CYCLIC PERMUTATIONS: DEGREES AND COMBINATORIAL TYPES

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ABSTRACT. This note will give elementary counts for the number of n-cycles in the symmetric group $\mathcal{S}_n$ with a given degree (a variant of descent number) and studies similar counting problems for the conjugacy classes of n-cycles under the action of the rotation subgroup of $\mathcal{S}_n$. This is achieved by relating such cycles to periodic orbits of an associated dynamical system acting on the circle. We also compute the mean and variance of the distribution of degree on n-cycles and show that this distribution is asymptotically normal as $n \to \infty$.

1. INTRODUCTION

The classical Eulerian numbers describe the distribution of descent number in the full symmetric group $\mathcal{S}_n$ and have been studied extensively for more than a century (see for example [Pe] and [St]). Understanding the distribution of descent number in a given conjugacy class of $\mathcal{S}_n$ is a more subtle problem that was first tackled in the late 1990's by Diaconis, McGrath, and Pitman [DMP] and by Fulman [F].

This note will consider a variant of the descent number of a permutation $\nu \in \mathcal{S}_n$ called the *degree*, defined by

$$\deg(\nu) = \#\big\{ i : \nu(i) > \nu(i+1) \big\}$$

where the integer i is taken modulo n. This modified version has the advantage of being invariant under the left and right action of the rotation subgroup of $\mathcal{S}_n$ generated by the cycle $(1\ 2\ \cdots\ n)$, and naturally occurs in the study of the combinatorial patterns of periodic orbits of covering maps of the circle (see [Mc] and [PZ]). Motivated by this connection, we will investigate the distribution of degree in the special conjugacy class $\mathcal{C}_n$ consisting of all n-cycles in $\mathcal{S}_n$. Let $N_{n,d}$ denote the number of $\nu \in \mathcal{C}_n$ with $\deg(\nu) = d$. In §4 we prove

Theorem 1.1. *For every $d \geq 1$,*

$$N_{n,d} = \sum_{k=1}^{d} (-1)^{d-k} \binom{n}{d-k} \Delta_n(k),$$

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where

$$\Delta_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big)\Big(\sum_{j=0}^{r-1} k^j\Big)$$

and μ is the Möbius function.

This is the analog of the alternating sum formula for Eulerian numbers. Our proof makes essential use of a count for the number of period n orbits of the linear endomorphism $\mathbf{m}_k(x) = kx \pmod{\mathbb{Z}}$ of the circle $\mathbb{R}/\mathbb{Z}$ that realize the combinatorics of degree d elements in $\mathcal{C}_n$, developed by Petersen and the author [PZ]. The $\Delta_n(k)$ for $k \geq 2$ can be interpreted as the number of period n points of $\mathbf{m}_k$ up to rotation by a $(k-1)$-st root of unity (see §4.1, §4.2 and §5.1).

It is well known that the descent number of a randomly chosen permutation in $\mathcal{S}_n$ has the mean $(n-1)/2$ and variance $(n+1)/12$ (see for example [Pi]). In §5 we prove

Theorem 1.2. *The degree of a randomly chosen cycle in $\mathcal{C}_n$ (with respect to the uniform measure) has the mean*

$$\frac{n}{2} - \frac{1}{n-1} \qquad \text{if } n \geq 3,$$

and variance

$$\frac{n}{12} + \frac{n}{(n-1)^2(n-2)} \qquad \text{if } n \geq 5.$$

The idea of the proof, inspired by the method of Fulman in [F], is to express the generating function of the $N_{n,d}$ in terms of the generating functions of the Eulerian numbers (the so-called Eulerian polynomials) for which the mean and variance are already known (see §5.2).

The following central limit theorem for the degree is also proved in §5:

Theorem 1.3. *When normalized by its mean and variance, the distribution of $\deg(\nu)$ for $\nu \in \mathcal{C}_n$ converges to the standard normal distribution as $n \to \infty$.*

Compare Fig. 1.

The central limit theorem for the distribution of descent number over $\mathcal{S}_n$ or a given conjugacy class has been known (see for example the papers of Bender [B] or Harper [H] for the former, and the recent work of Kim and Lee [KL1] for the latter). Our proof follows the strategy of [FKL] and is based on the idea of reducing convergence in distribution to pointwise convergence of the moment generating functions, as utilized in [KL2] (see §5.2).

Motivated by applications in dynamics, we also study the conjugacy classes of $\mathcal{C}_n$ under the action of the rotation subgroup of $\mathcal{S}_n$. Each such class is called a *combinatorial type* in $\mathcal{C}_n$. In §3 we count the number of n-cycles of a given

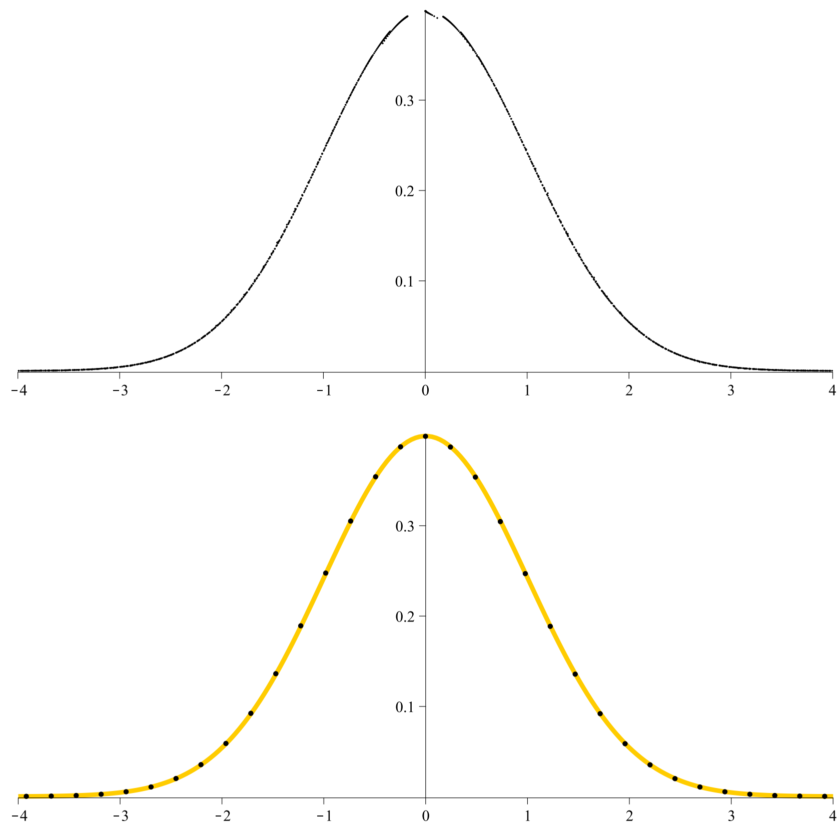

FIGURE 1. *The combined distributions* $\{N_{n,d}\}_{n \leq 100}$ *(top) and the distribution* $N_{200,d}$ *(bottom), normalized by their mean and variance. Here* $N_{n,d}$ *is the number of* $\nu \in \mathcal{C}_n$ *with* $\deg(\nu) = d$*. The continuous curve in yellow is the standard normal distribution.*

symmetry order and use it to derive a (known) formula for the number of distinct combinatorial types in $\mathcal{C}_n$ (see Theorems 3.1 and 3.3). This section is elementary and rather independent of the rest of the paper, except for §4.3 where we discuss the problem of counting the number of distinct combinatorial types of a given degree.

2. Preliminaries

Fix an integer $n \geq 2$. We denote by $\mathcal{S}_n$ the group of all permutations of $\{1, \ldots, n\}$ and by $\mathcal{C}_n$ the collection of all n-cycles in $\mathcal{S}_n$. Following the tradition of group theory, we represent $\nu \in \mathcal{C}_n$ by the symbol

$$(1 \;\; \nu(1) \;\; \nu^2(1) \;\; \cdots \;\; \nu^{n-1}(1)).$$

The ***rotation group*** $\mathcal{R}_n$ is the cyclic subgroup of $\mathcal{S}_n$ generated by the n-cycle

$$\rho := (1 \ 2 \ \cdots \ n).$$

Elements of $\mathcal{R}_n \cap \mathcal{C}_n$ are called ***rotation cycles***. Thus, $\nu \in \mathcal{C}_n$ is a rotation cycle if and only if $\nu = \rho^m$ for some integer $1 \leq m < n$ with $\gcd(m,n) = 1$. The reduced fraction m/n is called the ***rotation number*** of ρ^m.

