NP-completeness of the game *Kingdomino*™

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Abstract

Kingdomino™ is a board game designed by Bruno Cathala and edited by *Blue Orange* since 2016. The goal is to place 2×1 dominoes on a grid layout, and get a better score than other players. Each 1×1 domino cell has a color that must match at least one adjacent cell, and an integer number of crowns (possibly none) used to compute the score. We prove that even with full knowledge of the future of the game, in order to maximize their score at *Kingdomino*™, players are faced with an NP-complete optimization problem.

1 Introduction

Kingdomino™ is a 2-4 players game where players, turn by turn, place 2×1 dominoes on a grid layout (each player has its own board, independent of others). Each domino has a color on each of its two 1×1 cells, and when a player is given a domino to place on its board, he or she must do so with a color match along at least one of its edges. Also if a domino *can* be placed (with at least one possible color match), then it *must* be placed. Finally, each player starts from a 1×1 tower matching any color. The winner of the game is the player that has the maximum score among the competitors. The computation of score will be precised in Section 3, it is basically a weighted sum of the number of cells in each monochromatic connected components on the player's board.

The purpose of this article is to prove that, even with full knowledge of the future of the game (*i.e.* the sequence of dominoes he or she will have to place), a player is faced with an NP-complete optimization problem.

Section 2 reviews some results around games complexity and domino problems, Section 3 presents our theoretical modeling of *Kingdomino*™, Section 4 illustrates the combinatorial explosion of possibilities, and Section 5 proves the NP-hardness result.

2 Computational complexity of games and dominoes

Kingdomino™ has been studied in [8], where the authors compare different strategies to play the game, via numerical simulations.

Understanding the computational complexity of games has raised some interest in the computer science community, and many games have been proven to be complete for NP or co-NP. Examples include *Minesweeper* [14, 18], *SET* [15], *Hanabi* [3], *Nintendo* games [2] and *Candy Crush* [9, 20].

Domino tiling problems are a cornerstone for computer science, from undecidable ones [4, 11, 13] to simple puzzles [10, 12, 19]. Tiling some board with dominoes under constraints has already been seen to be NP-complete, and constructions vary according to the model definition [5, 6, 16, 17]. The model of [21] is close to *Kingdomino*^{TM}, its construction can be adapted to prove that starting from a board with some dominoes already placed, completing it to optimize one's score is NP-complete. The challenging part of the present work is to start from nothing else but the tower of *Kingdomino*^{TM}.

3 Model and problem statement

The way dominoes are chosen by players is at the heart of strategies one may elaborate to play *Kingdomino*^{TM}. In order to apply the theory of computational complexity to this game, we will consider a one player model, which concentrates on an even more essential aspect of the game: how to maximize one's score, which is the source of domino choices. Also, in the game *Kingdomino*^{TM} there is a fixed set of colors (6) and a fixed multiset of dominoes (48, some dominoes have more than one occurrence), but we will abstract these quantities to be any finite (multi)set. Definitions are illustrated on Figure 1.

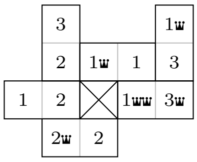


Figure 1: Example of K-tiling by ([2♛ | 2] , [1 | 2] , [1♛ | 1] , [1♛♛ | 3♛] , [3 | 1♛] , [2 | 3]), with score $0 + 1 + 4 + 9 + 2 + 0 = 16$ (regions are ordered from top to bottom, left to right).

A *domino* is a 2×1 rectangle, with one *color* among a finite set on each of its two 1×1 *cells*. A domino also has a number of *crowns* on each of its cells, used to compute the score. For convenience we consider colors to be integers, and represent a domino as follows: [1♛ | 2] for the domino with one cell of color 1 with one crown, and one cell of color 2 with no crown. A *tiling* is an overlap free placement of dominoes on the $\mathbb{Z}^2$ board, with a special cell at position $(0, 0)$ called the *tower*. Given a sequence of n dominoes $\tau = (\boxed{c_1 \mid c_2}, \dots, \boxed{c_{2n-1} \mid c_{2n}})$ possibly with crowns, a *K-tiling by τ* is a tiling respecting the following constraints defined inductively.

