrightarrow \mathbf{R}_{\geqslant 0}V_i$. Since $\mathcal{S}_i$ encloses Γ_i over σ_i and we identify $\mathbf{Z}_{\sigma_i} \subset \mathbf{Z}_\sigma$ as above,

$$\begin{aligned}\mathsf{adj}(\Gamma, \sigma) &= \langle \mathsf{adj}(\Gamma_1, \sigma_1)\rangle + \langle \mathsf{adj}(\Gamma_2, \sigma_2)\rangle + \langle [M^{(0)}]\rangle \\ &= \langle \mathsf{adj}(\mathcal{S}_1)\rangle + \langle \mathsf{adj}(\mathcal{S}_2)\rangle + \langle [M^{(0)}]\rangle \\ &= \mathsf{adj}(\mathcal{S}^{(0)}) + \langle [M^{(0)}]\rangle\end{aligned}$$

Beginning with $\mathcal{S}^{(0)}$ and $M^{(0)}$, in the following, we use graph-theoretic operations to construct a finite sequence of scaffolds $\mathcal{S}^{(n)}$ over σ and sets $M^{(n)} \subset V_1 \times V_2$ such that

$$\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}(\mathcal{S}^{(n)}) + \langle [M^{(n)}]\rangle$$

for each $n \geqslant 0$. The very last of these will satisfy $M^{(\infty)} = \varnothing$ and $\mathcal{S}^{(\infty)}$ will enclose Γ over σ. Moreover, by construction the hypergraph $\mathsf{H}(\mathcal{S}^{(\infty)})$ will coincide with H as defined in Theorem 7.24 up to relabelling of hyperedges.

7.7.2 Phase 1: eliminating non-radical joining edges

In order to construct $\mathcal{S}^{(n+1)}$ from $\mathcal{S}^{(n)}$, we will employ the following observation based on the same idea as Lemma 6.8. Recall that we assume throughout that $\leqslant_\sigma$ induces a total preorder on V.

Lemma 7.32 ("Triangle reduction")**.** *Let $\mathcal{S} = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|)$ be a scaffold on V. Let $u \to v$ be an oriented edge in Φ. Let $M \subset V \times V$, let $z \in V$, and suppose that $uz, vz \in M$. Let $M' := M \setminus \{vz\}$. Define a scaffold $\mathcal{S}' = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|')$ via*

$$\|h\|' = \begin{cases} \|h\|, & \text{if } h \neq uv, \\ \|uv\| \cup \{z\}, & \text{if } h = uv. \end{cases}$$

Then $\mathsf{adj}(\mathcal{S}) + \langle [M]\rangle = \mathsf{adj}(\mathcal{S}') + \langle [M']\rangle$.

Proof. By condition (S1) in Definition 7.7, we have $u \leqslant_\sigma v$. Condition (S2) allows us to choose a vertex $w \in \|uv\|$. Define

$$\begin{aligned} g &:= X^w v - X^{v+w-u} u \in \mathsf{adj}(\mathcal{S}), \\ g' &:= X^z v - X^{v+z-u} u \in \mathsf{adj}(\mathcal{S}'), \end{aligned}$$

and note that $\mathsf{adj}(\mathcal{S}') = \mathsf{adj}(\mathcal{S}) + \langle g'\rangle$. Further observe that

$$g' + [vz] = X^{v-u}[uz]. \tag{7.5}$$

We claim that $\langle g, [uz], [vz]\rangle = \langle g, g', [uz]\rangle$ over $\mathbf{Z}_\sigma$. To prove that, we consider two cases. First suppose that $w \leqslant_\sigma z$. Then $g' = X^{z-w} g$ over $\mathbf{Z}_\sigma$ and (7.5) implies that $\langle g, [uz], [vz]\rangle = \langle g, [uz]\rangle = \langle g, g', [uz]\rangle$. Next, suppose that $z \leqslant_\sigma w$. Then

$$g + X^{w-z}[vz] = X^{(v-u)+(w-z)}[uz]$$

over $\mathbf{Z}_\sigma$ whence $\langle g, [uz], [vz]\rangle = \langle [uz], [vz]\rangle$. Moreover, by (7.5) and since $g = X^{w-z}g'$ over $\mathbf{Z}_\sigma$, we have $\langle [uz], [vz]\rangle = \langle g', [uz]\rangle = \langle g, g', [uz]\rangle$. The claim now follows since

$$\begin{aligned}\mathsf{adj}(\mathcal{S}) + \langle [M]\rangle &= \mathsf{adj}(\mathcal{S}) + \langle g, [uz], [vz]\rangle + \langle [M']\rangle \\ &= \mathsf{adj}(\mathcal{S}) + \langle g, g', [uz]\rangle + \langle [M']\rangle \\ &= \mathsf{adj}(\mathcal{S}') + \langle [M']\rangle. \end{aligned}$$ ♦

Remark 7.33. Since $[zu] = -[uz]$, the preceding lemma remains true if uz is replaced by zu or vz is replaced by zv.

Invariants. Let $\prec_i$ be the natural partial order induced on V_i by the given orientation ori_i on Φ_i; see §7.4. Let $\prec := \prec_1 \times \prec_2$ be the product order on $V_1 \times V_2$. Recall that E_i denotes the edge set of Φ_i. Suppose that

- scaffolds $\mathcal{S}^{(0)}, \dots, \mathcal{S}^{(n)}$ on V,
- subsets $V_1 \times V_2 = M^{(0)} \supset M^{(1)} \supset \cdots \supset M^{(n)}$, and
- elements $v_i^{(0)}, \dots, v_i^{(n-1)} \in V_i$ for $i = 1, 2$

have been constructed and that the following conditions are satisfied for $\ell = 1, \dots, n$:

(M1) $\mathcal{S}^{(\ell)} = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|^{(\ell)})$.

(M2) $M^{(\ell)}$ is downward closed with respect to $\prec$.

(M3) $M^{(\ell)}$ contains all $\prec$-minimal elements of $V_1 \times V_2$.

(M4) $v_1^{(\ell-1)}v_2^{(\ell-1)} \in M^{(\ell-1)}$ and $M^{(\ell)} = M^{(\ell-1)} \setminus \{v_1^{(\ell-1)}v_2^{(\ell-1)}\}$.

(M5) For $i + j = 3$, if $u_iv_i \in E_i$, then $\|u_iv_i\|^{(\ell-1)} \subset \|u_iv_i\|^{(\ell)} \subset \|u_iv_i\|^{(\ell-1)} \cup \{v_j^{(\ell-1)}\}$.

(M6) There exist $i \in \{1, 2\}$ and an edge $u_i \to v_i^{(\ell-1)}$ in Φ_i with $v_j^{(\ell-1)} \in \|u_iv_i^{(\ell-1)}\|^{(\ell)}$ for $i + j = 3$.

(M7) $\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}(\mathcal{S}^{(\ell)}) + \langle [M^{(\ell)}]\rangle$.

Some comments on these conditions are in order. Formalising the strategy in §7.7.1, condition (M7) asserts that $\mathsf{adj}(\Gamma, \sigma)$, the module of primary interest to us, coincides with $\mathsf{adj}(\mathcal{S}^{(\ell)})$, except for an error measured by $M^{(\ell)}$. As outlined earlier, the objective of our construction is to eventually eliminate this error term.

Condition (M4) states that $M^{(\ell)}$ is obtained from $M^{(\ell-1)}$ by removing a single distinguished pair $v_1^{(\ell-1)}v_2^{(\ell-1)}$. In our construction, these distinguished pairs will be chosen among $\prec$-maximal elements of $M^{(\ell-1)}$; when working with the latter, (M2) will be crucial.

Condition (M1) asserts that $\mathcal{S}^{(\ell)}$ only differs from $\mathcal{S}^{(0)}$ in its support function. This is made more precise by (M5) which asserts that if e is any edge in Φ_i, then the $\|\cdot\|^{(\ell)}$-support of e coincides with its $\|\cdot\|^{(\ell-1)}$-support except possibly for the addition of the distinguished vertex $v_j^{(\ell-1)}$—here, $j \in \{1,2\}$ is the "index distinct from i", captured succinctly by the identity "$i+j=3$". Note that this does not yet rule out the possibility that $\|e\|^{(\ell)} = \|e\|^{(\ell-1)}$ for all edges e in Φ. However, by (M6), there is some edge e in some Φ_i such that $\|e\|^{(\ell)} = \|e\|^{(\ell-1)} \cup \{v_j^{(\ell-1)}\}$; in addition, there exists such an edge e such that $v_i^{(\ell-1)}$, the other vertex incident to the distinguished edge from (M4), is incident to e. Finally, (M3) will guarantee that after finitely many steps, $M^{(\ell)}$ stabilises at the set of minimal elements of $M^{(0)} = V_1 \times V_2$; this will conclude Phase 1.

Removing non-minimal pairs. Let R_i denote the set of $\prec_i$-minimal elements of V_i; note that this is precisely the set of roots of the connected components of Φ_i. Clearly, $R_1 \times R_2$ is the set of $\prec$-minimal elements of $V_1 \times V_2$.

Suppose that $M^{(n)} \supsetneq R_1 \times R_2$ and choose a non-minimal pair $v_1'v_2' \in M^{(n)} \setminus (R_1 \times R_2)$. Let v_1v_2 be any $\prec$-maximal element of $M^{(n)}$ with $v_1'v_2' \prec v_1v_2$; note that $v_1v_2 \notin R_1 \times R_2$. Define $v_1^{(n)}v_2^{(n)} := v_1v_2$ and $M^{(n+1)} := M^{(n)} \setminus \{v_1v_2\}$; clearly, $M^{(n+1)}$ is downward closed and $M^{(n+1)} \supset R_1 \times R_2$.

Next, we construct $\mathcal{S}^{(n+1)}$. Without loss of generality, suppose that $v_1 \notin R_1$. (If both $v_1 \notin R_1$ and $v_2 \notin R_2$, we proceed as in the following.) Let $u_1 \in V_1$ be the (unique) $\prec_1$-predecessor of v_1. Define $\|\cdot\|^{(n+1)} \colon E_1 \sqcup E_2 \to \mathcal{P}(V)$ via

$$\|h\|^{(n+1)} := \begin{cases} \|h\|^{(n)}, & \text{if } h \neq u_1v_1, \\ \|u_1v_1\|^{(n)} \cup \{v_2\}, & \text{if } h = u_1v_1. \end{cases}$$

Let $\mathcal{S}^{(n+1)} := (\Phi, \sigma, \mathrm{ori}, \|\cdot\|^{(n+1)})$. Then (M1)–(M6) are clearly satisfied for $\ell = n+1$.

Since $v_1v_2 \in M^{(n)}$ and $M^{(n)}$ is downward closed, u_1v_2 belongs to $M^{(n)}$ and also to $M^{(n+1)}$. It thus follows from Lemma 7.32 that (M7) is satisfied for $\ell = n+1$.

Changing support. Since each $M^{(n+1)}$ is a proper subset of $M^{(n)}$ and both of these are supersets of $R_1 \times R_2$, the above construction terminates after finitely many steps when $M^{(N)} = R_1 \times R_2$ for some $N \geqslant 0$. A key property of $\mathcal{S}^{(N)}$ is the following:

Lemma 7.34. *Let $i+j=3$. Then for each oriented edge $u_i \to v_i$ in Φ_i,*

$$\|u_iv_i\|^{(0)} \cup R_j \subset \|u_iv_i\|^{(N)} \subset \|u_iv_i\|^{(0)} \cup V_j.$$

Proof. The second inclusion is immediate from (M5). To prove the first inclusion, we assume, without loss of generality, that $i=1$ and $j=2$. (When $i=2$ and $j=1$, we only need to suitably reverse ordered pairs in the following.) Let $r_2 \in R_2$ be arbitrary. It suffices to show that $r_2 \in \|u_1v_1\|^{(N)}$. Since $v_1 \notin R_1$ (because $u_1 \prec_1 v_1$) and $M^{(N)} = R_1 \times R_2$, we have $v_1r_2 \in M^{(0)} \setminus M^{(N)}$. Hence, by (M4), for some $\ell \in \{1,\dots,N\}$, we have $v_1r_2 = v_1^{(\ell-1)}v_2^{(\ell-1)}$. As r_2 is a root (= $\prec_2$-minimal element) of one of the connected components of Φ_2, the edge $u_i \to v_i^{(\ell-1)}$ in (M6) has to be the given edge $u_1 \to v_1$. In particular, (M5)–(M6) imply that $r_2 = v_2^{(\ell-1)} \in \|u_1v_1\|^{(\ell)} \subset \|u_1v_1\|^{(N)}$. ♦

To proceed further, we need another lemma.

