

Enumerating partial Latin rectangles

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Abstract

This paper deals with distinct computational methods to enumerate the set $\text{PLR}(r, s, n; m)$ of $r \times s$ partial Latin rectangles on n symbols with m non-empty cells. For fixed r , s , and n , we prove that the size of this set is a symmetric polynomial of degree $3m$, and we determine the leading terms (the monomials of degree $3m$ through $3m - 9$) using inclusion-exclusion. For $m \leq 13$, exact formulas for these symmetric polynomials are determined using a chromatic polynomial method. Adapting Sade's method for enumerating Latin squares, we compute the exact size of $\text{PLR}(r, s, n; m)$, for all $r \leq s \leq n \leq 7$, and all $r \leq s \leq 6$ when $n = 8$. Using an algebraic geometry method together with Burnside's Lemma, we enumerate isomorphism, isotopism, and main classes when $r \leq s \leq n \leq 6$. Numerical results have been cross-checked where possible.

Keywords: Partial Latin rectangle, isomorphism, isotopism, main class, inclusion-exclusion chromatic polynomial algebraic geometry.

MSC[2010]: 05B15.

1 Introduction

Let $[n] := \{1, 2, \dots, n\}$. An $r \times s$ *partial Latin rectangle* $L = (l_{ij})$ on the symbol set $[n] \cup \{\cdot\}$ is an $r \times s$ matrix such that each row and each column has at most one copy of any symbol in $[n]$. Here, r , s , and n are arbitrary positive integers, and we admit the possibility that $n < \min\{r, s\}$. If $r = s = n$, then this constitutes a *partial Latin square* of order n . The cells containing the symbol \cdot are considered *empty*, and we say that l_{ij} is *undefined*. An *entry* of L is any triple $(i, j, l_{ij}) \in [r] \times [s] \times [n]$. The set of all entries of L is called its *entry set*, which is denoted $E(L)$. The *weight* of L is its number of non-empty cells, that is, the size of its entry set. Let $\text{PLR}(r, s, n; m)$ denote the set of $r \times s$ partial Latin rectangles on the symbol set $[n] \cup \{\cdot\}$ of weight m and let $\text{PLR}(r, s, n) = \cup_{0 \leq m \leq rs} \text{PLR}(r, s, n; m)$. Let $\text{PLS}(n; m) = \text{PLR}(n, n, n; m)$ be the set of partial Latin squares of weight m . For $m = n^2$, this is the set of *Latin squares of order n* .

For each positive integer $t \in \mathbb{Z}^+$, let S_t denote the symmetric group on the set $[t]$.

- The *isotopism group* $\mathfrak{I}_{r,s,n} := S_r \times S_s \times S_n$ acts on the set $\text{PLR}(r, s, n; m)$, with the *isotopism* $\Theta = (\alpha, \beta, \gamma)$ permuting the rows according to α , the columns according to β , and the symbols according to γ . This gives the *isotopic* partial Latin rectangle $L^\Theta \in \text{PLR}(r, s, n; m)$, whose entry set is $E(L^\Theta) = \{(\alpha(i), \beta(j), \gamma(l_{i,j})) : (i, j, l_{i,j}) \in E(L)\}$.
- The symmetric group S_n is isomorphic to the subgroup $\{(\alpha, \alpha, \alpha) : \alpha \in S_n\}$ of the isotopism group $\mathfrak{I}_{n,n,n}$ via the isomorphism $\alpha \mapsto (\alpha, \alpha, \alpha)$. In this regard, the *isomorphism group* S_n acts on the set $\text{PLS}(n; m)$, with $\alpha \in S_n$ mapping L to $L^{(\alpha, \alpha, \alpha)}$.
- Let $\pi \in S_3$ and $L \in \text{PLR}(d_1, d_2, d_3; m)$. The *parastrophic* partial Latin rectangle $L^\pi \in \text{PLR}(d_{\pi(1)}, d_{\pi(2)}, d_{\pi(3)}; m)$ is defined so that its entry set is $E(L^\pi) = \{(p_{\pi(1)}, p_{\pi(2)}, p_{\pi(3)}) : (p_1, p_2, p_3) \in E(L)\}$. The permutation π is said to be a *parastrophism*. Since parastrophisms may not preserve the dimensions of partial Latin rectangles, the *parastrophism group* $S_{r,s,n}$ is defined as the stabilizer of the ordered triple (r, s, n) under the action $(d_1, d_2, d_3) \xrightarrow{\pi} (d_{\pi(1)}, d_{\pi(2)}, d_{\pi(3)})$ by S_3 .
- The *paratopism group* $\mathfrak{P}_{r,s,n} := \mathfrak{I}_{r,s,n} \times S_{r,s,n}$ acts on the set $\text{PLR}(r, s, n; m)$ so that each *paratopism* (Θ, π) maps L to the *paratopic* partial Latin rectangle $L^{(\Theta, \pi)} = (L^\pi)^\Theta$. When $\pi = \text{Id}$ (i.e., the trivial permutation in S_3) the isotopism group arises as a normal subgroup of the paratopism group.

Orbits of $\text{PLR}(r, s, n; m)$ under the isotopism, isomorphism, and paratopism groups are equivalence classes, called *isotopism*, *isomorphism*, and *main* classes, respectively. The stabilizer subgroups under these groups are called *autotopism*, *automorphism*, and *autoparatopism* groups, respectively. Let $\text{PLR}((\Theta, \pi))$ and $\text{PLR}((\Theta, \pi); m)$ denote, respectively, the subsets of partial Latin rectangles in the sets $\text{PLR}(r, s, n)$ and $\text{PLR}(r, s, n; m)$ that admit an autoparatopism $(\Theta, \pi) \in \mathfrak{P}_{r,s,n}$.

The goal of this paper is to find methods for computing the size of $\text{PLR}(r, s, n; m)$, along with its equivalence class sizes. It is unrealistic to expect a succinct solution to both problems for arbitrary r, s, n , and m , since they include in particular the number of Latin squares of given order n , which is a long-standing research problem in combinatorics. This is known only for order $n \leq 11$ [38, 50]; see [58, 59, 66] for some related results on Latin rectangles. Currently, the number of partial Latin rectangles is known only for $r, s, n \leq 6$ [25, 26, 27]. To make progress, we need to restrict these parameters in some way. In particular, we enumerate (a) fixed-weight partial Latin rectangles, (b) partial Latin rectangles for small m , and (c) partial Latin rectangles for small r, s , and n .

The number of isotopism, isomorphism, and main classes of Latin squares has been determined [38, 47] for order $n \leq 11$, whereas for partial Latin rectangles, these numbers were computed [30] for $r, s, n \leq 6$. Adams, Bean, and Khodkar [1] enumerated main classes of partial Latin squares of order $n \leq 6$ that constitute critical sets. More recently, it has been obtained [18, 69] the number of main classes of partial Latin rectangles with at most 12 entries.

Autoparatopisms and asymmetry for partial Latin squares were studied in [33, 60] and several constructions of partial Latin rectangles with trivial autotopism groups for various autoparatopism groups was given in [32]. Computational methods to determine autotopism groups of partial Latin rectangles were compared in [16, 65]. For Latin squares of order $n \leq 17$, identifying when $\#\text{PLR}((\Theta, \pi)) \neq 0$ (throughout this paper $\#$ denotes the cardinality of a set) was done for isotopisms in [61] and paratopisms in [53], with prior work in [24, 28].

Symmetries of Latin squares and rectangles have been studied in a wide range of contexts, e.g., enumeration [54, 62, 63], subsquares [12, 52], the Alon-Tarsi Conjecture [19, 64], quasigroups and loops [7, 41, 42, 51], special kinds of symmetries [14, 29, 39, 68], and in their own right [8, 13, 22, 23].

They are beginning to find applications in secret sharing schemes [21, 67, 71], erasure codes [72], and graph coloring games [3, 4].

The remainder of the paper is organized as follows. The three following sections deal with distinct combinatorial methods that enable us to determine the size of the set $\text{PLR}(r, s, n; m)$: (a) Section 2: an inclusion-exclusion method that demonstrates $\#\text{PLR}(r, s, n; m)$ for fixed m is a symmetric polynomial of degree $3m$; (b) Section 3: a chromatic polynomial method that gives exact formulas for this symmetric polynomial, which we compute for $m \leq 13$; and (c) Section 4: an adaptation of Sade’s method (which efficiently enumerates Latin squares) to partial Latin rectangles, which enables us to determine explicitly the number $\#\text{PLR}(r, s, n; m)$ for all $r \leq s \leq n \leq 7$, and all $r \leq s \leq 6$ when $n = 8$. Section 5 describes an algebraic method for computing $\#\text{PLR}((\Theta, \pi); m)$ and also the number of isotopisms between two given partial Latin rectangles. In Section 6 we use the Orbit-Stabilizer Theorem and Burnside’s Lemma to compute the size of isomorphisms, isotopism and main classes. Section 7 describes the computational results and the implementations of the various methods. In Section 8 we comment how these computational results have been cross-checked in order to ensure their accuracy. A glossary of the most common symbols that are used throughout the paper is shown in Appendix A. To improve the readability of the paper, tables are in Appendix B.

2 Inclusion-exclusion method

For fixed $m \geq 1$, we find formulas for the size of $\text{PLR}(r, s, n; m)$ by modifying the method for enumerating partial orthomorphisms of finite cyclic groups given in [63]. At first, this is a surprising claim, as partial Latin rectangles and partial orthomorphisms are largely unrelated (unless we impose some symmetry, which we don’t in the context of this section). The similarity between these two types of objects is that both partial Latin rectangles of weight m and partial orthomorphisms with domain size m are equivalent to non-clashing m -sets of ordered triples (the difference is what constitutes a “clash”).

2.1 Generalized ordered partial Latin rectangles

Let $\mathcal{S}_m = \mathcal{S}(r, s, n; m)$ be the set of sequences $\mathbf{e} = (e_i)_{i=1}^m$, where each $e_i = (e_i[1], e_i[2], e_i[3])$ is a 3-tuple in $[r] \times [s] \times [n]$. From any $\mathbf{e} \in \mathcal{S}_m$, we construct an $r \times s$ matrix $M = M(\mathbf{e})$ as follows:

- We begin with each cell in M containing the empty multiset \emptyset .
- For $i \in [m]$, we add symbol $e_i[3]$ in the multiset in cell $(e_i[1], e_i[2])$.

For example, if $r = s = n = m = 3$ and $\mathbf{e} = ((1, 1, 1), (1, 2, 3), (1, 1, 1))$, then

$$M(\mathbf{e}) = \begin{array}{|c|c|c|} \hline \{1, 1\} & \{3\} & \emptyset \\ \hline \emptyset & \emptyset & \emptyset \\ \hline \emptyset & \emptyset & \emptyset \\ \hline \end{array}.$$

If it turns out that every non-empty multiset in M has cardinality 1 and there are no repeated elements in any row or column of M , then M is essentially a partial Latin rectangle (formally, we need to map $\emptyset \mapsto \cdot$ and $\{i\} \mapsto i$). For example, if $r = s = n = m = 3$ and $\mathbf{e} = ((1, 1, 1), (1, 2, 3), (2, 2, 2))$, then

$$M(\mathbf{e}) = \begin{array}{|c|c|c|} \hline \{1\} & \{3\} & \emptyset \\ \hline \emptyset & \{2\} & \emptyset \\ \hline \emptyset & \emptyset & \emptyset \\ \hline \end{array} \longleftrightarrow \begin{array}{|c|c|c|} \hline 1 & 3 & \cdot \\ \hline \cdot & 2 & \cdot \\ \hline \cdot & \cdot & \cdot \\ \hline \end{array}.$$

Thus, sequences in \mathcal{S}_m are generalized partial Latin rectangles consisting of m ordered entries.

Let \mathcal{A}_m be the subset of \mathcal{S}_m that gives rise to partial Latin rectangles. Hence,

$$|\mathcal{A}_m| = m! \#\text{PLR}(r, s, n; m)$$

because we can order the entries in a partial Latin rectangle in $m!$ ways. For fixed m , we define

$$f_m(r, s, n) := |\mathcal{A}_m|.$$

To find a formula for $f_m(r, s, n)$ by using inclusion-exclusion on the number of “clashes”, let

$$C_m := \{[i, j, k]: 1 \leq i < j \leq m \text{ and } k \in \{1, 2, 3\}\},$$

which we use to index the possible clashes in $\mathbf{e} \in \mathcal{S}_m$ as follows:

Clash $[i, j, 1]$: When $e_i[2] = e_j[2]$ and $e_i[3] = e_j[3]$. This would result in two copies of the same symbol in the same column in M (not necessarily in distinct cells).

Clash $[i, j, 2]$: When $e_i[1] = e_j[1]$ and $e_i[3] = e_j[3]$. This would result in two copies of the same symbol in the same row in M (not necessarily in distinct cells).

Clash $[i, j, 3]$: When $e_i[1] = e_j[1]$ and $e_i[2] = e_j[2]$. This would result in two (not necessarily distinct) symbols in the same cell in M .

