

# TAYLOR COEFFICIENTS OF THE JACOBI $\theta_3(q)$ FUNCTION

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ABSTRACT. We extend some results recently obtained by Dan Romik [16] about the Taylor coefficients of the theta function  $\theta_3(1)$  to the case  $\theta_3(q)$  of an arbitrary value of the elliptic modulus  $k$ . These results are obtained by carefully studying the properties of the cumulants associated to a  $\theta_3$  (or discrete normal) distributed random variable. This article also states some congruence conjectures about integers sequences that generalize the one studied by D. Romik.

## 1. INTRODUCTION

Recently, Dan Romik derived [16] some results about the Taylor coefficients of the Jacobi  $\theta_3$  function. One of the most unexpected results is the existence of an integer sequence  $d(n)$

$$1, -1, 51, 849, -26199, 1341999, 82018251 \dots$$

that allows us to compute <sup>1</sup> the moments of a discrete normal distribution as a finite sum, according to the formula

$$(1.1) \quad \frac{1}{\theta_3(e^{-\pi})} \sum_{p=-\infty}^{+\infty} p^{2n} e^{-\pi p^2} = \frac{1}{(4\pi)^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n)!}{2^{n-2j} (4j)! (n-2j)!} d(j) \Omega^j,$$

with the constant  $\Omega := \frac{\Gamma^8(\frac{1}{4})}{32\pi^8}$ .

One unanswered question in Romik's paper is the extension of this result to a discrete normal distribution with arbitrary parameter  $q$ . The main result of this article is the formula

$$\frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} p^{2n} q^{p^2} = \sum_{j=0}^n \binom{2n}{2j} \left(\frac{z}{2}\right)^{2j} R_{2j}(k) \left(-\frac{1}{2}\right)^{n-j} \sigma^{2n-2j} \frac{(2n-2j)!}{(n-j)!}$$

with  $q = e^{-\pi \frac{K'(k)}{K(k)}}$ ,  $\sigma^2 = \frac{K(k)^2}{\pi^2} \left[ \frac{E(k)}{K(k)} - (k')^2 \right]$ , and  $\{R_{2j}(k)\}$  a series of integer coefficients described in Theorem 1. The case  $k = \frac{1}{\sqrt{2}}$  reduces to Romik's identity.

This result can be interpreted as the calculation of the moments of a discretized Gaussian distribution. In this article we adopt a probabilistic language which is not necessary in this context; however, it simplifies the statement of most results. Let us denote by  $X_{\theta_3}$  a discrete normal random variable with parameter  $k$  :

$$\Pr \{X_{\theta_3} = n\} = \frac{1}{\theta_3(q)} q^{n^2}, \quad n \in \mathbb{Z},$$

with  $q = e^{-\pi \frac{K'(k)}{K(k)}}$ . It essentially samples the standard normal distribution at equally spaced integer points, and then renormalizes the resulting discrete distribution. Its moment generating function is defined by

$$\varphi_{\theta_3}(u) = \mathbb{E} e^{uX_{\theta_3}} = \frac{1}{\theta_3(q)} \sum_{n \in \mathbb{Z}} e^{nu} q^{n^2}$$

and its cumulants  $\kappa_n$  are defined as the Taylor coefficients of the cumulant generating function

$$\psi_{\theta_3}(u) = \log \varphi_{\theta_3}(u) = \sum_{n \geq 1} \kappa_n \frac{u^n}{n!}.$$

To obtain our main results, we first study the cumulants of  $\theta_3$  and show that they have a rich structure: not only they are related with Eisenstein and Lambert series, but they also possess a clean combinatorial interpretation in terms of restricted permutations.

<sup>1</sup>a different notation is used in [16] where  $\theta_3(x)$  denotes  $\sum_{n \in \mathbb{Z}} e^{-\pi x n^2}$

## 2. INTRODUCTION TO ELLIPTIC FUNCTIONS

Our approach relies heavily on properties of elliptic functions. We recall here some basic results and notations about those used in this article. Some useful references about elliptic functions are the classic [3, 15] and the more recent [4]. Additionally, [18] provides an extensive historical approach.

For a given elliptic modulus  $k$ , the complete elliptic integrals of the first and second kind are the functions

$$K(k) = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \quad E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt.$$

Expansion of the factors  $\frac{1}{\sqrt{1-k^2t^2}}$  and  $\sqrt{1-k^2t^2}$  in the respective integrals shows that they also be expressed as hypergeometric functions according to

$$K(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}, k^2\right), \quad E(k) = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}, k^2\right).$$

The complementary elliptic modulus is

$$k' = \sqrt{1-k^2}$$

and we adopt the usual notations

$$E'(k) = E(k') = E(\sqrt{1-k^2}), \quad K'(k) = K(k') = K(\sqrt{1-k^2}).$$

The nome  $q$  is defined as a function of the elliptic modulus  $k$  as

$$q = e^{-\pi \frac{K'(k)}{K(k)}}.$$

The Jacobi theta functions are

$$\theta_3(z, q) := \sum_{n \in \mathbb{Z}} q^{n^2} e^{i2nz}, \quad \theta_2(z, q) := \sum_{n \in \mathbb{Z}} q^{(n+\frac{1}{2})^2} e^{i(2n+1)z}$$

and

$$\theta_4(z, q) := \sum_{n \in \mathbb{Z}} (-1)^n q^{n^2} e^{i2nz}, \quad \theta_1(z, q) := \sum_{n \in \mathbb{Z}} (-1)^{n-\frac{1}{2}} q^{(n+\frac{1}{2})^2} e^{i(2n+1)z},$$

and we adopt the shortcut notation

$$\theta_i(q) := \theta_i(0, q), \quad 1 \leq i \leq 4.$$

They have infinite product representations [1, (10.7.7)]

$$(2.1) \quad \begin{aligned} \theta_1(z, q) &= -iq^{\frac{1}{4}} e^{iz} (q^2, q^2)_\infty (q^2 e^{i2z}, q^2)_\infty (q^2 e^{-i2z}, q^2)_\infty, \\ \theta_2(z, q) &= q^{\frac{1}{4}} e^{iz} (q^2, q^2)_\infty (-q^2 e^{i2z}, q^2)_\infty (-q^2 e^{-i2z}, q^2)_\infty, \end{aligned}$$

$$(2.2) \quad \begin{aligned} \theta_3(z, q) &= (q^2, q^2)_\infty (-q e^{i2z}, q^2)_\infty (-q e^{-i2z}, q^2)_\infty, \\ \theta_4(z, q) &= (q^2, q^2)_\infty (q e^{i2z}, q^2)_\infty (q e^{-i2z}, q^2)_\infty. \end{aligned}$$

The theta functions are related to the complete elliptic integral of the first kind by

$$(2.3) \quad \theta_2^2(q) = \frac{2}{\pi} k K(k), \quad \theta_3^2(q) = \frac{2}{\pi} K(k), \quad \theta_4^2(q) = \frac{2}{\pi} k' K(k).$$