Lemma 7.35 ("Support addition"). *Let $\mathcal{S} = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|)$ be a scaffold on V. Let $u \to v$ be an oriented edge of Φ. Let $w \in \|uv\|$, let $z \in V$, and suppose that $w \leqslant_\sigma z$. Define a scaffold $\mathcal{S}' = (\Phi, \sigma, \mathrm{ori}, \|\cdot\|')$ via*

$$\|h\|' = \begin{cases} \|h\|, & h \neq uv, \\ \|uv\| \cup \{z\}, & h = uv. \end{cases}$$

Then $\mathsf{adj}(\mathcal{S}) = \mathsf{adj}(\mathcal{S}')$.

Proof. Define

$$g := X^w v - X^{v+w-u} u \in \mathsf{adj}(\mathcal{S}),$$
$$g' := X^z v - X^{v+z-u} u \in \mathsf{adj}(\mathcal{S}')$$

so that $\mathsf{adj}(\mathcal{S}') = \mathsf{adj}(\mathcal{S}) + \langle g' \rangle$. By assumption, $g' = X^{z-w} g \in \mathsf{adj}(\mathcal{S})$ over $\mathbf{Z}_\sigma$. ♦

Define $\|\cdot\|^{(N+1)} \colon E_1 \sqcup E_2 \to \mathcal{P}(V)$ via

$$\|h\|^{(N+1)} := \begin{cases} \|h\|_1 \sqcup V_2, & \text{if } h \in E_1, \\ V_1 \sqcup \|h\|_2, & \text{if } h \in E_2. \end{cases}$$

Let $\mathcal{S}^{(N+1)} := (\Phi, \sigma, \mathrm{ori}, \|\cdot\|^{(N+1)})$ and $M^{(N+1)} := M^{(N)} = R_1 \times R_2$.

Corollary 7.36. $\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}\left(\mathcal{S}^{(N+1)}\right) + \langle [M^{(N+1)}] \rangle$.

Proof. By (M7), it suffices to show that $\mathsf{adj}(\mathcal{S}^{(N)}) = \mathsf{adj}(\mathcal{S}^{(N+1)})$. Let $i + j = 3$ and let h be any oriented edge of Φ_i. Let $z_j \in V_j$ be arbitrary. Let $r_j \in R_j$ be the root of the connected component of Φ_j which contains z_j. By condition (S1) in Definition 7.7, $r_j \leqslant_\sigma z_j$. By Lemmas 7.34–7.35, $\mathsf{adj}(\mathcal{S}^{(N)})$ remains unchanged after adding z_j to $\|h\|^{(N)}$. Repeated application gives the desired result. ♦

7.7.3 Phase 2: growing a tree from two forests

Constructing a scaffold enclosing $\Gamma_1 \vee \Gamma_2$. By assumption, there exists $v \in V = V_1 \sqcup V_2$ such that $v \leqslant_\sigma u$ for all $u \in V$. Without loss of generality, suppose that $v \in V_1$. Let a_1 be the root of the connected component of Φ_1 which contains v. By condition (S1) in Definition 7.7, $a_1 \leqslant_\sigma v$ so that a_1 too is a $\leqslant_\sigma$-minimum of V. Choose $b_2 \in R_2$ among the $\leqslant_\sigma$-minima of R_2. Define $\mathcal{S}^{(\infty)} := (\Phi^{(\infty)}, \sigma, \mathrm{ori}^{(\infty)}, \|\cdot\|^{(\infty)})$ as follows:

- $\Phi^{(\infty)}$ is the tree (!) with orientation $\mathrm{ori}^{(\infty)}$ obtained from $\Phi = \Phi_1 \oplus \Phi_2$ by inserting a directed edge $a_1 \to r_i$ for each $r_i \in (R_1 \setminus \{a_1\}) \cup R_2$. Note that this orientation is outgoing with a_1 as the root of $\Phi^{(\infty)}$.
- $\|\cdot\|^{(\infty)} \colon \mathrm{E}(\Phi^{(\infty)}) \to \mathcal{P}(V)$ is defined via

$$\|h\|^{(\infty)} = \begin{cases} V_1, & \text{if } h = a_1 r_2 \text{ for } r_2 \in R_2 \setminus \{b_2\}, \\ V_1 \sqcup V_2, & \text{if } h = a_1 b_2, \\ V_2, & \text{if } h = a_1 r_1 \text{ for } r_1 \in R_1 \setminus \{a_1\}, \\ \|h\|^{(N+1)}, & \text{otherwise.} \end{cases}$$

By our choice of a_1 as a $\leqslant_\sigma$-minimal element of V, we see that $\mathcal{S}^{(\infty)}$ is a scaffold on V.

Lemma 7.37. *$\mathcal{S}^{(\infty)}$ encloses $\Gamma = \Gamma_1 \vee \Gamma_2$ over σ.*

Proof. First note that $\Phi^{(\infty)}$ and Γ are both connected: the former by construction and the latter since it is a join of non-empty graphs. It thus only remains to show that $\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}(\mathcal{S}^{(\infty)})$. Let

$$F := \left\langle X^{w-a_1}[a_1, r] : r \in (R_1 \setminus \{a_1\}) \sqcup R_2,\ w \in \|a_1 r\|^{(\infty)} \right\rangle \leqslant \mathbf{Z}_\sigma V$$

and note that, by (7.4), $\mathsf{adj}(\mathcal{S}^{(\infty)}) = \mathsf{adj}(\mathcal{S}^{(N+1)}) + F$.

By condition (S1) in Definition 7.7 and since R_i consists of the roots of Φ_i, for each $v_i \in V_i$, there exists $r_i \in R_i$ with $r_i \leqslant_\sigma v_i$. Moreover, $a_1 \leqslant_\sigma v$ for each $v \in V$ and $b_2 \leqslant_\sigma v_2$ for each $v_2 \in V_2$ by our choices of a_1 and b_2. Hence, by the definition of $\|\cdot\|^{(\infty)}$,

$$F = \left\langle X^{b_2 - a_1}[a_1, r_1] : r_1 \in R_1 \setminus \{a_1\} \right\rangle + \left\langle [a_1, r_2] : r_2 \in R_2 \right\rangle.$$

On the other hand, setting $G := \langle [R_1 \times R_2] \rangle \leqslant \mathbf{Z}_\sigma V$, by Corollary 7.36, $\mathsf{adj}(\Gamma, \sigma) = \mathsf{adj}(\mathcal{S}^{(N+1)}) + G$. It thus suffices to show that $F = G$. Write $H := \langle [a_1, r_2] : r_2 \in R_2 \rangle \subset F \cap G$. For $r_1 \in R_1 \setminus \{a_1\}$ and $r_2 \in R_2$, since $a_1 \leqslant_\sigma r_1$ and $a_1 \leqslant_\sigma r_2$, we obtain the "triangle identity" (cf. Lemma 6.8)

$$[r_1, r_2] = X^{r_1 - a_1}[a_1, r_2] - X^{r_2 - a_1}[a_1, r_1]. \tag{7.6}$$

As $[a_1, r_2] \in H \subset F$ and $X^{r_2 - a_1}[a_1, r_1] = X^{r_2 - b_2} \cdot X^{b_2 - a_1}[a_1, r_1] \in F$, we obtain $G \subset F$. Conversely, by taking $r_2 = b_2$ in (7.6), we see that $X^{b_2 - a_1}[a_1, r_1] \in G$ whence $F \subset G$. ♦

Remark 7.38. The proof of Lemma 7.37 rested on the validity of the following conditions:

- $a_1 \in \|a_1 r_2\|^{(\infty)}$ for all $r_2 \in R_2$.
- $b_2 \in \|a_1 r_1\|^{(\infty)} \subset V_2$ for all $r_1 \in R_1 \setminus \{a_1\}$.

In particular, numerous alternative definitions of $\|\cdot\|^{(\infty)}$ are possible while maintaining the validity of Lemma 7.37. The crucial point of the definition that we chose—to be exploited in the upcoming final step of our proof of Theorem 7.24—is that, up to relabelling hyperedges, the specific choice that we made works uniformly in all possible cases. That is to say, it works uniformly for all possible choices of a_1 and b_2 and also in the case that all $\leqslant_\sigma$ elements of V belong to V_2 (in which case we choose $a_2 \in V_2$ and $b_1 \in V_1$ and proceed analogously to what we did above).

Finale. Recall that c_i denotes the number of connected components of Γ_i. Note that $c_i = |R_i|$ by condition (S2) in Definition 7.7. Hence, by unravelling the definition of $\|\cdot\|^{(\infty)}$ from above, we see that, up to relabelling of hyperedges, the hypergraph $\mathsf{H}(\mathcal{S}^{(\infty)})$ coincides with H in Theorem 7.24. In particular, H is a local model of Γ over our fixed but arbitrary cone σ from the beginning of this section. This completes the proof of Theorem 7.24. ♦

8 Cographs, hypergraphs, and cographical groups

As in §7, all graphs in this section are assumed to be simple.

The story so far. In §3.4 we attached a unipotent group scheme ("graphical group scheme") $\mathbf{G}_\Gamma$ to each graph Γ. For each compact DVR $\mathfrak{O}$ we expressed, in Corollary B, the class counting zeta functions of the group scheme $\mathbf{G}_\Gamma \otimes \mathfrak{O}$ in terms of the rational function $W_\Gamma^-(X,T)$ from Theorem A(ii):

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}}(s) = W_\Gamma^-\left(q,\, q^{|\mathrm{E}(\Gamma)|-s}\right).$$

For a *cograph* Γ, the Cograph Modelling Theorem (Theorem D) established that there exists an explicit *modelling hypergraph* $\mathsf{H} = \mathsf{H}(\Gamma)$ for Γ. This is a specific hypergraph on the same vertex set as Γ which satisfies

$$W_\Gamma^-(X,T) = W_{\mathsf{H}}(X,T),$$

where $W_{\mathsf{H}}(X,T)$ is the rational function associated with H in Theorem A(i).

Our proof of the Cograph Modelling Theorem was constructive. Indeed, cographs (save for isolated vertices) are disjoint unions or joins of smaller cographs. Given modelling hypergraphs of two cographs we constructed, in Proposition 7.23 and Theorem 7.24, modelling hypergraphs of their disjoint union and join, respectively; cf. Remark 7.27.

In §5, we carried out an extensive analysis of the rational functions $W_{\mathsf{H}}(X,T)$ associated with hypergraphs H resulting, in particular, in an explicit formula, viz. Theorem C. We also investigated the effects of taking disjoint unions and complete unions of hypergraphs. This ties in well with our constructive proof of the Cograph Modelling Theorem. Namely, by Proposition 7.23, the disjoint union $\mathsf{H}_1 \oplus \mathsf{H}_2$ of modelling hypergraphs H_1 and H_2 of cographs Γ_1 and Γ_2 is a model of the cograph $\Gamma_1 \oplus \Gamma_2$. Moreover, the modelling hypergraph of the join $\Gamma_1 \vee \Gamma_2$ can be described in terms of the complete union $\mathsf{H}_1 \circledast \mathsf{H}_2$ and the operations from §5.4; cf. Remark 7.25(i).

In the present section, we apply the results from §5 to class counting zeta functions of cographical group schemes.