Any $\mathbf{e} \in \mathcal{S}_m$ therefore has a corresponding set of clashes $C_{\mathbf{e}} \subseteq C_m$. For any $U \subseteq C_m$, define

$$\mathcal{B}_U := \{\mathbf{e} \in \mathcal{S}_m : U \subseteq C_{\mathbf{e}}\},$$

i.e., the sequences in \mathcal{S}_m that have the clashes in U (and possibly more clashes), and

$$\mathcal{D}_U := \{\mathbf{e} \in \mathcal{S}_m : U = C_{\mathbf{e}}\},$$

i.e., the sequences in \mathcal{S}_m that have precisely those clashes in U (and no more clashes). By definition,

$$\mathcal{D}_U = \mathcal{B}_U \setminus \bigcup_{\substack{V \subseteq C_m \\ V \supsetneq U}} \mathcal{B}_V.$$

Hence, by inclusion-exclusion,

$$\begin{aligned} |\mathcal{D}_U| &= |\mathcal{B}_U| - \left| \bigcup_{\substack{V \subseteq C_m \\ V \supsetneq U}} \mathcal{B}_V \right| \\ &= |\mathcal{B}_U| + \sum_{\substack{V \subseteq C_m \\ V \supsetneq U}} (-1)^{|V|-|U|} |\mathcal{B}_V| \\ &= \sum_{\substack{V \subseteq C_m \\ V \supseteq U}} (-1)^{|V|-|U|} |\mathcal{B}_V|. \end{aligned}$$

When $U = \emptyset$, we have $|\mathcal{D}_U| = |\mathcal{A}_m|$ and consequently the following lemma.

Lemma 2.1. *For all $m, r, s, n \geq 1$, we have*

$$f_m(r, s, n) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|.$$

2.2 Graph colorings

Our next goal is to find an equation for $|\mathcal{B}_V|$ in terms of the number of vertex colorings of an edge-colored graph, satisfying some additional constraints (neither vertex colorings nor edge colorings are required to be proper in the ordinary sense). Given $V \subseteq C_m$, we define a graph $G = G(V)$ with an edge coloring $\delta = \delta(V)$ by the following process. We start with the null graph on the vertex set $[m]$, and for each $[i, j, k] \in V$:

- I: If $k = 1$, then add a red edge between i and j .
- II: If $k = 2$, then add a blue edge between i and j .
- III: If $k = 3$, then add a green edge between i and j .
- IV: Replace any parallel edges resulting from I–III with a single black edge.

We denote the graph together with its edge coloring generated from V by $(G, \delta)_V$. An example of an edge-colored graph generated in this way is given in Figure 1.

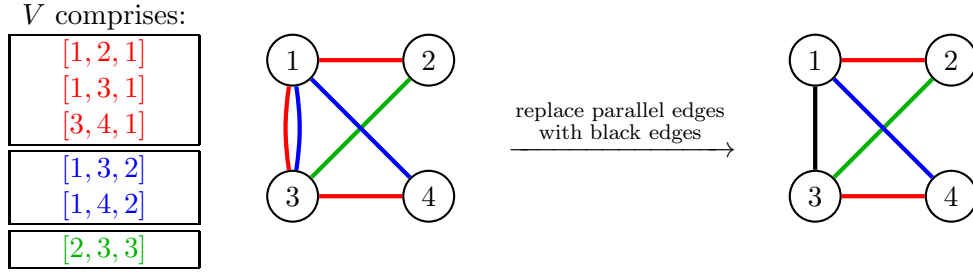


Figure 1: An example of the graph $G = G(V)$, and its edge coloring $\delta = \delta(V)$ (right), for the set of clashes $V \subseteq C_4$.

Sequences $\mathbf{e} \in \mathcal{B}_V$ are equivalent to a special type of vertex coloring ϕ of $(G, \delta)_V$, for which we assign to vertex $i \in [m]$ the color

$$(\phi_1(i), \phi_2(i), \phi_3(i)) := (e_i[1], e_i[2], e_i[3]) \in [r] \times [s] \times [n].$$

This coloring satisfies the properties:

- If there is a red edge between vertices i and j , then $\phi_2(i) = \phi_2(j)$ and $\phi_3(i) = \phi_3(j)$.
- If there is a blue edge between vertices i and j , then $\phi_1(i) = \phi_1(j)$ and $\phi_3(i) = \phi_3(j)$.
- If there is a green edge between vertices i and j , then $\phi_1(i) = \phi_1(j)$ and $\phi_2(i) = \phi_2(j)$.
- If there is a black edge between vertices i and j , then $\phi_1(i) = \phi_1(j)$, $\phi_2(i) = \phi_2(j)$ and $\phi_3(i) = \phi_3(j)$.

We call such a vertex coloring of $(G, \delta)_V$ *suitable*. Conversely, any suitable vertex coloring of $(G, \delta)_V$ with the vertex color set $[r] \times [s] \times [n]$ that satisfies the above four properties is a member of \mathcal{B}_V , thus giving the following lemma.

Lemma 2.2. *For all $V \subseteq C_m$, the set \mathcal{B}_V is the set of suitable vertex colorings of $(G, \delta)_V$, hence $|\mathcal{B}_V|$ is the number of suitable proper vertex colorings of $(G, \delta)_V$.*

We can find a simple formula (Lemma 2.3) for the number of suitable colorings of $(G, \delta)_V$ since each of the three coordinates can be accounted for separately. Let H_1 , H_2 , and H_3 respectively be the graphs formed by deleting the red, blue, and green edges from $(G, \delta)_V$, then ignoring the edge colors. For any graph H , let $c(H)$ denote the number of connected components in H .

Lemma 2.3. *The number of proper suitable colorings of $(G, \delta)_V$ is*

$$|\mathcal{B}_V| = r^{c(H_1)} s^{c(H_2)} n^{c(H_3)}.$$

Proof. In order to be a suitable vertex coloring, the vertices in each component of H_1 must be assigned colors in G that agree at the first coordinate. We can thus assign the first coordinates of a suitable vertex coloring in $r^{c(H_1)}$ ways. Similar claims hold for H_2 and H_3 . ■

We are now ready to make the following fundamental observation about the polynomials f_m .

Theorem 2.4. *For fixed m , we have that $f_m = f_m(r, s, n)$ is given by a 3-variable symmetric polynomial with integer coefficients of degree $3m$.*

Proof. Lemmas 2.1 and 2.3 imply that $f_m(r, s, n) = \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V|$ where $|\mathcal{B}_V|$ is given by $r^{c(H_1)} s^{c(H_2)} n^{c(H_3)}$ for the graph $(G, \delta)_V$. This ensures that $f_m(r, s, n)$ is a polynomial in variables r, s, n and has integer coefficients. The leading term is $(rsn)^m$, which arises when $V = \emptyset$; for all other $V \subseteq C_m$, we see $|\mathcal{B}_V|$ has degree less than $3m$. Finally, to verify that $f_m(r, s, n)$ is a symmetric polynomial, we observe that we can permute the colors red, blue, and green (or equivalently, permute the third coordinate of the elements in C_m). Each equivalence class under this action contributes

$$\begin{aligned} & r^{c(H_1)} s^{c(H_2)} n^{c(H_3)} + r^{c(H_1)} s^{c(H_3)} n^{c(H_2)} + r^{c(H_2)} s^{c(H_1)} n^{c(H_3)} \\ & + r^{c(H_2)} s^{c(H_3)} n^{c(H_1)} + r^{c(H_3)} s^{c(H_1)} n^{c(H_2)} + r^{c(H_3)} s^{c(H_2)} n^{c(H_1)} \end{aligned}$$

to the sum in Lemma 2.1, which is symmetric. We conclude that $f_m(r, s, n)$ is the sum of symmetric polynomials, and is also symmetric. ■

2.3 A simplified equation

For a 4-edge-colored graph (G, δ) , with possible edge colors red, blue, green, and black, let $|G|$ be the number of vertices in G , let $|E(G)|$ be the number of edges in G , and let $b(\delta)$ be the number of black edges in δ . There are $4^{|b(\delta)|}$ sets $V \subseteq C_m$ for which $(G, \delta) = (G, \delta)_V$, since a black edge can be formed in 4 possible ways: (a) when exactly two of properties I, II and III hold, or (b) when all three of properties I, II and III hold. From Lemmas 2.1 and 2.3, we have

$$\begin{aligned} f_m(r, s, n) &= \sum_{V \subseteq C_m} (-1)^{|V|} |\mathcal{B}_V| \\ &= \sum_{\substack{(G, \delta) \\ |G|=m}} \sum_{\substack{V \subseteq C_m: \\ G(V)=G, \\ \delta(V)=\delta}} (-1)^{|V|} |\mathcal{B}_V| \\ &= \sum_{\substack{(G, \delta) \\ |G|=m}} r^{c(H_1)} s^{c(H_2)} n^{c(H_3)} \sum_{\substack{V \subseteq C_m: \\ G(V)=G, \\ \delta(V)=\delta}} (-1)^{|V|}. \end{aligned}$$

From here, we use the following identity from [63]: For any (G, δ) , we have

$$\begin{aligned} \sum_{\substack{V \subseteq C_m: \\ G(V) = G, \\ \delta(V) = \delta}} (-1)^{|V|} &= \sum_{x \geq 0} \binom{b(\delta)}{x} (-1)^{|E(G)| + b(\delta) + x} 3^{b(\delta) - x} \\ &= (-1)^{|E(G)|} (-2)^{b(\delta)} \end{aligned}$$

using the Binomial Theorem. The local variable x counts the number of black edges where I, II and III all hold. This yields the following theorem:

Theorem 2.5. *For all $r, s, n, m \geq 1$, we have*

$$f_m(r, s, n) = \sum_{\substack{(G, \delta) \\ |G| = m}} (-1)^{|E(G)|} (-2)^{b(\delta)} r^{c(H_1)} s^{c(H_2)} n^{c(H_3)}.$$

The advantage of Theorem 2.5 is that it eliminates the need for accounting for clashes (via the variable V). Instead, we are now working solely with graphs. For computational purposes, it is easier to work with isomorphism classes of graphs (rather than labeled graphs).

We will also account for isolated vertices mathematically. For $v \geq 0$ and $e \geq 0$, let $\Gamma_{e,v}$ denote the set of unlabeled e -edge v -vertex graphs without isolated vertices (the set $\Gamma_{0,0}$ contains the empty graph, whereas $\Gamma_{e,1} = \emptyset$). We can split Theorem 2.5 according to e, v , and $\Gamma_{e,v}$ to give the following theorem.

Theorem 2.6. *For all $r, s, n, m \geq 1$, we have*

$$f_m(r, s, n) = (rsn)^m + \sum_{v \geq 2} \binom{m}{v} (rsn)^{m-v+1} \sum_{e \geq 1} (-1)^e \sum_{G \in \Gamma_{e,v}} \frac{v!}{|\text{Aut}(G)|} P(G)$$

where

$$P(G) = P(G; r, s, n) := \sum_{\delta} (-2)^{b(\delta)} r^{c(H_1)-1} s^{c(H_2)-1} n^{c(H_3)-1}$$

where the sum is over all edge colorings δ of G .

Proof. Given a graph $G \in \Gamma_{e,v}$, there are $\binom{m}{v} \frac{v!}{|\text{Aut}(G)|}$ labeled graphs on the vertex set $[m]$ that are isomorphic to G together with $m - v$ isolated vertices. Thus,

$$\begin{aligned} f_m(r, s, n) &= \sum_{\substack{(G, \delta) \\ |G| = m}} (-1)^{|E(G)|} (-2)^{b(\delta)} r^{c(H_1)} s^{c(H_2)} n^{c(H_3)} \\ &= \sum_{\substack{G \in \Gamma_{e,v} \\ v \geq 0 \\ e \geq 0}} \sum_{\delta} \binom{m}{v} \frac{v!}{|\text{Aut}(G)|} (-1)^e (-2)^{b(\delta)} (rsn)^{m-v} r^{c(H_1)} s^{c(H_2)} n^{c(H_3)}. \end{aligned}$$

We obtain the theorem by rearranging this equation. ■

The following corollary follows straightforwardly from Theorem 2.6.

Corollary 2.7. *For fixed $m \geq 1$, the polynomial $f_m(r, s, n)$ is divisible by rsn .*

Furthermore, we use the next result to reduce the required computation.

Lemma 2.8. *Let G_1 and G_2 be two graphs. Then,*

1. *if both graphs are disjoint, then, $P(G_1 \cup G_2) = rsn P(G_1)P(G_2)$; and*
2. *if both graphs meet at a single vertex, then $P(G_1 \cup G_2) = P(G_1)P(G_2)$.*

Finally, the following lemma is useful for finding which graphs have to be included when computing the leading terms in $f_m(r, s, n)$.

Lemma 2.9. *For any graph G on v vertices, the degree of $(rsn)^{m-v+1}P(G)$ in Theorem 2.6 is at most $3m - 2v + 2c(G)$.*

Proof. From Theorem 2.6, the degree of $(rsn)^{m-v+1}P(G)$ is at most $3m - 3v + \max_{\delta}(c(H_1) + c(H_2) + c(H_3))$. Let us show, by induction on the number of edges, that

$$c(H_1) + c(H_2) + c(H_3) \leq v + 2c(G) \tag{2.1}$$

for any 4-edge-coloring δ (with equality when all the edges are red, say). If G has no edges, then we have equality in (2.1). Next, assume (2.1) holds for some 4-edge-coloring, and add a colored edge xy . Adding this edge will not increase $c(H_1)$, $c(H_2)$, and $c(H_3)$, so (2.1) continues to hold unless possibly if adding xy affects $c(G)$. Adding xy decreases $c(G)$ by 1 if and only if x and y belong to separate components of G . In this case, x and y also belong to separate components of H_1 , H_2 , and H_3 . If xy is a red edge, then $c(H_2)$ and $c(H_3)$ both decrease by 1, and (2.1) holds for the new graph. The same argument works if xy is a blue or green edge. If xy is a black edge, then $c(H_1)$, $c(H_2)$, and $c(H_3)$ all decrease by 1, and (2.1) holds for the new graph. ■

3 Chromatic polynomial method

Let $R_{r,s}$ be the $r \times s$ rook's graph, i.e., the Cartesian product of K_r and K_s . The graph $R_{3,4}$ is drawn in Figure 2. Any partial Latin rectangle in $\text{PLR}(r, s, n; m)$ can be interpreted as a proper n -coloring of an m -vertex induced subgraph of $R_{r,s}$. An example of this correspondence is also given in Figure 2.