Jacobi's identity expresses the transformation of theta functions under the action of the modular group: denoting

$$q = e^{i\pi\tau}, \quad \tau = i \frac{K'(k)}{K(k)},$$

the invariance reads

$$\theta_3\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \theta_3(\tau)$$

or, expressed in terms of the elliptic modulus,

$$\frac{1}{\sqrt{K(k)}} \theta_3\left(e^{-\pi \frac{K'(k)}{K(k)}}\right) = \frac{1}{\sqrt{K(k')}} \theta_3\left(e^{-\pi \frac{K(k)}{K'(k)}}\right),$$

an identity that can be interpreted as the invariance of  $\theta_3$  under the change of parameter  $k \mapsto k' = \sqrt{1-k^2}$ . In the parameterization

$$\theta_3(e^{-\pi c}) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 c},$$

with  $c = \frac{K'(k)}{K(k)}$ , this invariance reads

$$\theta_3\left(e^{-\pi\frac{1}{c}}\right) = \sqrt{c}\theta_3\left(e^{-\pi c}\right).$$

The Eisenstein series  $G_{2k}(\tau)$  of weight  $2k$  with  $k \geq 2$  are defined as

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m + \tau n)^{2k}}.$$

The Weierstrass  $\wp$  elliptic function is defined as

$$\wp(z; \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left[ \frac{1}{(z + m\omega_1 + n\omega_2)^2} - \frac{1}{(m\omega_1 + n\omega_2)^2} \right].$$

The invariants  $g_2$  and  $g_3$  are defined by

$$g_2 = 60G_4, \quad g_3 = 140G_6.$$

### 3. THE INTEGER SEQUENCE $d(n)$

Our main result is an extension of Romik's identity (1.1) as follows. Let us introduce the Hermite polynomials  $H_n(x)$  defined by the generating function

$$\sum_{n=0}^{\infty} H_n(x) \frac{w^n}{n!} = e^{2xw - w^2},$$

and recall that we defined the variance

$$(3.1) \quad \sigma^2 := \frac{K(k)^2}{\pi^2} \left[ \frac{E(k)}{K(k)} - (k')^2 \right].$$

**Theorem 1.** *With  $H_n(x)$  denoting the  $n$ -th Hermite polynomial, we have*

$$(3.2) \quad \frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} q^{p^2} H_{2n}\left(\frac{p}{\sigma\sqrt{2}}\right) = \left(\frac{z^2}{2\sigma^2}\right)^n R_{2n}(k),$$

where  $\sigma$  is given by (3.1) and  $R_{2n}(k)$  are polynomials of degree  $n$  in the variable  $k^2$  with integer coefficients. The first cases of the  $R_{2n}(k)$  are

$$R_2(k) = 0, \quad R_4(k) = 2(kk')^2, \quad R_6(k) = -8(kk')^2(1 - 2k^2)$$

and

$$\begin{aligned} R_8(k) &= 4(kk')^2(8 - 33k^2 + 33k^4), \\ R_{10}(k) &= 32(kk')^2(4 - 27k^2 + 57k^4 - 38k^6), \\ R_{12}(k) &= 8(kk')^2(1555k^8 - 3110k^6 + 2187k^4 - 632k^2 + 64). \end{aligned}$$

*Proof.* Consider the random variable

$$Z = X_{\theta_3} + \iota N_{\sigma^2}$$

where  $N_{\sigma^2}$  is a Gaussian random variable whose variance  $\sigma^2 = \kappa_2$  coincides with the variance of  $X_{\theta_3}$ . Then

$$\kappa_2(Z) = \kappa_2(X_{\theta_3}) - \kappa_2(N_{\sigma^2}) = 0.$$

Moreover  $Z$  has odd moments equal to 0 and even moments given by

$$\begin{aligned} \mu_{2n}(Z) &= \mathbb{E}(X_{\theta_3} + \iota N_{\sigma^2})^{2n} = \frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} q^{p^2} \mathbb{E}(p + \iota N_{\sigma^2})^{2n} \\ &= \frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} q^{p^2} \mathbb{E}\left(p + \iota\sigma\sqrt{2}N_{\frac{1}{2}}\right)^{2n} \\ &= \frac{1}{\theta_3(q)} \left(\frac{\sigma}{\sqrt{2}}\right)^{2n} \sum_{p=-\infty}^{+\infty} q^{p^2} H_{2n}\left(\frac{p}{\sigma\sqrt{2}}\right), \end{aligned}$$

where we have used the representation for the Hermite polynomials

$$H_{2n}(w) = 2^{2n} \mathbb{E}\left(w + \iota N_{\frac{1}{2}}\right)^{2n}.$$

On the other hand, the moments of  $Z$  can be expressed in terms of complete Bell polynomials as

$$\mu_{2n}(Z) = B_{2n}(\kappa_1 = 0, \kappa_2 = 0, \kappa_3 = 0, \kappa_4, \dots, \kappa_{2n})$$

where  $\kappa_{2n}$ ,  $n \geq 2$ , is the order  $2n$  cumulant of  $Z$  (or  $X_{\theta_3}$ ). For example,

$$\mu_2(Z) = 0, \quad \mu_4(Z) = 2(kk')^2 \left(\frac{z}{2}\right)^4,$$

$$\mu_6(Z) = 8(kk')^2(2k^2 - 1) \left(\frac{z}{2}\right)^6.$$

Moreover, as will be shown in Theorem 5 later on, there exists a sequence of polynomials  $P_n(k)$  such that, for  $n \geq 2$ ,

$$\kappa_{2n}(m) = (-1)^n P_{2n-2}(k) \left(\frac{z}{2}\right)^{2n};$$

the polynomials are characterized in Theorem 5 as convolutions of Schett polynomials.

Since moreover the complete Bell polynomial  $B_{2n}$  is homogeneous of degree  $2n$ , we deduce that  $B_{2n}$  is proportional to  $z^{2n}$  up to factor that is a polynomial in  $k$ . Hence, for  $n \geq 2$ ,

$$\mu_{2n}(Z) = \left(\frac{z}{2}\right)^{2n} R_{2n}(k).$$

Moreover, since the polynomials  $P_{2n-2}$  and the complete Bell polynomials have integer coefficients, the corresponding polynomials  $R_{2n}$  have also integer coefficients.