8.1 Proofs of Theorems E–F

Proof of Theorem E. Let V be the set of vertices of the cograph Γ. Let $\mathsf{H} = \mathsf{H}(\Gamma)$ be a modelling hypergraph for Γ with hyperedge multiplicities $(\mu_I)_{I\subset V}$ as in Theorems C–D. Our proof of Theorem D in §7 shows that we may assume that $\sum_I \mu_I = n - c$, where n and c are the numbers of vertices and connected components of Γ, respectively; cf. Lemma 7.19. Let m be the number of edges of Γ. The bound $m \geqslant n - c$ then implies that for each summand of $W_{\mathsf{H}}(X, X^m T)$ in (1.3), the coefficients of T^k in $X - 1$ are non-negative. ♦

Proof of Theorem F. For the first part and the integrality of local poles, combine Corollary B, Theorem D, and Theorem 5.26. It remains to prove that the real parts of the poles of $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}_K}(s)$ are positive. Let Γ, V, H, $(\mu_I)_{I\subset V}$, m, and n be as in the proof of Theorem E above. As we argued there, $m-\sum_{I\cap J\neq\varnothing}\mu_I \geqslant 0$ for each $J\subset V$. In particular, $f(J) := |J| + m - \sum_{I\cap J\neq\varnothing} \mu_I > 0$ whenever $J\neq\varnothing$. Unless Γ is discrete, $f(\varnothing) = m > 0$. If $\Gamma = \Delta_n$ is discrete, then the real parts of the poles of $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Delta_n}\otimes\mathfrak{O}}(s) = 1/(1-q^{n-s})$ are equal to n which is positive since cographs are non-empty. ♦

8.2 Disjoint unions of hypergraphs and direct products of cographical groups

Much as for hypergraphs in §5.2, for arbitrary graphs Γ_1 and Γ_2, the rational function $W^{\pm}_{\Gamma_1\oplus\Gamma_2}(X,T)$ is the Hadamard product of $W^{\pm}_{\Gamma_1}(X,T)$ and $W^{\pm}_{\Gamma_2}(X,T)$. In particular, if R is the ring of integers of a number field or a compact DVR, then the class counting zeta function $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_1\oplus\Gamma_2}\otimes R}(s)$ is the Hadamard product of the Dirichlet series $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_1}\otimes R}(s)$ and $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_2}\otimes R}(s)$; this simply reflects the fact that class numbers of finite groups are multiplicative: $\mathrm{k}(H_1\times H_2) = \mathrm{k}(H_1)\times\mathrm{k}(H_2)$ for finite groups H_1 and H_2.

8.2.1 A special case: hypergraphs with disjoint supports and direct products of free class-2-nilpotent groups

We now apply §5.2.1 to study class counting zeta functions of cographical groups modelled by hypergraphs with disjoint supports.

Let $\mathbf{n} = (n_1,\dots,n_r)\in\mathbf{N}^r$. We write $n = \sum_{i=1}^r n_i$ and $\binom{\mathbf{n}}{2} = \sum_{i=1}^r \binom{n_i}{2}$. We consider the cographical group scheme associated with the cograph

$$\mathrm{K}_{\mathbf{n}} = \mathrm{K}_{n_1}\oplus\dots\oplus\mathrm{K}_{n_r};$$

see (3.5) and note that $\mathrm{K}_{\mathbf{n}}$ has $m = \binom{\mathbf{n}}{2}$ edges. These cographs are of specific group-theoretic interest since $\mathbf{G}_{\mathrm{K}_{\mathbf{n}}}(\mathbf{Z}) = F_{2,n_1}\times\dots\times F_{2,n_r}$ is the direct product of the free class-2-nilpotent groups on n_i generators; in particular, $\mathbf{G}_{\mathrm{K}_n}(\mathbf{Z}) = F_{2,n}$.

Remark 8.1. Conflicting notation in the literature notwithstanding, the cograph $\mathrm{K}_{\mathbf{n}}$ is not to be confused with the complete multipartite graph on disjoint, independent sets of cardinalities $n_1,\dots,n_r$; the latter graph will feature as $\Delta_{\mathbf{n}}$ in §8.3.1.

Combining Proposition 7.23 and Example 7.17, we see that $\mathrm{K}_{\mathbf{n}}$ is modelled by the hypergraph $\mathsf{H}(\mathrm{K}_{\mathbf{n}}) = \mathsf{BH}_{\mathbf{n},\mathbf{n-1}} = \bigoplus_{i=1}^r \mathsf{BH}_{n_i,n_i-1}$. By Corollary 5.14 (noting that $n_i - m_i = n_i - (n_i-1) = 1$ for all $i\in[r]$),

$$\begin{aligned}W^{-}_{\mathrm{K}_{\mathbf{n}}}(X,T) &= W_{\mathsf{H}(\mathrm{K}_{\mathbf{n}})}(X,T)\\ &= W_{\mathsf{BH}_{\mathbf{n},\mathbf{n-1}}}(X,T)\\ &= \sum_{y\in\widehat{\mathrm{WO}}_r}\left(\prod_{i\in\sup(y)}(1-X^{-n_i})\right)\prod_{J\in y}\mathrm{gp}\left(X^{|J|}T\right).\end{aligned}$$

Corollary 8.2. *Let* $\mathbf{n} = (n_1, \ldots, n_r) \in \mathbf{N}^r$. *For each compact* DVR $\mathfrak{O}$,

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_{\mathbf{n}}} \otimes \mathfrak{O}}(s) = W^{-}_{\mathrm{K}_{\mathbf{n}}}(q, q^{\binom{\mathbf{n}}{2} - s}) = \sum_{y \in \widehat{\mathrm{WO}}_r} \left(\prod_{i \in \sup(y)} \underline{(n_i)} \right) \prod_{J \in y} \mathrm{gp}\left(q^{\binom{\mathbf{n}}{2} + |J| - s} \right). \qquad \blacklozenge$$

Example 8.3.

(i) If $r = 1$ and $\mathbf{n} = (n)$, then $\binom{\mathbf{n}}{2} = \binom{n}{2}$, whence

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_n} \otimes \mathfrak{O}}(s) = \frac{1 - q^{\binom{n-1}{2} - s}}{\left(1 - q^{\binom{n}{2} - s}\right)\left(1 - q^{\binom{n}{2} + 1 - s}\right)},$$

in accordance with [47, Corollary 1.5]. There $\mathbf{G}_{\mathrm{K}_n}$ goes by the name $F_{\mathrm{n},\delta}$, where $n = 2\mathrm{n} + \delta$ with $\delta \in \{0, 1\}$. See Example 8.23 for a bivariate version of this formula.

(ii) If $r = 2$ and $\mathbf{n} = (n_1, n_2)$, then $n = n_1 + n_2$ and $\binom{\mathbf{n}}{2} = \binom{n_1}{2} + \binom{n_2}{2}$, whence

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_{(n_1,n_2)}} \otimes \mathfrak{O}}(s) = \zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_{n_1}} \otimes \mathfrak{O}}(s) \star \zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_{n_2}} \otimes \mathfrak{O}}(s) =$$
$$\frac{1 + q^{\binom{\mathbf{n}}{2} + 1 - s}\left(1 - q^{-n_1} - q^{-n_2} - q^{-n_1+1} - q^{-n_2+1} + q^{-n+1}\right) + q^{2\binom{\mathbf{n}}{2} - n + 3 - s}}{\left(1 - q^{\binom{\mathbf{n}}{2} - s}\right)\left(1 - q^{\binom{\mathbf{n}}{2} + 1 - s}\right)\left(1 - q^{\binom{\mathbf{n}}{2} + 2 - s}\right)}. \tag{8.1}$$

8.3 Complete unions of hypergraphs and free class-2-nilpotent products of cographical groups

The results of §5.3 on complete unions of hypergraphs have direct corollaries pertaining to class counting zeta functions of joins of graphs. Recall from §3.4 that for graphs Γ_1 and Γ_2, the graphical group $\mathbf{G}_{\Gamma_1 \vee \Gamma_2}(\mathbf{Z})$ is the free class-2-nilpotent product of $\mathbf{G}_{\Gamma_1}(\mathbf{Z})$ and $\mathbf{G}_{\Gamma_2}(\mathbf{Z})$.

Proposition 8.4. *Let* Γ_1 *and* Γ_2 *be cographs on* n_1 *and* n_2 *vertices, respectively. Then*

$$\begin{aligned} W^{-}_{\Gamma_1 \vee \Gamma_2}(X, T) = (&X^{1 - n_1 - n_2} T - 1 \\ &+ W^{-}_{\Gamma_1}(X, X^{-n_2} T)(1 - X^{-n_2} T)(1 - X^{1 - n_2} T) \\ &+ W^{-}_{\Gamma_2}(X, X^{-n_1} T)(1 - X^{-n_1} T)(1 - X^{1 - n_1} T)) \\ &/((1 - T)(1 - XT)). \end{aligned} \tag{8.2}$$

In particular, if Γ *is a cograph, then* $W^{-}_{\Gamma \vee \bullet}(X, T) = \frac{1 - X^{-1} T}{1 - XT} \cdot W^{-}_{\Gamma}(X, X^{-1} T)$.

Proof. We may assume that $\mathrm{V}(\Gamma_1) \cap \mathrm{V}(\Gamma_2) = \varnothing$. By Corollary 7.26, each Γ_i admits a model, H_i say. In particular, $W^{-}_{\Gamma_i}(X, T) = W_{\mathsf{H}_i}(X, T)$ for $i = 1, 2$ by Corollary 7.21.

Let $\mathsf{H} = (\mathsf{H}_1^{\square} \circledast \mathsf{H}_2^{\square})^{\mathbf{1}}$ (see Definition 5.22), where $\mathsf{H}_i^{\square} = (\mathsf{H}_i)^{\mathbf{0}^{(c_i - 1)}}$ and c_i is the number of connected components of Γ_i. By Theorem 7.24 and Remark 7.25, H is a model of

$\Gamma_1 \vee \Gamma_2$. Hence, by Corollary 7.21, $W^-_{\Gamma_1\vee\Gamma_2}(X,T) = W_{\mathsf{H}}(X,T)$. By Proposition 5.23 (applying (5.29) c_1 resp. c_2 times and (5.28) once),

$$W_{\mathsf{H}}(X,T) = \frac{1-X^{-1}T}{1-T}\cdot W_{\mathsf{H}_1\circledast\mathsf{H}_2}(X,X^{-1}T).$$

The claim now follows from Corollary 5.17 by substituting $n_i - 1$ for m_i in (5.21). This reflects the fact that the hypergraphs $\mathsf{H}_i^{\square}$ are “near squares”: they have n_i vertices and a total number of $n_i - 1$ hyperedges; see Remark 7.3. ♦

Let $\mathbf{G}_{\Gamma_1}$ and $\mathbf{G}_{\Gamma_2}$ be the cographical group schemes associated with the cographs Γ_1 and Γ_2. Let Γ_i have n_i vertices and m_i edges. For each compact DVR $\mathfrak{O}$, Proposition 8.4 now allows us to express

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_1\vee\Gamma_2}\otimes\mathfrak{O}} = W^-_{\Gamma_1\vee\Gamma_2}(q, q^{m_1+m_2+n_1n_2-s})$$

in terms of $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_1}\otimes\mathfrak{O}}(s)$ and $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma_2}\otimes\mathfrak{O}}(s)$. As a special case, we record the following.

Corollary 8.5. *Let Γ be a cograph with n vertices and m edges. Write $\bullet = \mathrm{K}_1 = \Delta_1$. Then for each compact* DVR $\mathfrak{O}$,

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma\vee\bullet}\otimes\mathfrak{O}}(s) = \frac{1-q^{m+n-1-s}}{1-q^{m+n+1-s}}\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Gamma}\otimes\mathfrak{O}}(s+1-m-n).$$ ♦

Remark 8.6. Via the functional equations $W^-_{\Gamma}(X^{-1},T^{-1}) = -X^nT\,W^-_{\Gamma}(X,T)$ in Corollary 1.4, the numbers n_1 and n_2 in Proposition 8.4—and hence the left-hand side of (8.2)—are already determined by the rational functions $W^-_{\Gamma_i}(X,T)$.

8.3.1 A special case: hypergraphs with codisjoint supports and free class-2-nilpotent products of abelian groups

We now apply the results and formulae developed in §5.3.1 to cographical groups modelled by hypergraphs with codisjoint supports.

As before, let $\mathbf{n} = (n_1,\dots,n_r) \in \mathbf{N}^r$, $n = \sum_{i=1}^r n_i$, and $\binom{\mathbf{n}}{2} = \sum_{i=1}^r \binom{n_i}{2}$. We consider the cographical group scheme associated with the cograph

$$\Delta_{\mathbf{n}} := \Delta_{n_1} \vee \ldots \vee \Delta_{n_r},$$

viz. the complete multipartite graph on disjoint, independent sets of cardinalities $n_1,\dots,n_r$. Note that $\Delta_{\mathbf{n}}$ has $m = \binom{n}{2} - \binom{\mathbf{n}}{2}$ edges. These cographs are of specific group-theoretic interest since $\mathbf{G}_{\Delta_{\mathbf{n}}}(\mathbf{Z}) = \mathbf{Z}^{n_1} \circledast \cdots \circledast \mathbf{Z}^{n_r}$ (see (3.11)) is the free class-2-nilpotent product of free abelian groups of ranks $n_1,\dots,n_r$. In particular, $\mathbf{G}_{\Delta_{\mathbf{1}^{(r)}}}(\mathbf{Z}) = F_{2,r}$ is the free class-2-nilpotent group on r generators.