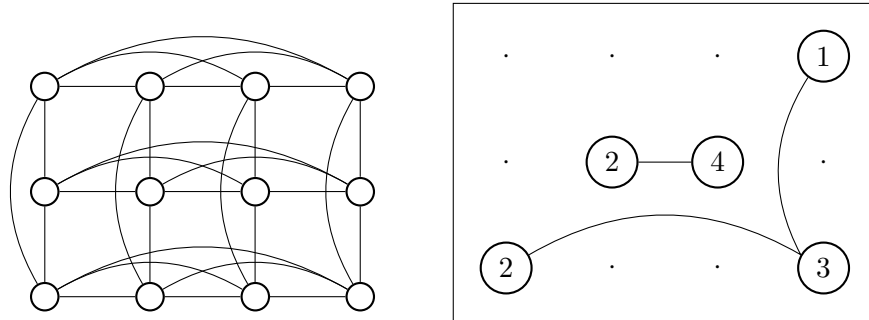


Figure 2: The graph $R_{3,4}$ along with a proper 4-coloring of an induced 5-vertex subgraph of $R_{3,4}$. This illustrates the corresponding partial Latin rectangle in $\text{PLR}(3, 4, 4; 5)$.

We can naturally think of (labeled) induced subgraphs of $R_{r,s}$ as $(0, 1)$ -matrices, with a 1 in cell (i, j) whenever vertex (i, j) is present. Under this equivalence, we talk of the *rows* and *columns*

of such graphs, and of the chromatic polynomial of their corresponding induced subgraphs. If Π denotes this chromatic polynomial, then

$$\#\text{PLR}(r, s, n; m) = \sum_M \Pi(M; n) \quad (3.1)$$

where the sum is over all $r \times s$ $(0, 1)$ -matrices M with exactly m ones (or equivalently over all m -vertex induced subgraphs of $R_{r,s}$).

Given any $r \times s$ $(0, 1)$ -matrix M , we define a *general block* of M to be a submatrix H in which: (a) every row and every column of H contains a 1; (b) in M , there are no 1's in the rows of H outside of H ; and (c) in M , there are no 1's in the columns of H outside of H . We define a *block* as a general block which has no proper submatrix which is general block in itself. Blocks correspond to components of the induced subgraph of $R_{r,s}$.

We act on the set of $r \times s$ $(0, 1)$ -matrices by permuting rows and columns. Under this action, we choose representatives from each orbit and call them *canonical*. After that, we define a function C such that $C(M)$ is the canonical matrix in the orbit of each $(0, 1)$ -matrix M . From M , we can also construct a multiset $\{C(M_i)\}_{i=1}^k$ where M_1, M_2, \dots, M_k are the blocks of M .

In the other direction, let $\mathcal{K}_{r,s,m,k}$ denote the set of multisets $\mathbf{K} = \{K_i\}_{i=1}^k$ of canonical blocks such that (a) the number of 1's in the blocks sum to m ; (b) the number of rows in the blocks sum to $\leq r$; and (c) the number of columns in the blocks sum to $\leq s$. Given any $\mathbf{K} \in \mathcal{K}_{r,s,m,k}$, we can arrange the blocks as follows:

K_1	\emptyset	\cdots	\emptyset	\emptyset
\emptyset	K_2	\cdots	\emptyset	\emptyset
\vdots	\vdots	\ddots	\vdots	\vdots
\emptyset	\emptyset	\cdots	K_k	\emptyset
\emptyset	\emptyset	\cdots	\emptyset	\emptyset

where \emptyset denotes an all-0 submatrix, so that there are r rows and c columns. Call this matrix $M(\mathbf{K})$. If we permute the rows and columns of this matrix, we generate every $r \times s$ $(0, 1)$ -matrix M that has $\{C(M_i)\}_{i=1}^k = \mathbf{K}$ some number of times, Γ say, by the Orbit-Stabilizer Theorem. It follows from (3.1) that

$$\#\text{PLR}(r, s, n; m) = \sum_{k \geq 0} \sum_{\mathbf{K} \in \mathcal{K}_{r,s,m,k}} \frac{r!s!}{\Gamma} \Pi(M(\mathbf{K}); n). \quad (3.2)$$

If $\mathbf{K} = \{K_i\}_{i=1}^k$, then

$$\Pi(M(\mathbf{K}); n) = \prod_{i=1}^k \Pi(K_i; n)$$

since each K_i corresponds to a disjoint component in the induced subgraph of $R_{r,s}$.

Let e_{row} and e_{col} denote the number of non-empty rows and columns in the matrix $M(\mathbf{K})$, respectively. If there are ℓ distinct matrices in the multiset \mathbf{K} , let k_i , for $i \in [\ell]$, be the number of copies of the i -th distinct matrix. The elements in the stabilizer of $M(\mathbf{K})$ are those which permute the all-0 rows and columns, permute the identical blocks amongst themselves, and stabilize each K_i individually. Thus since every $K_i \in \mathbf{K}$ is canonical,

$$\Gamma = (r - e_{\text{row}})!(s - e_{\text{col}})! \left(\prod_{i=1}^k |\text{Stab}(K_i)| \right) \left(\prod_{i=1}^{\ell} k_i! \right). \quad (3.3)$$

The stabilizer of a $(0, 1)$ -matrix $M = (M_{ij})$ under row and column permutations is isomorphic to the automorphism group of the vertex-colored bipartite graph G_M with vertex set

$$\overbrace{\{\mathbf{r}_1, \dots, \mathbf{r}_r\}}^{\text{color 1}} \cup \overbrace{\{\mathbf{c}_1, \dots, \mathbf{c}_s\}}^{\text{color 2}}$$

and edges $\mathbf{r}_i \mathbf{c}_j$ if and only if $M_{ij} = 1$.

To compute $\#\text{PLR}(r, s, n; m)$ for small m , we thus:

- Generate a list of possible blocks K with up to m ones, inequivalent under row and column permutations, and compute, for each block, the size of $\text{Aut}(G_K)$, and the chromatic polynomial of K . Table 1 lists the results of this computation for $m \leq 5$.
- Iterate through each $\mathbf{K} \in \cup_{k \geq 0} \mathcal{K}_{r,s,m,k}$, computing its contribution to (3.2) from the table generated in the first step.

To further reduce the computation, we only store blocks with no more rows than columns. This requires the modification of (3.3) to account for transposing the blocks. By ordering the set of all blocks, a multiset $\mathbf{K} = \{K_i\}_{i=1}^k$ is equivalent to a unique ordered sequence $(K_i)_{i=1}^k$. We use a $(0, 1)$ -sequence $(t_i)_{i=1}^k$ to keep track of which K_i we transpose, with 1 meaning “transpose” and 0 meaning “don’t transpose”. We define $(t_i)_{i=1}^k$ as *good* if (a) $t_i = 0$ whenever K_i is a square matrix, and (b) if $K_i = K_{i+1}$ and $t_i = 0$, then $t_{i+1} = 0$. We choose not to transpose square matrices at this stage, as it adds the task of identifying when the transpose of a matrix can be formed by permuting its rows and columns, hence we have condition (a). Condition (b) prevents overcounting in cases such as

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Define

$$\overline{K}_i = \begin{cases} K_i & \text{if } t_i = 0, \\ (K_i)^T & \text{if } t_i = 1. \end{cases}$$

Thus, (3.2) can be rephrased to give the following theorem.

Theorem 3.1. *For all $m \geq 1$, we have*

$$\#\text{PLR}(r, s, n; m) = \sum_{k \geq 0} \sum_{\mathbf{K} \in \mathcal{K}_{r,s,m,k}} \sum_{\substack{(t_i)_{i=1}^k \\ \text{good}}} [r]_{e_{\text{row}}} [s]_{e_{\text{col}}} \frac{\prod_{i=1}^k \Pi(\overline{K}_i; n)}{\left(\prod_{i=1}^k |\text{Aut}(G_{K_i})| \right) \left(\prod_{i=1}^{\ell} k_i! \right)}$$

where $[r]_{e_{\text{row}}} = r! / (r - e_{\text{row}})!$ and $[s]_{e_{\text{col}}} = s! / (s - e_{\text{col}})!$.

4 Sade’s method

Sade’s method [55] outstrips all other methods for finding the number of Latin squares [58]. Subsequent authors [9, 50, 49, 70] who found the number of Latin squares of orders $n \in \{8, 9, 10, 11\}$ implemented optimized computerized versions of Sade’s method. We generalize Sade’s method to partial Latin rectangles:

Lemma 4.1. *Let $L, M \in \text{PLR}(r, s, n)$. If*

1. *L and M have the same set of symbols in each column, or*
2. *L is isotopic to M ,*

then they can be extended in the same number of ways to $(r + 1) \times s$ partial Latin rectangles of a given weight m by adding an $(r + 1)$ -th row.

Proof. In the first case, the possible $(r + 1)$ -th rows for L and M are the same. In the second case, if $M = L^\Theta$, then the possible $(r + 1)$ -th rows of L^Θ are precisely the possible $(r + 1)$ -th rows of L after applying Θ . ■

Let $L, M \in \text{PLR}(r, s, n)$. We say that L and M are *Sade equivalent* if L is isotopic to a partial Latin rectangle L' such that the columns of L' and M have the same sets of symbols. Particularly, L and M must have the same weight. Thus, for instance, the following four partial Latin rectangles in $\text{PLR}(2, 3, 3; 4)$ are Sade equivalent.

$$\begin{bmatrix} 1 & 3 & 2 \\ \cdot & 2 & \cdot \end{bmatrix}, \quad \begin{bmatrix} 3 & 1 & 2 \\ 2 & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 & 2 \\ 2 & \cdot & \cdot \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 3 & \cdot \\ 1 & \cdot & 2 \end{bmatrix}.$$

Practically, we need a fast method for checking whether a large number of partial Latin rectangles are Sade equivalent. To this end, for each partial Latin rectangle $L \in \text{PLR}(r, s, n)$, we perform the following steps, which we illustrate for this example $\text{PLR}(3, 4, 3; 6)$:

$$\begin{bmatrix} 1 & \cdot & 2 & \cdot \\ 2 & 1 & 3 & \cdot \\ \cdot & \cdot & \cdot & 3 \end{bmatrix}. \tag{4.1}$$

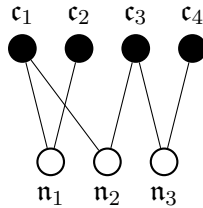
1. Construct a vertex-colored bipartite graph with vertex set

$$\overbrace{\{\mathbf{c}_1, \dots, \mathbf{c}_s\}}^{\text{color 1}} \cup \overbrace{\{\mathbf{n}_1, \dots, \mathbf{n}_n\}}^{\text{color 2}}$$

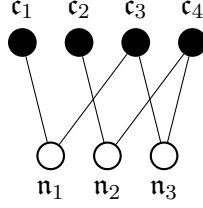
and edge set

$$\{\mathbf{c}_i \mathbf{n}_j : \text{symbol } j \text{ occurs in column } i \text{ in } L\}.$$

In our running example (4.1), we obtain:



2. Canonically label the graph in a way that preserves the vertex colors (to this end we use `nauty` [46], for which such a labeling is an internal procedure). In our running example, we obtain:



3. Find the submatrix of the adjacency matrix formed by the rows indexed by $\{c_i\}_{i=1}^s$ and columns indexed $\{n_i\}_{i=1}^n$, and read it as a binary number. We call the result the *Sade number*, denoted sn_L , of the partial Latin rectangle L . In our running example, we obtain:

0	0	0	0	1	0	0
0	0	0	0	0	1	0
0	0	0	0	1	0	1
0	0	0	0	0	1	1
1	0	1	0	0	0	0
0	1	0	1	0	0	0
0	0	1	1	0	0	0

which gives the Sade number of (4.1) as 100010101011 in binary, or 2219 in decimal.

We describe how to implement Sade’s method for partial Latin rectangles in Algorithm 1. Along with a partial Latin rectangle L itself, we store its Sade number sn_L and the number of Sade equivalent partial Latin rectangles, which we call the *Sade multiplier* sm_L . For each $i \in \{0, 1, \dots, s\}$, we maintain a database of Sade inequivalent $i \times s$ partial Latin rectangles $PLRs[i]$. We compute $PLRs[i]$ by extending the partial Latin rectangles in $PLRs[i - 1]$ in all possible ways, then filtering out Sade equivalent partial Latin rectangles.

Algorithm 1 Sade’s method for partial Latin rectangles

- 1: Set $PLRs[0] = \langle (L^0, 0, 1) \rangle$ where L^0 is the $0 \times s$ partial Latin rectangle
 - 2: **for** i from 1 to s **do**
 - 3: Set $PLRs[i] = \langle \rangle$
 - 4: **for all** $(L, sn_L, sm_L) \in PLRs[i - 1]$ **do**
 - 5: **for all** extensions L^{ext} of L to an $i \times s$ partial Latin rectangle **do**
 - 6: Compute its Sade number $sn_{L^{\text{ext}}}$
 - 7: Binary search for $sn_{L^{\text{ext}}}$ among the Sade numbers in $PLRs[i]$
 - 8: **if** $sn_{L^{\text{ext}}}$ is found **then**
 - 9: We have $sn_{L^{\text{ext}}} = sn_M$ for some $(M, sn_M, sm_M) \in PLRs[i]$
 - 10: Increase sm_M by sm_L
 - 11: **else**
 - 12: Insert $(L^{\text{ext}}, sn_{L^{\text{ext}}}, sm_L)$ into $PLRs[i]$
 - 13: **end if**
 - 14: **end for**
 - 15: **end for**
 - 16: **end for**
-

Importantly, $PLRs[i]$ is sorted according to Sade numbers. This enables the use of binary search when checking for equivalent partial Latin rectangles, thereby greatly reducing the number

of pairwise equivalence comparisons we need to make. We iterate through extensions of partial Latin rectangles by use of a backtracking algorithm.