First cases of  $\mu_{2n}(Z)$  are obtained by putting  $\sigma = 0$  in the expression of the moments of  $X_{\theta_3}$ :

$$\mu_2(Z) = 0, \quad \mu_4(Z) = 2(kk')^2 \left(\frac{z}{2}\right)^4,$$

$$\mu_6(Z) = 8(kk')^2(2k^2 - 1) \left(\frac{z}{2}\right)^6,$$

$$\mu_8(Z) = 4(kk')^2(8 - 33k^2 + 33k^4) \left(\frac{z}{2}\right)^8,$$

$$\mu_{10}(Z) = 32(kk')^2(4 - 27k^2 + 57k^4 - 38k^6) \left(\frac{z}{2}\right)^{10},$$

$$\mu_{12}(Z) = 8(kk')^2(1555k^8 - 3110k^6 + 2187k^4 - 632k^2 + 64) \left(\frac{z}{2}\right)^{12},$$

so that

$$R_2(k) = 0, \quad R_4(k) = 2(kk')^2, \quad R_6(k) = -8(kk')^2(1 - 2k^2)$$

and

$$R_8(k) = 4(kk')^2(8 - 33k^2 + 33k^4).$$

We deduce the identity

$$\sum_{p=-\infty}^{+\infty} q^{p^2} H_{2n} \left( \frac{p}{\sigma\sqrt{2}} \right) = \frac{2^n}{\sigma^{2n}} \theta_3(q) \left(\frac{z}{2}\right)^{2n} R_{2n}(k).$$

□

**Corollary 2.** *In the standard case  $k = \frac{1}{\sqrt{2}}$ , identity (3.2) coincides with Romik's identity [16, Proposition 10]*

$$\sum_{p=-\infty}^{+\infty} e^{-\pi p^2} H_{2n}(\sqrt{2\pi}p) = \begin{cases} 2^{2n} \theta_3(1) \Phi^{\frac{n}{2}} d\left(\frac{n}{2}\right) & n \equiv 0 \pmod{2} \\ 0 & n \equiv 1 \pmod{2}. \end{cases}$$

Moreover, Romik's sequence  $d(n) = 1, 1, -1, 51, \dots$  is related to the polynomials  $R_n(k)$  by

$$d(n) = 2^n R_{4n} \left( \frac{1}{\sqrt{2}} \right), \quad n \geq 1.$$

*Proof.* In the standard case  $k = \frac{1}{\sqrt{2}}$ ,

$$\Phi = 4\pi^2 \kappa_4 = \frac{\Gamma\left(\frac{1}{4}\right)^8}{2^7 \pi^4} = \frac{\pi^2}{8} \theta_3^8(1),$$

so that

$$4^n \Phi^{\frac{n}{2}} = 2^{\frac{n}{2}} \frac{\Gamma^{4n}\left(\frac{1}{4}\right)}{(4\pi^2)^n}$$

and

$$\frac{E\left(\frac{1}{\sqrt{2}}\right)}{K\left(\frac{1}{\sqrt{2}}\right)} - \frac{1}{2} = \frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)},$$

so that

$$\frac{8^n}{\left(\frac{E\left(\frac{1}{\sqrt{2}}\right)}{K\left(\frac{1}{\sqrt{2}}\right)} - \frac{1}{2}\right)^n} = 8^n \left(\frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^2}\right)^n = 8^n \frac{\Gamma^{4n}\left(\frac{1}{4}\right)}{(4\pi^2)^n}.$$

Moreover,  $\sigma^2 = \frac{1}{4\pi}$  so that  $\frac{1}{\sigma\sqrt{2}} = \frac{2\sqrt{\pi}}{\sqrt{2}} = \sqrt{2\pi}$  and we recover the  $\sqrt{2\pi}$  factor in Romik's identity.  $\square$

Our second main result is an explicit formula for the moments of a discrete normal random variable with parameter  $k$  as a finite sum. It is an equivalent rephrasing of Theorem 1.

**Theorem 3.** *With  $q = e^{-\pi \frac{K'(k)}{K(k)}}$  and  $\sigma$  given by (3.1), the moments of a discrete normal random variable can be computed using the sequence of polynomials  $\{R_{2n}\}$  according to the formula*

$$\frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} p^{2n} q^{p^2} = \sum_{j=0}^n \binom{2n}{2j} \frac{(2n-2j)!}{(n-j)!} \left(\frac{z}{2}\right)^{2j} R_{2j}(k) \left(-\frac{\sigma^2}{2}\right)^{n-j}.$$

The standard case  $k = \frac{1}{\sqrt{2}}$  corresponds to Romik's identity [16, Proposition 9]

$$\sum_{p=-\infty}^{+\infty} p^{2n} e^{-\pi p^2} = \frac{\theta_3(1)}{(4\pi)^n} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(2n)!}{2^{n-2j} (4j)! (n-2j)!} d(j) \Omega^j.$$

*Proof.* The proof is obtained by remarking that

$$Z + \imath N_{\sigma^2} = X_{\theta_3}$$

and taking the  $2n$  order moment. This is

$$\mathbb{E} X_{\theta_3}^{2n} = \frac{1}{\theta_3(q)} \sum_{p=-\infty}^{+\infty} p^{2n} q^{p^2}$$

on one side, and also

$$\mathbb{E} (Z + \imath N_{\sigma^2})^{2n} = \sum_{j=0}^n \binom{2n}{2j} \mathbb{E} Z^{2j} (-1)^{n-j} \sigma^{2n-2j} \mathbb{E} N_1^{2n-2j}$$

with

$$\mathbb{E} N_1^{2n-2j} = \frac{1}{2^n} \frac{(2n)!}{n!},$$

so that

$$\sum_{p=-\infty}^{+\infty} p^{2n} q^{p^2} = \theta_3(q) \sum_{j=0}^n \binom{2n}{2j} \mathbb{E} Z^{2j} \left(-\frac{1}{2}\right)^{n-j} \sigma^{2n-2j} \frac{(2n-2j)!}{(n-j)!}.$$

Substituting

$$\mu_{2n}(Z) = \left(\frac{z}{2}\right)^{2n} R_{2n}(k),$$

we deduce the result.  $\square$

#### 4. THE CUMULANTS

We first study the cumulants of the  $\theta_3$  random variable. A careful characterization of their properties will allow us to derive some results about the moment generating function of  $\theta_3$  itself.