By Remark 7.25(ii) and using the notation in Definition 5.22 and (3.4), $\Delta_{\mathbf{n}}$ is modelled by $\mathsf{H}(\Delta_{\mathbf{n}}) = (\mathsf{PH}_{\mathbf{n},\mathbf{n}-\mathbf{1}})^{\mathbf{1}^{(r-1)}}$. By Proposition 5.23 (applying (5.28) $r-1$ times),

$$W_{\mathsf{H}(\Delta_{\mathbf{n}})}(X,T) = \frac{1-X^{1-r}T}{1-T}W_{\mathsf{PH}_{\mathbf{n},\mathbf{n}-\mathbf{1}}}(X,X^{1-r}T).$$

Combining Corollary B with the explicit formula for $W_{\mathsf{PH}_{\mathbf{n},\mathbf{n-1}}}(X,T)$ in Corollary 5.19 (substituting $n-r$ for m there), we obtain the following.

Proposition 8.7. *Let* $\mathbf{n} = (n_1,\dots,n_r) \in \mathbf{N}^r$ *and* $m = \binom{n}{2} - \binom{\mathbf{n}}{2}$. *For each compact* DVR $\mathfrak{O}$,

$$\begin{aligned}\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Delta_\mathbf{n}}\otimes\mathfrak{O}}(s) &= W^-_{\Delta_\mathbf{n}}(q,q^{m-s}) = W_{\mathsf{H}(\Delta_\mathbf{n})}(q,q^{m-s}) = \frac{1-q^{1-r+m-s}}{1-q^{m-s}} W_{\mathsf{PH}_{\mathbf{n},\mathbf{n-1}}}(q,q^{1-r+m-s}) \\ &= \frac{1}{(1-q^{m-s})\,(1-q^{1+m-s})} \times \\ &\quad \left(1 - q^{1-n+m-s}\left(1-\sum_{i=1}^{r}\frac{(q^{n_i}-1)(q^{n_i-1}-1)}{1-q^{2n_i-n+m-s}}\right)\right). \end{aligned} \tag{8.3}$$ ♦

Example 8.8. Proposition 8.7 unifies and generalises a number of known formulae.

(i) If $r=1$ and $\mathbf{n}=(n)$, then $m = \binom{n}{2} - \binom{\mathbf{n}}{2} = 0$, confirming the trivial formula

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Delta_n}\otimes\mathfrak{O}}(s) = \zeta^{\mathrm{cc}}_{\mathbf{G}_a^n\otimes\mathfrak{O}}(s) = \frac{1}{1-q^{n-s}},$$

where $\mathbf{G}_a$ denotes the additive group scheme.

(ii) If $\mathbf{n} = \mathbf{1}^{(r)} \in \mathbf{N}^r$, then $\binom{\mathbf{n}}{2} = 0 = q^{n_i-1}-1$. Proposition 8.7 thus reconfirms

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Delta_{\mathbf{1}^{(r)}}}\otimes\mathfrak{O}}(s) = \zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_r}\otimes\mathfrak{O}}(s) = \frac{1-q^{\binom{r-1}{2}-s}}{\left(1-q^{\binom{r}{2}-s}\right)\left(1-q^{\binom{r}{2}+1-s}\right)};$$

see Example 8.3(i).

(iii) If $r=2$ and $\mathbf{n}=(N,N)$, then $m = \binom{n}{2} - \binom{\mathbf{n}}{2} = N^2$ and $2n_i = n = 2N$, whence

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\Delta_N\vee\Delta_N}\otimes\mathfrak{O}}(s) = \frac{(1-q^{N(N-1)-s})(1-q^{N(N-1)+1-s}) + q^{N^2-s}(1-q^{-N})(1-q^{-N+1})}{(1-q^{N^2-s})^2(1-q^{1+N^2-s})},$$

in accordance with [47, Corollary 1.5], where $\mathbf{G}_{\Delta_N\vee\Delta_N}$ goes by the name G_N.

8.4 Kite graphs

In this section we introduce and study a specially well-behaved class of cographs admitting, in particular, ask zeta functions of "Riemann-type"; see Theorem 8.19. Throughout, $\bullet = \mathrm{K}_1 = \Delta_1$ denotes a fixed simple graph on one vertex.

Definition 8.9. A **kite graph** is any graph belonging to the class Kites which is recursively defined to be minimal subject to the following conditions:

(i) $\bullet \in$ Kites.

(ii) If $\Gamma \in \mathsf{Kites}$, then $\bullet \vee \Gamma \in \mathsf{Kites}$ and $\Gamma \oplus \bullet \in \mathsf{Kites}$.

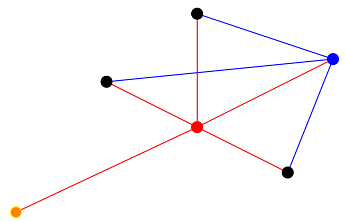
(iii) If $\Gamma \in \mathsf{Kites}$ and Γ is isomorphic to a graph Γ', then $\Gamma' \in \mathsf{Kites}$.

As we will see in Theorem 8.19 below, ask zeta functions associated with (negative adjacency representations of) kite graphs admit a particularly nice and explicit description.

Note that every kite graph is a cograph. Further note that any kite graph contains at most one connected component consisting of more than one vertex and that such a component is a kite graph itself.

Further note that the **Z**-points of cographical group schemes associated with kite graphs form exactly the class of those torsion-free finitely generated groups of nilpotency class at most 2 which contains **Z** and which is closed under taking direct and free class-2-nilpotent products with **Z**.

Example 8.10. The following is an example of a connected kite graph:



Note that the central vertex is connected to all other vertices. Its removal results in a disconnected graph consisting of one isolated vertex and another component which is a star graph on four vertices. The above graph is therefore isomorphic to

$$\Big(((\bullet \oplus \bullet \oplus \bullet) \vee \bullet) \oplus \bullet\Big) \vee \bullet$$

and is thus a kite graph.

Remark 8.11. Neither the so-called Krackhardt kite graph nor the graph on five vertices called "kite" on [11, p. 18] are kite graphs in the sense of Definition 8.9.

We seek to parameterise kite graphs in a useful fashion. Let $k_1, k_2, \dots$ be a sequence of non-negative integers. Define $\mathrm{Kite}()$ to be the empty graph and recursively define

$$\mathrm{Kite}(k_1, \dots, k_{c+1}) := \begin{cases} \mathrm{Kite}(k_1, \dots, k_c) \oplus \Delta_{k_{c+1}}, & \text{if } c \text{ is even,} \\ \mathrm{Kite}(k_1, \dots, k_c) \vee \mathrm{K}_{k_{c+1}}, & \text{if } c \text{ is odd.} \end{cases}$$

Clearly, $\mathrm{Kite}(k_1, \dots, k_c)$ is a kite graph for each $c \geqslant 1$ and choice of $k_1, \dots, k_c$, provided that at least one k_i is positive. (The empty graph is neither a kite graph nor a cograph.)

Example 8.12. $\mathrm{Kite}(n) = \Delta_n$ and $\mathrm{Kite}(1, n-1) = \mathrm{K}_n$.

Recall that a **composition** of a non-negative integer n is a sequence $k = (k_1, \dots, k_c)$ of positive integers with $n = k_1 + \dots + k_c$. We tacitly identify compositions and infinite sequences $k = (k_1, k_2, \dots)$ such that $k_i = 0$ for some i and, in addition, $k_j = 0$ whenever $i < j$ and $k_i = 0$.

Proposition 8.13.

(i) Every kite graph on n vertices is isomorphic to $\mathrm{Kite}(k_1,\dots,k_c)$ *for some composition* $(k_1,\dots,k_c)$ *of n.*

(ii) Let k and k' be compositions of positive integers. Then $\mathrm{Kite}(k)$ *and* $\mathrm{Kite}(k')$ *are isomorphic if and only if $k = k'$.*

Proof. Given a label ℓ and rooted labelled trees $\mathsf{T}_1,\dots,\mathsf{T}_u$, let $(\ell,\mathsf{T}_1,\dots,\mathsf{T}_u)$ denote the rooted tree whose root, v say, has label ℓ and such that the descendant trees of v are precisely the trees $\mathsf{T}_1,\dots,\mathsf{T}_u$. For notational convenience, we identify a label ℓ and the rooted labelled tree (ℓ). We see that kite graphs with vertex set $\{1,\dots,n\}$ are precisely those cographs with cotrees (see §7.1) of the form

$$\Big\langle \dots \Big\{ \oplus, [\vee, (\oplus, 1, 2, \dots, k_1), k_1+1, \dots, k_1+k_2], k_1+k_2+1, \dots, k_1+k_2+k_3 \Big\}, \dots \Big\rangle,$$

where $(k_1,k_2,\dots)$ is a composition of n and we used different types of parentheses for clarity. The uniqueness of cotrees of cographs (see §7.1) now implies both claims. ♦

Example 8.14 (Example 8.10, part II)**.** The kite graph in Example 8.10 is isomorphic to

$$((\Delta_3 \vee \mathrm{K}_1) \oplus \Delta_1) \vee \mathrm{K}_1 \approx \mathrm{Kite}(3,1,1,1).$$

Corollary 8.15. *Let $n \geqslant 2$. Up to isomorphism, there are precisely 2^{n-1} kite graphs on n vertices. Among these, precisely 2^{n-2} are connected.*

Proof. By construction, for a composition $k = (k_1,\dots,k_c)$, the graph $\mathrm{Kite}(k)$ is connected if and only if c is even. As is well-known, there are precisely 2^{n-1} compositions of n and it is easily verified that precisely half of these have even length. ♦

Let $k = (k_1,k_2,\dots)$ be a composition of a positive integer. For $t \geqslant 1$, let $k(t) := \sum\limits_{i=1}^{t} k_i$ and $k[t] := \sum\limits_{i=t}^{\infty} (-1)^{i+1} k_i$; note that $k[t] = 0 = k_t$ for $t \gg 0$. The following is easily proved by induction.

Lemma 8.16.

$$|\mathrm{V}(\mathrm{Kite}(k))| = \sum_{i=1}^{\infty} k_i, \quad |\mathrm{E}(\mathrm{Kite}(k))| = \sum_{i=1}^{\infty} \binom{k(2i)}{2} - \binom{k(2i-1)}{2}. \qquad ♦$$

Recall from (3.9) the notion of the staircase hypergraph $\Sigma\mathsf{H}_{\mathbf{m}}$ associated with a vector $\mathbf{m} = (m_0,\dots,m_n) \in \mathbf{N}_0^{n+1}$.

Proposition 8.17. *Every kite graph admits a staircase hypergraph as a model.*

Proof. We proceed by induction on the length c of the composition $k = (k_1, \dots, k_c)$ representing a given kite graph. For $c = 1$, note that $\Sigma\mathsf{H}_{(0,\dots,0)}$ is a model of $\mathrm{Kite}(k_1) = \Delta_{k_1}$. Next, supposing that $\mathrm{Kite}(k_1, \dots, k_{r-1})$ admits a model of the form $\Sigma\mathsf{H}_{\mathbf{m}}$, we obtain a model $\Sigma\mathsf{H}_{\mathbf{m}'}$ of $\mathrm{Kite}(k_1, \dots, k_r)$ by repeated application of Proposition 7.29. ♦

Example 8.18 (Example 8.10, part III)**.** The staircase hypergraph $\mathsf{H} := \Sigma\mathsf{H}_{(0,1,2,0,0,1,1)}$ with incidence matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is a model of the kite graph $\mathrm{Kite}(3,1,1,1)$ in Example 8.10.

Combining Proposition 8.17 with Proposition 5.9, we see that the rational function $W^-_{\mathrm{Kite}(k)}(X,T)$ associated with a kite graph $\mathrm{Kite}(k)$ is of a particularly simple form. The following theorem, which is the main result of the present section, spells this out.