- The tiling with only the tower at position $(0, 0)$ is a K-tiling by $\varnothing$ (case $n = 0$).
- Given a K-tiling by the $n - 1$ first dominoes of τ, the last domino of colors c_{2n-1} and c_{2n} can be placed on a pair of adjacent positions, if and only if at least one of its two cells is adjacent to a cell of the same color that has already been placed, or is adjacent to the tower. It is lost (not placed) if and only if it can be placed

nowhere. This gives a K-tiling by the n dominoes of τ.

Hence dominoes must be placed in the order given by the sequence τ. Note that the definition of K-tiling does not take into consideration the crowns. They are only used to compute the score, as we will explain now.

The *score* of a K-tiling is the sum, for each monochromatic connected component (called *region*), of its number of cells times the number of crowns it contains. Note that a color may give rise to more than one region (as on the example of Figure 1), and that a *region* scores no point if it contains no crown. We will say that some cell (resp. domino) must be *connected* to some region, to mean that it (resp. one of its two cells) must belong to this monochromatic connected component.

We are ready to state the problem.

K-tiling problem
input: a sequence of dominoes τ and an integer s.
question: is there a K-tiling by τ with score at least s?

Given a tiling where each domino of the sequence τ is identified (a potential solution, *aka* certificate), one verifies domino after domino that it is indeed a K-tiling by τ, and computes the score to verify that it is indeed at least s, in polynomial time. Hence **K-tiling problem** belongs to NP.

4 Counting K-tilings

To give an idea of the combinatorial explosion one faces when playing *Kingdomino*^{TM} or when deciding some **K-tiling problem** instance, we propose in Table 1 to count the number of possible K-tilings for some small sequences of dominoes. These results were obtained by numerical simulations.

dominoes	([1/1♛], [1/1], [1/1], [1/1])	([1/1♛], [2/2♛], [3/3♛], [4/4♛])	([2♛/1♛], [4♛/3♛], [6♛/5♛], [8♛/7♛])
1st (score 2)	2 (24)	2 (24)	4 (24)
2nd (score 4)	19 (752)	13 (400)	52 (400)
3rd (score 6)	253 (35448)	63 (4032)	504 (4032)
4th (score 8)	3529 (2176064)	141 (18048)	2256 (18048)

Table 1: Number of K-tilings reaching the maximum possible score, for some small sequences of dominoes. Rotations and axial symmetries are counted only once, and the positions of crowns are not taken into account (in parenthesis the full counts are given).

Table 1 may be compared to the number of domino tilings of a $2n \times 2n$ square, appearing in the *Online Encyclopedia of Integer Sequences* under reference `A004003` [1]: 1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000, ...

5 NP-hardness of K-tiling problem

In this section we prove the main result of the article.

Theorem 1. K-tiling problem *is* NP*-hard.*

Proof. We make a polynomial time many-one reduction from **4-Partition problem**, which is known to be strongly NP-complete [7]. This is important, since we encode the instance of **4-Partition problem** in unary into an instance of **K-tiling problem** (basically, with $28x$ domino cells for each item of size x).

4-Partition problem
input: n items of integer sizes $x_1, \dots, x_n$, and $m = \frac{n}{4}$ bins of size k, with n a multiple of four, $x_i > 0$ for all i, and $\sum_{i=1}^{n} x_i = km$.
question: is it possible to pack[1] the n items into the m bins, with exactly four items (whose sizes thus sum to k) per bin?

Given such an instance of **4-Partition problem**, we first multiply by 28 all item and bin sizes (for technical reasons to be explained later) and consider the equivalent instance with n items of strictly positive integer sizes $x_1 \leftarrow 28x_1, \dots, x_n \leftarrow 28x_n$ and m bins of size $k \leftarrow 28k$ (for convenience we keep the initial notations with x and k). We then construct (in polynomial time from a unary encoding) the following sequence of dominoes τ:

1. *guardians* $\boxed{1\text{w} \mid 1}, \boxed{2 \mid 2}, \boxed{3 \mid 3}, \boxed{4 \mid 4}$,
2. *square* $(\boxed{1 \mid 1})^{18m^2-6}$,
3. *contour* $\boxed{1 \mid 5\text{w}}, \boxed{5 \mid 6\text{w}}, \boxed{6 \mid 7\text{w}}, \boxed{7 \mid 8\text{w}}, \boxed{8 \mid 1}, \boxed{8 \mid 1}, \boxed{8 \mid 2}, \boxed{8 \mid 9\text{w}}$,
   $(\boxed{9 \mid 9})^{9m-8}, \boxed{9 \mid 10\text{w}}$,
   $\boxed{10 \mid 11\text{w}}, \boxed{11 \mid 12\text{w}}, \boxed{12 \mid 13\text{w}}, \boxed{13 \mid 14\text{w}}, \dots, \boxed{3m+10 \mid 3m+11\text{w}}$,
   $\boxed{3m+11 \mid 3m+12\text{w}}, \boxed{3m+12 \mid 5}$,
4. *guide* $(\boxed{6 \mid 7})^{18m^2+12m}$,
5. *arms* $(\boxed{10 \mid 11})^{\frac{k}{4}+2}, (\boxed{13 \mid 14})^{\frac{k}{4}+2}, (\boxed{16 \mid 17})^{\frac{k}{4}+2}, \dots, (\boxed{3m+10 \mid 3m+11})^{\frac{k}{4}+2}$
6. *anchors* $(\boxed{12 \mid 3m+13})^2, (\boxed{15 \mid 3m+13})^2, \dots, (\boxed{3m+9 \mid 3m+13})^2$,
7. *items* for each x_i we have $\boxed{3m+13 \mid 3m+13+i\text{w}}, (\boxed{3m+13+i \mid 3m+13+i})^{\frac{x_i}{2}-1}$,
8. *zippers* $\boxed{11 \mid 3m+13+n+1\text{w}}, \boxed{3m+13+n+1 \mid 13}$,
   $\boxed{14 \mid 3m+13+n+2\text{w}}, \boxed{3m+13+n+2 \mid 16}$,
   $\dots$,
   $\boxed{3m+8 \mid 4m+13+n\text{w}}, \boxed{4m+13+n \mid 3m+10}$,

[1] Of course an item cannot be split.

and the target score $s = 72m^2 + 54m + \frac{k}{2}(3m+1) + 1$. This is our instance of **K-tiling problem**. Let us now prove that there exists a packing of the n items into the m bins of size k with four items per bin if and only if there exists a K-tiling by τ with score at least s.

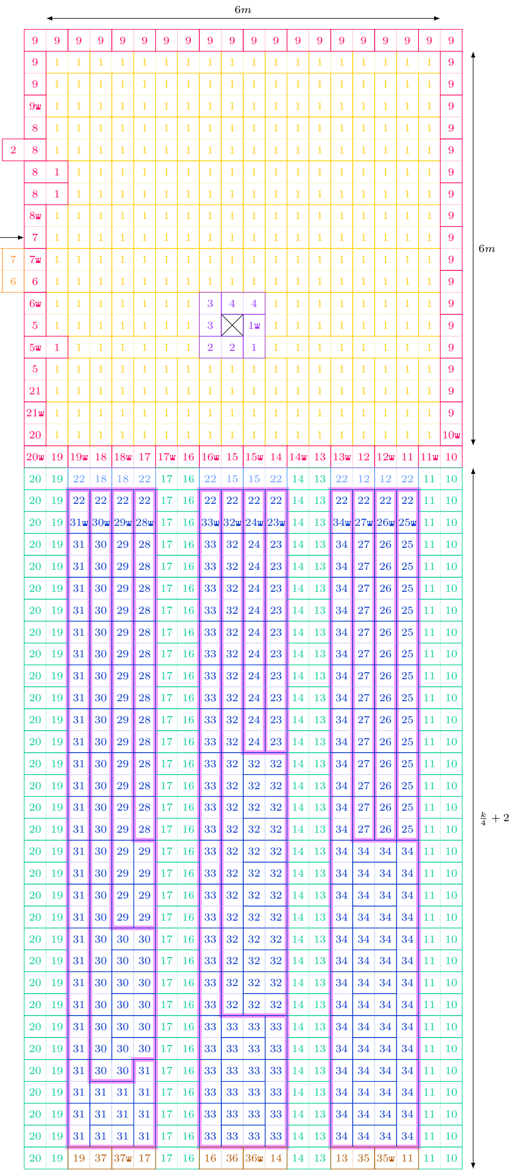
$\boxed{\Rightarrow}$ Suppose there exists a packing of the n items into the m bins of size k with exactly four items per bin, and let X_j be the set of items in bin j. We construct the following K-tiling by τ (see Figure 2).