The rotation group $\mathcal{R}_n$ acts on $\mathcal{C}_n$ by conjugation. We refer to each orbit of this action as a ***combinatorial type*** in $\mathcal{C}_n$. The combinatorial type of an n-cycle ν is denoted by $[\nu]$. It is easy to see that ν is a rotation cycle if and only if $[\nu]$ consists of ν only. In fact, if $\rho\nu\rho^{-1} = \nu$, then $\nu = \rho^m$ where $m = \nu(n)$.

2.1. **The symmetry order.** The combinatorial type of $\nu \in \mathcal{C}_n$ can be explicitly described as follows. Let

$$\mathcal{G}_\nu := \left\{ \rho^i : \rho^i \nu \rho^{-i} = \nu \right\}$$

be the stabilizer group of ν under the action of $\mathcal{R}_n$. We call the order of $\mathcal{G}_\nu$ the ***symmetry order*** of ν and denote it by $\operatorname{sym}(\nu)$. If $r := n/\operatorname{sym}(\nu)$, it follows that $\mathcal{G}_\nu$ is generated by the power ρ^r and the combinatorial type of ν is the r-element set

$$[\nu] = \left\{ \nu, \rho\nu\rho^{-1}, \ldots, \rho^{r-1}\nu\rho^{-(r-1)} \right\}.$$

Since $\operatorname{sym}(\rho\nu\rho^{-1}) = \operatorname{sym}(\nu)$, we can define the symmetry order of a combinatorial type unambiguously as that of any cycle representing it:

$$\operatorname{sym}([\nu]) := \operatorname{sym}(\nu).$$

Evidently there are no 2- or 3-cycles of symmetry order 1, and there is no 4-cycle of symmetry order 2. By contrast, it is not hard to see that for every $n \geq 5$ and every divisor s of n there is a $\nu \in \mathcal{C}_n$ with $\operatorname{sym}(\nu) = s$.

Of the $(n-1)!$ elements of $\mathcal{C}_n$, precisely $\varphi(n)$ are rotation cycles. Here φ is Euler's totient function defined by

$$\varphi(n) := \# \{ m \in \mathbb{Z} : 1 \leq m \leq n \text{ and } \gcd(m,n) = 1 \}.$$

If $\nu_1, \ldots, \nu_T$ are representatives of the distinct combinatorial types in $\mathcal{C}_n$, then

$$(n-1)! = \sum_{\nu_i \in \mathcal{R}_n} \#[\nu_i] + \sum_{\nu_i \notin \mathcal{R}_n} \#[\nu_i] = \varphi(n) + \sum_{\nu_i \notin \mathcal{R}_n} \#[\nu_i].$$

When n is a prime number, we have $\varphi(n) = n-1$ and each $\#[\nu_i]$ in the far right sum is n. In this case the number of distinct combinatorial types in $\mathcal{C}_n$ is given by

$$T = (n-1) + \frac{(n-1)! - (n-1)}{n} = \frac{(n-1)! + (n-1)^2}{n}. \tag{2.1}$$

Observe that T being an integer gives a simple proof of Wilson's theorem according to which $(n-1)! = -1 \pmod{n}$ whenever n is prime.

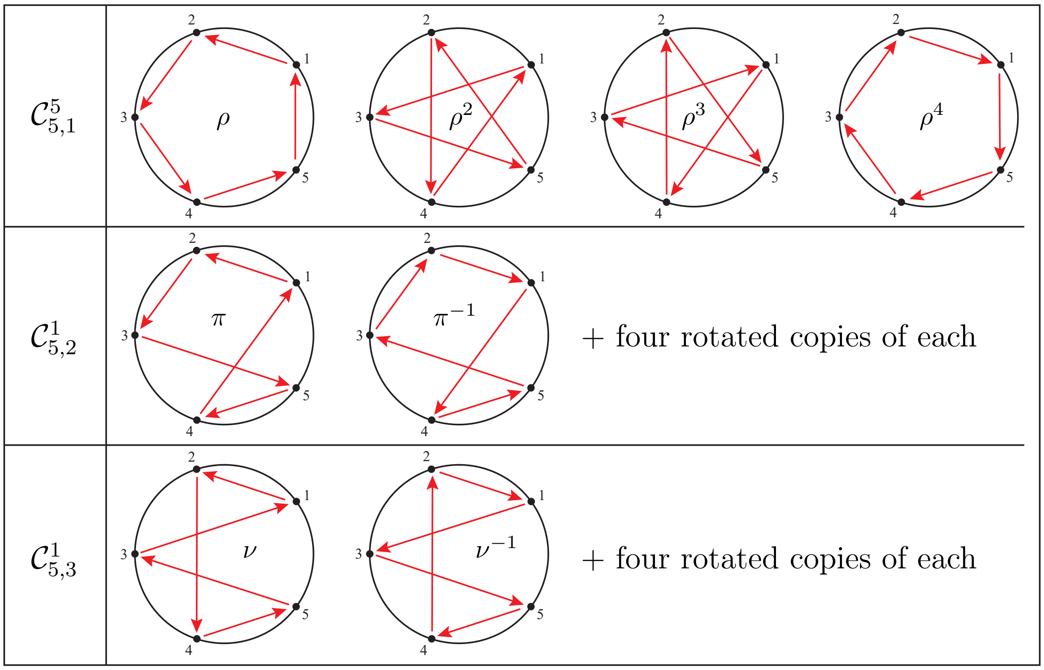

FIGURE 2. *The decomposition of $\mathcal{C}_5$ into subsets $\mathcal{C}_{5,d}^s$ of cycles with degree d and symmetry order s, where the only admissible pairs are $(d, s) = (1, 5), (2, 1), (3, 1)$. See Examples 2.1 and 2.2.*

Example 2.1. The $4! = 24$ cycles in $\mathcal{C}_5$ fall into $(4! + 4^2)/5 = 8$ distinct combinatorial types. The 4 rotation cycles

$$\rho = (1\ 2\ 3\ 4\ 5) \qquad \rho^2 = (1\ 3\ 5\ 2\ 4)$$
$$\rho^3 = (1\ 4\ 2\ 5\ 3) \qquad \rho^4 = (1\ 5\ 4\ 3\ 2)$$

(of rotation numbers $1/5, 2/5, 3/5, 4/5$) form 4 distinct combinatorial types. The remaining 20 cycles have symmetry order 1, so they fall into 4 combinatorial types each containing 5 elements. These types are represented by

$$\pi = (1\ 2\ 3\ 5\ 4) \qquad \pi^{-1} = (1\ 4\ 5\ 3\ 2)$$
$$\nu = (1\ 2\ 4\ 5\ 3) \qquad \nu^{-1} = (1\ 3\ 5\ 4\ 2).$$

Compare Fig. 2.

2.2. **Descent number vs. degree.** A permutation $\nu \in \mathcal{S}_n$ has a ***descent*** at $i \in \{1, \ldots, n-1\}$ if $\nu(i) > \nu(i+1)$. The total number of such i is called the ***descent number*** of ν and is denoted by $\operatorname{des}(\nu)$:

$$\operatorname{des}(\nu) := \#\big\{1 \leq i \leq n-1 : \nu(i) > \nu(i+1)\big\}$$

Note that $0 \leq \mathrm{des}(\nu) \leq n-1$. The descent number is a basic tool in enumerative combinatorics (see for example [St]).

In this paper we will be working with a rotationally invariant version of the descent number called ***degree***.[1] It simply amounts to counting $i = n$ as a descent if $\nu(n) > \nu(1)$:

$$\deg(\nu) := \begin{cases} \mathrm{des}(\nu) & \text{if } \nu(n) < \nu(1) \\ \mathrm{des}(\nu) + 1 & \text{if } \nu(n) > \nu(1). \end{cases}$$

The terminology comes from the following topological characterization (see [Mc] and [PZ]): Take any set $\{x_1, \ldots, x_n\}$ of distinct points on the circle in positive cyclic order. Then $\deg(\nu)$ is the minimum degree of a continuous covering map $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which acts on this set as the permutation ν in the sense that $f(x_i) = x_{\nu(i)}$ for all i.

Example 2.2. The cycle $\nu = (1\ 2\ 4\ 5\ 3) \in \mathcal{C}_5$ has descents at $i = 2$, $i = 4$ and $i = 5$, so $\deg(\nu) = 3$. The eight representative cycles in $\mathcal{C}_5$ described in Example 2.1 have the following degrees:

$$\begin{aligned} &\deg(\rho) = \deg(\rho^2) = \deg(\rho^3) = \deg(\rho^4) = 1, \\ &\deg(\pi) = \deg(\pi^{-1}) = 2, \\ &\deg(\nu) = \deg(\nu^{-1}) = 3. \end{aligned}$$

Compare Fig. 2.

The following statement summarizes the basic properties of the degree for cycles. For a proof, see [PZ].

Theorem 2.3. *Let* $\nu \in \mathcal{C}_n$ *with* $\mathrm{sym}(\nu) = s$ *and* $\deg(\nu) = d$.