#### 4.1. The cumulants as Lambert series.

**Theorem 4.** *The cumulants of  $X_{\theta_3}$  can be expressed as the Lambert series*

$$(4.1) \quad \kappa_{2n} = \sum_{k \geq 1} \frac{(-1)^{k-1} k^{2n-1}}{\sinh(ck\pi)}, \quad n \geq 1,$$

and

$$\kappa_{2n+1} = 0, \quad n \geq 0.$$

*Proof.* Since

$$\theta_3(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i2nz}$$

the moment generating function for  $X_{\theta_3}$  is

$$\mathbb{E}e^{zX_{\theta_3}} = \frac{1}{\theta_3(0, q)} \sum_{n=-\infty}^{\infty} q^{n^2} e^{nz} = \frac{\theta_3\left(\frac{z}{2i}, q\right)}{\theta_3(0, q)}.$$

Using the infinite product representation (2.2) for the function  $\theta_3$  gives

$$\frac{\theta_3\left(\frac{z}{2i}, q\right)}{\theta_3(0, q)} = \prod_{p \geq 0} \frac{(1 + e^z q^{2p+1})}{(1 + q^{2p+1})} \frac{(1 + e^{-z} q^{2p+1})}{(1 + q^{2p+1})}.$$

Defining the function

$$f(z) := \sum_{p \geq 0} \log(1 + e^z q^{2p+1}),$$

the cumulants of  $X_{\theta_3}$  can be computed as the Taylor coefficients of

$$\log \frac{\theta_3\left(\frac{z}{2i}, q\right)}{\theta_3(0, q)} = f(z) + f(-z) - 2f(0).$$

Since

$$\begin{aligned} f(z) &= \sum_{p \geq 0} \log(1 + e^z q^{2p+1}) = \sum_{p \geq 0} \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} e^{kz} q^{k(2p+1)} \\ &= \sum_{p \geq 0} \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \sum_{n \geq 0} \frac{(kz)^n}{n!} q^{k(2p+1)} \\ &= \sum_{n \geq 0} \frac{z^n}{n!} \sum_{k \geq 1} (-1)^{k+1} k^{n-1} \sum_{p \geq 0} q^{k(2p+1)}, \end{aligned}$$

we deduce

$$f(z) = \sum_{n \geq 0} \frac{z^n}{n!} \sum_{k \geq 1} (-1)^{k+1} k^{n-1} \frac{q^k}{1 - q^{2k}}$$

and

$$f(0) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} \frac{q^k}{1 - q^{2k}}.$$

The cumulant generating function is then

$$\log \frac{\theta_3\left(\frac{z}{2i}, q\right)}{\theta_3(0, q)} = 2 \sum_{n \geq 1} \frac{z^{2n}}{2n!} \sum_{k \geq 1} (-1)^{k+1} k^{2n-1} \frac{q^k}{1 - q^{2k}},$$

and the cumulants are identified as

$$\begin{aligned} \kappa_{2n+1} &= 0, \quad n \geq 0, \\ \kappa_{2n} &= 2 \sum_{k \geq 1} \frac{(-1)^{k-1} k^{2n-1}}{q^{-k} - q^k}, \quad n \geq 1. \end{aligned}$$

With  $q = e^{-c\pi}$ , this is the desired result.  $\square$

4.2. **The cumulants as Schett polynomials.** The Schett polynomials  $X_n(x, y, z)$  are defined by the recurrence

$$X_n = \left( yz \frac{d}{dx} + zx \frac{d}{dy} + xy \frac{d}{dz} \right) X_{n-1}$$

with initial condition

$$X_0(x, y, z) = x.$$

The first values are

$$\begin{aligned} X_0(x, y, z) &= x, \quad X_1(x, y, z) = yz, \\ X_2(x, y, z) &= x(y^2 + z^2), \quad X_3(x, y, z) = yz(y^2 + z^2 + 4x^2), \\ X_4(x, y, z) &= x(y^4 + z^4) + 4x^3(y^2 + z^2) + 14xy^2z^2. \end{aligned}$$

We will be interested in the evaluations  $X_{2n+1}(0, k, ik')$ , with first values

$$X_1 = ikk', \quad X_3 = ikk'(2k^2 - 1), \quad X_5 = ikk'(k^4 - 14k^2(k')^2 + (k')^4).$$

These polynomials appear naturally in the expression of the cumulants as follows: denote

$$P_{2p}(k) = \sum_{n=0}^{p-1} \binom{2p}{2n+1} X_{2n+1}(0, k, ik') X_{2p-2n-1}(0, k, ik'), \quad p \geq 1,$$

the polynomial obtained as a self-convolution of Schett polynomials. The first values are

$$\begin{aligned} P_0(k) &= 0, \quad P_2(k) = -2(kk')^2, \\ P_4(k) &= -8(kk')^2(2k^2 - 1), \\ P_6(k) &= -16(kk')^2(2 - 17k^2 + 17k^4). \end{aligned}$$

Then we have

**Theorem 5.** *With  $z = \theta_3^2(q) = \frac{2}{\pi}K(k)$ , the cumulants are expressed as*

$$(4.2) \quad \kappa_{2n} = (-1)^{n-1} \left( \frac{z}{2} \right)^{2n} P_{2n-2}(k), \quad n \geq 2.$$

*Proof.* It can be shown that a moment generating function for the Schett polynomials  $X_n(0, a, b)$  is

$$\text{sn}(u; a, b) = \frac{1}{ab} \sum_{n \geq 0} X_n(0, a, b) \frac{u^n}{n!}.$$

Since this is an odd function of  $u$ , we rewrite this equivalently as

$$\frac{1}{ia} \text{sn} \left( iau, \frac{b}{a} \right) = \frac{1}{ab} \sum_{n \geq 0} X_{2n+1}(0, a, b) \frac{u^{2n+1}}{(2n+1)!},$$

so that, choosing  $a = k$  and  $b = ik'$ ,

$$\frac{1}{ik} \text{sn} \left( iku, \frac{ik'}{k} \right) = \frac{1}{ikk'} \sum_{n \geq 0} X_{2n+1}(0, k, ik') \frac{u^{2n+1}}{(2n+1)!}.$$

We deduce

$$-\frac{1}{k^2} \text{sn}^2 \left( iku, \frac{ik'}{k} \right) = -\frac{1}{(kk')^2} \sum_{p \geq 0} \frac{u^{2p+2}}{(2p+2)!} \sum_{n=0}^p \binom{2p+2}{2n+1} X_{2n+1}(0, k, ik') X_{2p-2n+1}(0, k, ik').$$

Recalling the definition of  $P_{2p}(k)$ , we deduce

$$\frac{1}{k^2} \text{sn}^2 \left( iku, \frac{ik'}{k} \right) = \frac{1}{(kk')^2} \sum_{p \geq 0} \frac{u^{2p+2}}{(2p+2)!} P_{2p+2}(k).$$

Comparing to the generating function (see section 4.4 below)

$$\frac{4}{(kk')^2} \sum_{m \geq 0} \frac{(-1)^m 2^{2m}}{z^{2m+2}} \kappa_{2m+2} \frac{u^{2m}}{(2m)!} = \frac{1}{k^2} \text{sn}^2 \left( iuk, i \frac{k'}{k} \right)$$

produces

$$\frac{4}{(kk')^2} \frac{(-1)^m 2^{2m}}{z^{2m+2}} \kappa_{2m+2} = \frac{1}{(kk')^2} P_{2m}(k),$$

or

$$\kappa_{2n} = (-1)^{n-1} \left(\frac{z}{2}\right)^{2n} P_{2n-2}(k).$$

□

For example, with  $\frac{z}{2} = \frac{K(k)}{\pi}$ ,

$$\kappa_4(k) = 2(kk')^2 \left(\frac{z}{2}\right)^4, \quad \kappa_6(m) = 8(kk')^2(1-2k^2) \left(\frac{z}{2}\right)^6,$$

$$\kappa_8(k) = 16(kk')^2(2-17k^2+17k^4) \left(\frac{z}{2}\right)^8,$$

$$\kappa_{10}(k) = 128(kk')^2(1-33k^2+93k^4-62k^6) \left(\frac{z}{2}\right)^{10}.$$