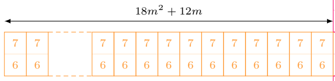
1. Place the four *guardians* dominoes around the tower.
2. Around this create a square of size $6m \times 6m$ with the *square* dominoes $\boxed{1 \mid 1}$, leaving three dents empty on the left border of the square at the fifth, twelfth and thirteenth positions from the bottom left corner (the square has area $36m^2$, minus 9 cells already taken by the *guardians* dominoes and the tower, minus 3 dents, hence exactly the $18m^2 - 6$ *square* dominoes are required).
3. Make a path clockwise around the square with the *contour* dominoes in this order, filling the three dents with cells of color 1, leaving the cell of color 2 outside, and starting with the first dent at the fifth position above the bottom left corner of the square (the contour has length $4(6m+1)$, corresponding exactly to the $12m+4$ *contour* dominoes with four dents).
4. Stack all *guide* dominoes on the left of the $\boxed{7\text{♛} \mid 6}$ domino of the border.
5. Stack all *arms* dominoes, color by color, below the corresponding dominoes of the border. Observe that they match exactly one domino over three of the bottom border, creating $m+1$ stacks of length $\frac{k}{4}+2$, and therefore m bins of size $k+8$ in between.
6. Place a pair of *anchors* dominoes per bin, matching the existing colors, as on Figure 2.
7. Place *items* dominoes corresponding to items of X_j in bin j, filling k cells of each bin and leaving the last row of four cells empty.
8. Close each bin with the corresponding *zippers* dominoes (anchors *dominoes* fill four cells, consequently the *items* dominoes leave four cells in each bin, exactly the number of cells required for the *zipper* dominoes to match colors onto the arms on both ends[2]).

The score of this K-tiling is s, as detailed on Figure 2.

$\boxed{\Leftarrow}$ This is the challenging part of the proof, where we will argue that the construction of a K-tiling by τ with score at least s is compelled to have the structure described above and illustrated on Figure 2, which corresponds to solving the **4-Partition problem** instance. The proofs of some claims are postponed.

[2]Note that the pattern of placement sketched on Figure 2 can be extended to pack each bin with *items* dominoes corresponding to any four items of sizes summing to k, and leaving four cells on the bottom end for the *zippers* dominoes.





dominoes	score [general form]
soldiers	2 [2]
square	312 [$2(18m^2 - 6)$]
contour	79 [$4(6m + 1) + 3$]
guide	396 [$2(18m^2 + 12m)$]
arms	256 [$2(\frac{k}{4} + 2)(m + 1)$]
anchors	6 [$2m$]
items	348 [$m(k - 4)$]
zippers	12 [$4m$]
total	1411 [$72m^2 + 54m + \frac{k}{2}(3m + 1) + 1$]

Figure 2: To reduce the height of this figure, original sizes have only been multiplied by 4 instead of 28. A K-tiling by τ with score s (hence solving the **K-tiling problem** instance), from a solution to the **4-Partition problem** instance with $n = 12$, $m = 3$, $k = 120$ (originally $k = 30$), and item sizes $12, 12, 16, 16, 16, 16, 24, 40, 40, 48, 48, 72$ (originally $3, 3, 4, 4, 4, 4, 6, 10, 10, 12, 12, 18$): first bag $72 + 16 + 16 + 16$, second bag $48 + 48 + 12 + 12$, third bag $40 + 40 + 24 + 16$. Dominoes *soldiers* in purple, *square* in yellow, *contour* in red, *guide* in orange, *arms* in green, *anchors* in light blue, *items* in dark blue (groups are highlighted in pink), and *zippers* in brown. Note that the anchor color $3m + 13$ (on which groups of item dominoes can match) equals 22 on this exemple.