Other practical improvements can be made:

- In enumerating up to 8×8 partial Latin rectangles, the Sade number will be less than 2^{64} , which can thus be stored as 64-bit unsigned integers.
- When processing the last few rows, we can forgo Sade's method and instead use a simple backtracking algorithm to count the number of extensions up to completions of each partial Latin rectangle.
- Although $\#\text{PLR}(r, s, n; m) = \#\text{PLR}(r, n, s; m)$, it is significantly faster to compute the value of $\#\text{PLR}(r, s, n; m)$ when $s \leq n$.

5 Algebraic geometry method

In this section, we review how sets of partial Latin rectangles are identified with the algebraic sets of certain ideals. This follows the idea of Bayer [10] and Adams and Loustanaunau [2] to solve the problem of n -coloring a graph by means of algebraic geometry, since every Latin square of order n is equivalent to an n -colored bipartite graph $K_{n,n}$ [44]. Much more recently, this algebraic method has been adapted to solve sudokus [6, 34, 56], enumerate quasigroup rings derived from partial Latin squares [20], enumerate partial Latin rectangles that admit a given autotopism [25, 26, 27, 28] or autoparatopism [32], and also the number of isotopisms between two given partial Latin rectangles [30], thereby enabling us to compute in Section 6 the numbers of equivalence classes by means of the Orbit-Stabilizer Theorem and Burnside's Lemma. See [15, 43] for more details on algebraic geometry.

Let $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field \mathbb{K} . An *ideal* of $\mathbb{K}[\mathbf{x}]$ is any subset $I \subseteq \mathbb{K}[\mathbf{x}]$ that (a) contains the zero polynomial; (b) is closed under polynomial addition; and (c) is closed under multiplication by polynomials in $q \in \mathbb{K}[\mathbf{x}]$. The ideal generated by a finite set of polynomials $\{p_1, \dots, p_m\} \subseteq \mathbb{K}[\mathbf{x}]$ is defined as

$$\langle p_1, \dots, p_m \rangle := \{p \in \mathbb{K}[\mathbf{x}] : p = \sum_{i=1}^m q_i p_i, \text{ where } q_i \in \mathbb{K}[\mathbf{x}], \text{ for all } i \leq m\}.$$

The *algebraic set* of I is the set of points

$$\mathcal{V}(I) := \{(a_1, \dots, a_n) \in k^n : p(a_1, \dots, a_n) = 0, \text{ for all } p \in I\}.$$

For this paper, interest in this topic arises from the polynomial ring $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_{111}, \dots, x_{rsn}]$ and the ideal

$$\begin{aligned} I_{r,s,n;m} := & \langle x_{ijk}^2 - x_{ijk} : (i, j, k) \in [r] \times [s] \times [n] \rangle \\ & + \langle x_{ijk} x_{i'jk} : (i, j, k) \in [r] \times [s] \times [n], i' \in [r], i < i' \rangle \\ & + \langle x_{ijk} x_{ij'k} : (i, j, k) \in [r] \times [s] \times [n], j' \in [s], j < j' \rangle \\ & + \langle x_{ijk} x_{ijk'} : (i, j, k) \in [r] \times [s] \times [n], k' \in [n], k < k' \rangle \\ & + \langle m - \sum_{i \in [r]} \sum_{j \in [s]} \sum_{k \in [n]} x_{ijk} \rangle. \end{aligned}$$

There is a bijection between partial Latin rectangles $L = (l_{ij}) \in \text{PLR}(r, s, n; m)$ and elements of the algebraic set of $I_{r,s,n;m}$: we have $l_{ij} = k$ whenever $x_{ijk} = 1$, and l_{ij} is undefined otherwise. More specifically:

- Having $x_{ijk}^2 - x_{ijk} = 0$ implies that the algebraic set is contained in $\{0, 1\}^{rsn}$.
- Having $x_{ijk}x_{i'jk} = 0$ implies that the symbol k does not appear twice in the column j .
- Having $x_{ijk}x_{ij'k} = 0$ implies that the symbol k does not appear twice in the row i .
- Having $x_{ijk}x_{ijk'} = 0$ implies that there is at most one symbol in the cell (i, j) .
- Having $\sum_{i \in [r]} \sum_{j \in [s]} \sum_{k \in [n]} x_{ijk} = m$ implies that the weight of the partial Latin rectangle L is m .

Since the algebraic set $\mathcal{V}(I_{r,s,n;m})$ is finite and the ideal $I_{r,s,n;m} \cap \mathbb{Q}[x_{ijk}]$ is generated by the polynomial $x_{ijk}^2 - x_{ijk}$, which is contained in $I_{r,s,n;m}$, Seidenberg's Lemma and [43, Theorem 3.7.19] imply

$$\#\text{PLR}(r, s, n; m) = |\mathcal{V}(I_{r,s,n;m})| = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{r,s,n;m}).$$

This fact has recently been used in [27] to compute $\#\text{PLR}(r, s, n; m)$, for all $r, s, n \leq 6$ and $m \leq rs$.

This algebraic geometry enumeration method can be generalized to include cases in which a certain autoparatopism is imposed as follows.

Theorem 5.1. *Let $\Theta = (\delta_1, \delta_2, \delta_3) \in \mathfrak{I}_{r,s,n}$ and $\pi \in S_3$. Define*

$$I_{(\Theta, \pi); m} := I_{r,s,n;m} + \langle x_{i_1 i_2 i_3} - x_{\delta_{\pi(1)}(i_{\pi(1)}) \delta_{\pi(2)}(i_{\pi(2)}) \delta_{\pi(3)}(i_{\pi(3)})} : i_1 \in [r], i_2 \in [s], i_3 \in [n] \rangle.$$

Then, the set $\text{PLR}((\Theta, \pi); m)$ has a natural bijection with $\mathcal{V}(I_{(\Theta, \pi); m})$ and

$$\#\text{PLR}((\Theta, \pi); m) = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{(\Theta, \pi); m}).$$

Proof. Since $\mathcal{V}(I_{(\Theta, \pi); m}) \subseteq \mathcal{V}(I_{r,s,n;m})$, any point in $\mathcal{V}(I_{(\Theta, \pi); m})$ corresponds to a partial Latin rectangle $L = (l_{i_1 i_2}) \in \text{PLR}(r, s, n; m)$ as previously described. The addition of the polynomials $x_{i_1 i_2 i_3} - x_{\delta_{\pi(1)}(i_{\pi(1)}) \delta_{\pi(2)}(i_{\pi(2)}) \delta_{\pi(3)}(i_{\pi(3)})}$ implies that if $l_{i_1 i_2}$ is defined and $l_{i_1 i_2} = i_3$, then $l_{\delta_{\pi(1)}(i_{\pi(1)}) \delta_{\pi(2)}(i_{\pi(2)})} = \delta_{\pi(3)}(i_{\pi(3)})$. Thus, (Θ, π) is an autoparatopism of L . ■

A similar approach enables us to determine the set of isotopisms between two partial Latin rectangles as follows.

Theorem 5.2. *Let $\mathbb{Q}[\mathbf{x}] = \mathbb{Q}[x_{11}, \dots, x_{rr}, y_{11}, \dots, y_{ss}, z_{11}, \dots, z_{nn}]$ be a polynomial ring in $r^2 + s^2 + n^2$ variables. The set $\mathfrak{I}(P, Q)$ of isotopisms between two partial Latin rectangles $P = (p_{ij})$ and $Q = (q_{ij})$ in $\text{PLR}(r, s, n)$ has a natural bijection with the algebraic set of the ideal*

$$\begin{aligned} I_{P,Q} = & \langle x_{ij}^2 - x_{ij} : i, j \in [r] \rangle + \langle y_{ij}^2 - y_{ij} : i, j \in [s] \rangle + \langle z_{ij}^2 - z_{ij} : i, j \in [n] \rangle \\ & + \langle 1 - \sum_{j \in [r]} x_{ij} : i \in [r] \rangle + \langle 1 - \sum_{i \in [r]} x_{ij} : j \in [r] \rangle \\ & + \langle 1 - \sum_{j \in [s]} y_{ij} : i \in [s] \rangle + \langle 1 - \sum_{i \in [s]} y_{ij} : j \in [s] \rangle \\ & + \langle 1 - \sum_{j \in [n]} z_{ij} : i \in [n] \rangle + \langle 1 - \sum_{i \in [n]} z_{ij} : j \in [n] \rangle \\ & + \langle x_{ik} y_{jl} (z_{p_{ij} q_{kl}} - 1) : i, k \in [r], j, l \in [s], \text{ such that } p_{ij}, q_{kl} \in [n] \rangle \\ & + \langle x_{ik} y_{jl} : i, k \in [r], j, l \in [s], p_{ij} \text{ and/or } q_{kl} \text{ is undefined} \rangle. \end{aligned}$$

Consequently, the number of isotopisms from P to Q is given by

$$\#\mathfrak{I}(P, Q) = \dim_{\mathbb{Q}}(\mathbb{Q}[\mathbf{x}]/I_{P,Q}).$$

Proof. The first three subideals of $I_{P,Q}$ imply $\mathcal{V}(I_{P,Q}) \subseteq \{0, 1\}^{r^2+s^2+n^2}$. Every isotopism $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$ uniquely corresponds to a zero $(x_{11}^{\Theta}, \dots, x_{rr}^{\Theta}, y_{11}^{\Theta}, \dots, y_{ss}^{\Theta}, z_{11}^{\Theta}, \dots, z_{nn}^{\Theta})$ in $\mathcal{V}(I_{P,Q})$, where $x_{ij}^{\Theta} = 1$, (resp., $y_{ij}^{\Theta} = 1$ and $z_{ij}^{\Theta} = 1$) if $\alpha(i) = j$ (resp., $\beta(i) = j$ and $\gamma(i) = j$) and 0, otherwise. Specifically, the fourth and fifth subideals of $I_{P,Q}$ (in the statement of Theorem 5.2) imply α is a permutation of S_r (the fourth one ensures the injectivity, while the fifth one ensures surjectivity). Similarly, the next two pairs of subideals imply β and γ are permutations of S_s and S_n , respectively, and hence, Θ is an isotopism of $\text{PLR}(r, s, n)$. The last two subideals imply Θ is a bijection between the entry sets $E(P)$ and $E(Q)$. Further, since $I_{P,Q} \cap \mathbb{Q}[x_{ijk}] = \langle x_{ijk}^2 - x_{ijk} \rangle \subseteq I_{P,Q}$, Seidenberg's Lemma and Theorem 3.7.19 in [43] imply the theorem statement. ■

6 Counting equivalence classes of partial Latin rectangles

Theorem 5.2 can be used to determine not only the size of the autotopism group $\mathfrak{I}(P, P)$ of a partial Latin rectangle P , but also that of its autoparatopism group $\mathfrak{P}(P, P)$, because

$$\#\mathfrak{P}(P, P) = \sum_{\pi \in S_3} \#\mathfrak{I}(P, P^{\pi}).$$

The following result shows how the computation of both values enables us to determine the size of the isotopism and main classes containing P by means of the Orbit-Stabilizer Theorem.

Theorem 6.1. *Let $P \in \text{PLR}(r, s, n)$. Then,*

1. *the number of partial Latin rectangles that are isotopic to P , i.e., the size of the isotopism class containing P , is*

$$\frac{r! s! n!}{\#\mathfrak{I}(P, P)};$$

2. *the number of partial Latin rectangles that are paratopic to P , i.e., the size of the main class containing P , is*

$$\#S_{r,s,n} \frac{r! s! n!}{\#\mathfrak{P}(P, P)};$$

and

3. *the number of isotopism classes in the main class of P is*

$$\#S_{r,s,n} \frac{\#\mathfrak{I}(P, P)}{\#\mathfrak{P}(P, P)}.$$

Proof. The first two claims follow from the Orbit-Stabilizer Theorem. For the third claim, we observe that paratopic partial Latin rectangles have autotopism groups of the same size, because Θ is an autotopism of P if and only if $\Lambda^{-1}\Theta\Lambda$ is an autotopism of P^{Λ} for any paratopism Λ . They thus also have isotopism classes of the same size, which partition the main class, so the first two claims imply the third. ■

From here on, let $\text{Isom}(n; m)$, $\text{Isot}(r, s, n; m)$ and $\text{MC}(r, s, n; m)$ respectively denote the set of isomorphism classes of $\text{PLS}(n; m)$ and the sets of isotopism and main classes of $\text{PLR}(r, s, n; m)$. The following result follows straightforwardly from Burnside's Lemma and the fact of acting the isomorphism, isotopism, and paratopism groups on the set of partial Latin rectangles of a given order.

Theorem 6.2. *Let $r, s, n \geq 1$ and $m \leq rs$. Then,*

1. *the number of isomorphism classes in $\text{PLS}(n; m)$ is*

$$\#\text{Isom}(n; m) = \frac{1}{n!} \sum_{\pi \in S_n} \#\text{PLR}(((\pi, \pi, \pi), \text{Id}); m);$$

2. *the number of isotopism classes in $\text{PLR}(r, s, n; m)$ is*

$$\#\text{Isot}(r, s, n; m) = \frac{1}{r! s! n!} \sum_{\Theta \in \mathcal{J}_{r,s,n}} \#\text{PLR}((\Theta, \text{Id}); m);$$

and

3. *the number of main classes in $\text{PLR}(r, s, n; m)$ is*

$$\#\text{MC}(r, s, n; m) = \frac{1}{r! s! n! \#\mathcal{S}_{r,s,n}} \sum_{(\Theta, \pi) \in \mathfrak{P}_{r,s,n}} \#\text{PLR}((\Theta, \pi); m).$$

In practice, it is not necessary to compute all the values $\#\text{PLR}((\Theta, \pi); m)$ in order to determine $\#\text{Isom}(n; m)$, $\#\text{Isot}(r, s, n; m)$, or $\#\text{MC}(r, s, n; m)$ in Theorem 6.2. The following lemma implies that $\#\text{PLR}((\Theta, \pi); m)$ only depends on the conjugacy class of the corresponding paratopism. Recall that two permutations $\alpha, \beta \in S_n$ are *conjugate* if there exists a third permutation $\gamma \in S_n$ such that $\alpha = \gamma^{-1}\beta\gamma$, which naturally generalizes to isotopisms and paratopisms under componentwise conjugacy. In what follows, conjugacy is denoted \sim .