In the standard case  $k = \frac{1}{\sqrt{2}}$ ,

$$\frac{E\left(\frac{1}{\sqrt{2}}\right)}{K\left(\frac{1}{\sqrt{2}}\right)} - \frac{1}{2} = \frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)}$$

and

$$z = \theta_3^2(1) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^{\frac{3}{2}}}$$

so that we recover the variance

$$\frac{z^2}{4} \left[ \frac{E(k)}{K(k)} - (k')^2 \right] = \frac{1}{4} \frac{\Gamma^4\left(\frac{1}{4}\right)}{4\pi^3} \frac{4\pi^2}{\Gamma^4\left(\frac{1}{4}\right)} = \frac{1}{4\pi}.$$

Theorem 5 provides a refinement of a result of Shaun Cooper and Heung Yeung Lam [8, Thm. 0.3], which is equivalent to expressing the cumulants as

$$\kappa_{2n} = z^{2n} (kk')^2 p_{n-2}(k^2), \quad n \geq 2,$$

for some polynomial  $p_{n-2}$  of degree  $n-2$  with rational coefficients. The previous result shows the link between these polynomials and Schett polynomials.

Cooper and Lam in fact considered *sixteen* different families of Eisenstein series and wrote each as a prefactor times a polynomial of restricted degree, which they did not characterize further. We have provided the explicit polynomial in one of these cases, but have not attempted to identify it in the other fifteen.

**4.3. The cumulants as Eisenstein series.** We now use a result by Ling [12] to provide an alternate expression for the cumulants as Eisenstein series.

**Theorem 6.** *With  $c = \frac{K'(k)}{K(k)}$ , the even cumulants of the  $\theta_3$  distribution are, for  $n \geq 1$ ,*

$$\begin{aligned} \kappa_{2n} &= \frac{2(-1)^{n+1}(2n-1)!}{\pi^{2n}} \sum_{\substack{n_1 \geq 1 \\ n_2 \in \mathbb{Z}}} \frac{1}{(2n_1-1+ic(2n_2-1))^{2n}} \\ &= \frac{(-1)^{n+1}(2n-1)!}{\pi^{2n}} \sum_{n_1, n_2 \in \mathbb{Z}} \frac{1}{(2n_1-1+ic(2n_2-1))^{2n}}. \end{aligned}$$

*Proof.* The first expression is [12, Eq.(14)]. It is obtained using the Mittag-Leffler expansion

$$\tanh(\pi x) = \frac{8x}{\pi} \sum_{m \geq 1} \frac{1}{(2m-1)^2 + 4x^2}$$

together with the partial fraction decomposition

$$\sum_{p \geq 1} (-1)^p e^{-2\pi p x} = -\frac{1}{2} + \frac{i}{\pi} \sum_{m \geq 1} \left( \frac{1}{2m-1+2ix} - \frac{1}{2m-1-2ix} \right).$$

The second identity is deduced from the first by symmetry. □



**4.4. The cumulant generating function.** The cumulant generating function for the discrete normal random variable can be expressed in terms of the Jacobi elliptic function  $\text{sd}(u, k)$  using a result by Milne [14, Eq.2.43]:

$$(4.3) \quad \text{sd}^2(u, k) = -\frac{1}{k^2} + \frac{1}{(kk')^2} \frac{E(k)}{K(k)} - \frac{8}{(kk')^2} \sum_{m \geq 0} \frac{(-1)^m 2^{2m}}{z^{2m+2}} \left( \sum_{r \geq 1} \frac{(-1)^{r-1} r^{2m+1} q^r}{1 - q^{2r}} \right) \frac{u^{2m}}{(2m)!}$$

with the notation

$$z = \frac{2}{\pi} K(k) = \theta_3^2(q), \quad q = e^{-\frac{\kappa(k')}{K(k)}}.$$

A consequence of this representation is as follows: since

$$\text{sd}^2(u, k) = u^2 + O(u^4),$$

we deduce

$$-\frac{1}{k^2} + \frac{1}{(kk')^2} \frac{E(k)}{K(k)} - \frac{8}{(kk')^2} \frac{1}{z^2} \kappa_2 = 0$$

which provides the value of the variance  $\sigma^2 = \kappa_2$  as expressed by (3.1).

**4.5. A combinatorial interpretation of the cumulants.** In the concluding Open Problems section of his recent article [16], Dan Romik asked for a combinatorial interpretation of the sequence  $d(n)$ . We were not able to find such an interpretation, but we can provide one for the sequence of cumulants as follows.

**4.5.1. Dumont's results.** Consider a permutation  $\sigma \in \mathfrak{S}_n$  and denote  $\sigma^{-1}$  its inverse. A cycle peak of  $\sigma$  is an integer  $k$  such that  $2 \leq k \leq m$  and

$$\sigma(k) \neq k, \quad \sigma(k) < k \quad \text{and} \quad \sigma^{-1}(k) < k.$$

Dumont [10] provides the example

$$\sigma = (134)(2)(56),$$

for which the cycle peaks are 4 and 6. Denote  $P_{n,i,j}$  the number of permutations  $\sigma \in \mathfrak{S}_n$  that have  $i$  odd cycle peaks and  $j$  even cycle peaks. The link with elliptic functions is given by the following result:

**Theorem 7.** [9, Corollaire 8.7] *The coefficient of  $a^{2j} b^{2n-2j} \frac{u^{2n}}{(2n)!}$  in the Taylor expansion of the function  $\frac{1}{2} \text{sn}^2(u, a, b)$  is equal to the number  $P_{2n,1,j}$ .*

Notice that the first terms in the Taylor expansion of  $\frac{1}{2} \text{sn}^2(u, a, b)$  are

$$\frac{u^2}{2!} + 4(a^2 + b^2) \frac{u^4}{4!} + 8(2a^4 + 13a^2b^2 + 2b^4) \frac{u^6}{6!} + \dots$$

so that the coefficient of  $\frac{u^{2n}}{(2n)!}$  is an homogeneous polynomial in  $(a, b)$  of degree  $n - 2$  and not  $n$  as suggested by Dumont's result. The correct statement is in fact:

**Theorem 8.** *The coefficient of  $a^{2j-2} b^{2n-2j} \frac{u^{2n}}{(2n)!}$  in the Taylor expansion of the function  $\frac{1}{2} \text{sn}^2(u, a, b)$  is equal to the number  $P_{2n,1,j}$ .*

For  $n = 1$  the only 2 permutations are  $(1)(2)$  and  $(2, 1)$ . The first has zero even cycle peaks and zero odd cycle peaks, so that  $P_{2,1,1} = 0$ , while the second has one odd cycle peak and zero even cycle peaks so that  $P_{2,1,0} = 1$ . The coefficient of  $a^{-2} b^0 \frac{u^2}{2!}$  is  $P_{2,1,1} = 0$  and the coefficient of  $a^0 b^0 \frac{u^2}{2!}$  is  $P_{2,1,0} = 1$ .

