A GENERALIZATION OF BALANCED TABLEAUX AND MARRIAGE PROBLEMS WITH UNIQUE SOLUTIONS

BRIAN T. CHAN

Abstract. We consider families of finite sets that we call shellable and that have been characterized by Chang and by Hirst and Hughes as being the families of sets that admit unique solutions to Hall's marriage problem. We prove that shellable families can be characterized by using a generalized notion of hook-lengths. Then, we introduce a natural generalization of standard skew tableaux and Edelman and Greene's balanced tableaux, then prove existence results about such a generalization using our characterization of shellable families.

1. Introduction

Hall's Marriage Theorem is a combinatorial theorem that characterises when a finite family of sets has a system of distinct representatives, which is also called a transversal. Hall [10] proved that such a family has a system of distinct representatives if and only if this family satisfies the marriage condition. This theorem is known to be equivalent to at least six other theorems [22] which include Dilworth's Theorem, Menger's Theorem, and the Max-Flow Min-Cut Theorem.

Hall Jr. proved [11] that Hall's Marriage Theorem also holds for arbitrary families of finite sets. Afterwards, Chang [4] noted how Hall Jr.'s work in [11] can be used to characterize marriage problems with unique solutions. Specifically, the families of sets that admit marriage problems with unique solutions were characterized [4]. Later on, Hirst and Hughes proved that such a characterization of marriage problems with unique solutions can be derived by only using a subsystem of second order arithmetic known as RCA_0 [13], and they showed that their work in [13] can also be extended to marriage problems with a fixed finite number of solutions [12]. In this paper, we call the families of finite sets that admit marriage problems with unique solutions *shellable* and give a new characterization of these families of sets by generalizing the notion of standard Young tableaux and Edelman and Greene's balanced tableaux.

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Standard skew tableaux are well-known and intensively studied in algebraic combinatorics, for example [14, 19, 20, 26]. Moreover, another class of tableaux was introduced by Edelman and Greene in [5, 6], where they defined balanced tableaux on partition shapes. In investigating the number of maximal chains in the weak Bruhat order of the symmetric group, Edelman and Greene proved [5, 6] that the number of balanced tableaux of a given partition shape equals the number of standard Young tableaux of that shape. Since then, connections to random sorting networks [2], the Lascoux-Schützenberger tree [17], and a generalization of balanced tableaux pertaining to Schubert polynomials [7] have been explored.

Lastly, properties of products of hook-lengths have recently enjoyed some attention by Pak et.al. [18, 21] and by Swanson [27]. In particular, an inequality between products of hook-lengths and products of dual hook-lengths was derived [18, 21, 27]. We introduce a generalization of standard Young tableaux and balanced tableaux for skew shapes, show, in Corollary 4.21 using our characterization of marriage problems with unique solutions, that the number of such generalizations that can exist is given by a product of hook-lengths, and show in Theorem 4.22, as a consequence, that the average number of tableaux that belongs to such a generalization is given by the hook-length formula. Afterwards, we indicate how our generalization of standard Young tableaux and balanced tableaux can be analysed using Naruse's Formula for skew tableaux and how our results can be extended to skew shifted shapes [9, 18, 20] and likely to certain d-complete posets [9, 20].

2. Preliminaries

In this section, we give the preliminaries that will be needed for this paper. Throughout this paper, let $\mathbb{N}$ denote the set of positive integers, let $\mathbb{N}_0$ denote the set of non-negative integers, and for all positive integers n, define $[n] = \{1, 2, \ldots, n\}$.

Let X and Y be sets. If $n \in \mathbb{N}$ and if X_1, X_2, $\ldots$, X_n are sets, then the *Cartesian product* $X_1 \times X_2 \times \cdots \times X_n$ *of* X_1, X_2, $\ldots$, *and* X_n is the set of ordered n-tuples $\{(r_1, r_2, \ldots, r_n) : \forall i \in [n], r_i \in X_i\}$. If $X_1 = X_2 = \cdots = X_n$ and $X = X_1$, then write $X^n = X_1 \times X_2 \times \cdots \times X_n$.

We use the terms function and map interchangeably. Let X and Y be sets, and let $f : X \to Y$ be a function. Then for all $X' \subseteq X$, let $f(X') = \{f(r) : r \in X'\}$, and, for all $Y' \subseteq Y$, let $f^{-1}(Y') = \{r \in Y : f(r) \in Y'\}$. If f is injective, then write $f^{-1}(r) = f^{-1}(\{r\})$ for all $r \in f(X)$. For all $X' \subseteq X$, let the *restriction of* f *to* X', which we denote by $f|_{X'}$, be the function $g : X' \to Y$ defined by $g(r) = f(r)$ for all $r \in X'$.