Suppose there exists a K-tiling by τ with score s. First notice that s is an upper bound on the score one can obtain with a K-tiling by τ, as it is the sum for each color of the number of cells of this color times the number of crowns on cells of this color. It must therefore correspond to a K-tiling with one region per color, except for colors $2, 3, 4$ and $3m + 13$ which have no crown. This will be the main assumption that will guide us as we study the dominoes chronologically. Also remark that all dominoes must be placed: at the beginning colors $2, 3, 4$ can always match the *tower*, and afterwards colors $2, 3m + 13$ appear only on dominoes with another color bringing some necessary points to the sum s.

1. There is no choice but to place the four *guardians* dominoes on the four sides of the tower. As a consequence, we don't have to treat the particular case of the tower anymore.
2. All *square* dominoes are monochromatic, hence they form one large region of color 1 (remark that they can, and therefore must, all be placed). We have now $9 + 2(18m^2 - 6) = 36m^2 - 3$ cells occupied by some dominoes or the tower.
3. For the *contour* dominoes the three cells of color 1 must be connected to the unique region of color 1 since this color will not appear anymore. Let a *pseudo-path* be a path on the grid which can contain some large connected component, *i.e.* a path plus some domino cells connected to it. The length of a pseudo-path is the length of the shortest path it contains from its first to its last cell.

   Claim 1. *All* contour *dominoes must be placed (to reach score s), and they must form a pseudo-path that cycles, of length at most* $4(6m + 1)$.

   It is connected to the region of color 1 via three dominoes $\boxed{1 \mid 5♛}$, $\boxed{8 \mid 1}$, $\boxed{8 \mid 1}$:
   - two of them are intended to frame the $\boxed{6 \mid 7♛}$ domino,
   - and the last one is for parity of cells number, since we have an odd number of occupied cells so far that is intended to form a square of even side length.

   The pseudo-cycle of *contour* dominoes may have the region of color 1 either inside its outer face, or inside its inner face (in this case the pseudo-cycle surrounds the region of color 1).
4. The *guide* dominoes enforces that the cycle of *contour* dominoes surrounds the region of color 1.

   Claim 2. *The* guide *dominoes must be stacked one after the other in a straight segment, rooted at the analogous* contour *domino*

   Claim 3. *The pseudo-cycle of* contour *dominoes must have the region of color* 1 *inside its inner face.*

   After this step we have $36m^2 - 3$ occupied cells surrounded by a cycle of $4(6m+1)$ cells with three additional cells (with color 1) inside the cycle, hence $(6m)^2$ cells inside the cycle which is just long enough to make a square of side $6m + 2$ around it. However, any other shape would either require a too long path, or leave a too small area inside.

<u>**Intermediate conclusion:**</u> at this point we have a square of *contour* cells with the tower, *guardians* and *square* dominoes inside, three dents of color 1 inside, one dent of color 2 outside, and a stack of *guide* dominoes starting from the corresponding $\boxed{6 \mid 7♛}$ *contour* domino (the reader can refer to Figure 2 for an illustration).

We will argue thereafter why the contour is well aligned, with the third line of *contour* dominoes all on the same side of the square.

5. The *arm* dominoes must create m bins.

   Claim 4. *The* arm *dominoes must be placed into* $m+1$ *bicolor stacks of length* $\frac{k}{4}+2$ *starting from the corresponding* contour *dominoes (separated by four positions), and joined with a pair of* zipper *dominoes.*

   This also explains why *contour* dominoes are well aligned, with the third line of *contour* dominoes all on the same side of the square.

   <u>**Intermediate conclusion:**</u> after these dominoes the K-tiling by τ with score s must have created m bins (with $m+1$ arms) of size $4 \times (\frac{k}{4}+2)$. This size explains why the bin size (and consequently all item sizes) of the original **4-Partition problem** instance is converted to a multiple of 4.