We deduce the following result.

**Theorem 9.** *With  $z = \theta_3^2(q) = \frac{2}{\pi} K(k)$ , the cumulant  $\kappa_{2n+2}$  is equal to*

$$\kappa_{2n+2} = \frac{z^{2n+2}}{2^{2n+1}} \sum_{j=0}^n (-1)^{j-1} k^{2j+2} (k')^{2n-2j+2} P_{2n,1,j}.$$

*Proof.* First we notice that the function  $\text{sn}(u; a, b)$  as defined by Dumont [9] is related to the classical Jacobi  $\text{sn}(u; k)$  function by

$$\text{sn}(u; a, b) = \frac{1}{ia} \text{sn}\left(iau; \frac{b}{a}\right).$$

Next we relate the cumulant generating function  $\text{sd}$  to the function  $\text{sn}$ : this can be done by applying first the Jacobi real transformation [15, 13.34]

$$\text{sd}(u, k) = \frac{1}{k} \text{sc}\left(uk, \frac{1}{k}\right),$$

and by noticing that the  $sc$  function is related to the  $sn$  function by the Jacobi imaginary transformation [15, 13.25]

$$sc(u, k) = -i \operatorname{sn}(iu, k').$$

We deduce

$$sd(u, k) = \frac{-i}{k} \operatorname{sn}\left(iuk, i\frac{k'}{k}\right)$$

and

$$sd^2(u, k) = -\frac{1}{k^2} \operatorname{sn}^2\left(iuk, i\frac{k'}{k}\right).$$

Now Dumont's result is the generating function

$$\sum_{n \geq 1} \frac{u^{2n}}{(2n)!} \sum_{j=0}^n b^{2j} c^{2n-2j} P_{2n,1,j} = \frac{1}{2} \operatorname{sn}^2(u, b, c) = -\frac{1}{2b^2} \operatorname{sn}^2\left(ibu, \frac{c}{b}\right).$$

Choosing  $c = ik', b = k$  produces

$$\sum_{n \geq 1} \sum_{j=0}^n \frac{(-1)^{n-j} u^{2n}}{(2n)!} k^{2j} (k')^{2n-2j} P_{2n,1,j} = -\frac{1}{2k^2} \operatorname{sn}^2\left(iku, \frac{ik'}{k}\right),$$

whereas the cumulant generating function is

$$-\frac{4}{(kk')^2} \sum_{m \geq 0} \frac{(-1)^m 2^{2m}}{z^{2m+2}} \kappa_{2m+2} \frac{u^{2m}}{(2m)!} = sd^2(u, k) = -\frac{1}{k^2} \operatorname{sn}^2\left(iuk, i\frac{k'}{k}\right).$$

Identifying the coefficient of  $\frac{u^{2m}}{(2m)!}$  in each expression produces

$$-\frac{4}{(kk')^2} \frac{(-1)^m 2^{2m}}{z^{2m+2}} \kappa_{2m+2} = 2 \sum_{j=0}^m (-1)^{m-j} k^{2j} (k')^{2m-2j} P_{2m,1,j}$$

or

$$\kappa_{2m+2} = \frac{z^{2m+2}}{2^{2m+1}} \sum_{j=0}^m (-1)^{j-1} k^{2j+2} (k')^{2m-2j+2} P_{2m,1,j}.$$

□

**Corollary 10.** *In the standard case  $k = k' = \frac{1}{\sqrt{2}}$ , we have  $K\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}$ ,  $z = \frac{\Gamma^2\left(\frac{1}{4}\right)}{2\pi^{\frac{3}{2}}}$  and the expansion of cumulants simplifies to*

$$\kappa_{2n+2} = \frac{\Gamma^{4n+2}\left(\frac{1}{4}\right)}{2^{3n+4}\pi^{\frac{3}{2}}} \sum_{j=0}^n (-1)^{j-1} P_{2n,1,j}.$$

#### 4.6. Some additional remarks about the standard case.

(1) The sequence  $\{Q_n\}$  defined by

$$(4.4) \quad Q_{2n} = \sum_{j=0}^{n-1} (-1)^{j-1} P_{2n-2,1,j}$$

counts the difference between the number of permutations with one odd cycle peak and an odd number of even cycle peaks and the number of permutations with one odd cycle peak and an even number of even cycle peaks. Moreover

$$Q_4 = 2, \quad Q_6 = 0, \quad Q_8 = -144, \quad Q_{10} = 0, \quad Q_{12} = 96768.$$

(2) The sequence  $\{\frac{1}{2}Q_{2n}\}$  appears as OEIS A260779 and coincides with the sequence of Taylor coefficients of the reciprocal of Weierstrass'  $\wp$  function in the lemniscatic case. More precisely,

$$\frac{1}{\wp(u, \{g_2 = 4, g_3 = 0\})} = 2 \frac{u^2}{2!} - 144 \frac{u^6}{6!} + 96768 \frac{u^{10}}{10!} + \dots$$

It was first studied by Hurwitz [11]. It is also proportional by a factor  $(-12)^n$  to the sequence OIES A144849 of Taylor coefficients of the square of the sine lemniscate function

$$\operatorname{sl}(u) = \frac{1}{\sqrt{2}} \operatorname{sd}\left(u\sqrt{2}, \frac{1}{\sqrt{2}}\right).$$

- (3) The result of Thm 9 can be restated as follows: in the standard case  $k = \frac{1}{\sqrt{2}}$ , the cumulants are, for  $n \geq 2$  and with  $z = \frac{\Gamma^2(\frac{1}{4})}{2\pi^{\frac{3}{2}}}$ ,

$$\kappa_{2n} = \left( \frac{z}{2\sqrt{2}} \right)^{2n} Q_{2n}$$

with  $Q_{2n} \in \mathbb{Z}$  given by (4.4). First cases are

$$\begin{aligned} Q_4 &= 2, \quad \kappa_4 = 2 \left( \frac{z}{2\sqrt{2}} \right)^4 = \frac{1}{2^9} \frac{\Gamma^8(\frac{1}{4})}{\pi^6}, \\ Q_6 &= 0, \quad \kappa_6 = 0, \\ Q_8 &= -144, \quad \kappa_8 = -144 \left( \frac{z}{2\sqrt{2}} \right)^6. \end{aligned}$$

## 5. SYMMETRIES

The sequences of polynomials  $\{P_{2n}(k)\}$  and  $\{R_{2n}(k)\}$  and the sequences of moments and cumulants associated with the discrete normal distribution exhibit a natural symmetry with respect to the transformation  $k \mapsto k' = \sqrt{1-k^2}$  of the elliptic modulus, as expressed in the following theorem.