(i) $1 \leq d \leq n-2$ *if* $n \geq 3$.

(ii) $d = 1 \Longleftrightarrow s = n \Longleftrightarrow \nu$ *is a rotation cycle.*

(iii) s *is a divisor of* $d - 1$.

(iv) $\deg(\rho\nu) = \deg(\nu\rho) = \deg(\rho\nu\rho^{-1}) = d$.

By (iv), the degree of a combinatorial type is well-defined:

$$\deg([\nu]) := \deg(\nu).$$

[1] What we define as the "degree" in this paper is called the "descent number" in [PZ].

2.3. **Decompositions of $\mathcal{C}_n$.** Fix $n \geq 3$ and consider the following cross sections of $\mathcal{C}_n$ by the symmetry order and degree:

$$\begin{aligned} \mathcal{C}_n^s &:= \{\nu \in \mathcal{C}_n : \mathrm{sym}(\nu) = s\} \\ \mathcal{C}_{n,d} &:= \{\nu \in \mathcal{C}_n : \deg(\nu) = d\} \\ \mathcal{C}_{n,d}^s &:= \mathcal{C}_n^s \cap \mathcal{C}_{n,d}. \end{aligned}$$

Observe that in our notation the symmetry order always appears as a superscript and the degree as a subscript after n. By Theorem 2.3,

$$\mathcal{C}_n^n = \mathcal{C}_{n,1} = \mathcal{C}_{n,1}^n = \mathcal{C}_n \cap \mathcal{R}_n$$

and we have the decompositions

$$\begin{aligned} \mathcal{C}_n &= \bigcup_{s|n} \mathcal{C}_n^s = \bigcup_{d=1}^{n-2} \mathcal{C}_{n,d} \\ \mathcal{C}_n^s &= \bigcup_{j=1}^{\lfloor (n-3)/s \rfloor} \mathcal{C}_{n,js+1}^s && \text{if } s|n, \ s < n \\ \mathcal{C}_{n,d} &= \bigcup_{s|\gcd(n,d-1)} \mathcal{C}_{n,d}^s && \text{if } 2 \leq d \leq n-2. \end{aligned}$$

Hence the cardinalities

$$\begin{aligned} N_n^s &:= \#\, \mathcal{C}_n^s \\ N_{n,d} &:= \#\, \mathcal{C}_{n,d} \\ N_{n,d}^s &:= \#\, \mathcal{C}_{n,d}^s \end{aligned}$$

satisfy the following relations:

$$N_n^n = N_{n,1} = N_{n,1}^n = \varphi(n)$$

$$(n-1)! = \sum_{s|n} N_n^s = \sum_{d=1}^{n-2} N_{n,d}$$

$$N_n^s = \sum_{j=1}^{\lfloor (n-3)/s \rfloor} N_{n,js+1}^s \qquad \text{if } s|n, \ s < n \tag{2.2}$$

$$N_{n,d} = \sum_{s|\gcd(n,d-1)} N_{n,d}^s \qquad \text{if } 2 \leq d \leq n-2. \tag{2.3}$$

Let us also consider the counts for the corresponding combinatorial types

$$\begin{aligned} T_n &:= \# \{[\nu] : \nu \in \mathcal{C}_n\} \\ T_n^s &:= \# \{[\nu] : \nu \in \mathcal{C}_n^s\} \\ T_{n,d} &:= \# \{[\nu] : \nu \in \mathcal{C}_{n,d}\} \\ T_{n,d}^s &:= \# \{[\nu] : \nu \in \mathcal{C}_{n,d}^s\}. \end{aligned}$$

Evidently

$$T_{n,d}^s = \frac{s}{n}\, N_{n,d}^s \qquad \text{and} \qquad T_n^s = \frac{s}{n}\, N_n^s$$

and we have the following relations:

$$\begin{aligned} T_n^n &= T_{n,1} = T_{n,1}^n = \varphi(n) \\ T_n &= \frac{1}{n} \sum_{s|n} s N_n^s \\ T_{n,d} &= \frac{1}{n} \sum_{s|\,\gcd(n,d-1)} s N_{n,d}^s \qquad \text{if } 2 \leq d \leq n-2. \end{aligned} \tag{2.4}$$

Of course knowing the joint distribution $N_{n,d}^s$ would allow us to count all the N's and T's. However, finding an closed formula for $N_{n,d}^s$ seems to be difficult (a sample computation can be found in §4.3). In §3.1 we derive a formula for N_n^s by a direct count which in turn leads to a formula for T_n (see Theorems 3.1 and 3.3). In §4.2 we find a formula for $N_{n,d}$ indirectly by relating cycles in $\mathcal{C}_{n,d}$ to periodic orbits of an associated dynamical system acting on the circle (see Theorem 4.5).

3. The distribution of symmetry order

3.1. **The numbers N_n^s.** We begin with the simplest of our counting problems, that is, finding a formula for N_n^s. We will make use of the ***Möbius inversion formula***

$$g(m) = \sum_{k|m} f(k) \quad \Longleftrightarrow \quad f(m) = \sum_{k|m} \mu(k)\, g\Big(\frac{m}{k}\Big) \tag{3.1}$$

on a pair of arithmetical functions f, g. Here μ is the Möbius function uniquely determined by the conditions $\mu(1) := 1$ and $\sum_{k|m} \mu(k) = 0$ for $m > 1$. Applying (3.1) to the relation

$$m = \sum_{k|m} \varphi(k)$$

gives the classical identity

$$\varphi(m) = \sum_{k|m} \frac{m}{k}\, \mu(k) = \sum_{k|m} k\mu\Big(\frac{m}{k}\Big). \tag{3.2}$$

Theorem 3.1. *For every $n \geq 2$ and every divisor s of n,*

$$N_n^s = \frac{1}{n} \sum_{j | \frac{n}{s}} \mu(j)\, \varphi(sj)\, (sj)^{\frac{n}{sj}} \left(\frac{n}{sj}\right)! \tag{3.3}$$

When $s = n$ the formula reduces to $N_n^n = (1/n)\mu(1)\varphi(n)n = \varphi(n)$ which agrees with our earlier count.

Proof. Set $r := n/s$. We have $\rho^r \nu \rho^{-r} = \nu$ if and only if $\mathrm{sym}(\nu)$ is a multiple of s if and only if $\nu \in \mathcal{C}_n^{n/j}$ for some $j|r$. Denoting ν by $(\nu_1\ \nu_2\ \cdots\ \nu_n)$, this condition can be written as

$$(\rho^r(\nu_1)\ \rho^r(\nu_2)\ \cdots\ \rho^r(\nu_n)) = (\nu_1\ \nu_2\ \cdots\ \nu_n),$$

which holds if and only if there is an integer m such that

$$\rho^r(\nu_i) = \nu_{\rho^m(i)} \qquad \text{for all } i. \tag{3.4}$$

The rotations $\rho^r : i \mapsto i + r$ and $\rho^m : i \mapsto i + m \pmod n$ have orders $n/\gcd(r, n) = n/r$ and $n/\gcd(m, n)$ respectively. By (3.4), these orders are equal, hence

$$r = \gcd(m, n).$$

Setting $t := m/r$ gives $\gcd(t, s) = 1$, so there are at most $\varphi(s)$ possibilities for t and therefore for m. The action of the rotation ρ^m partitions $\mathbb{Z}/n\mathbb{Z}$ into r disjoint orbits each consisting of s elements and these r orbits are represented by $1, \ldots, r$. In fact, if

$$i + \ell m = i' + \ell' m \pmod n \quad \text{for some } 1 \leq i, i' \leq r \text{ and } 1 \leq \ell, \ell' \leq s,$$

then $i - i' = m(\ell' - \ell) \pmod n$ so $i = i' \pmod r$ which gives $i = i'$. Moreover, $\ell m = \ell' m \pmod n$ so $\ell t = \ell' t \pmod s$. Since $\gcd(t, s) = 1$, this implies $\ell = \ell' \pmod s$ which shows $\ell = \ell'$.