6. Each pair of *anchors* dominoes must be placed so that each color already present in the contour (from 12 to $3m+9$) form one region because these are the last dominoes with these colors. So there is a pair of *anchor dominoes* at the rear of each bin. Remark that color $3m+13$ has no crown hence it can be split into multiple regions. The purpose of this color is to be an anchor inside each bin, intended for groups of *items* dominoes to match.

7. For each item x_i we have a group of *items* dominoes, where the first domino of color $3m+13$ allows to match a bin anchor, and then all other dominoes of the group will form one region from this anchored domino (with the unique color $3m+13+i$ for each i), for a total of x_i cells.

   Claim 5. *For each bin, at most four groups of* items *dominoes can match its anchors.*

   <u>**Intermediate conclusion:**</u> As all n groups of *items* dominoes must be placed on the board to reach score s, we must have exactly four groups of *items* dominoes in each of the $m = \frac{n}{4}$ bins (corresponding to the values of four items from the **4-Partition problem** instance).

8. The *zippers* dominoes have the purpose of closing bins, with one pair of zippers matching the colors of each bin's arms. They must join the two arms of each bin with a path of length four (because of the unique pair of cell colors $3m+13+n+1$ to $4m+13+n$). However this is possible if and only if no *items* dominoes exceed a volume of $4 \times (\frac{k}{4}+1)$ inside the bin (leaving the last row of four positions of each

bin for the zipper), *i.e.* each bin contains four groups of *items* dominoes for a sum of at most k cells (anchors already occupy four cells). Observe that when they contain a total of at most $\frac{k}{2}$ dominoes, it is always possible to place four groups of *items* dominoes in a bin and leave the last row for a pair of *zippers* dominoes (as on the example of Figure 2).

Conclusion: to reach score s with a K-tiling by τ, a player must close the zipper on top of each of the m bins and therefore trap inside each of them four items of sum at most k, for a total of $4m = n$ items, therefore solving the **4-Partition problem** instance. ☐

Proof of Claim 1. For the pseudo-path that cycles, by induction on *contour* dominoes:

- either there is no occurrence of a color after the *contour* dominoes hence it must directly form one region (case of dominoes with color 8, and color 5 in the last *contour* domino which must close the cycle; *e.g.* when placing domino $\boxed{3m+12 \mid 5}$ its cell of color 5 must match the existing region),
- or there is no other placed occurrence of one of its colors apart from the previous *contour* domino (case of all other *contour* dominoes; *e.g.* when placing domino $\boxed{6 \mid 7♛}$ its cell of color 6 must match domino $\boxed{5 \mid 6♛}$),
- and the group of $\boxed{9 \mid 9}$ dominoes forms one region along the path.

For the length calculation, see Figure 3. ☐

Proof of Claim 2. It follows from the fact that there are no other dominoes of colors 6 nor 7 after the *guide* dominoes. Details are given on Figure 4. ☐

Proof of Claim 3. Observe that if the pseudo-cycle of *contour* dominoes have the region of color 1 on its outer face, then the $18m^2 + 12m$ $\boxed{6 \mid 7}$ dominoes would have been placed inside the pseudo-cycle (at most $\frac{4(6m+1)}{2}$ of them) or inside the region of color 1 (at most $18m^2 - 6$ of them), but they are too numerous so it would be impossible to have simultaneously a unique region for colors 6 and 7 (see Figure 5). ☐

Proof of Claim 4. The placement describe in the claim is the only way to get one region per color $10, 11, 13, 14, 16, 17, \ldots, 3m + 10, 3m + 11$. Details are given on Figure 6. ☐

Proof of Claim 5. Each item has size at least 28 and therefore corresponds to at least 14 dominoes, however after placing a pair of *anchors* dominoes and four of these minimum size groups of 14 *items* dominoes in any possible way, no anchor cell of color $3m + 13$ is available for a fifth group of *items* dominoes (see details on Figure 7). This argument explains why all item and bin sizes of the original **4-Partition problem** instance have been multiplied by 28 (and not simply by 4): so that each group of *items* dominoes is large enough to enforce that at most four per bin can match the anchor. ☐