**Theorem 11.** For  $k' = \sqrt{1-k^2}$  and  $n \geq 1$ ,

$$P_{2n}(k') = (-1)^{n-1} P_{2n}(k),$$

and

$$R_{2n}(k') = (-1)^n R_{2n}(k),$$

so that

$$R_{4n}(k') = R_{4n}(k).$$

Moreover, the cumulants are related as

$$\kappa_{2n}(k') = \left( i \frac{K(k')}{K(k)} \right)^{2n} \kappa_{2n}(k).$$

For  $n \geq 2$ , the moments  $\mu_{2n}(k)$  and  $\mu_{2n}(k')$  are related as

$$\mu_{2n}(k') = \sum_{j=0}^n \binom{2n}{2j} \frac{(2n-2j)!}{(n-j)!} \left( i \frac{K(k')}{K(k)} \right)^{2j} \left( \frac{\delta}{\sqrt{2}} \right)^{2n-2j} \mu_{2j}(k),$$

with  $\delta^2 = \sigma^2(k') - \sigma^2(k)$ . The variances are related as

$$(5.1) \quad \frac{\sigma^2(k)}{K^2(k)} + \frac{\sigma^2(k')}{K^2(k')} = \frac{1}{2\pi} \frac{1}{K(k)K(k')}.$$

*Proof.* From (4.2),

$$\kappa_{2n}(k') = (-1)^{n-1} \left( \frac{z'}{2} \right)^{2n} P_{2n-2}(k') = (-1)^{n-1} \left( \frac{z'}{2} \right)^{2n} (-1)^n P_{2n-2}(k),$$

with

$$\frac{z'}{2} = \frac{K(k')}{\pi} = \frac{K(k')}{K(k)} \frac{z}{2},$$

so that

$$\kappa_{2n}(k') = - \left( \frac{K(k')}{K(k)} \right)^{2n} \left( \frac{z}{2} \right)^{2n} P_{2n-2}(k) = (-1)^n \left( \frac{K(k')}{K(k)} \right)^{2n} \kappa_{2n}(k).$$

Using Legendre's identity

$$K(k)E(k') + K(k')E(k) - K(k)K(k') = \frac{\pi}{2}$$

and the expression of the variance  $\sigma^2(k)$  in (3.1), we deduce (5.1).  $\square$

*Remark 12.* An equivalent statement of the previous result is as follows: for  $X_k$  and  $X_{k'}$  two discrete normal random variables with respective elliptic moduli  $k$  and  $k'$ , and two standard Gaussian random variables  $N$  and  $N'$ , the random variables

$$X_{k'} + \sigma_{k'}^2 N' \quad \text{and} \quad \left( i \frac{K(k')}{K(k)} \right) X_k + \sigma_k^2 N$$

have the same moments and cumulants.

## 6. A NUMERICAL APPROACH

A numerical toolbox [19] written by D. Zeilberger allows us to compute the sequence  $d(n)$  (and many other quantities related to Romik's paper) under the Maple environment. The first 200 values of  $d(n)$  are precomputed while the higher-order ones are computed using a recurrence formula derived by D. Romik [16, Thm 7] that requires the computation of the Taylor coefficients of the two functions

$$U(t) = \frac{{}_2F_1\left(\frac{3}{4}, \frac{3}{4}; \frac{3}{2}; 4t\right)}{{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; 4t\right)}, \quad V(t) = \sqrt{{}_2F_1\left(\frac{1}{4}, \frac{1}{4}; \frac{1}{2}; 4t\right)}.$$

We propose here another method based on the prior computation of the cumulants using a quadratic recurrence, and on a linear recurrence between the cumulants and the moments.

**Theorem 13.** *The cumulants of the standard discrete normal distribution satisfy the recurrence, for  $n \geq 2$ ,*

$$(6.1) \quad \kappa_{4n} = -6 \sum_{j=0}^{n-2} \binom{4n-4}{4j+2} \kappa_{4j+4} \kappa_{4n-4j-4}$$

with initial condition  $\kappa_4 = \frac{1}{2^9} \frac{\Gamma^8(\frac{1}{4})}{\pi^6}$ .

The moments can then be recursively computed in terms of the cumulants using (see [17]), for  $n \geq 2$ ,

$$(6.2) \quad \mu_n = \kappa_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} \kappa_m \mu_{n-m}.$$

*Proof.* The cumulants are related to the OEIS sequence  $\{A_{4n}\}$  A144849

$$A_0 = 1, \quad A_1 = 6, \quad A_2 = 336 \dots$$

as follows:

$$(6.3) \quad \kappa_{4n} = \left(\frac{\kappa_4}{2}\right)^n Q_{4n}, \quad n \geq 1,$$

with

$$Q_4 = 2, \quad Q_8 = -144, \quad Q_{12} = 96768 \dots$$

The sequence  $\left\{\frac{Q_{4n}}{2}\right\}$  is OEIS A260779 and is related to OEIS A144849 by

$$Q_{4n} = 2(-12)^{n-1} A_{n-1}.$$

Therefore,

$$(6.4) \quad \kappa_{4n} = \left(\frac{\kappa_4}{2}\right)^n 2(-12)^{n-1} A_{n-1}.$$

The sequence  $\{A_n\}$  satisfies the recurrence

$$A_{n+1} = \sum_{j=0}^n \binom{4n+4}{4j+2} A_j A_{n-j}.$$

Substituting (6.4) yields the recurrence (6.1). □

For the computation of the moments, the mixed recurrence (6.2) between moments and cumulants can also be replaced by a direct formula for the moments in terms of the cumulants, at the price of the computation of a determinant (see [17]).