Lemma 6.3. *Let (Θ_1, π_1) and (Θ_2, π_2) be two conjugate paratopisms in $\mathfrak{P}_{r,s,n}$. Then,*

$$\#\text{PLR}((\Theta_1, \pi_1); m) = \#\text{PLR}((\Theta_2, \pi_2); m).$$

Proof. Since $(\Theta_1, \pi_1) \sim (\Theta_2, \pi_2)$, there exists $(\Theta_3, \pi_3) \in \mathfrak{P}_{r,s,n}$ such that $(\Theta_2, \pi_2) = (\Theta_3, \pi_3)^{-1}(\Theta_1, \pi_1)(\Theta_3, \pi_3)$. Let $L \in \text{PLR}((\Theta_1, \pi_1); m)$. Then, $(L^{(\Theta_3, \pi_3)})^{(\Theta_2, \pi_2)} = (L^{(\Theta_1, \pi_1)})^{(\Theta_3, \pi_3)} = L^{(\Theta_3, \pi_3)}$ and so, $L^{(\Theta_3, \pi_3)} \in \text{PLR}((\Theta_2, \pi_2); m)$. The result holds because $\text{PLR}((\Theta_2, \pi_2); m) = \{L^{(\Theta_3, \pi_3)} : L \in \text{PLR}((\Theta_1, \pi_1); m)\}$. \blacksquare

The following result is shown using similar reasoning to that used by Mendis and Wanless in the proof of Theorem 2.2 in [52] for paratopisms of Latin squares.

Theorem 6.4. *Two paratopisms $((\alpha_1, \alpha_2, \alpha_3), \pi_1)$ and $((\beta_1, \beta_2, \beta_3), \pi_2)$ in $\mathfrak{P}_{r,s,n}$ are conjugate if and only if there is a length preserving bijection η from the cycles of π_1 to those of π_2 such that, if η maps a cycle (a_1, \dots, a_k) to a cycle (b_1, \dots, b_k) , both of them in the symmetric group S_3 , then $\alpha_{a_1} \cdots \alpha_{a_k} \sim \beta_{b_1} \cdots \beta_{b_k}$.*

As a consequence, any paratopism $((\alpha, \beta, \gamma), (12)) \in \mathfrak{P}_{r,r,n}$ is conjugate to $((\text{Id}, \alpha\beta, \gamma), (12)) \in \mathfrak{P}_{r,r,n}$, and any $((\alpha, \beta, \gamma), (123)) \in \mathfrak{P}_{r,r,r}$ is conjugate to both $((\text{Id}, \text{Id}, \alpha\beta\gamma), (123)) \in \mathfrak{P}_{r,r,r}$ and $((\text{Id}, \text{Id}, \alpha\beta\gamma), (132)) \in \mathfrak{P}_{r,r,r}$.

Proof. If $r = s = n$, or if $\pi_1 = \pi_2$, the proof of Theorem 2.2 in [52] suffices to prove the theorem. Otherwise, up to equivalence, we have $\pi_1 = (12)$ and $\pi_2 = \text{Id}$ and $r = s \neq n$. Clearly, η does not exist. The two paratopisms are not conjugate since conjugation in $\mathfrak{P}_{r,s,n}$ preserves the conjugacy class of the parastrophe permutation. \blacksquare

Theorem 6.5. *For any paratopism $((\alpha, \beta, \gamma), (12)) \in \mathfrak{P}_{r,r,n}$, we have*

$$\begin{aligned} & \#\text{PLR}(((\alpha, \beta, \gamma), (12)); m) = \#\text{PLR}(((\beta, \alpha, \gamma), (12)); m) \\ & = \#\text{PLR}(((\alpha, \gamma, \beta), (13)); m) = \#\text{PLR}(((\beta, \gamma, \alpha), (13)); m) \\ & = \#\text{PLR}(((\gamma, \alpha, \beta), (23)); m) = \#\text{PLR}(((\gamma, \beta, \alpha), (23)); m). \end{aligned}$$

Moreover, any $((\alpha, \beta, \gamma), (12)) \in \mathfrak{P}_{r,r,n}$ is conjugate to a paratopism $((\text{Id}, \beta', \gamma'), (12))$, where $\alpha\beta \sim \beta'$ and $\gamma \sim \gamma'$. If π is a 3-cycle, then any paratopism $((\alpha, \beta, \gamma), \pi) \in \mathfrak{P}_{r,r,r}$ is conjugate to a paratopism $((\text{Id}, \text{Id}, \gamma), (123)) \in \mathfrak{P}_{r,s,n}$, where $\alpha\beta\gamma \sim \gamma'$.

Proof. We have $((\alpha, \beta, \gamma), (12)) \in \text{Atop}(L)$ if and only if $((\alpha, \gamma, \beta), (13)) \in \text{Atop}(L^{(23)})$. Consequently, $\#\text{PLR}(((\alpha, \beta, \gamma), (12)); m) = \#\text{PLR}(((\alpha, \gamma, \beta), (13)); m)$. Lemma 6.3 implies

$$\#\text{PLR}(((\alpha, \beta, \gamma), (12)); m) = \#\text{PLR}(((\beta, \alpha, \gamma), (12)); m).$$

And we similarly prove the other equalities. Theorem 6.4 implies the conjugacy claims. \blacksquare

Conjugacy in symmetric groups constitutes an equivalence relation in which each conjugacy class is characterized by the common cycle structure of their elements. Recall that the *cycle structure* of a permutation $\pi \in S_m$ is the expression $z_\pi := m^{d_m} \cdots 1^{d_1}$, where d_i^π denotes the number of cycles of length i in the unique cycle decomposition of the permutation π . Thus, for instance, the cycle structure of the permutation $(12)(345)(78)(9)$ is $3^1 2^2 1^1$. From here on, we denote the set of cycle structures of the symmetric group S_m as \mathcal{CS}_m . The number of permutations in S_m with cycle structure $m^{d_m} \cdots 1^{d_1} \in \mathcal{CS}_m$

$$\frac{m!}{\prod_{i \in [m]} d_i! i^{d_i}}, \quad (6.1)$$

as in [5, Theorem 13.2]. Further, the *cycle structure* of an isotopism $(\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$ is defined as the triple $(z_\alpha, z_\beta, z_\gamma)$ formed by the respective cycle structures of α , β , and γ . Keeping in mind Lemma 6.3 and Theorem 6.5, the following values are well-defined:

- $\Delta_m(z_1, z_2, z_3) := \#\text{PLR}((\Theta, \text{Id}); m)$, for any $\Theta \in \mathfrak{I}_{r,s,n}$ with cycle structure (z_1, z_2, z_3) ;
- $\Delta_m^{(12)}(z_2, z_3) := \#\text{PLR}((\Theta, (12)); m)$, for any $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$ where $r = s$, such that $z_{\alpha\beta} = z_2$ and $z_\gamma = z_3$; and
- $\Delta_m^{(123)}(z_3) := \#\text{PLR}((\Theta, (123)); m)$, for any $\Theta = (\alpha, \beta, \gamma) \in \mathfrak{I}_{r,s,n}$ where $r = s = n$, such that $z_{\alpha\beta\gamma} = z_3$.

Given a cycle structure $z \in \mathcal{CS}_m$, define $d_i^z := d_i^\pi$ for any permutation $\pi \in S_m$ with cycle structure z . The next theorem follows straightforwardly from Theorem 6.2, Lemma 6.3 and (6.1).

Theorem 6.6. *Let $r, s, n \geq 1$ and $m \leq rs$. Then,*

1. the number of isomorphism classes in $\text{PLS}(n; m)$ is

$$\#\text{Isom}(n; m) = \sum_{z \in \mathcal{CS}_n} \frac{\Delta_m(z, z, z)}{\prod_{i \in [n]} d_i^{z_i} i^{d_i^{z_i}}},$$

and

2. the number of isotopism classes in $\text{PLR}(r, s, n; m)$ is

$$\#\text{Isot}(r, s, n; m) = \sum_{\substack{z_1 \in \mathcal{CS}_r \\ z_2 \in \mathcal{CS}_s \\ z_3 \in \mathcal{CS}_n}} \frac{\Delta_m(z_1, z_2, z_3)}{\prod_{\substack{i \in [r] \\ j \in [s] \\ k \in [n]}} d_i^{z_1^i} d_j^{z_2^j} d_k^{z_3^k} i^{d_i^{z_1^i}} j^{d_j^{z_2^j}} k^{d_k^{z_3^k}}}.$$

In practice, it is not necessary perform computations for all possible triples $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$ to determine the number of isotopism classes in statement 2 of Theorem 6.6. The following lemma gives necessary and sufficient conditions for $\text{PLR}((\Theta, \text{Id}))$ to contain a non-empty partial Latin rectangle. This generalizes in a natural way a pair of similar results referred to Latin squares [61, Lemma 3.6] and partial Latin squares [25, Lemma 2.2].

Lemma 6.7. *Let $\Theta \in \mathfrak{I}_{r,s,n}$ be an isotopism of cycle structure $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$. The set $\text{PLR}((\Theta, \text{Id}))$ contains at least one non-empty partial Latin rectangle if and only if there exists a triple $(i, j, k) \in [r] \times [s] \times [n]$ such that the following two conditions are satisfied:*

1. $\text{lcm}(i, j) = \text{lcm}(i, k) = \text{lcm}(j, k) = \text{lcm}(i, j, k)$, and
2. z_1 has an i -cycle, z_2 has a j -cycle, and z_3 has a k -cycle.

Parastrophisms preserve the number of isotopism and main classes of partial Latin rectangles of a given order. Thus, in practice, it is enough to focus on the case $r \leq s \leq n$ to determine the number of isotopism classes in $\text{PLR}(r, s, n; m)$, whereas the number of main classes splits into three cases: (a) $r < s < n$; (b) $r = s < n$; and (c) $r = s = n$. In (a), the parastrophism group $S_{r,s,n}$ is only formed by the trivial permutation $\text{Id} \in S_3$ and hence, the number of main classes coincides with that of isotopism classes. In order to deal with (b) and (c), and keeping in mind Theorem 6.5, let us define the following two sets for each pair of permutations $\beta, \gamma \in S_r$

$$C_1(\beta, \gamma) := \{(\alpha, \beta', \gamma') \in \mathfrak{I}_{r,r,n} : \alpha\beta' \sim \beta \text{ and } \gamma' \sim \gamma\}.$$

$$C_2(\gamma) := \{(\alpha, \beta, \gamma') \in \mathfrak{I}_{r,r,r} : \alpha\beta\gamma' \sim \gamma\}.$$

The next result holds straightforwardly from Theorem 6.2, Lemma 6.3, and (6.1).

Theorem 6.8. *Let $r, n \geq 1$ and $m \leq r^2$. The following statements hold.*

1. If $n \neq r$, then the number of main classes in $\mathcal{PLR}(r, r, n; m)$ is

$$\#\text{MC}(r, r, n; m) = \frac{\#\text{Isot}(r, r, n; m)}{2} + \frac{1}{2r!^2 n!} \sum_{(\beta, \gamma) \in S_s \times S_n} |C_1(\beta)| \Delta_m^{(12)}(z_{\beta, z_{\gamma}}).$$

2. The number of main classes in $\mathcal{PLR}(r, r, r; m)$ is

$$\begin{aligned} \#\text{MC}(r, r, r; m) &= \frac{\#\text{Isot}(r, r, r; m)}{6} + \frac{1}{2r!^3} \sum_{(\beta, \gamma) \in S_s \times S_n} |C_1(\beta)| \Delta_m^{(12)}(z_{\beta, z_{\gamma}}) \\ &\quad + \frac{1}{3r!^3} \sum_{\gamma \in S_n} |C_2(\gamma)| \Delta_m^{(123)}(z_{\gamma}). \end{aligned}$$

7 Computational results

Inclusion-exclusion method For small graphs G , Tables 2 and 3 list the polynomial $P(G) = P(G; r, s, n)$ in Theorem 2.6. These polynomials were computed using a C++ program, using **geng** (packaged with **nauty** [45, 46, 48]) to generate a list of isolated-vertex-free non-isomorphic graphs (e.g. “**geng -d1 3**” generates 3-vertex isomorphism class representatives with minimum degree 1) and **bliss** [40] to compute their automorphism group size. The notation \overline{abc} is shorthand for the sum of the monic monomials with variables r, s , and n and exponents a, b , and c . For example, $\overline{210} = r^2s + r^2n + s^2r + s^2n + n^2r + n^2s$ and $2\overline{100} = 2(r + s + n)$.

By Lemma 2.9, substituting the data in Tables 2 and 3 into the formula in Theorem 2.6 gives a formula for $f_m(r, s, n)$ containing all terms of degree $\geq 3m - 9$; unlisted graphs G have $v - c(G) \geq 5$, and thus contribute to terms in the polynomial with degree at most $3m - 10$. In this regard, the following result generalizes Theorem 4.7 in [27], which only deals with the case $m \leq 6$.