Now (3.4) shows that for each of the $\varphi(s)$ choices of m, the cycle ν is completely determined by the integers $\nu_1, \ldots, \nu_r$, and different choices of m lead to different cycles. We may always assume $\nu_1 = 1$. This leaves $n - s$ choices for ν_2 (corresponding to the elements of $\{1, \ldots, n\}$ that are not in the orbit of $\nu_1 = 1$ under ρ^m), $n - 2s$ choices for ν_3, … and $n - (r-1)s = s$ choices for ν_r. Thus, the total number of choices for ν is

$$\varphi(s)(n-s)(n-2s)\cdots s = \varphi(s)\, s^{r-1}\, (r-1)! = \frac{1}{n}\varphi(s)\, s^r\, r!$$

This proves

$$\sum_{j|r} N_n^{n/j} = \frac{1}{n}\varphi\left(\frac{n}{r}\right)\left(\frac{n}{r}\right)^r r!$$

$n \backslash s$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	1													
3	0	–	2												
4	4	0	–	2											
5	20	–	–	–	4										
6	108	6	4	–	–	2									
7	714	–	–	–	–	–	6								
8	4992	40	–	4	–	–	–	4							
9	40284	–	30	–	–	–	–	–	6						
10	362480	380	–	–	16	–	–	–	–	4					
11	3628790	–	–	–	–	–	–	–	–	–	10				
12	39912648	3768	312	60	–	8	–	–	–	–	–	4			
13	479001588	–	–	–	–	–	–	–	–	–	–	–	12		
14	6226974684	46074	–	–	–	–	36	–	–	–	–	–	–	6	
15	87178287120	–	3880	–	192	–	–	–	–	–	–	–	–	–	8

TABLE 1. *The distributions N_n^s for $2 \leq n \leq 15$.*

An application of the Möbius inversion formula (3.1) then gives

$$N_n^s = N_n^{n/r} = \frac{1}{n} \sum_{j|r} \mu(j)\, \varphi\Big(\frac{nj}{r}\Big) \Big(\frac{nj}{r}\Big)^{\frac{r}{j}} \Big(\frac{r}{j}\Big)!$$

$$= \frac{1}{n} \sum_{j|\frac{n}{s}} \mu(j)\, \varphi(sj)\, (sj)^{\frac{n}{sj}} \Big(\frac{n}{sj}\Big)! \qquad \square$$

Table 1 shows the values of N_n^s for $2 \leq n \leq 15$. Notice that $N_2^1 = N_3^1 = N_4^2 = 0$ but all other values are positive. Moreover, as n gets larger the distribution N_n^s appears to be overwhelmingly concentrated at $s = 1$. This is quantified in the following

Theorem 3.2. $N_n^1 \sim (n-1)!$ *as* $n \to \infty$.

This justifies the intuition that the chance of a randomly chosen n-cycle having any non-trivial rotational symmetry tends to zero as $n \to \infty$.

Proof. The formula (3.3) with $s = 1$ gives

$$nN_n^1 = n! + \mu(n)\varphi(n)n + \sum_j \mu(j)\, \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)!$$

or

$$\frac{N_n^1}{(n-1)!} = 1 + \frac{\mu(n)\varphi(n)}{(n-1)!} + \frac{1}{n!} \sum_j \mu(j)\, \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)!$$

where the sums are taken over all divisors j of n with $1 < j < n$. We need only check that the term on the far right tends to 0 as $n \to \infty$. If $j|n$ and

$1 < j < n$, then $j \le \lfloor n/2 \rfloor$ and $n/j \le \lfloor n/2 \rfloor$. Hence,

$$\varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \le j^{\frac{n}{j}+1} \left(\frac{n}{j}\right)! \le \left\lfloor \frac{n}{2} \right\rfloor^{\lfloor n/2 \rfloor + 1} \left\lfloor \frac{n}{2} \right\rfloor ! \tag{3.5}$$

The Stirling formula $k! \sim \sqrt{2\pi k}\; k^k\, e^{-k}$ gives the elementary estimate

$$\frac{k^k\; k!}{(2k)!} \le \text{const.} \left(\frac{e}{4}\right)^k .$$

Applying this to (3.5) for $k = \lfloor n/2 \rfloor$, we obtain

$$\frac{1}{n!}\, \varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \le \text{const.}\, n \left(\frac{e}{4}\right)^{\frac{n}{2}} .$$

Thus,

$$\frac{1}{n!} \left| \sum_j \mu(j)\, \varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \right| \le \frac{1}{n!} \sum_j \varphi(j)\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \le \text{const.}\, n^2 \left(\frac{e}{4}\right)^{\frac{n}{2}},$$

which tends to 0 as $n \to \infty$. □

3.2. **The numbers T_n.** The count (3.3) leads to the following formula for the number of distinct combinatorial types of n-cycles. It turns out that this formula is not new: It appears in the *On-line Encyclopedia of Integer Sequences* as the number of 2-colored patterns of an $n \times n$ chessboard [Sl].

Theorem 3.3. *For every $n \ge 2$,*

$$T_n = \frac{1}{n^2} \sum_{j|n} (\varphi(j))^2\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \tag{3.6}$$

Observe that for prime n the formula reduces to

$$T_n = \frac{1}{n^2}\big((\varphi(1))^2\, n! + (\varphi(n))^2\, n\big) = \frac{1}{n}((n-1)! + (n-1)^2)$$

which agrees with our derivation in (2.1). Table 2 shows the values of T_n for $2 \le n \le 20$.

Proof. By (2.4) and (3.3),

$$T_n = \frac{1}{n} \sum_{s|n} s N_n^s = \frac{1}{n^2} \sum_{s|n} \sum_{j|\frac{n}{s}} s\mu(j)\, \varphi(sj)\, (sj)^{\frac{n}{sj}} \left(\frac{n}{sj}\right)!$$

The sum interchange formula

$$\sum_{s|n} \sum_{j|\frac{n}{s}} f(j,s) = \sum_{j|n} \sum_{s|j} f\left(\frac{j}{s}, s\right)$$

n	T_n
2	1
3	2
4	3
5	8
6	24
7	108
8	640
9	4492
10	36336
11	329900
12	3326788
13	36846288
14	444790512
15	5811886656
16	81729688428
17	1230752346368
18	19760413251956
19	336967037143596
20	6082255029733168

TABLE 2. *The values of T_n for $2 \leq n \leq 20$.*

then gives

$$\begin{aligned} T_n &= \frac{1}{n^2} \sum_{j|n} \sum_{s|j} s\mu\Big(\frac{j}{s}\Big) \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \\ &= \frac{1}{n^2} \sum_{j|n} \Big(\sum_{s|j} s\mu\Big(\frac{j}{s}\Big)\Big)\, \varphi(j)\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \\ &= \frac{1}{n^2} \sum_{j|n} (\varphi(j))^2\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \qquad \text{(by (3.2))}, \end{aligned}$$

as required. □

It is evident from Table 2 that the sequence $\{T_n\}$ grows rapidly as $n \to \infty$.

Theorem 3.4. *$T_n \sim \dfrac{n!}{n^2} \sim (n-2)!$ as $n \to \infty$.*

Proof. This is easy to verify. By (3.6),

$$\frac{n^2\, T_n}{n!} = 1 + \frac{(\varphi(n))^2}{(n-1)!} + \frac{1}{n!} \sum_{j} (\varphi(j))^2\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)!$$

where the sum is taken over all divisors j of n with $1 < j < n$. The same estimate as in the proof of Theorem 3.2 shows that for such j,

$$\frac{1}{n!} (\varphi(j))^2\, j^{\frac{n}{j}} \Big(\frac{n}{j}\Big)! \leq \text{const.}\, n^2 \Big(\frac{e}{4}\Big)^{\frac{n}{2}}.$$

Thus,

$$\frac{1}{n!}\sum_j (\varphi(j))^2\, j^{\frac{n}{j}} \left(\frac{n}{j}\right)! \le \text{const.}\, n^3 \left(\frac{e}{4}\right)^{\frac{n}{2}}$$

which tends to 0 as $n \to \infty$. $\square$

Remark 3.5. The ratio $n^2 T_n/n!$ tends to 1 at a much faster rate than geometric. In fact, a slightly more careful estimate gives the improved (but not optimal) bound

$$\frac{n^2\, T_n}{n!} = 1 + O\Big(\Big(\frac{3}{n}\Big)^{\frac{n}{2}}\Big) \qquad \text{as } n \to \infty.$$

4. The distribution of degree

We now turn to the problem of counting n-cycles with a given degree, using the dynamics of a family of linear endomorphisms of the circle.

Conventions 4.1. (i) It will be convenient to extend the definition of $N_{n,d}$ to all $d \ge 1$ by setting $N_{n,d} = 0$ if $d \ge n-1$.