Let $\mathcal{F}$ be a family $h : I \to X$ of sets. Then we define a *member* F of $\mathcal{F}$ to be an ordered pair $(i, h(i))$ where $i \in I$. *When we use subscripts to describe the members* F *of a family* $h : I \to X$ *of sets, the subscripts do not necessarily have to be elements of* I. We write $F_1, F_2, \dots \in \mathcal{F}$ if $F_k \in \mathcal{F}$ for all k. Moreover, two members $F_1 = (i_1, h(i_1))$ and $F_2 = (i_2, h(i_2))$ of $\mathcal{F}$ are *different* if $i_1 \neq i_2$. If $F = (i, h(i))$ is a member of $\mathcal{F}$, then we write $r \in F$ if $r \in h(i)$. We write $r_1, r_2, \dots \in F$ if $r_k \in F$ for all k. For any set Y, define a *function* $f : \mathcal{F} \to Y$ *from* $\mathcal{F}$ *to* Y to be a function $g : I \to Y$, and for all members $F \in \mathcal{F}$, write $f(F) = g(i)$ if $i \in I$ satisfies $F = (i, h(i))$. We also call $f : \mathcal{F} \to Y$ a *map from* $\mathcal{F}$ *to* Y. Such a function $f : \mathcal{F} \to Y$ is *injective* if $F_1, F_2 \in \mathcal{F}$ and $f(F_1) = f(F_2)$ implies that F_1 and F_2 are not different.

Let $\mathcal{F}$ be a family $h : I \to X$ of sets. When describing the members $F = (i, h(i))$ of such families, we will write $h(i)$ instead of the ordered pair $(i, h(i))$. We will also use set-theoretic notation to describe families of sets by writing $\mathcal{F} = \{F : F \in \mathcal{F}\}$. For instance, if $\mathcal{F}$ is the family of sets defined by $h : \{1, 2\} \to \{\{1\}\}$, then we write $\mathcal{F} = \{\{1\}, \{1\}\}$, where the members $(1, \{1\})$ and $(2, \{1\})$ of $\mathcal{F}$ are both denoted by $\{1\}$. Moreover, we write $|\mathcal{F}| = |I|$, and say that $|\mathcal{F}|$ is the *number of members of* $\mathcal{F}$. A family $\mathcal{F}$ of sets is *finite* if $|\mathcal{F}|$ is finite. Lastly, if $\mathcal{F}$ is a family $h : I \to X$ of sets, then write $\bigcup_{F \in \mathcal{F}} F = \bigcup_{r \in I} h(r)$.

For all $n \in \mathbb{N}$, a *partition* λ *of* n, written $\lambda \vdash n$, is a weakly decreasing sequence of positive integers whose sum is n. We write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ to denote such a partition, where $\lambda_i \in \mathbb{N}$ for all $1 \leq i \leq \ell$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_\ell = n$. For instance, $(3, 2, 2)$ is a partition of 7 and $(3, 2, 1)$ is a partition of 6. We also let $\emptyset$ denote the *empty partition*, which we define to be the only partition of 0. Whether the symbol $\emptyset$ refers to the empty set or to the empty partition can be easily determined from context.

We will define Young diagrams in the following way (cf. [1], also cf. [16, 23, 25, 26].) A *Young diagram* is the empty set or a finite subset X of $\mathbb{N}^2$ such that for some $i, j \in \mathbb{N}$, $(1, j) \in X$ and $(i, 1) \in X$. We call the elements of a Young diagram X the *cells of* X. Lastly, given a Young diagram X, define, for all $i \in \mathbb{N}$, *row* i *of* X to be the following subset of cells

$$\{r \in X : \exists j \in \mathbb{N} \text{ such that } r = (i, j)\}$$

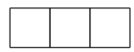
and, for all $j \in \mathbb{N}$, define *column* j *of* X to be the following subset of cells

$$\{r \in X : \exists i \in \mathbb{N} \text{ such that } r = (i, j)\}.$$

Sometimes, when we mention a cell $r = (i, j)$ in a Young diagram, we write (i, j) instead of r.

In order for us to follow the conventions used in the literature [1, 16, 23, 25, 26], we will always depict Young diagrams by using an array of boxes where each such box has unit area and where each such box contains an element of $\mathbb{N}^2$ at its centre. Moreover, we also follow conventions in the literature by doing the following. We will, when depicting a Young diagram X, always draw row $i+1$ of X beneath row i and we will always draw column $j+1$ to the right of column j.