Theorem 7.1. *Let m be a positive integer. Then, $f_m(r, s, n) = (rsn)^m + \binom{m}{2}(rsn)^{m-1}(2 - \overline{100}) + \binom{m}{3}(rsn)^{m-2}(14 - 12\overline{100} + 6\overline{110} + 2\overline{200}) + \binom{m}{4}(rsn)^{m-3}(198 - 228\overline{100} + 198\overline{110} - 84\overline{111} + 72\overline{200} - 36\overline{210} - 12\overline{211} + 6\overline{221} - 6\overline{300} + 3\overline{311}) + \binom{m}{5}(rsn)^{m-4}(-6360\overline{100} + 7440\overline{110} - 6080\overline{111} + 2880\overline{200} - 2520\overline{210} + 820\overline{211} + 480\overline{220} + 360\overline{221} - 180\overline{222} - 480\overline{300} + 240\overline{310} + 160\overline{311} - 80\overline{321} + 24\overline{400} - 20\overline{411}) + \binom{m}{6}(rsn)^{m-5}(-13170\overline{211} + 17340\overline{221} - 15990\overline{222} + 7580\overline{311} - 7050\overline{321} + 3300\overline{322} + 1520\overline{331} + 180\overline{332} - 90\overline{333} - 1740\overline{411} + 870\overline{421} + 90\overline{422} - 45\overline{432} + 130\overline{511} - 15\overline{522}) + \binom{m}{7}(rsn)^{m-6}(-10920\overline{322} + 15540\overline{332} - 15120\overline{333} + 7350\overline{422} - 7140\overline{432} + 3570\overline{433} + 1680\overline{442} - 2100\overline{522} + 1050\overline{532} + 210\overline{622}) + \binom{m}{8}(rsn)^{m-7}(-3360\overline{433} + 5040\overline{443} - 5040\overline{444} + 2520\overline{533} - 2520\overline{543} + 1260\overline{544} + 630\overline{553} - 840\overline{633} + 420\overline{643} + 105\overline{733}) + \text{some polynomial of degree } \leq 3m - 10.$*

Theorem 7.1 is exact for $m \leq 5$ since Tables 2 and 3 contain all graphs with no isolated vertices with up to 5 vertices, and when $v \geq 6$, graphs make a zero contribution to Theorem 2.6 since the binomial $\binom{m}{v} = 0$.

Chromatic polynomial method We use Theorem 3.1 to compute exact formulas for $f_m(r, s, n)$ for all $m \leq 13$ which are available from [31]. They corroborate in particular the formulas shown in [27] for $m \leq 6$. The authors acknowledge the use of **GAP** [35], the **GAP** package **GRAPE** [57] (which uses **nauty**), and the Tutte polynomial software **tutte_bhkk** [11] (available from github.com/thorehusfeldt/tutte_bhkk) for these computations.

Sade’s method We implement Algorithm 1 in C++ using **nauty** for graph isomorphism and **GMP** [37] for arbitrary precision arithmetic, which we use to compute $\#\text{PLR}(r, s, n; m)$ for all $r, s, n \leq 7$, and for $r, s \leq 6$ when $n = 8$ (for all $0 \leq m \leq rs$). Our computations for $r, s, n \leq 6$ corroborate Tables 2 through 5 in [27]. The remaining cases are listed here in Tables 4 through 8.

Algebraic geometry method We implement Theorem 5.1 in **Singular** [17] and **Minion** [36] to determine the values $\Delta(z_1, z_2, z_3)$, for all $(z_1, z_2, z_3) \in \mathcal{CS}_r \times \mathcal{CS}_s \times \mathcal{CS}_n$ satisfying the conditions of Lemma 6.7, when $r, s, n \leq 6$. Theorem 6.6 has then be applied to obtain the corresponding numbers of isomorphism and isotopism classes of partial Latin rectangles, as listed in Tables 9 through 12.

The number of main classes of partial Latin rectangles in $\text{PLR}(r, s, n)$ according to their weights is given in Tables 13 and 14 when $2 \leq r \leq s \leq n \leq 6$. We include only the cases in which r, s , and n are not pairwise distinct; otherwise, the number of main classes and isotopism classes coincide.

Constructive enumeration It is also possible to enumerate constructively the number of isotopism and main classes in $\text{PLR}(r, s, n; m)$. We simply extend all representative weight- $(m - 1)$ partial Latin rectangles by one entry in all possible ways, and throw away those that belong to the same class as an already discovered partial Latin rectangle. To compare isotopism and main class equivalence, it is enough, for instance, to generate a graph similar to those proposed in [47].

For fixed $m \geq 1$, provided $r \geq m$, $s \geq m$, and $n \geq m$, the number of isotopism classes and main classes in the set $\text{PLR}(r, s, n; m)$ do not vary with r , s and n , which amounts to adding empty rows, empty columns, or unused symbols. To compute these numbers, we use the above constructive method, but allow the possibility of introducing new rows, columns, and/or symbols when extending weight- $(m - 1)$ partial Latin rectangles. We perform this enumeration for $m \leq 11$, and the results are given in Table 15. The results for main classes is consistent with those independently obtained in [18, 69], and moreover, [18] also computes the number of main classes for $m = 12$.

Direct constructive enumeration of isomorphism classes is infeasible, since the numbers grow too quickly. Moreover, isotopic partial Latin rectangles may have different-sized isomorphism classes¹, so we cannot easily derive the number of isomorphism classes within a isotopism class, which thwarts modifying the approach we use for enumerating isotopism classes to enumerating isomorphism classes. Instead, using an algebraic geometry method like in Section 5, we enumerate isomorphism classes for $m \leq 6$ in [30, Table 2].

8 Verification

The authors have made efforts to ensure the numbers and formulas presented here are as bug-free as possible; we document these efforts in this section. First, the various source codes used and their output are available from [31]. Next, where feasible, computations have been independently performed, using different techniques and different software. Where possible, we have also cross-checked the results of the enumeration methods.

- The computation of $\#\text{PLR}(r, s, n; m)$ for all $r, s, n \leq 7$ has been performed using both the algebraic geometry method (except for $(s, n) \in \{(6, 7), (7, 7)\}$) and Sade’s method.
- The number of isotopism classes and main classes has been computed using both the algebraic geometry method (for $r, s, n \leq 6$) and constructive enumeration (for $r, s, n \leq 5$).
- For $m \leq 13$, the results of the computation of $\#\text{PLR}(r, s, n; m)$ have been cross-checked against the computed polynomials $f_m(r, s, n)$. Thus for 854 quadruples $(r, s, n; m)$ the computations agreed exactly.

In addition to cross-checking computational results, we check the divisibility of the numbers computed using the following theorem. More specifically, we check the exact formulas for $f_m(r, s, n)$, for $m \leq 13$, satisfy Theorem 8.1 whenever $k \in \{1, \dots, 10\}$ and $r, s, n \in \{k + 1, \dots, k + 10\}$.

Theorem 8.1. *For all $r, s, n, m \geq 1$ and $k \geq 0$, we have*

$$\begin{aligned} \#\text{PLR}(r, s, n; m) &\equiv \#\text{PLR}(k, s, n; m) \pmod{r - k} \text{ when } r \geq k + 1, \\ \#\text{PLR}(r, s, n; m) &\equiv \#\text{PLR}(r, k, n; m) \pmod{s - k} \text{ when } s \geq k + 1, \text{ and} \\ \#\text{PLR}(r, s, n; m) &\equiv \#\text{PLR}(r, s, k; m) \pmod{n - k} \text{ when } n \geq k + 1. \end{aligned}$$

¹In the context of Latin squares, this led to [47, Th. 2(i)] being false; which is acknowledged in a corrected version of [47] on McKay’s website http://users.cecs.anu.edu.au/~bdm/papers/ls_final_corr.pdf.

Proof. Firstly, we prove the second claim. We act on $\text{PLR}(r, s, n; m)$ by permuting the columns using the group G of isotopisms generated by $(\text{Id}, \beta, \text{Id})$ for an $(m - k)$ -cycle β . By the Orbit-Stabilizer Theorem, orbits have size $m - k$ unless they contain partial Latin rectangles that admit a non-trivial autotopism in G . This is only possible if the columns permuted by β are empty. Hence, the orbits of size less than $m - k$ together form $\text{PLR}(r, k, n; m)$ by deleting the columns permuted by β . The same argument works for rows and symbols, which gives the first and third claims. ■

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A Glossary of symbols

S_n	The symmetric group on n elements.
$\mathcal{I}_{r,s,n}$	The isotopism group $S_r \times S_s \times S_n$.
$S_{r,s,n}$	The parastrophism group defined in Section 1.
$\mathfrak{P}_{r,s,n}$	The paratopism group $S_r \times S_s \times S_n \rtimes S_{r,s,n}$.
$\text{PLR}(r, s, n)$	The set of $r \times s$ partial Latin rectangles on $[n] \cup \{\cdot\}$.
$\text{PLR}(r, s, n; m)$	The subset of partial Latin rectangles in $\text{PLR}(r, s, n)$ of weight m .
$\text{PLR}((\Theta, \pi))$	The set of partial Latin rectangles having (Θ, π) as autoperatopism.
$\text{PLR}((\Theta, \pi); m)$	The subset of partial Latin rectangles in $\text{PLR}((\Theta, \pi))$ of weight m .
$\text{PLS}(n; m)$	The set of partial Latin squares of order n and weight m .
$\text{Isom}(n; m)$	The set of isomorphism classes of $\text{PLS}(n; m)$.
$\text{Isot}(r, s, n; m)$	The set of isotopism classes of $\text{PLR}(r, s, n; m)$.
$\text{MC}(r, s, n; m)$	The set of main classes of $\text{PLR}(r, s, n; m)$.

B Tables

















block K	induced subgraph	$ \text{Aut}(G_K) $	$\Pi(K; n)$
$\boxed{1}$		1	n
$\boxed{1 \ 1}$		2	$n(n-1)$
$\boxed{1 \ 1 \ 1}$		6	$n(n-1)(n-2)$
$\begin{matrix} \boxed{1 \ 1} \\ \boxed{1 \ 0} \end{matrix}$		1	$n(n-1)^2$
$\boxed{1 \ 1 \ 1 \ 1}$		24	$n(n-1)(n-2)(n-3)$
$\begin{matrix} \boxed{1 \ 1 \ 1} \\ \boxed{1 \ 0 \ 0} \end{matrix}$		2	$n(n-1)^2(n-2)$
$\begin{matrix} \boxed{1 \ 1 \ 0} \\ \boxed{1 \ 0 \ 1} \end{matrix}$		2	$n(n-1)^3$
$\begin{matrix} \boxed{1 \ 1} \\ \boxed{1 \ 1} \end{matrix}$		4	$n(n-1)(n^2-3n+3)$
$\boxed{1 \ 1 \ 1 \ 1 \ 1}$		120	$n(n-1)(n-2)(n-3)(n-4)$
$\begin{matrix} \boxed{1 \ 1 \ 1 \ 1} \\ \boxed{1 \ 0 \ 0 \ 0} \end{matrix}$		6	$n(n-1)^2(n-2)(n-3)$
$\begin{matrix} \boxed{1 \ 1 \ 1 \ 0} \\ \boxed{1 \ 0 \ 0 \ 1} \end{matrix}$		2	$n(n-1)^3(n-2)$
$\begin{matrix} \boxed{1 \ 1 \ 1} \\ \boxed{1 \ 1 \ 0} \end{matrix}$		2	$n(n-1)(n-2)(n^2-3n+3)$
$\begin{matrix} \boxed{1 \ 1 \ 1} \\ \boxed{1 \ 0 \ 0} \\ \boxed{1 \ 0 \ 0} \end{matrix}$		4	$n(n-1)^2(n-2)^2$
$\begin{matrix} \boxed{1 \ 1 \ 1} \\ \boxed{1 \ 0 \ 0} \\ \boxed{0 \ 1 \ 0} \end{matrix}$		2	$n(n-1)^3(n-2)$
$\begin{matrix} \boxed{1 \ 1 \ 0} \\ \boxed{1 \ 0 \ 1} \\ \boxed{1 \ 0 \ 0} \end{matrix}$		2	$n(n-1)^3(n-2)$
$\begin{matrix} \boxed{1 \ 1 \ 0} \\ \boxed{1 \ 0 \ 1} \\ \boxed{0 \ 1 \ 0} \end{matrix}$		1	$n(n-1)^4$

Table 1: The blocks K with ≤ 5 ones, along with $|\text{Aut}(G_K)|$ and its chromatic polynomial $\Pi(K; n)$.














G	v	e	$c(G)$	$ \text{Aut}(G) $	$P(G) = P(G; r, s, n)$
	2	1	1	2	$\overline{100} - 2$
	3	2	1	2	$P(\bullet\text{---}\bullet)^2$
	3	3	1	6	$\overline{200} - 2$
	4	2	2	8	$\overline{111} P(\bullet\text{---}\bullet)^2$
	4	3	1	6	$P(\bullet\text{---}\bullet)^3$
	4	3	1	2	$P(\bullet\text{---}\bullet)^3$
	4	4	1	2	$P(\bullet\text{---}\bullet)P(\bullet\text{---}\bullet)$
	4	4	1	8	$\overline{300} + 6\overline{110} - 12\overline{100} + 16$
	4	5	1	4	$\overline{300} + 2\overline{110} - 4\overline{100} + 4$
	4	6	1	24	$\overline{300} - 2$
	5	3	2	4	$\overline{111} P(\bullet\text{---}\bullet)^3$
	5	4	2	12	$\overline{111} P(\bullet\text{---}\bullet)P(\bullet\text{---}\bullet)$
	6	3	3	48	$\overline{222} P(\bullet\text{---}\bullet)^3$

Table 2: The polynomial $P(G)$ for v -vertex graphs G with $v - c(G) \leq 3$.