Theorem 8.19. *Let $k = (k_1, k_2, \dots)$ be a composition of a positive integer. Then*

$$W^-_{\mathrm{Kite}(k)}(X,T) = \frac{1}{1 - X^{k[1]}T} \prod_{i=1}^{\infty} \frac{(1 - X^{k[2i+1]-k_{2i}+1}T)(1 - X^{k[2i+1]-k_{2i}}T)}{(1 - X^{k[2i+1]+1}T)(1 - X^{k[2i+1]}T)}. \tag{8.4}$$

Proof. Straightforward induction along blocks

$$(k_1, k_2, \dots, k_{2\rho-1}, k_{2\rho}) \rightsquigarrow (k_1, k_2, \dots, k_{2\rho+1}, k_{2\rho+2})$$

using Corollary 7.30. ♦

Example 8.20 (Example 8.10, part IV)**.** Consider the graph $\mathrm{Kite}(k)$ for $k = (3,1,1,1)$ from Example 8.10. Here, $k[1] = 2$ and $k[3] = k[5] = 0$. Theorem 8.19 thus asserts that

$$\begin{aligned} W^-_{\mathrm{Kite}(3,1,1,1)}(X,T) &= \frac{1}{1 - X^2T} \cdot \frac{(1-T)(1-X^{-1}T)}{(1-XT)(1-T)} \cdot \frac{(1-T)(1-X^{-1}T)}{(1-XT)(1-T)} \\ &= \frac{(1-X^{-1}T)^2}{(1-XT)^2(1-X^2T)}. \end{aligned}$$

For kite graphs, we can strengthen Theorem F. Recall that $|\mathrm{E}(\mathrm{Kite}(k))|$ is given by Lemma 8.16.

Theorem 8.21. *Let $k = (k_1, k_2, \dots)$ be a composition of a positive integer. Then for every number field K with ring of integers $\mathcal{O}_K$, the abscissa of convergence of the class counting zeta function $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{Kite}(k)} \otimes \mathcal{O}_K}(s)$ is equal to*

$$\alpha(\mathrm{Kite}(k)) = |\mathrm{E}(\mathrm{Kite}(k))| + \max\{k[1] + 1, k[2i+1] + 2 : i \in \mathbf{N}\}.$$

The function $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{Kite}(k)} \otimes \mathcal{O}_K}(s)$ may be meromorphically continued to all of $\mathbf{C}$.

Proof. By Theorem 8.19, the Euler product

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{Kite}(k)}\otimes\mathcal{O}_K}(s) = \prod_{v\in\mathcal{V}_K} W^{-}_{\mathrm{Kite}(k)}(q_v, q_v^{|\mathrm{E}(\mathrm{Kite}(k))|-s})$$

is a product of finitely many translates of the Dedekind zeta function $\zeta_K(s)$ and inverses of such translates. The abscissa of convergence is then readily read off from (8.4). ♦

Example 8.22 (Example 8.10, part V)**.** The graphical group scheme $\mathbf{G}_{\mathrm{Kite}(3,1,1,1)}$ has the property that, for each number field K,

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{Kite}(3,1,1,1)}\otimes\mathcal{O}_K}(s) = \frac{\zeta_K(s-9)^2\zeta_K(s-10)}{\zeta_K(s-7)^2},$$

with global abscissa of convergence $\alpha(\mathrm{Kite}(3,1,1,1)) = 11 = 8 + \max\{2+1, 0+2\}$, in accordance with Theorem 8.21.

8.5 Bivariate conjugacy class zeta functions associated with cographical group schemes

For a finite group G, let $\mathrm{cc}_n(G)$ denote its number of conjugacy classes of size n and let $\xi^{\mathrm{cc}}_G(s) := \sum_{n=1}^{\infty} \mathrm{cc}_n(G)n^{-s}$ be the associated Dirichlet polynomial; note that $\mathrm{k}(G) = \xi^{\mathrm{cc}}_G(0)$.

Let $\mathbf{G}$ be a unipotent group scheme over the ring of integers $\mathcal{O} = \mathcal{O}_K$ of a number field K; see [66, §2.1]. For a place $v \in \mathcal{V}_K$, let $\mathfrak{P}_v \in \mathrm{Spec}(\mathcal{O})$ be the associated prime ideal with residue field size $q_v = |\mathcal{O}/\mathfrak{P}_v|$ and let $\mathcal{O}_v = \varprojlim_k \mathcal{O}/\mathfrak{P}_v^k$.

In [46, Definition 1.2], Lins defined the **bivariate conjugacy class zeta function**

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}}(s_1, s_2) = \sum_{0\neq I\lhd\mathcal{O}} \xi^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/I)}(s_1)\cdot|\mathcal{O}/I|^{-s_2} = \prod_{v\in\mathcal{V}_K} \mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}_v}(s_1, s_2) \tag{8.5}$$

associated with $\mathbf{G}$, where the Euler factors are given by

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}_v}(s_1, s_2) = \sum_{i=0}^{\infty} \xi^{\mathrm{cc}}_{\mathbf{G}(\mathcal{O}/\mathfrak{P}_v^i)}(s_1)(q_v^{-s_2})^i. \tag{8.6}$$

For all but (possibly) finitely many places $v \in \mathcal{V}_K$, the Euler factors (8.6) are rational functions in q^{-s_1} and q^{-s_2}; see [46, Theorem 1.2]. Both these local and the global zeta functions (8.5) refine the class counting zeta functions defined in §1.3. Indeed, as observed in [46, §1.2],

$$\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}}(0, s) = \zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}}(s). \tag{8.7}$$

(Lins used slightly different notation for these functions; see Remark 8.24.) Just as univariate ask zeta functions may be expressed in terms of carefully designed *univariate* p-adic integrals, Lins expressed bivariate conjugacy zeta functions in terms of suitably defined *bivariate* p-adic integrals; see [46, §4].

We record here, in all brevity, that expressing class counting zeta functions $\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}}(s) = W_{\mathsf{H}}(q, q^{|\mathrm{E}(\Gamma)|-s})$ associated with a cographical group scheme $\mathbf{G}_\Gamma$ in terms of the ask zeta function of a modelling hypergraph H is compatible with Lins's bivariate refinement of class counting zeta functions. The reason for this is the common *multivariate* origin of all the p-adic integrals involved.

To be more precise, let $\mathsf{H} = (V, E, |\cdot|)$ be a hypergraph with incidence representation η. For each compact DVR $\mathfrak{O}$, we define the **bivariate ask zeta function**

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s_1, s_2) = (1-q^{-1})^{-1} \int\limits_{\mathfrak{O}V \times \mathfrak{O}} |y|^{(s_1+1)|E|+s_2-|V|-1} \prod_{e\in E} \|x_e; y\|^{-s_1-1} \, \mathrm{d}\mu_{\mathfrak{O}V\times\mathfrak{O}}(x,y);$$

note that $\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s) = \zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(0, s)$; cf. Proposition 3.4. One may define bivariate ask zeta functions in greater generality but we shall not need this here.

Generalising (5.2) (for $D = \varnothing$), we may express the bivariate ask zeta functions in terms of the *multivariate* function $\mathsf{Z}_{V,\varnothing}(\boldsymbol{s})$ (see (5.1)):

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s_1, s_2) = (1-q^{-1})^{-1}\, \mathsf{Z}_{V,\varnothing}\left((s_1+1)|E| + s_2 - |V| - 1; (\mu_I(-s_1-1))_{I\subset V}\right).$$

As in §4.4, there exists a rational function $W_{\mathsf{H}}(X, T_1, T_2) \in \mathbf{Q}(X, T_1, T_2)$ such that

$$\zeta^{\mathsf{ask}}_{\eta^{\mathfrak{O}}}(s_1, s_2) = W_{\mathsf{H}}(X, T_1, T_2);$$

of course, $W_{\mathsf{H}}(X, T) = W_{\mathsf{H}}(X, 1, T)$. We may then use the multivariate nature of (5.8) to deduce a "trivariate" analogue of the formula for $W_{\mathsf{H}}(X, T)$ given in Corollary 5.6.

If H is a model of a cograph Γ, then Lins's bivariate conjugacy class zeta function is recovered via the formula

$$\mathscr{Z}^{\mathrm{cc}}_{\mathbf{G}_\Gamma\otimes\mathfrak{O}}(s_1, s_2) = W_{\mathsf{H}}(q, q^{-s_1}, q^{|\mathrm{E}(\Gamma)|-s_2});$$

this is based on a trivariate form of Theorem D.

Example 8.23. For the block hypergraph $\mathsf{BH}_{n,m}$, we readily obtain

$$W_{\mathsf{BH}_{n,m}}(X, T_1, T_2) = \frac{1 - X^{-m} T_1^m T_2}{(1-T_2)(1 - X^{n-m} T_1^m T_2)}, \tag{8.8}$$

generalising the bivariate formula given in Example 5.10(i). Using the fact that $\mathsf{BH}_{n,n-1}$ is a model of the complete graph K_n (see Example 7.17) we may use this formula (with $n = 2\mathrm{n} + \delta$ and $m = n - 1$) to recover Lins's formula (cf. [47, Theorem 1.4])

$$\mathscr{Z}^{\mathrm{cc}}_{F_{\mathrm{n},\delta}\otimes\mathfrak{O}}(s_1, s_2) = \mathscr{Z}^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_n}\otimes\mathfrak{O}}(s_1, s_2) = \frac{1 - q^{\binom{n-1}{2} - (n-1)s_1 - s_2}}{\left(1 - q^{\binom{n}{2} - s_2}\right)\left(1 - q^{\binom{n}{2} + 1 - (n-1)s_1 - s_2}\right)}.$$

As predicted by (8.7), setting $(s_1, s_2) = (0, s)$, we recover the formula in Example 8.3(i).

Remark 8.24.

(i) Lins's notation [46, 47] differs slightly from ours. Her $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}(R)}(s_1, s_2)$ is what we called $\mathcal{Z}^{\mathrm{cc}}_{\mathbf{G}\otimes R}(s_1, s_2)$, for various rings R. Our *class counting zeta function* $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes R}(s)$ goes by the name *class number zeta function* $\zeta^{\mathrm{k}}_{\mathbf{G}(R)}(s)$ in Lins's work. We note that our $\zeta^{\mathrm{cc}}_{\mathbf{G}\otimes\mathcal{O}}(s)$ may not only be obtained by suitably specialising Lins's bivariate conjugacy class zeta function as in (8.7) but also by specialising her **bivariate representation zeta function** $\mathcal{Z}^{\mathrm{irr}}_{\mathbf{G}(\mathcal{O})}(s_1, s_2)$. The latter is defined analogously to (8.5) by enumerating the ordinary irreducible characters of the finite groups $\mathbf{G}(\mathcal{O}/I)$ by degree (rather than conjugacy classes by cardinality); see [46, (1.2)].

As is apparent from our discussion here, the techniques developed and employed in our study of class counting zeta functions of (co)graphical group schemes are slanted towards counting conjugacy classes rather than irreducible characters.

(ii) Many of our results about univariate local and global class counting zeta functions associated with (co)graphical group schemes have bivariate analogues. The bivariate version of Theorem 5.26, for instance, describes the domain of convergence of the bivariate ask zeta functions from above. General analytic properties of bivariate conjugacy class and representation zeta functions associated with unipotent group schemes over number fields are studied in [48].

(iii) Beyond cographs, using suitable bivariate versions of Theorem A(ii) and Corollary B, we may strengthen Corollary 1.3 as follows: for each simple graph Γ and $k \geqslant 1$, the number of conjugacy classes of $\mathbf{G}_\Gamma(\mathbf{F}_q)$ of size q^k is given by a polynomial in q.

9 Further examples

In this section, we collect a number of further examples of the function $W^{\pm}_{\Gamma}(X, T)$ for graphs Γ beyond the infinite families covered in §8.

9.1 Computer calculations: Zeta

Our constructive proof of Theorem A (see §§4.4, 6.4) leads to algorithms for computing the rational functions $W_{\mathsf{H}}(X, T)$ and $W^{\pm}_{\Gamma}(X, T)$ associated with a hypergraph and graph, respectively, complementing the formulae derived in §5. In detail, given a hypergraph H, Proposition 3.4 expresses $W_{\mathsf{H}}(X, T)$ in terms of a combinatorially defined p-adic integral. The latter can be expressed as a univariate specialisation of the integrals studied in [54, §3]. In particular, [54, Proposition 3.9] and [58, §6] together provide practical means for computing $W_{\mathsf{H}}(X, T)$. Behind the scenes, these techniques rely on algorithms due to Barvinok and Woods [4] for computing with rational generating functions enumerating lattice points in polyhedra.

Regarding the case of a graph Γ, the inductive proof of Theorem 6.4 in §6.4 readily translates into a recursive algorithm for computing $W^{\pm}_{\Gamma}(X, T)$. For the base case, combine Proposition 6.5 and Proposition 4.8 with [54, §3] and [58, §6] as above.