(ii) We follow the customary practice of setting

$$\binom{a}{b} = 0 \qquad \text{if } \ b < 0 \ \text{ or } \ 0 < a < b.$$

4.1. The circle endomorphisms $\mathbf{m}_k$.

For each integer $k \ge 2$, consider the multiplication-by-k map of the circle $\mathbb{R}/\mathbb{Z}$ defined by

$$\mathbf{m}_k(x) := kx \pmod{\mathbb{Z}}.$$

Let $\mathcal{O} = \{x_1, x_2, \ldots, x_n\}$ be a period n orbit of $\mathbf{m}_k$, where the representatives are labeled so that $0 < x_1 < x_2 < \cdots < x_n < 1$. We say that $\mathcal{O}$ ***realizes*** the cycle $\nu \in \mathcal{C}_n$ if

$$\mathbf{m}_k(x_i) = x_{\nu(i)} \qquad \text{for all } i.$$

We say that $\mathcal{O}$ realizes a combinatorial type $[\nu]$ in $\mathcal{C}_n$ if it realizes the cycle $\rho^i \nu \rho^{-i}$ for some i. For example, the periodic orbit

$$\left\{ x_1 = \frac{16}{242}, x_2 = \frac{48}{242}, x_3 = \frac{86}{242}, x_4 = \frac{144}{242}, x_5 = \frac{190}{242} \right\}$$

of the tripling map $\mathbf{m}_3$ realizes $\nu = (1\ 2\ 4\ 5\ 3) \in \mathcal{C}_5$ and therefore it realizes the combinatorial type $\{\nu, \rho\nu\rho^{-1}, \rho^2\nu\rho^{-2}, \rho^3\nu\rho^{-3}, \rho^4\nu\rho^{-4}\}$.

It follows from the topological interpretation of the degree in §2.2 that if an orbit of $\mathbf{m}_k$ realizes $\nu \in \mathcal{C}_{n,d}$, then necessarily $k \ge d$. Conversely, if $\nu \in \mathcal{C}_{n,d}$ and $k \ge \max\{d, 2\}$, there are always period n orbits of $\mathbf{m}_k$ that realize the combinatorial type $[\nu]$. In fact, by translating the realization problem to finding the steady-state of a regular Markov chain, the following result is proved in [PZ]:

Theorem 4.2. *If $\nu \in \mathcal{C}^s_{n,d}$ and $k \geq \max\{d, 2\}$, there are precisely*
$$\frac{k-1}{s}\binom{n+k-d-1}{n-1}$$
period n orbits of $\mathbf{m}_k$ that realize the combinatorial type $[\nu]$.

The following corollary is immediate:

Corollary 4.3. *For every $k \geq 2$ and $d \geq 1$, the number of period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}_{n,d}$ is*
$$\frac{k-1}{n}\binom{n+k-d-1}{n-1} N_{n,d}.$$

Proof. The claim is trivial if $d > k$ since in this case the number of such orbits and the binomial coefficient $\binom{n+k-d-1}{n-1}$ are both 0. If $2 \leq d \leq k$, then by Theorem 4.2 for each divisor s of $\gcd(n, d-1)$ there are
$$\frac{k-1}{s}\binom{n+k-d-1}{n-1} T^s_{n,d} = \frac{k-1}{n}\binom{n+k-d-1}{n-1} N^s_{n,d}$$
period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}^s_{n,d}$. The result then follows from (2.3) by summing over all such s. Finally, since $\mathcal{C}^n_{n,1} = \mathcal{C}_{n,1}$, Theorem 4.2 shows that there are
$$\frac{k-1}{n}\binom{n+k-2}{n-1} T_{n,1} = \frac{k-1}{n}\binom{n+k-2}{n-1} N_{n,1}$$
period n orbits of $\mathbf{m}_k$ that realize some $\nu \in \mathcal{C}_{n,1}$. $\square$

4.2. **The numbers $N_{n,d}$.** For $k \geq 2$ let $P_n(k)$ denote the number of periodic points of $\mathbf{m}_k$ of period n. The periodic points of $\mathbf{m}_k$ whose period is a divisor of n are precisely the $k^n - 1$ solutions of the equation $k^n x = x \pmod{\mathbb{Z}}$. Thus,
$$\sum_{r|n} P_r(k) = k^n - 1 \tag{4.1}$$
and the Möbius inversion formula gives
$$P_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big)(k^r - 1). \tag{4.2}$$
Introduce the integer-valued quantity
$$\Delta_n(k) := \begin{cases} \dfrac{P_n(k)}{k-1} & \text{if } k \geq 2 \\ \varphi(n) & \text{if } k = 1. \end{cases}$$
When $k \geq 2$ we can interpret $\Delta_n(k)$ as the number of period n points of $\mathbf{m}_k$ up to the rotation of the form $x \mapsto x + j/(k-1) \pmod{\mathbb{Z}}$. This is because $\mathbf{m}_k$ and

the rotation $x \mapsto x + 1/(k-1) \pmod{\mathbb{Z}}$ commute, so x is has period n under $\mathbf{m}_k$ if and only if $x + 1/(k-1)$ does.

By (4.2), for every $k \geq 2$,

$$\Delta_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big) \frac{k^r - 1}{k-1} = \sum_{r|n} \mu\Big(\frac{n}{r}\Big) \Big(\sum_{j=0}^{r-1} k^j \Big).$$

If $k = 1$, the sum on the far right reduces to $\sum_{r|n} r\mu(n/r)$ which is equal to $\varphi(n)$ by (3.2). It follows that

$$\Delta_n(k) = \sum_{r|n} \mu\Big(\frac{n}{r}\Big) \Big(\sum_{j=0}^{r-1} k^j \Big) \qquad \text{for all } k \geq 1. \tag{4.3}$$

Since $\mathbf{m}_k$ has $P_n(k)/n$ period n *orbits* altogether, Corollary 4.3 shows that for every $k \geq 2$,

$$\frac{k-1}{n} \sum_{d=1}^{n-2} \binom{n+k-d-1}{n-1} N_{n,d} = \frac{P_n(k)}{n}$$

or

$$\sum_{d=1}^{n-2} \binom{n+k-d-1}{n-1} N_{n,d} = \Delta_n(k). \tag{4.4}$$

This is in fact true for every $k \geq 1$ (the case $k = 1$ reduces to $N_{n,1} = \Delta_n(1) = \varphi(n)$).

Remark 4.4. Since the summand in (4.4) is zero unless $1 \leq d \leq \min(n-2, k)$, we can replace the upper bound of the sum by k.

Theorem 4.5. *For every $d \geq 1$,*

$$N_{n,d} = \sum_{i=1}^{d} (-1)^{d-i} \binom{n}{d-i} \Delta_n(i). \tag{4.5}$$

In particular, the theorem claims vanishing of the sum if $d \geq n-1$. Table 3 shows the values of $N_{n,d}$ for $2 \leq n \leq 12$.

$n \backslash d$	1	2	3	4	5	6	7	8	9	10
2	1									
3	2									
4	2	4								
5	4	10	10							
6	2	42	54	22						
7	6	84	336	252	42					
8	4	208	1432	2336	980	80				
9	6	450	5508	16548	14238	3402	168			
10	4	950	19680	99250	153860	77466	11320	350		
11	10	1936	66616	534688	1365100	1233760	389224	36784	682	
12	4	3972	217344	2671560	10568280	15593376	8893248	1851096	116580	1340

TABLE 3. *The distributions* $N_{n,d}$ *for* $2 \leq n \leq 12$.

Proof. This is a form of inversion for binomial coefficients. Use (4.4) to write

$$\begin{aligned}
&\sum_{i=1}^{d} (-1)^{d-i} \binom{n}{d-i} \Delta_n(i) \\
&= \sum_{i=1}^{d} \sum_{j=1}^{n-2} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} N_{n,j} \\
&= \sum_{i=1}^{d} \sum_{j=1}^{i} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} N_{n,j} \qquad \text{(by Remark 4.4)} \\
&= \sum_{j=1}^{d} \left(\sum_{i=j}^{d} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} \right) N_{n,j}.
\end{aligned}$$

Thus, (4.5) is proved once we check that

$$\sum_{i=j}^{d} (-1)^{d-i} \binom{n}{d-i} \binom{n+i-j-1}{n-1} = 0 \qquad \text{for } j < d. \tag{4.6}$$

Introduce the new variables $a := i - j$ and $b := d - j > 0$ so (4.6) takes the form

$$\sum_{a=0}^{b} (-1)^{a} \binom{n}{b-a} \binom{n+a-1}{n-1} = 0. \tag{4.7}$$

The identity

$$\binom{n}{b-a} \binom{n+a-1}{n-1} = \frac{n}{b} \binom{n+a-1}{b-1} \binom{b}{a}$$

shows that (4.7) is in turn equivalent to

$$\sum_{a=0}^{b}(-1)^a\binom{n+a-1}{b-1}\binom{b}{a}=0. \tag{4.8}$$

To prove (4.8), consider the binomial expansion

$$P(x):=x^{n-1}(x+1)^b=\sum_{a=0}^{b}\binom{b}{a}x^{n+a-1}$$

and differentiate it $b-1$ times with respect to x to get

$$P^{(b-1)}(x)=(b-1)!\sum_{a=0}^{b}\binom{n+a-1}{b-1}\binom{b}{a}x^{n+a-b}.$$

Since P has a zero of order b at $x=-1$, we have $P^{(b-1)}(-1)=0$ and (4.8) follows. □