**Theorem 14.** *A determinant representation of the moments of the standard discrete normal distributions is*

$$\mu_n = (-1)^{n-1} (n-1)! \det \begin{bmatrix} \frac{\kappa_2}{1!} & 1 & 0 & 0 & 0 & \dots & 0 \\ \frac{\kappa_3}{2!} & \frac{\kappa_1}{2 \cdot 0!} & 1 & 0 & 0 & & \\ \frac{\kappa_4}{3!} & \frac{\kappa_2}{2 \cdot 1!} & \frac{\kappa_1}{3 \cdot 0!} & 1 & 0 & & \\ \frac{\kappa_5}{4!} & \frac{\kappa_3}{2 \cdot 2!} & \frac{\kappa_2}{3 \cdot 1!} & \frac{\kappa_1}{4 \cdot 0!} & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & 0 \\ & & & & & & 1 \\ \frac{\kappa_n}{(n-1)!} & \frac{\kappa_{n-2}}{2(n-3)!} & \frac{\kappa_{n-3}}{3(n-4)!} & \frac{\kappa_{n-4}}{4(n-5)!} & & & \frac{\kappa_1}{(n-1) \cdot 0!} \end{bmatrix}, \quad n \geq 2.$$

*Proof.* The moments are related to the cumulants via [17]

$$\frac{\kappa_n}{(n-1)!} = \frac{\mu_n}{(n-1)!} - \sum_{m=0}^{n-2} \frac{\kappa_{m+1}}{m!} \frac{\mu_{n-m-1}}{(n-m-2)!} \frac{1}{n-m-1}.$$

With the notation

$$(6.5) \quad \tilde{\kappa}_n = \frac{\kappa_n}{(n-1)!}, \quad \tilde{\mu}_n = \frac{\mu_n}{(n-1)!},$$

this is expressed as the linear system

$$\begin{bmatrix} \tilde{\kappa}_2 \\ \tilde{\kappa}_3 \\ \tilde{\kappa}_4 \\ \tilde{\kappa}_5 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{2}\tilde{\kappa}_1 & 1 & 0 & 0 & \\ -\frac{1}{2}\tilde{\kappa}_2 & -\frac{1}{3}\tilde{\kappa}_1 & 1 & 0 & \\ -\frac{1}{2}\tilde{\kappa}_3 & -\frac{1}{3}\tilde{\kappa}_2 & -\frac{1}{4}\tilde{\kappa}_1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \tilde{\mu}_2 \\ \tilde{\mu}_3 \\ \tilde{\mu}_4 \\ \tilde{\mu}_5 \\ \vdots \end{bmatrix}.$$

Using Cramer's formula, we deduce

$$\begin{aligned} \tilde{\mu}_n &= \det \begin{bmatrix} \tilde{\kappa}_2 & 1 & 0 & 0 & 0 & \cdots \\ \tilde{\kappa}_3 & -\frac{\tilde{\kappa}_1}{2} & 1 & 0 & 0 & \\ \tilde{\kappa}_4 & -\frac{\tilde{\kappa}_2}{2} & -\frac{\tilde{\kappa}_1}{3} & 1 & 0 & \\ \tilde{\kappa}_5 & -\frac{\tilde{\kappa}_3}{2} & -\frac{\tilde{\kappa}_2}{3} & -\frac{\tilde{\kappa}_1}{4} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \tilde{\kappa}_n & -\frac{\tilde{\kappa}_{n-2}}{2} & -\frac{\tilde{\kappa}_{n-3}}{3} & -\frac{\tilde{\kappa}_{n-4}}{4} & \cdots & 1 \end{bmatrix} \\ &= (-1)^{n-1} \det \begin{bmatrix} \tilde{\kappa}_2 & 1 & 0 & 0 & 0 & \cdots \\ \tilde{\kappa}_3 & \frac{\tilde{\kappa}_1}{2} & 1 & 0 & 0 & \\ \tilde{\kappa}_4 & \frac{\tilde{\kappa}_2}{2} & \frac{\tilde{\kappa}_1}{3} & 1 & 0 & \\ \tilde{\kappa}_5 & \frac{\tilde{\kappa}_3}{2} & \frac{\tilde{\kappa}_2}{3} & \frac{\tilde{\kappa}_1}{4} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \tilde{\kappa}_n & \frac{\tilde{\kappa}_{n-2}}{2} & \frac{\tilde{\kappa}_{n-3}}{3} & \frac{\tilde{\kappa}_{n-4}}{4} & \cdots & 1 \end{bmatrix}. \end{aligned}$$

Substituting (6.5) yields

$$\mu_n = (-1)^{n-1} (n-1)! \det \begin{bmatrix} \frac{\kappa_2}{1!} & 1 & 0 & 0 & 0 & \cdots \\ \frac{\kappa_3}{2!} & \frac{\kappa_1}{2 \cdot 0!} & 1 & 0 & 0 & \\ \frac{\kappa_4}{3!} & \frac{\kappa_2}{2 \cdot 1!} & \frac{\kappa_1}{3 \cdot 0!} & 1 & 0 & \\ \frac{\kappa_5}{4!} & \frac{\kappa_3}{2 \cdot 2!} & \frac{\kappa_2}{3 \cdot 1!} & \frac{\kappa_1}{4 \cdot 0!} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ \frac{\kappa_n}{(n-1)!} & \frac{\kappa_{n-2}}{2(n-3)!} & \frac{\kappa_{n-3}}{3(n-4)!} & \frac{\kappa_{n-4}}{4(n-5)!} & \cdots & \frac{1}{(n-1) \cdot 0!} \end{bmatrix}.$$

□

Finally, notice that an expression of the cumulants as a sum over partitions is given by [17]

$$\mu_n = \sum_{\pi \vdash n} \kappa_\pi$$

with, for  $(\pi_1, \dots, \pi_k)$  a partition of  $n$ ,  $\kappa_\pi = \prod_{j=1}^k \kappa_{\pi_j}$ .

## 7. A CONJECTURE

We look now at the equivalent of Romik's sequence  $d(n)$  for some values of  $k \neq \frac{1}{\sqrt{2}}$ . Let us define the parameterized sequence

$$d_k(n) = R_{4n}(k)$$

so that  $d(n) = d_{\frac{1}{\sqrt{2}}}(n)$ , and conjecture its integrality for some values of the elliptic modulus  $k$ . The following table shows the sequences  $\alpha_m d_k(m)$ , where  $\alpha_m$  is a properly chosen prefactor, for all values  $k = \frac{1}{\sqrt{p}}$  with  $2 \leq p \leq 7$ . We

conjecture that these sequences are integral for any  $m \geq 0$ . More generally, we conjecture that for any fixed integer value of  $p$ , there are rational constants  $C_p, D_p$  such that

$$\frac{(C_p p^2)^m}{D_p} d_{\frac{1}{\sqrt{p}}}(m) \in \mathbb{Z}.$$

$k$	$\alpha_m$	sequence $\alpha_m d_k(m), m \geq 0$
$\frac{1}{\sqrt{3}}$	$\left(\frac{3}{2}\right)^{2m}$	1, 3, 7, 2953, 291969, 12470011, -148801721
$\frac{1}{2}$	$\frac{2^{3m}}{2}$	1, 29, 43, 116171, 78138169, 40042714493
$\frac{1}{\sqrt{5}}$	$\frac{5^{2m}}{2^{3m}}$	1, 17, 105, 4521, 1802457, 535169097
$\frac{1}{\sqrt{6}}$	$\frac{2^m 3^{2m}}{5}$	1, 123, 8059, 724877, 1686624921, 3594330803003
$\frac{1}{\sqrt{7}}$	$\frac{14^{2m}}{3}$	1, 97, 5959, 293923, 294067681, 490927058857

We note that none of these sequences appears in the online encyclopedia OEIS.

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