Based on the steps just outlined, the first author's software package Zeta [61] for the computer algebra system SageMath [67] includes implementations of algorithms for computing the rational functions $W_{\mathsf{H}}(X,T)$ and $W_\Gamma^{\pm}(X,T)$ in Theorem A. In practice, these algorithms often substantially outperform the previously existing functionality for computing ask zeta functions based on [57] that is available in Zeta; note, however, that the present algorithms are only applicable in the context of ask zeta functions associated with graphs and hypergraphs. We used them, for instance, to compute the rational functions $W_\Gamma^-(X,T)$ for all simple graphs on at most seven vertices.

In the remainder of this section, we record a number of explicit examples of the functions $W_{\mathsf{H}}(X,T)$ and $W_\Gamma^{\pm}(X,T)$ computed with the help of Zeta.

9.2 Graphs on at most four vertices

Table 1 lists both types of rational functions $W_\Gamma^{\pm}(X,T)$ for all 18 (isomorphism classes of non-empty) simple graphs on at most four vertices; any entry "%" in the column $W_\Gamma^+(X,T)$ indicates that $W_\Gamma^-(X,T) = W_\Gamma^+(X,T)$ for the specific graph Γ in question. All graphs in Table 1, save for the path P_4, are cographs. In particular, 17 of the formulae in Table 1 could, in principle, be derived from Theorems C–D. We further note that all but the following graphs in Table 1 are kite graphs: $\mathrm{K}_2 \oplus \mathrm{K}_2$, P_4, and C_4; see Question 10.5.

9.3 Graphs on five vertices

In this subsection, we list $W_\Gamma^-(X,T)$ for all 34 simple graphs on five vertices. By Corollary 7.30(i) and using Table 1, it suffices to consider simple graphs on five vertices without isolated vertices; there are precisely 23 of these and their associated rational functions $W_\Gamma^-(X,T)$ are listed in Table 2. We chose not to include the (often bulky) corresponding rational functions $W_\Gamma^+(X,T)$. In Table 2, cographs are flagged and kite graphs are labelled as such.

9.4 Paths and cycles on at most nine vertices

Recall that P_n and C_n denote the path and cycle graph on n vertices, respectively; see (3.7)–(3.8). The rational functions $W_{\mathrm{P}_n}^-(X,T)$ and $W_{\mathrm{C}_n}^-(X,T)$ for $n \leqslant 9$ are given in Tables 3 and 4. For a group-theoretic interpretation in the case of paths, as in §1.2, let U_n denote the group scheme of upper unitriangular $n \times n$ matrices. As we noted in §1.4, the graphical group scheme $\mathbf{G}_{\mathrm{P}_n}$ is isomorphic to the maximal quotient $\mathrm{U}_{n+1,3} := \mathrm{U}_{n+1}/\boldsymbol{\gamma}_3(\mathrm{U}_{n+1})$ of U_{n+1} of class at most 2. The class numbers of the finite groups $\mathrm{U}_{n+1,3}(\mathbf{F}_q)$ were determined by Marjoram [50, Theorem 7]. For $n \leqslant 9$, in accordance with Corollary B, his general formulae agree with the coefficients of T in the expansions of the rational functions $W_{\mathrm{P}_n}^-(X, X^{n-1}T)$ recorded in Table 3.

9.5 The numerator of $W^+_{(\mathrm{K}_3 \oplus \mathrm{K}_3)\vee \mathrm{K}_2}$

We may now finish Example 1.6: Table 5 records the numerator of $W_\Gamma^+(X,T)$ in (1.7).

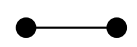
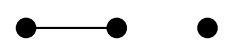
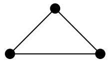
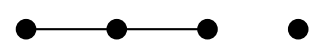
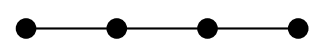
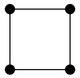
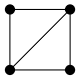
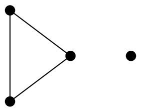
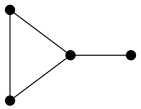
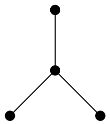
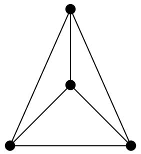
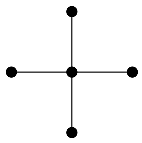
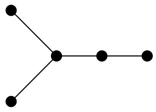
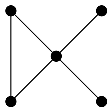
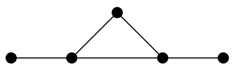
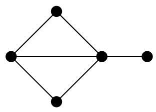
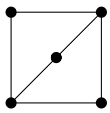
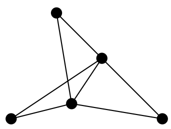
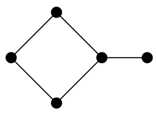
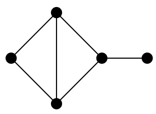
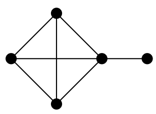
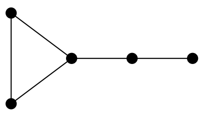
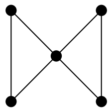
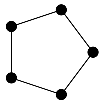
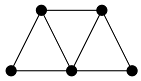
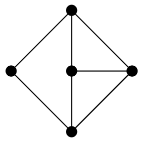
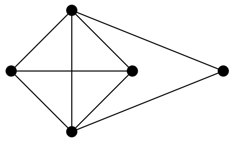
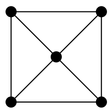
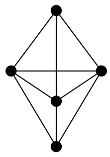
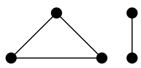
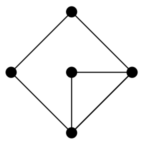
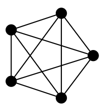
Γ	$W_\Gamma^-(X,T)$	$W_\Gamma^+(X,T)$
	$1/(1-XT)$	%
	$1/(1-X^2T)$	%
	$\frac{1-X^{-1}T}{(1-T)(1-XT)}$	%
	$1/(1-X^3T)$	%
	$\frac{1-T}{(1-XT)(1-X^2T)}$	%
	$\frac{1-X^{-1}T}{(1-XT)^2}$	%
	$\frac{1-X^{-2}T}{(1-T)(1-XT)}$	$(T^2+T+1-3X^{-1}T^2-6X^{-1}T+6X^{-2}T^2+3X^{-2}T-X^{-3}T^3-X^{-3}T^2-X^{-3}T)/(1-T)^4$
	$1/(1-X^4T)$	%
	$\frac{1-XT}{(1-X^2T)(1-X^3T)}$	
	$\frac{XT-2T+1+X^{-1}T^2-2X^{-1}T+X^{-2}T}{(1-X^2T)(1-XT)(1-T)}$	%
	$\frac{1-T}{(1-X^2T)^2}$	
	$\frac{-XT^2+3T^2-T+1-X^{-1}T^3+X^{-1}T^2-3X^{-1}T+X^{-2}T}{(1-T)(1-XT)^3}$	%
	$\frac{T+1-2X^{-1}T-2X^{-2}T+X^{-3}T^2+X^{-3}T}{(1-T)^2(1-XT)}$	%
	$\frac{(1-X^{-1}T)(1-X^{-2}T)}{(1-T)^2(1-XT)}$	$(T^3+2T^2+3T+1-3X^{-1}T^3-7X^{-1}T^2-4X^{-1}T+3X^{-2}T^3-3X^{-2}T+4X^{-3}T^3+7X^{-3}T^2+3X^{-3}T-X^{-4}T^4-3X^{-4}T^3-2X^{-4}T^2-X^{-4}T)/((1-X^{-1}T)(1-T)^2(1-XT^2))$
	$\frac{(1-X^{-1}T)}{(1-XT)(1-X^2T)}$	$(X^2T^2-3XT^2+XT-T^3+6T^2-6T+1-X^{-1}T^2+3X^{-1}T-X^{-2}T)/(1-XT)^4$
	$\frac{(1-X^{-1}T)^2}{(1-T)(1-XT)^2}$	$(-XT^4-XT^3+4T^4+4T^3-3T^2+2T+1-8X^{-1}T^4+2X^{-1}T^3+2X^{-1}T^2-8X^{-1}T+X^{-2}T^5+2X^{-2}T^4-3X^{-2}T^3+4X^{-2}T^2+4X^{-2}T-X^{-3}T^2-X^{-3}T)/((1-T)^3(1-XT)(1-XT^2))$
	$\frac{1-X^{-1}T}{(1-XT)(1-X^2T)}$	%
	$\frac{1-X^{-3}T}{(1-T)(1-XT)}$	$(1+3X^{-1}T^2+5X^{-1}T+3X^{-2}T^3-8X^{-2}T^2-14X^{-2}T-10X^{-3}T^3+10X^{-3}T+14X^{-4}T^3+8X^{-4}T^2-3X^{-4}T-5X^{-5}T^3-3X^{-5}T^2-X^{-6}T^4)/((1-X^{-1}T^2)(1-X^{-1}T)(1-T)^2)$

Table 1: Graphs on at most four vertices and their ask zeta functions

Table 2: Graphs without isolated vertices on at most five vertices and their negative ask zeta functions

Γ	comment	$W^-_\Gamma(X,T)$
	Kite(4, 1)	$\frac{(1-X^{-1}T)}{(1-XT)(1-X^3T)}$
	no cograph	$\frac{XT-2T+1+X^{-1}T^2-2X^{-1}T+X^{-2}T}{(1-XT)^2(1-X^2T)}$
	Kite(1, 1, 2, 1)	$\frac{(1-X^{-1}T)(1-T)}{(1-XT)^2(1-X^2T)}$
	no cograph (P_5)	$(-2XT^2 + XT + 4T^2 - 2T + 1 - X^{-1}T^3 + 2X^{-1}T^2 - 4X^{-1}T - X^{-2}T^2 + 2X^{-2}T)/(1 - XT)^4$
	no cograph	$\frac{-XT^2+3T^2-T+1-X^{-1}T^3+X^{-1}T^2-3X^{-1}T+X^{-2}T}{(1-XT)^4}$
	Kite(2, 1, 1, 1)	$\frac{(1-X^{-1}T)^2}{(1-XT)^3}$
	cograph	$\frac{T+1-X^{-1}T-2X^{-2}T-X^{-3}T+X^{-4}T^2+X^{-4}T}{(1-X^{-1}T)(1-XT)^2}$
	Kite(3, 2)	$\frac{(1-X^{-2}T)(1-X^{-1}T)}{(1-T)(1-XT)^2}$
	no cograph	$\frac{-XT^2+T^2+1+3X^{-1}T^2-3X^{-1}T-X^{-2}T^3-X^{-2}T+X^{-3}T}{(1-T)(1-XT)^3}$
	no cograph	$\frac{T+1-2X^{-1}T-2X^{-2}T+X^{-3}T^2+X^{-3}T}{(1-T)(1-XT)^2}$
	Kite(1, 2, 1, 1)	$\frac{(1-X^{-1}T)(1-X^{-2}T)}{(1-T)(1-XT)^2}$
	no cograph	$\frac{-XT^2+T^2+1+3X^{-1}T^2-3X^{-1}T-X^{-2}T^3-X^{-2}T+X^{-3}T}{(1-T)(1-XT)^3}$

	cograph	$\frac{T+1-2X^{-1}T-2X^{-2}T+X^{-3}T^2+X^{-3}T}{(1-XT)^2(1-T)}$
	no cograph (C_5)	$(T^2+3T+1-5X^{-1}T^2-5X^{-1}T+5X^{-2}T^2-5X^{-2}T+5X^{-3}T^2+5X^{-3}T-X^{-4}T^3-3X^{-4}T^2-X^{-4}T)/((1-T)^3(1-XT))$
	no cograph	$\frac{1-X^{-1}T^2-X^{-1}T+3X^{-2}T^2-3X^{-2}T+X^{-3}T^2+X^{-3}T-X^{-4}T^3}{(1-T)^3(1-XT)}$
	cograph	$\frac{1-X^{-1}T^2+X^{-2}T^2-3X^{-2}T+3X^{-3}T^2-X^{-3}T+X^{-4}T-X^{-5}T^3}{(1-XT)(1-X^{-1}T)(1-T)^2}$
	Kite$(1,1,1,2)$	$\frac{(1-X^{-2}T)^2}{(1-T)^2(1-XT)}$
	cograph	$\frac{1+X^{-1}T-2X^{-2}T-2X^{-3}T+X^{-4}T+X^{-5}T^2}{(1-X^{-1}T)(1-T)(1-XT)}$
	Kite$(2,3)$	$\frac{(1-X^{-3}T)(1-X^{-2}T)}{(1-X^{-1}T)(1-T)(1-XT)}$
	cograph	$\frac{-X^3T^2+X^2T^2-XT^3+3XT^2-3T+1-X^{-1}T+X^{-2}T}{(1-XT)^2(1-X^2T)^2}$
	cograph	$\frac{XT-T+1-2X^{-1}T+X^{-2}T^2-X^{-2}T+X^{-3}T}{(1-T)(1-XT)(1-X^2T)}$
	no cograph	$(T+1-2X^{-1}T^2-2X^{-1}T+4X^{-2}T^2-4X^{-2}T+2X^{-3}T^2+2X^{-3}T-X^{-4}T^3-X^{-4}T^2)/((1-T)^3(1-XT))$
	Kite$(1,4) = K_5$	$\frac{1-X^{-4}T}{(1-T)(1-XT)}$