As an application of Theorem 4.5, we record the following result which will be invoked in §5:

Theorem 4.6. *The generating function* $G_n(x) := \sum_{d=1}^{n-2} N_{n,d}\, x^d$ *has the expansion*

$$G_n(x)=(1-x)^n\sum_{i\geq 1}\Delta_n(i)\,x^i. \tag{4.9}$$

This should be viewed as an equality between formal power series. It is a true equality for $|x|<1$ where the series on the right converges absolutely.[2]

Proof. For each $d\geq 1$ the coefficient of x^d in the product

$$(1-x)^n\sum_{i\geq 1}\Delta_n(i)\,x^i=\sum_{j=0}^{n}(-1)^j\binom{n}{j}x^j\ \cdot\ \sum_{i\geq 1}\Delta_n(i)\,x^i$$

is $\sum_{i=1}^{d}(-1)^{d-i}\binom{n}{d-i}\,\Delta_n(i)$. This is $N_{n,d}$ by (4.5). □

4.3. **The numbers $T_{n,d}$.** Our counts for the numbers N_n^s and $N_{n,d}$ lead to the system of linear equations (2.2) and (2.3) on the $N_{n,d}^s$, but such systems are typically under-determined. Thus, additional information is needed to find the $N_{n,d}^s$ and therefore $T_{n,d}$. The following example serves to illustrates this point, where we use the dynamics of $\mathbf{m}_k$ to obtain this additional information.

[2]This is because $\Delta_n(i)$ grows like i^{n-1} for fixed n as $i\to\infty$; compare Lemma 5.5.

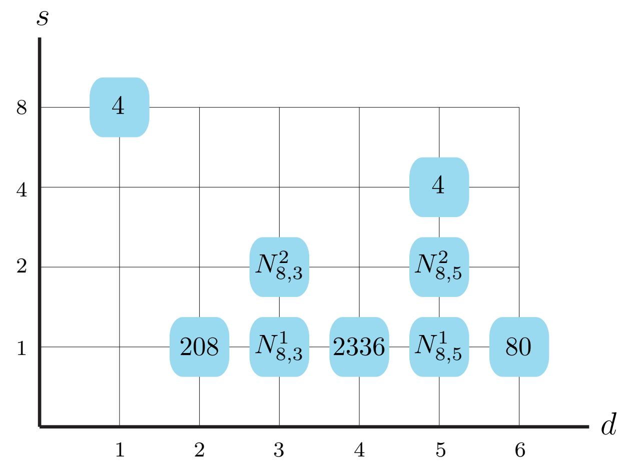

FIGURE 3. *Computation of the joint distribution $N^s_{8,d}$ in Example 4.7.*

Example 4.7. For $n = 8$ there are nine admissible pairs

$$(d, s) = (1, 8), (2, 1), (3, 1), (3, 2), (4, 1), (5, 1), (5, 2), (5, 4), (6, 1).$$

We record the values of $N^s_{8,d}$ on a grid as shown in Fig. 3. By (2.2) and (2.3), the values along the s-th row add up to N^s_8 and those along the d-th column add up to $N_{8,d}$, both available from Tables 1 and 3. This immediately gives five of the required nine values:

$$N^8_{8,1} = 4, \quad N^1_{8,2} = 208, \quad N^1_{8,4} = 2336, \quad N^4_{8,5} = 4, \quad N^1_{8,6} = 80.$$

Moreover, it leads to the system of linear equations

$$\begin{cases} N^1_{8,3} + N^2_{8,3} & = 1432 \\ N^1_{8,5} + N^2_{8,5} & = 976 \\ N^1_{8,3} + N^1_{8,5} & = 2368 \\ N^2_{8,3} + N^2_{8,5} & = 40 \end{cases} \tag{4.10}$$

on the remaining four unknowns which has rank 3 and therefore does not determine the solution uniquely. An additional piece of information can be obtained by considering the period 8 orbits of $\mathbf{m}_3$ which realize cycles in $\mathcal{C}^2_{8,3}$ (see [PZ], especially Theorem 6.6, for the results supporting the following claims). Every such orbit is ***self-antipodal*** in the sense that it is invariant under the 180° rotation $x \mapsto x + 1/2$ of the circle $\mathbb{R}/\mathbb{Z}$. It follows that x belongs to such orbit if and only if it satisfies

$$3^4 x = x + \frac{1}{2} \pmod{\mathbb{Z}}.$$

$n \backslash d$	1	2	3	4	5	6	7	8	9	10
2	1									
3	2									
4	2	1								
5	4	2	2							
6	2	7	10	5						
7	6	12	48	36	6					
8	4	26	182	292	126	10				
9	6	50	612	1844	1582	378	20			
10	4	95	1978	9925	15408	7753	1138	35		
11	10	176	6056	48608	124100	112160	35384	3344	62	
12	4	331	18140	222654	880848	1299448	741260	154258	9732	113

TABLE 4. *The distributions $T_{n,d}$ for $2 \leq n \leq 12$. The entries in red cannot be obtained from the sole knowledge of the N_n^s and $N_{n,d}$ in Tables 1 and 3.*

This is equivalent to x being rational of the form

$$x = \frac{2j-1}{160} \pmod{\mathbb{Z}} \quad \text{for some } 1 \leq j \leq 80.$$

Of the 10 orbits of $\mathbf{m}_3$ thus determined, 4 realize rotation cycles in $\mathcal{C}^8_{8,1}$ and the remaining 6 realize cycles in $\mathcal{C}^2_{8,3}$. Moreover, by Theorem 4.2 every combinatorial type in $\mathcal{C}^2_{8,3}$ is realized by a *unique* orbit of $\mathbf{m}_3$. It follows that $N^2_{8,3} = 4T^2_{8,3} = 24$. Now from (4.10) we obtain

$$N^1_{8,3} = 1408, \quad N^2_{8,3} = 24, \quad N^1_{8,5} = 960, \quad N^2_{8,5} = 16$$

and therefore

$$T_{8,1} = 4, \quad T_{8,2} = 26, \quad T_{8,3} = 182, \quad T_{8,4} = 292, \quad T_{8,5} = 126, \quad T_{8,6} = 10.$$

Observe that $T_8 = \sum_{d=1}^{6} T_{8,d} = 640$, in agreement with the value in Table 2 coming from formula (3.6).

Table 4 shows the result of similar but often more complicated dynamical arguments to determine $T_{n,d}$ for n up to 12. It would be desirable to develop a general method (and perhaps a closed formula) to compute these numbers for arbitrary n.

5. A STATISTICAL VIEW OF THE DEGREE

5.1. **Classical Eulerian numbers.** The numbers $N_{n,d}$ are the analogs of the ***Eulerian numbers*** $A_{n,d}$ which tally the permutations of descent number d in the full symmetric group $\mathcal{S}_n$:[3]

$$A_{n,d} := \#\big\{\nu \in \mathcal{S}_n : \operatorname{des}(\nu) = d\big\}.$$

For each n the index d now runs from 0 to $n-1$, with $A_{n,0} = A_{n,n-1} = 1$. The Eulerian numbers occur in many contexts, including areas outside of combinatorics, and have been studied extensively (for an excellent account, see [Pe]). Here are a few of their basic properties:

- *Symmetry*:
$$A_{n,d} = A_{n,n-d-1}.$$
- *Linear recurrence relation*:
$$A_{n,d} = (d+1)A_{n-1,d} + (n-d)A_{n-1,d-1}.$$
- *Worpitzky's identity*:
$$\sum_{d=0}^{n-1} \binom{n+k-d-1}{n} A_{n,d} = k^n \qquad \text{for all } k \geq 1 \tag{5.1}$$
- *Alternating sum formula*:
$$A_{n,d} = \sum_{i=1}^{d+1} (-1)^{d-i+1} \binom{n+1}{d-i+1} i^n. \tag{5.2}$$
- *Carlitz's identity*: The generating function $A_n(x) := \sum_{d=0}^{n-1} A_{n,d}\, x^d$ (also known as the n-th "Eulerian polynomial") satisfies
$$A_n(x) = (1-x)^{n+1} \sum_{i \geq 1} i^n\, x^{i-1}. \tag{5.3}$$

The last three formulas reveal a remarkable similarity between the sequences $N_{n,d}$ and $A_{n-1,d-1}$. In fact, (4.4) is the analog of Worpitzky's identity (5.1) for $A_{n-1,d-1}$ once $\Delta_n(k)$ is replaced with k^{n-1}. Similarly, (4.5) is the analog of the alternating sum formula (5.2) for $A_{n-1,d-1}$ when we replace $\Delta_n(i)$ with i^{n-1}. Finally, (4.9) is the analog of Carlitz's identity (5.3) for $\sum_{d=1}^{n-1} A_{n-1,d-1}\, x^d = xA_{n-1}(x)$, again replacing $\Delta_n(i)$ with i^{n-1}.