G	v	e	$c(G)$	$ \text{Aut}(G) $	$P(G) = P(G; r, s, n)$
	5	4	1	24	$P(\bullet\bullet)^4$
	5	4	1	2	$P(\bullet\bullet)^4$
	5	5	1	4	$P(\triangle)P(\bullet\bullet)^2$
	5	5	1	2	$P(\curvearrowright)P(\bullet\bullet)$
	5	5	1	2	$P(\triangle)P(\bullet\bullet)^2$
	5	6	1	2	$P(\curvearrowright)P(\bullet\bullet)$
	5	6	1	12	$P(\curvearrowright)P(\bullet\bullet)$
	5	7	1	12	$\overline{400} + 3\overline{210} + 6\overline{111} + 3\overline{211} - 6\overline{200} - 12\overline{110} + 12\overline{100} - 8$
	5	4	1	2	$P(\bullet\bullet)^4$
	5	5	1	2	$P(\triangle)P(\bullet\bullet)^2$
	5	6	1	8	$P(\triangle)^2$
	5	5	1	10	$\overline{400} + 10\overline{210} - 20\overline{200} - 30\overline{110} + 40\overline{100} - 32$
	5	6	1	2	$\overline{400} + 4\overline{210} + 4\overline{211} - 8\overline{200} - 6\overline{110} + 4\overline{110} + 4$
	5	7	1	2	$\overline{400} + 2\overline{210} - 4\overline{200} - 2\overline{110} + 4$
	5	6	1	2	$P(\curvearrowright)P(\bullet\bullet)$
	5	7	1	6	$P(\curvearrowright)P(\bullet\bullet)$
	5	8	1	4	$\overline{400} + \overline{210} - 2\overline{200} - 2\overline{100} + 4$
	5	7	1	4	$\overline{400} + \overline{210} - 2\overline{200} + 4\overline{110} - 10\overline{100} + 16$
	5	8	1	8	$\overline{400} + 4\overline{110} - 8\overline{100} + 10$
	5	9	1	12	$\overline{400} + 2\overline{110} - 4\overline{100} + 4$
K_5	5	10	1	120	$\overline{400} - 2$
	6	4	2	12	$\overline{111}P(\bullet\bullet)^4$
	6	4	2	8	$\overline{111}P(\bullet\bullet)^4$
	6	4	2	4	$\overline{111}P(\bullet\bullet)^4$
	6	5	2	4	$\overline{111}P(\triangle)P(\bullet\bullet)^2$
	6	5	2	16	$\overline{111}P(\curvearrowright)P(\bullet\bullet)^2$
	6	5	2	12	$\overline{111}P(\triangle)P(\bullet\bullet)^2$
	6	6	2	8	$\overline{111}P(\curvearrowright)P(\bullet\bullet)$
	6	6	2	72	$\overline{111}P(\triangle)^2$
	6	7	2	48	$\overline{111}P(\curvearrowright)P(\bullet\bullet)$
	7	4	3	16	$\overline{222}P(\triangle)P(\bullet\bullet)^2$
	7	5	3	48	$\overline{222}P(\triangle)P(\bullet\bullet)^2$
	8	3	4	384	$\overline{333}P(\bullet\bullet)^4$

Table 3: The polynomial $P(G)$ for v -vertex graphs G with $v - c(G) = 4$.

		#PLR($r, s, 7; m$)															
		$r.s.7$															
m	1.1.7	1.2.7	1.3.7	1.4.7	1.5.7	1.6.7	2.2.7	2.3.7	2.4.7	2.5.7	2.6.7	3.3.7	3.4.7	3.5.7	3.6.7		
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	14	21	28	35	42	28	42	56	70	84	63	84	105	126		
2		42	126	252	420	630	266	672	1260	2030	2982	1638	3024	4830	7056		
3			210	840	2100	4200	1008	5208	14784	31920	58800	22974	61488	128730	232680		
4				840	4200	12600	1302	20538	98364	299460	712530	190890	783972	2216340	5048190		
5					2520	15120		39060	378000	1739640	5549040	971838	6583248	26030340	76284180		
6						5040		28140	815640	6291600	28239960	3026772	37230984	214773720	829360980		
7								900480	13876800	93703680	5560380	142536240	1263691800	6610206960			
8								390600	17711400	198840600	5477220	365911560	5328892800	39009259800			
9									11718000	259408800	2212980	613495680	16053853200	171041026800			
10									3059280	194125680		637509600	34161276240	556100475840			
11										73422720		369109440	50271606000	1331142603840			
12										10679760		90296640	49395578400	2316314150640			
13													30542853600	2873671668000			
14													10629360000	2470492936800			
15													1573165440	1411231731840			
16														501894973440			
17														99105431040			
18														8211571200			
Total	8	57	358	1961	9276	37633	2605	93661	2599185	54730201	864744637	17464756	2263521961	199463431546	11785736969413		

Table 4: The values of $\#PLR(r, s, 7; m)$; continued in Tables 5 and 6.

		#PLR($r, s, 7; m$)						
		$r.s.7$						
m	4.4.7	4.5.7	4.6.7	5.5.7	5.6.7	6.6.7		
0	1	1	1	1	1	1	1	
1	112	140	168	175	210	252		
2	5544	8820	12852	14000	20370	29610		
3	160608	331800	594720	680400	1213800	2158800		
4	3040464	8342040	18654300	22520400	49851900	109648350		
5	39789792	148690080	421288560	539486640	1501095960	4129786080		
6	371511504	1945492080	7103917800	9705007200	34417437600	119886474960		
7	2518935552	19094265120	91553898240	134286297600	616139899200	2752801934400		
8	12508115256	142468484760	915820562160	1452407800200	8762762710800	50916808769400		
9	45551970240	814365132000	7182549494400	12413692800600	100228554703200	768744893767200		
10	121055555040	3578117047680	44440902031680	84446936458080	930070756954080	9567352024458480		
11	231977692800	12080294553600	217628184896640	459215324652000	7044643080720000	98867148338165760		
12	313967041920	31220730777600	844055906319360	2000199369924000	43729884582552000	852898268432422800		
13	290077079040	61311770150400	2588575554835200	6978471536484000	223002930233664000	6166083869012592000		
14	172656368640	90439590528000	6253755470524800	19466170012296000	935144001957312000	37457508269996136000		
15	59253304320	98519956738560	11829864008309760	43255879780478400	3223730533876070400	191508486724243180800		
16	8859553920	77323490294400	17371923533959680	76143045893544000	9122624741349504000	824650379018257377600		
17		42126214233600	19575543305041920	105358455643896000	21136648322563200000	2990529904515892704000		
18		14995766822400	16668621405273600	113411691586368000	39946627806672384000	9125647379336687472000		
19		3114811929600	10507596032102400	93673159102656000	61276248720916992000	23396431711383803520000		
20		284634362880	4768455577697280	58277161295539200	75809634470217446400	50284894217671092470400		
21			1496705022167040	26615217299328000	75050036025947136000	90334790420061996748800		
22			305223851842560	8591796855936000	58874808204632448000	135149758456395303936000		
23			36075091046400	1844984711808000	36156622400801280000	167647764880657152000000		
24			1862525145600	235436435136000	17119090026206784000	171521836534811629440000		
25				13481774369280	6126452955671086080	143844493310595330785280		
26					1613475264781900800	98168168535490134466560		
27					300888959183769600	54057999485833839820800		
28					37371505393152000	23779699801418663424000		
29					2759601374208000	8256726182294360064000		
30					91288879718400	2230046357199562137600		
31						459939642510304051200		
32						70680227381503488000		
33						7813153251735552000		
34						587441307350016000		
35						27048481121894400		
36						583662346444800		
Total	1258840124753	435973408185561	92518523839617121	556422824213480176	407007072002505214801	982388579887448747338333		

Table 5: The values of $\#PLR(r, s, 7; m)$; continuing from Table 4 and continued in Table 6.

$\#PLR(r, 7, 7; m)$							
$r, 7, 7$							
m	1.7.7	2.7.7	3.7.7	4.7.7	5.7.7	6.7.7	7.7.7
0	1	1	1	1	1	1	1
1	49	98	147	196	245	294	343
2	882	4116	9702	17640	27930	40572	55566
3	7350	97608	381318	969024	1971270	3498600	5661558
4	29400	1450302	9983358	36434244	96693660	211737330	407626002
5	52920	14173740	184571730	996695280	3508057980	9577064700	22091837670
6	35280	93118620	2493017100	20589037560	97824178200	336641627700	937499611860
7	5040	413327040	25114127220	329058167760	2151220104600	9441643402800	31995541817820
8		1229208120	191003176980	4136301605520	37983532771800	215279839870200	895147467758460
9		2396605680	1103575119780	41356003473120	545522619369000	4045906316281200	20823534145010940
10		2949266880	4851540242640	331382137961280	6433667717868960	6332609806263360	407161408673448240
11		2154479040	16187551364880	2138171372830080	62740410283404000	832012118607983040	6747928605026748720
12		845696880	40729136096880	11136703296096000	508448168895240000	9231573671794519920	95414556472688784240
13		149516640	76460194354320	46850306414526720	3435977823932808000	86898548210325012000	1157011091919371520720
14		9344160	105451973716320	158998861707477120	19404150304485744000	696350429909011332000	12080739048610887859680
15			104574912049440	434071037204501760	91672146486194601600	4762047565503866736000	108953982522887641120800
16			72399498706080	948939730997852160	362335335766117560000	27837450938084937912000	850783267970671119386400
17			33593118763680	1650710438532288000	1197163519923384216000	139234645196548772976000	5762016601975755442288800
18			9863841496320	2265820889356362240	3300872875576140816000	596053661819256139968000	33886035458720657756006400
19			1690904920320	2427993422686218240	7575781755486572208000	2183386963441728494016000	173162378870925255394329600
20			151342732800	2003724888642247680	14423117979384567129600	6838094015249132148105600	769084553432477123576582400
21			5411750400	1251798397787105280	22679370383067115200000	18286225276118547108614400	2968275252277138334102611200
22				579227971312972800	29296238052876891264000	41677467136757620221715200	99494010250665781152960038400
23				193034752263198720	30887319756977889408000	80769545715156012215424000	2893638003716064863828851200
24				46693646858846720	26372495140248144384000	132713702198233518483072000	72925724298137854106413908000
25				6860934701107200	18066619214217787207680	184255530135693426746542080	158995953693483311073403284480
26				655091210188800	9820001352274125465600	215290252940027718278891520	299284107563352671166958064640
27				34832706048000	4178611218476036966400	210725753874837580720373760	485219498022119678183816647680
28				782137036800	1369620935962581657600	171867215661546141628416000	675695410918704050010696483840
29					339042043904814028800	116095014971808867619430400	805661431870384140528231336960
30					61869133685050675200	64501823805957475751116800	819586476449488450769559091200
31					8073669012853248000	29244277165012229350195200	708500236904008865987686041600
32					723625308177408000	10722854433456169179033600	518142986889857917315006003200
33					41905003262976000	3147095680488347470848000	318986018230847365565350041600
34					1401648095232000	730644033010729291776000	164410170607527803740951142400
35					20449013760000	132359825819298115584000	70519394490467460516228096000
36						18410847956245011456000	2500698392495926034582016000
37						1928283614905632768000	7279610083869038651882496000
38						148277402084431872000	1726520284027400861325312000
39						8074157388079104000	331008922550326911141888000
40						293524209893376000	50895029497370545118822400
41						6359357620224000	6228531360821639220019200
42						61479419904000	602622105599612348006400
43							45850824283578118963200
44							2734157863261981900800
45							127631489644560384000
46							4668091942993920000
47							134218380312576000
48							3012491575296000
49							61479419904000
Total	130922	10256288925	467281806581416	12027068084311265945	170054389801866987652126	1289970420801370588662084277	51751662330606275236665748739420

Table 6: The values of $\#PLR(r, 7, 7; m)$.

$\#PLR(r, s, 8; m)$															
$r, s, 8$															
m	1.1.8	1.2.8	1.3.8	1.4.8	1.5.8	1.6.8	2.2.8	2.3.8	2.4.8	2.5.8	2.6.8	3.3.8	3.4.8	3.5.8	3.6.8
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	8	16	24	32	40	48	32	48	64	80	96	72	96	120	144
2		56	168	336	560	840	352	888	1664	2680	3936	2160	3984	6360	9288
3			336	1344	3360	6720	1568	8064	22848	49280	90720	35328	94272	197040	355776
4				1680	8400	25200	2408	37800	180432	548240	1302840	346248	1413216	3981600	9049320
5					6720	40320		86688	835968	3837120	12216960	2104704	14107968	55460160	161935200
6						20160		76272	2212224	17025120	76258560	7925232	95977056	548866080	2107810320
7								3050880	47040000	317197440	17823456	447552000	3921099840	20355148800	1078103200
8								1681680	77053200	866199600	21748608	1417799376	20342463120	147462199920	1078103200
9									67267200	1501920000	10997952	2973054336	76474164480	805219282560	1078103200
10									23782080	1555303680		3916301760	206198475840	3312619813440	1078103200
11										856154880		2911668480	390865104000	10212300322560	1078103200
12										189564480		925505280	503426851200	23343747174720	1078103200
13													415516953600	38901656989440	1078103200
14													196521292800	46071395395200	1078103200
15													40197104640	37300347786240	1078103200
16														19394880744960	1078103200
17														5775185848320	1078103200
18														742119920640	1078103200
Total	9	73	529	3393	19081	93289	4361	209761	7985761	236605001	5376213193	60983761	12703477825	1854072020881	186029569786849

Table 7: The values of $\#PLR(r, s, 8; m)$; continued in Table 8.