Γ	$W_{\Gamma}^{-}(X,T)$
P_1	$1/(1-XT)$
P_2	$(1-X^{-1}T)/((1-T)(1-XT))$
P_3	$(1-X^{-1}T)/((1-XT)^2$
P_4	$(-XT^2+3T^2-T+1-X^{-1}T^3+X^{-1}T^2-3X^{-1}T+X^{-2}T)/((1-XT)^3(1-T))$
P_5	$(-2XT^2+XT+4T^2-2T+1-X^{-1}T^3+2X^{-1}T^2-4X^{-1}T-X^{-2}T^2+2X^{-2}T)/(1-XT)^4$
P_6	$(X^2T^4-X^2T^3+X^2T^2-6XT^4+11XT^3-13XT^2+3XT+7T^4-12T^3+20T^2-6T+1-X^{-1}T^5+6X^{-1}T^4-20X^{-1}T^3+12X^{-1}T^2-7X^{-1}T-3X^{-2}T^4+13X^{-2}T^3-11X^{-2}T^2+6X^{-2}T-X^{-3}T^3+X^{-3}T^2-X^{-3}T)/((1-T)(1-XT)^5)$
P_7	$(X^3T^3+3X^2T^4-12X^2T^3+7X^2T^2-12XT^4+41XT^3-40XT^2+7XT+9T^4-28T^3+45T^2-12T+1-X^{-1}T^5+12X^{-1}T^4-45X^{-1}T^3+28X^{-1}T^2-9X^{-1}T-7X^{-2}T^4+40X^{-2}T^3-41X^{-2}T^2+12X^{-2}T-7X^{-3}T^3+12X^{-3}T^2-3X^{-3}T-X^{-4}T^2)/((1-XT)^6)$
P_8	$(-X^4T^5+X^4T^4-X^3T^6+14X^3T^5-28X^3T^4+14X^3T^3+10X^2T^6-84X^2T^5+200X^2T^4-150X^2T^3+31X^2T^2-25XT^6+185XT^5-496XT^4+462XT^3-161XT^2+14XT+11T^6-76T^5+310T^4-374T^3+189T^2-26T+1-X^{-1}T^7+26X^{-1}T^6-189X^{-1}T^5+374X^{-1}T^4-310X^{-1}T^3+76X^{-1}T^2-11X^{-1}T-14X^{-2}T^6+161X^{-2}T^5-462X^{-2}T^4+496X^{-2}T^3-185X^{-2}T^2+25X^{-2}T-31X^{-3}T^5+150X^{-3}T^4-200X^{-3}T^3+84X^{-3}T^2-10X^{-3}T-14X^{-4}T^4+28X^{-4}T^3-14X^{-4}T^2+X^{-4}T-X^{-5}T^3+X^{-5}T^2)/((1-XT)^7(1-T))$
P_9	$(X^5T^5-16X^4T^5+26X^4T^4-4X^3T^6+110X^3T^5-282X^3T^4+109X^3T^3+25X^2T^6-376X^2T^5+1162X^2T^4-798X^2T^3+109X^2T^2-46XT^6+559XT^5-2042XT^4+1962XT^3-486XT^2+26XT+8T^6-94T^5+918T^4-1526T^3+582T^2-50T+1-X^{-1}T^7+50X^{-1}T^6-582X^{-1}T^5+1526X^{-1}T^4-918X^{-1}T^3+94X^{-1}T^2-8X^{-1}T-26X^{-2}T^6+486X^{-2}T^5-1962X^{-2}T^4+2042X^{-2}T^3-559X^{-2}T^2+46X^{-2}T-109X^{-3}T^5+798X^{-3}T^4-1162X^{-3}T^3+376X^{-3}T^2-25X^{-3}T-109X^{-4}T^4+282X^{-4}T^3-110X^{-4}T^2+4X^{-4}T-26X^{-5}T^3+16X^{-5}T^2-X^{-6}T^2)/((1-XT)^8)$

Table 3: Ask zeta functions associated with paths on at most nine vertices

Γ	$W^-_\Gamma(X,T)$
C_3	$(1-X^{-2}T)/((1-XT)(1-T))$
C_4	$(T+1-2X^{-1}T-2X^{-2}T+X^{-3}T^2+X^{-3}T)/((1-XT)(1-T)^2)$
C_5	$(T^2+3T+1-5X^{-1}T^2-5X^{-1}T+5X^{-2}T^2-5X^{-2}T+5X^{-3}T^2+5X^{-3}T-X^{-4}T^3-3X^{-4}T^2-X^{-4}T)/((1-XT)(1-T)^3)$
C_6	$(T^3+8T^2+8T+1-6X^{-1}T^3-33X^{-1}T^2-15X^{-1}T+13X^{-2}T^3+28X^{-2}T^2-5X^{-2}T-5X^{-3}T^3+28X^{-3}T^2+13X^{-3}T-15X^{-4}T^3-33X^{-4}T^2-6X^{-4}T+X^{-5}T^4+8X^{-5}T^3+8X^{-5}T^2+X^{-5}T)/((1-XT)(1-T)^4)$
C_7	$(T^4+17T^3+41T^2+17T+1-7X^{-1}T^4-98X^{-1}T^3-168X^{-1}T^2-35X^{-1}T+21X^{-2}T^4+189X^{-2}T^3+175X^{-2}T^2-28X^{-3}T^4-70X^{-3}T^3+70X^{-3}T^2+28X^{-3}T-175X^{-4}T^3-189X^{-4}T^2-21X^{-4}T+35X^{-5}T^4+168X^{-5}T^3+98X^{-5}T^2+7X^{-5}T-X^{-6}T^5-17X^{-6}T^4-41X^{-6}T^3-17X^{-6}T^2-X^{-6}T)/((1-XT)(1-T)^5)$
C_8	$(T^5+33T^4+158T^3+158T^2+33T+1-8X^{-1}T^5-236X^{-1}T^4-924X^{-1}T^3-676X^{-1}T^2-76X^{-1}T+28X^{-2}T^5+660X^{-2}T^4+1884X^{-2}T^3+860X^{-2}T^2+24X^{-2}T-54X^{-3}T^5-772X^{-3}T^4-1128X^{-3}T^3-12X^{-3}T^2+46X^{-3}T+46X^{-4}T^5-12X^{-4}T^4-1128X^{-4}T^3-772X^{-4}T^2-54X^{-4}T+24X^{-5}T^5+860X^{-5}T^4+1884X^{-5}T^3+660X^{-5}T^2+28X^{-5}T-76X^{-6}T^5-676X^{-6}T^4-924X^{-6}T^3-236X^{-6}T^2-8X^{-6}T+X^{-7}T^6+33X^{-7}T^5+158X^{-7}T^4+158X^{-7}T^3+33X^{-7}T^2+X^{-7}T)/((1-XT)(1-T)^6)$
C_9	$(T^6+60T^5+516T^4+1015T^3+516T^2+60T+1-9X^{-1}T^6-504X^{-1}T^5-3798X^{-1}T^4-6192X^{-1}T^3-2358X^{-1}T^2-153X^{-1}T+36X^{-2}T^6+1770X^{-2}T^5+10974X^{-2}T^4+13896X^{-2}T^3+3603X^{-2}T^2+87X^{-2}T-84X^{-3}T^6-3141X^{-3}T^5-14154X^{-3}T^4-11760X^{-3}T^3-1287X^{-3}T^2+60X^{-3}T+117X^{-4}T^6+2268X^{-4}T^5+3456X^{-4}T^4-3456X^{-4}T^3-2268X^{-4}T^2-117X^{-4}T-60X^{-5}T^6+1287X^{-5}T^5+11760X^{-5}T^4+14154X^{-5}T^3+3141X^{-5}T^2+84X^{-5}T-87X^{-6}T^6-3603X^{-6}T^5-13896X^{-6}T^4-10974X^{-6}T^3-1770X^{-6}T^2-36X^{-6}T+153X^{-7}T^6+2358X^{-7}T^5+6192X^{-7}T^4+3798X^{-7}T^3+504X^{-7}T^2+9X^{-7}T-X^{-8}T^7-60X^{-8}T^6-516X^{-8}T^5-1015X^{-8}T^4-516X^{-8}T^3-60X^{-8}T^2-X^{-8}T)/((1-XT)(1-T)^7)$