There is also an analogy between the $N_{n,d}$ and the restricted Eulerian numbers

$$B_{n,d} := \#\big\{\nu \in \mathcal{C}_n : \operatorname{des}(\nu) = d\big\}. \tag{5.4}$$

In the beautiful paper [DMP] which is motivated by the problem of riffle shuffles of a deck of cards, the authors obtain exact formulas for the distribution of

[3]The numbers $A_{n,d}$ are denoted by $\left\langle {n \atop d} \right\rangle$ in [GKP] and by $A(n, d+1)$ in [St].

descents in a given conjugacy class of $\mathcal{S}_n$. As a special case, their formulas show that

$$B_{n,d} = \sum_{i=1}^{d+1} (-1)^{d-i+1} \binom{n+1}{d-i+1} f_n(i),$$

where

$$f_n(i) := \frac{1}{n} \sum_{r|n} \mu\Big(\frac{n}{r}\Big) i^r$$

is the number of aperiodic circular words of length n from an alphabet of i letters. One cannot help but notice the similarity between the above formula for $B_{n-1,d-1}$ and (4.5), and between $f_n(i)$ and $\Delta_n(i)$ in (4.3).

5.2. **Asymptotic normality.** The statistical behavior of classical Eulerian numbers is well understood. For example, it is known that the distribution $\{A_{n,d}\}_{0 \leq d \leq n-1}$ is unimodal with a peak at $d = \lfloor n/2 \rfloor$. Moreover, the descent number of a randomly chosen permutation in $\mathcal{S}_n$ (with respect to the uniform measure) has the mean and variance

$$\begin{aligned} \tilde{\mu}_n &:= \frac{1}{n!} \sum_{d=0}^{n-1} d A_{n,d} = \frac{n-1}{2} \\ \tilde{\sigma}_n^2 &:= \frac{1}{n!} \sum_{d=0}^{n-1} (d - \tilde{\mu}_n)^2 A_{n,d} = \frac{n+1}{12}. \end{aligned}$$

These computations can be expressed in terms of the generating functions A_n introduced in §5.1:

$$\frac{A_n'(1)}{n!} = \frac{n-1}{2} \tag{5.5}$$

$$\frac{A_n''(1)}{n!} + \frac{A_n'(1)}{n!} - \left(\frac{A_n'(1)}{n!}\right)^2 = \frac{n+1}{12}. \tag{5.6}$$

When normalized by its mean and variance, the distribution $\{A_{n,d}\}_{0 \leq d \leq n-1}$ converges to the standard normal distribution as $n \to \infty$ (see [B], [H], [Pi]). This is the central limit theorem for Eulerian numbers. In fact, we have the error bound

$$\sup_{x \in \mathbb{R}} \left| \frac{1}{n!} \sum_{d \leq \tilde{\sigma}_n x + \tilde{\mu}_n} A_{n,d} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt \right| = O(n^{-1/2}).$$

Similar results hold for the restricted Eulerian numbers $B_{n,d}$ defined in (5.4). In [F], Fulman shows that the mean and variance of $\operatorname{des}(\nu)$ for a randomly chosen $\nu \in \mathcal{C}_n$ are also $(n-1)/2$ and $(n+1)/12$ provided that $n \geq 3$ and $n \geq 4$ respectively. More generally, he shows that the k-th moment of $\operatorname{des}(\nu)$

for $\nu \in \mathcal{C}_n$ is equal to the k-th moment of $\mathrm{des}(\nu)$ for $\nu \in \mathcal{S}_n$ provided that $n \geq 2k$. From this result one can immediately conclude that the normalized distribution $B_{n,d}$ is also asymptotically normal as $n \to \infty$.

Below we will prove corresponding results for the distribution of degree for randomly chosen n-cycles.

Theorem 5.1. *The mean and variance of* $\deg(\nu)$ *for a randomly chosen* $\nu \in \mathcal{C}_n$ *(with respect to the uniform measure) are*

$$\mu_n := \frac{1}{(n-1)!} \sum_{d=1}^{n-2} d N_{n,d} = \frac{n}{2} - \frac{1}{n-1} \qquad (n \geq 3),$$

$$\sigma_n^2 := \frac{1}{(n-1)!} \sum_{d=1}^{n-2} (d-\mu_n)^2 N_{n,d} = \frac{n}{12} + \frac{n}{(n-1)^2(n-2)} \qquad (n \geq 5).$$

Proof. The argument is inspired by the method of [F, Theorem 2]. We begin by using the formula (4.3) for $\Delta_n(i)$ in the equation (4.9) to express the generating function G_n in terms of the Eulerian polynomials A_j in (5.3):

$$\begin{aligned} G_n(x) &= (1-x)^n \sum_{i\geq 1} \sum_{r|n} \sum_{j=0}^{r-1} \mu\left(\frac{n}{r}\right) i^j x^i \\ &= (1-x)^n \sum_{i\geq 1} \sum_{j=0}^{n-1} i^j x^i + (1-x)^n \sum_{i\geq 1} \sum_{\substack{r|n \\ r<n}} \sum_{j=0}^{r-1} \mu\left(\frac{n}{r}\right) i^j x^i \\ &= \sum_{j=0}^{n-1} x(1-x)^{n-j-1} A_j(x) + \sum_{\substack{r|n \\ r<n}} \sum_{j=0}^{r-1} \mu\left(\frac{n}{r}\right) x(1-x)^{n-j-1} A_j(x). \end{aligned} \tag{5.7}$$

If $n \geq 3$, every index j in the double sum in (5.7) is $\leq n-3$, so the polynomial in x defined by this double sum has $(1-x)^2$ as a factor. It follows that for $n \geq 3$,

$$G_n(x) = xA_{n-1}(x) + x(1-x)A_{n-2}(x) + O((1-x)^2)$$

as $x \to 1$. This gives

$$G_n'(1) = A_{n-1}'(1) + A_{n-1}(1) - A_{n-2}(1),$$

so by (5.5)

$$\mu_n = \frac{G_n'(1)}{(n-1)!} = \frac{n-2}{2} + 1 - \frac{1}{n-1} = \frac{n}{2} - \frac{1}{n-1}.$$

Similarly, if $n \geq 5$, every index j in the double sum in (5.7) is $\leq n-4$, so the polynomial defined by this double sum has $(1-x)^3$ as a factor. It follows that for $n \geq 5$,

$$G_n(x) = xA_{n-1}(x) + x(1-x)A_{n-2}(x) + x(1-x)^2 A_{n-3}(x) + O((1-x)^3)$$

as $x \to 1$. This gives

$$G''_n(1) = A''_{n-1}(1) + 2A'_{n-1}(1) - 2A'_{n-2}(1) - 2A_{n-2}(1) + 2A_{n-3}(1).$$

A straightforward computation using (5.5) and (5.6) then shows that

$$\sigma_n^2 = \frac{G''_n(1)}{(n-1)!} + \frac{G'_n(1)}{(n-1)!} - \left(\frac{G'_n(1)}{(n-1)!}\right)^2 = \frac{n}{12} + \frac{n}{(n-1)^2(n-2)},$$

as required. □

Remark 5.2. More generally, the expression (5.7) shows that for fixed k and large enough n,

$$G_n(x) = \sum_{j=0}^{k} x(1-x)^j A_{n-j-1}(x) + O((1-x)^{k+1})$$

as $x \to 1$. Differentiating this k times and evaluating at $x = 1$, we obtain the relation

$$G_n^{(k)}(1) = \sum_{j=0}^{k} (-1)^j \left(\binom{k}{j} j!\ A_{n-j-1}^{(k-j)}(1) + \binom{k}{j+1} (j+1)!\ A_{n-j-1}^{(k-j-1)}(1) \right)$$

which in theory links the moments of $\deg(\nu)$ for $\nu \in \mathcal{C}_n$ to the moments of $\operatorname{des}(\nu)$ for $\nu \in \mathcal{S}_j$ for $n-k \leq j \leq n-1$.

Numerical evidence suggest that the distribution $\{N_{n,d}\}_{1 \leq d \leq n-2}$ is also unimodal and reaches a peak at $d = \lfloor n/2 \rfloor$. Theorem 5.3 below asserts that when normalized by its mean and variance, the distribution $\{N_{n,d}\}_{1 \leq d \leq n-2}$ converges to normal as $n \to \infty$. In particular, the asymmetry of the numbers $N_{n,d}$ relative to d will asymptotically disappear. These facts are illustrated in Fig. 1.