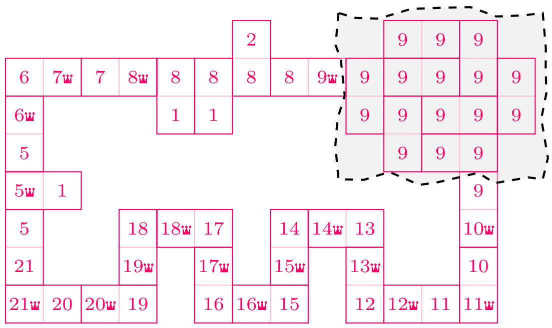


Figure 3: *Contour* dominoes must form a pseudo-path that cycles, with one large region of color 9 that may not be, strictly speaking, a path (dashed, containing the $9m-8$ $\boxed{9 \mid 9}$ dominoes), one region of color 8, and one region of color 5. Colors 1 and 2 are not part of the path, hence dominoes $\boxed{1 \mid 5\mathtt{w}}$ $\boxed{8 \mid 1}$, $\boxed{8 \mid 1}$ and $\boxed{8 \mid 2}$ count only one in its length, which is therefore at most (domino by domino, if regions of colors 5, 8 and 9 are as long as possible) $1+2+2+2+1+1+1+2+2(9m-8)+2+2(3m+1)+4 = 4(6m+1)$.

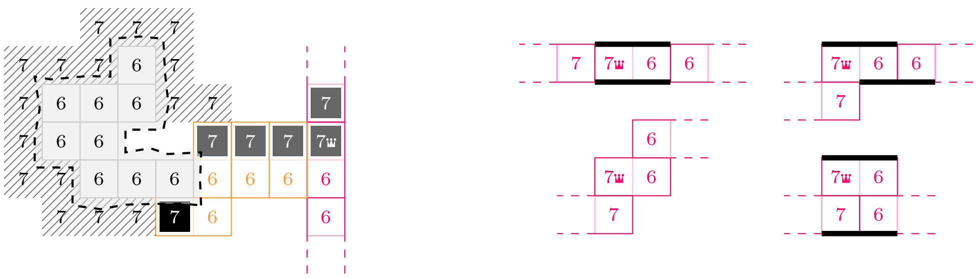


Figure 4: Left: when a first domino is not correctly stacked (fourth domino in this example), without loss of generality with color 7 disconnected, then a path of color 7 must join the two regions of this color (black and grey regions, example path hatched), going around the region of 6 (because there are no more dominoes with color 6 nor 7 after *guide* dominoes). However this path must contain some 90° angle, leaving some cell of color 7 with no neighbor of color 6, which is impossible with only $\boxed{6 \mid 7}$ dominoes. The same argument applies to all cases presented on the right. Right: possible ways to arrange the $\boxed{6 \mid 7\mathtt{w}}$ *contour* domino and the cells of colors 6 and 7 around it (up to rotation, axial symmetry, and swap of colors 6 and 7), with possible stack positions highlighted.

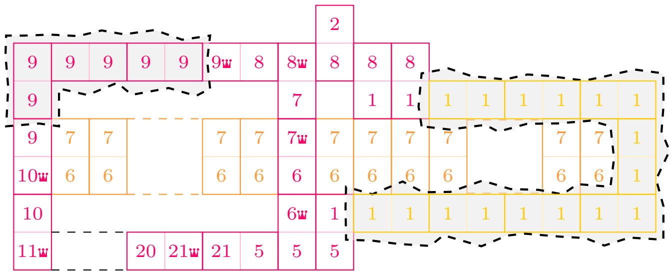


Figure 5: If the pseudo-cycle of *contour* dominoes has the region of color 1 on its outer face, then one cannot stack all the $18m^2+12m$ $\boxed{6\ \vert\ 7}$ dominoes (the region of 1 contains $2(18m^2-6)$ cells, and the contour contains $4(m+1)$ cells).