Table 4: Ask zeta functions associated with cycles on at most nine vertices

$1+5X^{-2}T^{2}+18X^{-2}T-4X^{-3}T^{2}+12X^{-4}T^{3}+4X^{-5}T^{4}-60X^{-3}T+44X^{-4}T^{2}-129X^{-5}T^{3}-56X^{-6}T^{4}-21X^{-7}T^{5}+83X^{-4}T-359X^{-5}T^{2}+345X^{-6}T^{3}+105X^{-7}T^{4}+28X^{-8}T^{5}+32X^{-9}T^{6}-106X^{-5}T+707X^{-6}T^{2}-623X^{-7}T^{3}+210X^{-8}T^{4}+265X^{-9}T^{5}-95X^{-10}T^{6}+74X^{-11}T^{7}+92X^{-6}T-477X^{-7}T^{2}+1415X^{-8}T^{3}-626X^{-9}T^{4}-500X^{-10}T^{5}+137X^{-11}T^{6}-497X^{-12}T^{7}+18X^{-13}T^{8}-43X^{-7}T+59X^{-8}T^{2}-2179X^{-9}T^{3}+409X^{-10}T^{4}-602X^{-11}T^{5}-570X^{-12}T^{6}+1329X^{-13}T^{7}-150X^{-14}T^{8}+X^{-15}T^{9}+8X^{-8}T+33X^{-9}T^{2}+1872X^{-10}T^{3}-6X^{-11}T^{4}+2093X^{-12}T^{5}+1360X^{-13}T^{6}-1815X^{-14}T^{7}+425X^{-15}T^{8}-100X^{-16}T^{9}+93X^{-10}T^{2}-947X^{-11}T^{3}-301X^{-12}T^{4}-2468X^{-13}T^{5}-1348X^{-14}T^{6}+1086X^{-15}T^{7}-608X^{-16}T^{8}+553X^{-17}T^{9}-8X^{-18}T^{10}-124X^{-11}T^{2}+35X^{-12}T^{3}-459X^{-13}T^{4}+2585X^{-14}T^{5}+806X^{-15}T^{6}+741X^{-16}T^{7}+1094X^{-17}T^{8}-1174X^{-18}T^{9}+117X^{-19}T^{10}-3X^{-20}T^{11}+56X^{-12}T^{2}+379X^{-13}T^{3}+2070X^{-14}T^{4}-1762X^{-15}T^{5}+268X^{-16}T^{6}-1682X^{-17}T^{7}-2568X^{-18}T^{8}+1093X^{-19}T^{9}-546X^{-20}T^{10}+10X^{-21}T^{11}-9X^{-13}T^{2}-258X^{-14}T^{3}-2141X^{-15}T^{4}-196X^{-16}T^{5}-1798X^{-17}T^{6}+88X^{-18}T^{7}+3403X^{-19}T^{8}-458X^{-20}T^{9}+1253X^{-21}T^{10}+74X^{-22}T^{11}+37X^{-15}T^{3}+688X^{-16}T^{4}+2069X^{-17}T^{5}+2381X^{-18}T^{6}+1478X^{-19}T^{7}-1543X^{-20}T^{8}+401X^{-21}T^{9}-1607X^{-22}T^{10}-379X^{-23}T^{11}-15X^{-24}T^{12}+26X^{-16}T^{3}+315X^{-17}T^{4}-2168X^{-18}T^{5}-1973X^{-19}T^{6}-1887X^{-20}T^{7}-2220X^{-21}T^{8}-887X^{-22}T^{9}+974X^{-23}T^{10}+428X^{-24}T^{11}+120X^{-25}T^{12}-9X^{-17}T^{3}-353X^{-18}T^{4}+667X^{-19}T^{5}+529X^{-20}T^{6}+1568X^{-21}T^{7}+4496X^{-22}T^{8}+1568X^{-23}T^{9}+529X^{-24}T^{10}+667X^{-25}T^{11}-353X^{-26}T^{12}-9X^{-27}T^{13}+120X^{-19}T^{4}+428X^{-20}T^{5}+974X^{-21}T^{6}-887X^{-22}T^{7}-2220X^{-23}T^{8}-1887X^{-24}T^{9}-1973X^{-25}T^{10}-2168X^{-26}T^{11}+315X^{-27}T^{12}+26X^{-28}T^{13}-15X^{-20}T^{4}-379X^{-21}T^{5}-1607X^{-22}T^{6}+401X^{-23}T^{7}-1543X^{-24}T^{8}+1478X^{-25}T^{9}+2381X^{-26}T^{10}+2069X^{-27}T^{11}+688X^{-28}T^{12}+37X^{-29}T^{13}+74X^{-22}T^{5}+1253X^{-23}T^{6}-458X^{-24}T^{7}+3403X^{-25}T^{8}+88X^{-26}T^{9}-1798X^{-27}T^{10}-196X^{-28}T^{11}-2141X^{-29}T^{12}-258X^{-30}T^{13}-9X^{-31}T^{14}+10X^{-23}T^{5}-546X^{-24}T^{6}+1093X^{-25}T^{7}-2568X^{-26}T^{8}-1682X^{-27}T^{9}+268X^{-28}T^{10}-1762X^{-29}T^{11}+2070X^{-30}T^{12}+379X^{-31}T^{13}+56X^{-32}T^{14}-3X^{-24}T^{5}+117X^{-25}T^{6}-1174X^{-26}T^{7}+1094X^{-27}T^{8}+741X^{-28}T^{9}+806X^{-29}T^{10}+2585X^{-30}T^{11}-459X^{-31}T^{12}+35X^{-32}T^{13}-124X^{-33}T^{14}-8X^{-26}T^{6}+553X^{-27}T^{7}-608X^{-28}T^{8}+1086X^{-29}T^{9}-1348X^{-30}T^{10}-2468X^{-31}T^{11}-301X^{-32}T^{12}-947X^{-33}T^{13}+93X^{-34}T^{14}-100X^{-28}T^{7}+425X^{-29}T^{8}-1815X^{-30}T^{9}+1360X^{-31}T^{10}+2093X^{-32}T^{11}-6X^{-33}T^{12}+1872X^{-34}T^{13}+33X^{-35}T^{14}+8X^{-36}T^{15}+X^{-29}T^{7}-150X^{-30}T^{8}+1329X^{-31}T^{9}-570X^{-32}T^{10}-602X^{-33}T^{11}+409X^{-34}T^{12}-2179X^{-35}T^{13}+59X^{-36}T^{14}-43X^{-37}T^{15}+18X^{-31}T^{8}-497X^{-32}T^{9}+137X^{-33}T^{10}-500X^{-34}T^{11}-626X^{-35}T^{12}+1415X^{-36}T^{13}-477X^{-37}T^{14}+92X^{-38}T^{15}+74X^{-33}T^{9}-95X^{-34}T^{10}+265X^{-35}T^{11}+210X^{-36}T^{12}-623X^{-37}T^{13}+707X^{-38}T^{14}-106X^{-39}T^{15}+32X^{-35}T^{10}+28X^{-36}T^{11}+105X^{-37}T^{12}+345X^{-38}T^{13}-359X^{-39}T^{14}+83X^{-40}T^{15}-21X^{-37}T^{11}-56X^{-38}T^{12}-129X^{-39}T^{13}+44X^{-40}T^{14}-60X^{-41}T^{15}+4X^{-39}T^{12}+12X^{-40}T^{13}-4X^{-41}T^{14}+18X^{-42}T^{15}+5X^{-42}T^{14}+X^{-44}T^{16}$

Table 5: Numerator $F(X,T)$ of $W^{+}_{(\mathrm{K}_3\oplus\mathrm{K}_3)\vee\mathrm{K}_2}$ in (1.7)

10 Open problems

Inspired by the theoretical results and the explicit formulae in this article, we raise a number of further questions (beyond Questions 1.7 and 1.8) pertaining to the topics covered here.

10.1 The algebra of graphs

As we mentioned in §8.2, the effect of taking disjoint unions of graphs corresponds to taking Hadamard products of the rational functions $W^{\pm}_{\Gamma}(X,T)$. In particular, for arbitrary simple graphs Γ_1 and Γ_2, the rational function $W^{-}_{\Gamma_1\oplus\Gamma_2}(X,T)$ does not depend on the individual graphs Γ_1 and Γ_2 but only on the rational functions attached to these.

In the special case that Γ_1 and Γ_2 are both *cographs*, Proposition 8.4 similarly expresses $W^{-}_{\Gamma_1\vee\Gamma_2}(X,T)$ in terms of the $W^{-}_{\Gamma_i}(X,T)$. As we mentioned in Remark 8.6, even though the formula in Proposition 8.4 involves the numbers of vertices of Γ_1 and Γ_2, these numbers can be recovered from the corresponding rational functions via the functional equation in Corollary 1.4.

Question 10.1 (Joins)**.** Do the conclusions of Proposition 8.4 hold for all simple graphs?

The smallest graph which is not a cograph is the path P_4 on four vertices; the rational function $W^{-}_{\mathrm{P}_4}(X,T)$ is recorded in Table 1 (and also in Table 3). Using Zeta, we find that

$$\begin{aligned} W^{-}_{\mathrm{P}_4\vee\mathrm{P}_4}(X,T) = (&1+2X^{-3}T-2X^{-4}T-6X^{-5}T-X^{-6}T^2+2X^{-6}T-2X^{-7}T^2\\ &+X^{-7}T+6X^{-8}T^2+2X^{-9}T^2-2X^{-10}T^2\\ &-X^{-13}T^3)/((1-X^{-3}T)^2(1-T)(1-XT)). \end{aligned}$$

A routine calculation shows that, even though P_4 is not a cograph, $W^{-}_{\mathrm{P}_4\vee\mathrm{P}_4}(X,T)$ is indeed correctly calculated by (8.2) for $\Gamma_1=\Gamma_2=\mathrm{P}_4$.

While we defined cographs in terms of disjoint unions and joins, either one of these two operations could be replaced by complements. For instance, the class of cographs is the smallest class of graphs that contains a single vertex and which is closed under taking disjoint unions and complements. Write $\hat{\Gamma}$ for the complement of a simple graph Γ.

Question 10.2 (Complements)**.** Is there an involution $W(X,T)\mapsto\hat{W}(X,T)$ of rational generating functions in T such that $W^{-}_{\hat{\Gamma}}(X,T)=\hat{W}^{-}_{\Gamma}(X,T)$ for each simple graph Γ?

10.2 Connections with statistics on Weyl groups

We noted in §5.2 that the class of functions $W_{\mathsf{H}}(X,T)$ attached to hypergraphs is closed under Hadamard products. However, it remains an open problem to exploit the Hadamard factorisation (5.15) in a way that improves upon the general Theorem C. For disjoint unions $\mathsf{BH}_{\mathbf{n},\mathbf{m}}$ of block hypergraphs, we achieved this in (5.18). Even in the special case $\mathbf{n}=\mathbf{m}$, the latter formula seems to admit substantial further improvements. We illustrate this for $\mathbf{n}=\mathbf{m}=(a,\dots,a)$. Recall from [57, §5.4] the definitions of the statistics N and $\mathrm{d_B}$ on the group $\mathrm{B}_r=\{\pm1\}\wr\mathrm{S}_r$ of signed permutations of degree r.

Proposition 10.3. *Let* $\boldsymbol{a} = (a, \dots, a) \in \mathbf{N}^r$. *Then*

$$W_{\mathsf{BH}_{\boldsymbol{a},\boldsymbol{a}}}(X,T) = \frac{\sum_{\sigma \in \mathrm{B}_r} (-X^{-a})^{\mathrm{N}(\sigma)} T^{\mathrm{d_B}(\sigma)}}{(1-T)^{r+1}}. \tag{10.1}$$

Proof. The proof of [57, Corollary 5.17] of the case $a = 1$ (a consequence of a result due to Brenti [12, Theorem 3.4]) carries over to a general a; just replace $-q^{-1}$ by $-q^{-a}$. ♦

The intriguing shape of the numerator of the right-hand side of (10.1) prompts the following.

Question 10.4. Is there an interpretation of the rational functions

(a) $W_{\mathsf{BH}_{\mathbf{n},\mathbf{n}}}(X,T)$ in (5.18) for $\mathbf{n} \in \mathbf{N}^r$,

(b) $W_{\mathsf{BH}_{\mathbf{n},\mathbf{m}}}(X,T)$ in (5.18) for $\mathbf{n}, \mathbf{m} \in \mathbf{N}^r$, or possibly even

(c) $W_{\mathsf{H}(\boldsymbol{\mu})}(X,T)$ in (5.12) for general $\boldsymbol{\mu} \in \mathbf{N}_0^{\mathcal{P}(V)}$

in terms of statistics on Weyl (or more general reflection) groups?

Note that (b) includes, as a special case, the problem of finding an interpretation of the class counting zeta functions $\zeta^{\mathrm{cc}}_{\mathbf{G}_{\mathrm{K}_\mathbf{n}} \otimes \mathfrak{O}}(s)$ in terms of permutation statistics; see §8.2.1.

10.3 Analytic properties

The determination of the rational functions $W^-_\Gamma(X,T)$ for all simple graphs on at most seven vertices inspired us to raise the following.

Question 10.5 (Characterising kites)**.** Let Γ be a simple graph. Are the following properties equivalent?

(a) Γ is a kite graph.

(b) $W^-_\Gamma(X,T)$ is a product of factors of the form $(1 - X^A T^B)^{\pm 1}$.

The implication (a)→(b) in Question 10.5 follows from Theorem 8.19. Let γ_- be the negative adjacency representation of Γ. Then (b) is equivalent to $\zeta^{\mathsf{ask}}_{\gamma_-}(s)$ factoring as a product of factors $\zeta(Bs - A)^{\pm 1}$, where ζ denotes the Riemann zeta function. In particular, class counting zeta functions of cographical group schemes associated with kite graphs over rings of integers of number fields admit meromorphic continuation to $\mathbf{C}$.

One may speculate that the condition in Question 10.5(b) is not just sufficient but also necessary for the Euler product

$$\zeta^{\mathrm{cc}}_{\mathbf{G}_\Gamma \otimes \mathcal{O}_K}(s) = \prod_{v \in \mathcal{V}_K} W^-_\Gamma(q_v, q_v^{|\mathrm{E}(\Gamma)|-s})$$

associated with an arbitrary simple graph Γ and number field K with ring of integers $\mathcal{O}_K$ to admit meromorphic continuation to the whole complex plane. Indeed, consider an

Euler product $\prod_{v\in\mathcal{V}_K} f(q_v, q_v^{-s})$, where $f(X,T)\in\mathbf{Z}[X,T]$ is a fixed polynomial. Then a general conjecture based on work of Estermann [28] and Kurokawa [40, 41] predicts that such an Euler product admits meromorphic continuation to all of $\mathbf{C}$ if and only if $f(X,T)$ is a product of unitary polynomials; cf. [22, Conjecture 1.11] for details and related work.

In a similar spirit, recall from Theorem 5.26 that for a hypergraph H with incidence representation η, we denoted the common abscissa of convergence of $\zeta^{\mathsf{ask}}_{\eta^{\mathcal{O}_K}}(s)$ for each number field K by $\alpha(\mathsf{H})$. By [57, Theorem 4.20], there exists a positive real number $\delta(\mathsf{H})$, independent of K, such that the function $\zeta^{\mathsf{ask}}_{\eta^{\mathcal{O}_K}}(s)$ can be meromorphically continued to the domain $\{s\in\mathbf{C} : \mathrm{Re}(s) > \alpha(\mathsf{H}) - \delta(\mathsf{H})\}$.

Question 10.6. What is the largest value of $\delta(\mathsf{H})$ (if such a value exists)? When does $\zeta^{\mathsf{ask}}_{\eta^{\mathcal{O}_K}}(s)$ admit meromorphic continuation to all of $\mathbf{C}$ for each number field K?

By Proposition 5.9, staircase hypergraphs have the latter property.

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