$\#PLR(r, s, 8; m)$							
m	$r, s, 8$						
	4.4.8	4.5.8	4.6.8	5.5.8	5.6.8	6.6.8	
0	1	1	1	1	1	1	1
1	128	160	192	200	240	288	
2	7296	11600	16896	18400	26760	38880	
3	245376	505920	905664	1035200	1844160	3275040	
4	5440032	14863920	33150960	39944000	88164000	193314600	
5	84155904	312224640	880629120	1123820160	3111238080	8513683200	
6	938106624	4857854400	17614343040	23931230400	84205144800	290863660800	
7	7674293760	57240046080	271706198400	395240496000	1792941696000	7913434233600	
8	46492403328	517971847680	3284306156880	5147427465600	30572202805200	174666634178400	
9	208994118144	3629706339840	31445326617600	53468641900800	422804060918400	3170532224025600	
10	693958185984	19775273602560	240125846929920	446402183619840	4786206531503040	47813256027210240	
11	1682575630848	83790652431360	1468289383142400	3010123202150400	44641974866227200	603661256780037120	
12	2918423765376	275253395880960	7199937958106880	16435824668659200	344653334210505600	6417774203497977600	
13	3499852769280	69631888996800	28293978724945920	72712045906752000	2208972817416960000	57702985271076096000	
14	2737429309440	1341720153849600	88858361994393600	260310345154272000	11772041299608844800	440141451429993062400	
15	1248707174400	1937078037135360	221915691372533760	751830550246218240	52182303997547888640	2854234197294902231040	
16	250631700480	2044882328832000	437480320642485120	1743112037427264000	192252531989997360000	15755740879827344094720	
17		1520755813785600	673828360242831360	3220935488443008000	587611818336920832000	74072163230186875084800	
18		749652906240000	799632015238103040	4696696282529664000	1485550685585627136000	296515269372479241369600	
19		218552140800000	717288915035750400	5332415365638144000	3093542054560661760000	1009795856630569892352000	
20		28375521914880	473551206050119680	4627930686056294400	5277311702476213478400	2921119683107942455372800	
21			221248992118210560	2991698111646720000	7323172199654814720000	7162134609634048893542400	
22			68717472783482880	1386151005947904000	8192740332265767936000	14840974457028794640384000	
23			12643342010449920	432666484604928000	7305506991725193216000	25896289714957972638720000	
24			1036744153804800	81104713998336000	5116406448359066112000	37882633011208622775936000	
25					2759923954130172641280	46209895877069315283271680	
26					1116269083866463027200	46698712018213236924579840	
27					325482156403465420800	38792083589752166760038400	
28					64264706091590860800	26235781041173371579699200	
29					7655207985266688000	14276150397241050415104000	
30					413733776530636800	6157077288600135234355200	
31						2063745518966035159449600	
32						523353569391869239296000	
33						96567369329870143488000	
34						12182875723557568512000	
35						937008615326102323200	
36						33087582858697113600	
Total	13295767306401	8920365218163361	3753438773423308993	25624385022295308521	4291466146209454592201	271169169298945362007111849	

Table 8: The values of $\#PLR(r, s, 8; m)$; continued from Table 7. The present authors did not compute $\#PLR(r, 7, 8; m)$ nor $\#PLR(r, 8, 8; m)$.

		#Isom($n; m$)					
		n					
m		1	2	3	4	5	6
0		1	1	1	1	1	1
1	1		4	5	5	5	5
2			10	50	84	93	94
3			4	221	1120	2112	2548
4			1	525	10128	43955	85234
5				651	60092	674957	2508483
6				415	239302	7679384	59110661
7				136	639098	65404265	1103309385
8				20	1148454	422142208	16466869051
9				5	1374447	2080853035	198621450446
10					1082019	7867483199	1953036511736
11					548440	22843744418	15756857221135
12					176137	50867669444	104784604156741
13					35473	86544642569	576125696499417
14					4696	111836743580	2623564948795633
15					403	108882205792	9901507463165937
16					35	79051125332	30959687376379661
17						42275685836	80100291981771263
18						16420711804	171118574787473668
19						4563456676	300957676311237853
20						894429087	434125855232450974
21						122238972	511227919780309083
22						11569016	488771341028032846
23						759296	376957644290919036
24						33736	232788472371575258
25						1411	114149339445885218
26							44033009520708974
27							13227534274721732
28							3061826358557444
29							540473537486248
30							72090555296085
31							7217657260917
32							540810639064
33							30364554576
34							1285684592
35							40649375
36							1130531
Total	2	20	2029	5319934	534759300183	2815323435872410905	

Table 9: Number of isomorphism classes in $PLS(n; m)$ for $n \leq 6$, according to weight m .

		#Isot($2, s, n; m$)														
		$2.s.n$														
m		2.2.2	2.2.3	2.2.4	2.2.5	2.2.6	2.3.3	2.3.4	2.3.5	2.3.6	2.4.4	2.4.5	2.4.6	2.5.5	2.5.6	2.6.6
0		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
3		1	2	2	2	2	5	5	5	5	5	5	5	5	5	5
4		1	2	3	3	3	6	9	9	9	15	15	15	15	15	15
5							2	4	5	5	10	13	13	19	19	19
6							1	3	4	5	13	19	22	31	37	47
7											3	7	9	22	30	45
8											2	5	8	20	35	69
9														5	12	40
10														2	7	35
11																7
12																4
Total		8	10	11	11	11	20	27	29	30	54	70	78	125	166	292

Table 10: Number of isotopism classes in $PLR(2, s, n; m)$ for $2 \leq s \leq n \leq 6$, according to weight m .

$\# \text{Isot}(3, s, n; m)$										
m	$3.s.n$									
	3.3.3	3.3.4	3.3.5	3.3.6	3.4.4	3.4.5	3.4.6	3.5.5	3.5.6	3.6.6
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	4	4	4	4	4	4	4	4	4	4
3	11	11	11	11	11	11	11	11	11	11
4	18	25	25	25	36	36	36	36	36	36
5	23	42	49	49	78	91	91	109	109	109
6	15	52	71	77	174	237	254	330	356	389
7	6	33	70	82	215	430	502	858	1012	1212
8	1	11	34	52	192	585	855	1770	2568	3782
9	1	4	13	23	91	491	962	2683	5168	10001
10					30	257	740	2689	7706	21857
11					4	71	298	1794	7988	35822
12					2	12	70	709	5446	42768
13								177	2301	34916
14								19	530	19078
15								3	62	6441
16										1315
17										133
18										16
Total	81	184	279	325	839	2227	3825	11194	33299	177892

Table 11: Number of isotopism classes in $\text{PLR}(3, s, n; m)$ for $3 \leq s \leq n \leq 6$, according to weight m .

#Isot($r, s, n; m$)												
m	$r.s.n$											
	4.4.4	4.4.5	4.4.6	4.5.5	4.5.6	4.6.6	5.5.5	5.5.6	5.6.6	6.6.6		
0	1	1	1	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1	1	
2	4	4	4	4	4	4	4	4	4	4	4	
3	11	11	11	11	11	11	11	11	11	11	11	
4	52	52	52	52	52	52	52	52	52	52	52	
5	139	160	160	187	187	187	221	221	221	221	221	
6	507	668	707	882	935	997	1158	1227	1306	1396	1396	
7	1161	2103	2395	3713	4223	4826	6310	7127	8064	9130	9130	
8	2136	5678	7754	14266	19080	25524	33293	43322	56110	72145	72145	
9	2429	10739	19067	42940	72764	121508	150964	241958	380083	583339	583339	
10	2004	14881	36957	99301	230072	515040	554285	1174047	2384388	4627607	4627607	
11	975	13865	50826	168900	565202	1797295	1594532	4699600	12974453	33362634	33362634	
12	364	8970	50244	210285	1064946	5054807	3539461	15159299	59361654	210409407	210409407	
13	72	3664	32727	187214	1498530	11135187	6017824	38833501	223569607	1129335392	1129335392	
14	18	995	13973	117985	1557518	19016101	7772366	78368607	686354327	5091624997	5091624997	
15	2	141	3268	51094	1166309	24794117	7568187	123670028	1706058231	19140028219	19140028219	
16	2	22	411	14960	616603	24415585	5493206	151457082	3417379856	59761963636	59761963636	
17				2814	220158	17834146	2939617	142614087	5488132262	154544375137	154544375137	
18				332	50723	9492300	1141472	102078688	7025903964	330108625102	330108625102	
19				24	6591	3575605	317980	54746803	7119415871	580559388329	580559388329	
20				3	428	926317	62319	21601198	5662138638	837440466326	837440466326	
21						156463	8676	6121385	3498117999	986167409118	986167409118	
22						16759	823	1203460	1658503251	942850011453	942850011453	
23							960	69	155952	594594494	727157075193	727157075193
24							56	6	12023	158425032	449054224783	449054224783
25								2	486	30703736	220195944263	220195944263
26										4220807	84941236104	84941236104
27										396518	25516234965	25516234965
28										24531	5906586539	5906586539
29										886	1042616896	1042616896
30										40	139114631	139114631
31											13928529	13928529
32											1048656	1048656
33											59130	59130
34											2846	2846
35											109	109
36											22	22
Total	9878	61955	218558	914969	7074338	118883849	37202840	742190170	37349106398	5431010366323	5431010366323	

Table 12: Number of isotopism classes in $PLR(r, s, n; m)$ for $4 \leq r \leq s \leq n \leq 6$, according to weight m .

#MC($r, s, n; m$)																
r, s, n																
m	2.2.2	2.2.3	2.2.4	2.2.5	2.2.6	2.3.3	2.4.4	2.5.5	2.6.6	3.3.3	3.3.4	3.3.5	3.3.6	3.4.4	3.5.5	3.6.6
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	2	3	3	3	3	3	3	3	3	2	3	3	3	3	3	3
3	1	2	2	2	2	4	4	4	4	5	8	8	8	8	8	8
4	1	2	3	3	3	5	11	11	11	8	18	18	18	24	24	24
5						2	8	14	14	9	28	33	33	49	67	67
6						1	10	22	32	7	34	46	50	104	191	224
7							3	17	32	4	23	46	54	128	477	667
8							2	16	48	1	9	24	36	116	963	2018
9								5	31	1	4	12	20	59	1444	5233
10								2	27					22	1452	11309
11									7					4	986	18435
12									4					2	411	21995
13															115	18028
14															16	9969
15															3	3468
16																765
17																94
18																14
Total	6	9	10	10	10	17	43	96	215	39	129	192	224	521	6162	92323

Table 13: Number of main classes in $\text{PLR}(r, s, n; m)$ for $2 \leq r \leq s \leq n \leq 6$ with $r \leq 3$, according to weight m . We omit the cases when r, s , and n are pairwise distinct.

#MC($r, s, n; m$)										
r, s, n										
m	4.4.4	4.4.5	4.4.6	4.5.5	4.6.6	5.5.5	5.5.6	5.6.6	6.6.6	
0	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	2	3	3	3	3	2	3	3	3	2
3	5	8	8	8	8	5	8	8	8	5
4	18	34	34	34	34	18	34	34	34	18
5	39	97	97	111	111	59	131	131	131	59
6	121	376	399	489	551	256	677	717	717	306
7	253	1135	1293	1969	2548	1224	3748	4225	4225	1747
8	442	2987	4070	7392	13142	5997	22209	28670	28670	12799
9	495	5579	9847	21936	61657	26188	122390	191855	191855	99715
10	420	7694	18939	50420	259637	94479	590423	1197283	1197283	779295
11	218	7170	25943	85477	902847	269456	2356900	6500092	6500092	5583650
12	96	4686	25682	106351	2535116	595649	7593131	29710547	29710547	35131875
13	25	1944	16768	94754	5579487	1010706	19438925	111845936	111845936	188377998
14	8	561	7283	59910	9524578	1304319	39216773	343290367	343290367	848955581
15	2	88	1742	26146	12416456	1270356	61876720	853215848	853215848	3190714878
16	2	19	262	7790	12227832	923128	75774954	1708965453	1708965453	9961645532
17				1533	8934309	495565	71353021	2744426741	2744426741	25759586139
18				200	4758913	193531	51076956	3513371841	3513371841	55021427957
19				19	1795458	54746	27401728	3560137618	3560137618	96764408110
20				3	467000	11052	10816787	2831458432	2831458432	139578978645
21					79692	1693	3069771	1749363542	1749363542	164367335977
22					8815	192	604997	829461470	829461470	157147744329
23					566	26	79301	297418767	297418767	121198141862
24					44	4	6249	79274342	79274342	74846573994
25						2	312	15377110	15377110	36702176578
26								2119455	2119455	14158650257
27								200664	200664	4253618044
28								12830	12830	984869538
29								527	527	173933415
30								33	33	23245431
31										2336988
32										179057
33										10603
34										640
35										40
36										12
Total	2148	32383	112372	464547	59568806	6239377	371406150	18677574543	18677574543	905214521078

Table 14: Number of main classes in $\text{PLR}(r, s, n; m)$ for $4 \leq r \leq s \leq n \leq 6$, according to weight m . We omit the cases when r, s , and n are pairwise distinct.

m	$\#\text{Isot}(r, s, n; m)$	$\#\text{MC}(r, s, n; m)$
0	1	1
1	1	1
2	4	2
3	11	5
4	52	18
5	221	59
6	1396	306
7	9719	1861
8	85145	15097
9	860347	146893
10	10071270	1693416
11	133048009	22239872
12	?	327670703

Table 15: Number of isotopism classes and main classes in $\text{PLR}(r, s, n; m)$ when $r \geq m$, $s \geq m$, and $n \geq m$, according to weight m . The second column is Sloane's oeis.org/A286317.