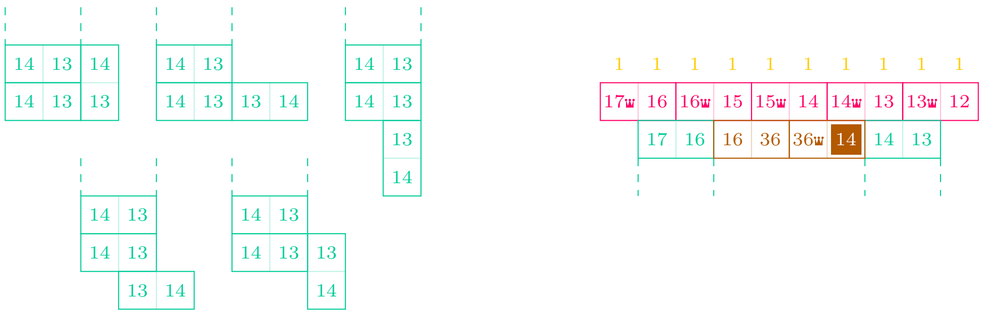


Figure 6: This case is similar to the stack of *guide* dominoes presented on Figure 4, except that there will be one more cell of each color, in *zipper* dominoes (appart from 10 and $3m+11$). Domino $\boxed{13\ \vert\ 14}$ is taken as an example. Left: possible ways to misplace an *arm* domino when some *arm* dominoes are already well stacked. In this case, at least two extra cells of the disconnectedd color are required in order to have one region of each color at the end. Right: the only possibility to exploit the extra cell (14 in brown) would be to shift the stack of *arm* dominoes as illustrated. However, *zipper* dominoes are forced to form paths of length four because of color 36, consequently to also have one region of color 16 it is neccesary to shift the next stack of *arm* dominoes and join it as illustrated with the pair of *zipper* dominoes, but then the pair of *anchor* dominoes with color 14 would be lost, which is not permitted to reach score s.

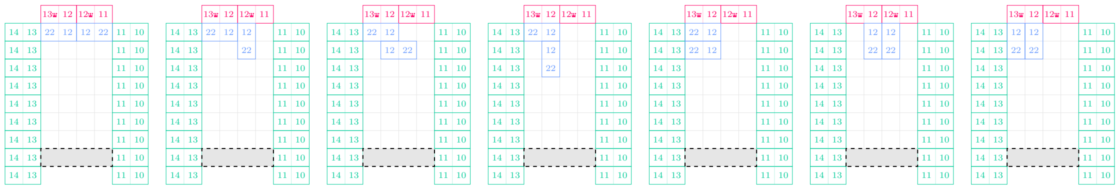


Figure 7: Up to axial symmetry, seven different ways to place a pair of anchors in a bin (colors of the first bin are taken as an example). After placing the first group of at least 14 *items* dominoes, at least one position on the eighth row of the bin (dashed) is occupied (one can simply count available positions); after placing the second group, at least a second position of the eighth row is occupied; after the third group a third one; and after the fourth group the eighth row of the bin is full of *items* dominoes. However, after these four groups of *items* dominoes, cells of anchor color $3m+13$ cannot exceed the seventh row (the third row after the pair of *anchors* dominoes, plus one for each group of *items* dominoes), consequently no more group of *items* dominoes can match an anchor color and take place inside the bin.

6 Conclusion

Theorem 1 establishes that *Kingdomino*^{TM} shares the feature of many fun games: it requires to solve instances of an NP-complete problem. Finding efficient moves is therefore[3] a computationally hard task, and players may feel glad to encounter good solutions.

As we have seen in Section 4, the number of possible K-tilings may grow rapidly. The main difficulty in the elaboration of the NP-hardness reduction to the **K-tiling problem**, has been to find an initial sequence of dominoes which imposes a rigid structure (with very few possible K-tiling reaching a maximum score), and still allows to be continued in order to implement some strong NP-complete problem (given by the instance from the reduction).

Our modeling of the game *Kingdomino*^{TM} abstracts various aspects of the game (as board games are finite, this is necessary), and our construction in Theorem 1 is frugal in terms of crowns, but it is opulent in terms of colors (in order to ease the argumentations). As an opening, one may ask: is the **K-tiling problem** still NP-hard if the number of colors is bounded?

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[3] Unless $\mathsf{P} = \mathsf{NP}$.

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