Consider the sequence of normalized random variables

$$X_n := \frac{1}{\sigma_n} \big(\deg|_{\mathcal{C}_n} - \mu_n \big).$$

Let $\mathcal{N}(0,1)$ denote the normally distributed random variable with the mean 0 and variance 1.

Theorem 5.3. *$X_n \to \mathcal{N}(0,1)$ in distribution as $n \to \infty$.*

The proof follows the strategy of [FKL] and makes use of the following recent result of [KL2] which is a variant of a classical theorem of Curtiss. Recall that the ***moment generating function*** M_X of a random variable X is the expected value of e^{sX}:

$$M_X(s) := \mathbb{E}(e^{sX}) \qquad\qquad (s \in \mathbb{R}).$$

Lemma 5.4 ([KL2])**.** *Let $\{X_n\}_{n\geq 1}$ and Y be random variables and assume that $\lim_{n\to\infty} M_{X_n}(s) = M_Y(s)$ for all s in some non-empty open interval in $\mathbb{R}$. Then $X_n \to Y$ in distribution as $n \to \infty$.*

The proof of Theorem 5.3 via Lemma 5.4 will depend on two preliminary estimates.

Lemma 5.5. *For every $\varepsilon > 0$ there are constants $n(\varepsilon), i(\varepsilon) > 0$ such that*

$$\Delta_n(i) \begin{cases} \leq (1+\varepsilon)\, i^{n-1} & \text{if } n \geq 2 \text{ and } i \geq i(\varepsilon) \\ \geq (1-\varepsilon)\, i^{n-1} & \text{if } n \geq n(\varepsilon) \text{ and } i \geq 2. \end{cases}$$

Proof. By (4.1),

$$\Delta_n(i) \leq \sum_{r|n} \Delta_r(i) = \frac{i^n - 1}{i-1}.$$

The upper bound follows since $(i^n-1)/(i-1) < (1+\varepsilon)i^{n-1}$ for all n if i is large enough depending on ε.

For the lower bound, first note that the inequality $(i^r-1)/(i-1) \leq 2i^{r-1}$ holds for all $r \geq 1$ and all $i \geq 2$. Thus, by (4.3), we can estimate

$$\begin{aligned} \Delta_n(i) &\geq \frac{i^n-1}{i-1} - \sum_{\substack{r|n \\ r<n}} \frac{i^r-1}{i-1} \geq i^{n-1} - \sum_{\substack{r|n \\ r<n}} 2i^{r-1} \\ &\geq i^{n-1} - 2\sum_{r=1}^{\lfloor n/2 \rfloor} i^{r-1} \geq i^{n-1} - 2\,\frac{i^{n/2}-1}{i-1} \\ &\geq i^{n-1} - 4i^{n/2-1}. \end{aligned}$$

The last term is bounded below by $(1-\varepsilon)i^{n-1}$ for all i if n is large enough depending on ε. □

Lemma 5.6 ([FKL])**.** *For every $0 < x < 1$ and $n \geq 1$,*

$$\frac{(n-1)!\, x}{(\log(1/x))^n} \leq \sum_{i\geq 2} i^{n-1} x^i \leq \frac{(n-1)!}{x(\log(1/x))^n}.$$

Proof. By elementary calculus,

$$\sum_{i\geq 2} i^{n-1} x^i \leq \int_0^\infty u^{n-1} x^{u-1}\, du = \frac{(n-1)!}{x(\log(1/x))^n}$$

and

$$\sum_{i\geq 2} i^{n-1}x^i \geq \int_0^\infty u^{n-1}x^{u+1}\,du = \frac{(n-1)!\,x}{(\log(1/x))^n}. \qquad \square$$

Proof of Theorem 5.3. By Lemma 5.4 it suffices to show that $\lim_{n\to\infty} M_{X_n}(s) = M_{\mathcal{N}(0,1)}(s) = e^{s^2/2}$ for all negative values of s. Fix an $s<0$ and set $0 < x := e^{s/\sigma_n} < 1$ (we will think of x as a function of n, with $\lim_{n\to\infty} x = 1$). Notice that by Theorem 5.1

$$\mu_n = \frac{n}{2} + O(n^{-1}) \quad \text{and} \quad \sigma_n^2 = \frac{n}{12} + O(n^{-2}) \quad \text{as } n\to\infty. \tag{5.8}$$

Using (4.9), we can write

$$\begin{aligned} M_{X_n}(s) = \mathbb{E}(e^{sX_n}) &= \frac{e^{-s\mu_n/\sigma_n}}{(n-1)!}\,G_n(e^{s/\sigma_n}) = \frac{x^{-\mu_n}}{(n-1)!}\,G_n(x) \\ &= \frac{x^{1-\mu_n}(1-x)^n\varphi(n)}{(n-1)!} + \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{i\geq 2}\Delta_n(i)x^i. \end{aligned}$$

As the first term is easily seen to tend to zero, it suffices to show that

$$H_n := \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{i\geq 2}\Delta_n(i)x^i \xrightarrow{n\to\infty} e^{s^2/2}. \tag{5.9}$$

By (5.8) we have the estimate

$$1-x = -\frac{s}{\sigma_n} - \frac{s^2}{2\sigma_n^2} + O(n^{-3/2}).$$

This, combined with the basic expansion

$$\log\left(\frac{1-x}{\log(1/x)}\right) = -\frac{1}{2}(1-x) - \frac{5}{24}(1-x)^2 + O((1-x)^3),$$

shows that

$$\left(\frac{1-x}{\log(1/x)}\right)^n = \exp\left(\frac{ns}{2\sigma_n} + \frac{ns^2}{24\sigma_n^2} + O(n^{-1/2})\right). \tag{5.10}$$

Take any $\varepsilon > 0$ and find $n(\varepsilon)$ from Lemma 5.5. Then, if $n \geq n(\varepsilon)$,

$$\begin{aligned}
H_n &\geq \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\,(1-\varepsilon)\sum_{i\geq 2} i^{n-1}x^i \\
&\geq (1-\varepsilon)\,x^{1-\mu_n}\left(\frac{1-x}{\log(1/x)}\right)^n && \text{(by Lemma 5.6)} \\
&= (1-\varepsilon)\exp\left(\frac{s(1-\mu_n)}{\sigma_n}+\frac{ns}{2\sigma_n}+\frac{ns^2}{24\sigma_n^2}+O(n^{-1/2})\right) && \text{(by (5.10))} \\
&= (1-\varepsilon)\exp\left(\frac{s(1+O(n^{-1}))}{\sigma_n}+\frac{s^2}{2+O(n^{-3})}+O(n^{-1/2})\right) && \text{(by (5.8)).}
\end{aligned}$$

Taking the lim inf as $n \to \infty$ and then letting $\varepsilon \to 0$, we obtain

$$\liminf_{n\to\infty} H_n \geq e^{s^2/2}.$$

Similarly, take any $\varepsilon > 0$, find $i(\varepsilon)$ from Lemma 5.5 and use the basic inequality $\Delta_n(i) \leq (i^n - 1)/(i-1) \leq 2i^{n-1}$ for all $n, i \geq 2$ to estimate

$$\begin{aligned}
H_n &= \frac{x^{-\mu_n}(1-x)^n}{(n-1)!}\left(\sum_{2\leq i<i(\varepsilon)}+\sum_{i\geq i(\varepsilon)}\right)\Delta_n(i)x^i \\
&\leq \frac{2x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{2\leq i<i(\varepsilon)} i^{n-1}x^i + \frac{(1+\varepsilon)x^{-\mu_n}(1-x)^n}{(n-1)!}\sum_{i\geq i(\varepsilon)} i^{n-1}x^i.
\end{aligned}$$

The first term is a polynomial in x and is easily seen to tend to zero as $n \to \infty$. The second term is bounded above by

$$\begin{aligned}
&(1+\varepsilon)\,x^{-1-\mu_n}\left(\frac{1-x}{\log(1/x)}\right)^n && \text{(by Lemma 5.6)} \\
=&(1+\varepsilon)\exp\left(\frac{s(-1-\mu_n)}{\sigma_n}+\frac{ns}{2\sigma_n}+\frac{ns^2}{24\sigma_n^2}+O(n^{-1/2})\right) && \text{(by (5.10))} \\
=&(1+\varepsilon)\exp\left(\frac{s(-1+O(n^{-1}))}{\sigma_n}+\frac{s^2}{2+O(n^{-3})}+O(n^{-1/2})\right) && \text{(by (5.8))}
\end{aligned}$$

Taking the lim sup as $n \to \infty$ and then letting $\varepsilon \to 0$, we obtain

$$\limsup_{n\to\infty} H_n \leq e^{s^2/2}.$$

This verifies (5.9) and completes the proof. □

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