Example 2.1. *If $X_1 = \{(1,1),(1,2),(1,3)\}$, $X_2 = \{(1,1),(1,3),(1,6),(1,7)\}$, and $X_3 = \{(1,1),(1,3),(2,2),(3,1),(3,3)\}$, then the Young diagram X_1 is depicted by*



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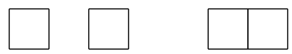
the Young diagram X_2 is depicted by



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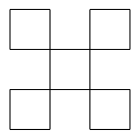
and the Young diagram X_3 is depicted by



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Moreover, row 1 of X_1 is X_1, column j of X_1, where $1 \le j \le 3$, is $\{(1,j)\}$, row 1 of X_2 is X_2, column j of X_2, where $j \in \{1,3,6,7\}$, is $\{(1,j)\}$, row 1 of X_3 is $\{(1,1),(1,3)\}$, row 2 of X_3 is $\{(2,2)\}$, row 3 of X_3 is $\{(3,1),(3,3)\}$, column 1 of X_3 is $\{(1,1),(3,1)\}$, column 2 of X_3 is $\{(2,2)\}$, and column 3 of X_3 is $\{(1,3),(3,3)\}$.

3. Results relating to the marriage condition

In this section, we consider families of sets that satisfy Hall's marriage condition. We introduce a generalized notion of hook-lengths for such families. Then, we establish an existence result based on such generalized hook-lengths that gives a new characterization of marriage problems with unique solutions. Afterwards, we prove a corollary that complements this existence result.

Definition 3.1. *(Hall, [10]) Let $n \in \mathbb{N}$, and let $\mathcal{F}$ be a finite family of subsets of $[n]$. Then a* transversal *of $\mathcal{F}$ is an injective function $t : \mathcal{F} \to [n]$ such that $t(F) \in F$ for all $F \in \mathcal{F}$.*

Informally, a transversal maps each F to one of its elements.

Definition 3.2. *(Hall, [10]) Let $n \in \mathbb{N}$, and let $\mathcal{F}$ be a finite family of subsets of $[n]$. Then $\mathcal{F}$ satisfies the* marriage condition *if for all subfamilies $\mathcal{F}'$ of $\mathcal{F}$,*

$$|\mathcal{F}'| \le \left| \bigcup_{F \in \mathcal{F}'} F \right|.$$

Example 3.3. *A simple example illustrating both Definition 3.1 and Definition 3.2 is as follows. Let $n = 5$, and let*

$$\mathcal{F} = \{\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\{1,2,3,4,5\}\}.$$

Then $\mathcal{F}$ satisfies the marriage condition. For example, if $\mathcal{F}'$ is the subfamily of $\mathcal{F}$ defined by $\mathcal{F}' = \{\{1\},\{1,2,3\}\}$, then $|\mathcal{F}'| = 2$ and $|\{1\} \cup \{1,2,3\}| = 3$. The map $t : \mathcal{F} \to \{1,2,3,4,5\}$ defined by $t([k]) = k$ for all $1 \le k \le 5$ is a transversal of $\mathcal{F}$.

One could interpret the above example as evidence to the possibility that a family of sets of $[n]$ has a transversal if and only if it satisfies the marriage condition. It turns out that this is always true. The following is known as *Hall's Marriage Theorem.*

Theorem 3.4. *(Hall, [10]) Let $n \in \mathbb{N}$, and let $\mathcal{F}$ be a family of non-empty subsets of $[n]$. Then $\mathcal{F}$ has a transversal if and only if $\mathcal{F}$ satisfies the marriage condition.*

In order to use the families of sets in Hall's Marriage Theorem, we will define more structure on the objects being considered. Definition 3.5 represents the local conditions and generalized hook-lengths mentioned in Section 1; how this relates to hook-lengths will become clear in the next section. Recall the notation we use for functions in Section 2.

Definition 3.5. *Let $n \in \mathbb{Z}$, let $\mathcal{F}$ be a family of non-empty subsets of $[n]$, and let t be a transversal of $\mathcal{F}$. Then a* configuration f of t *is a function $f : [n] \to \mathbb{N}$ such that for all $F \in \mathcal{F}$,*

$$f(t(F)) \leq |F|.$$

Moreover, a permutation $\sigma : [n] \to [n]$ satisfies f *if the following holds for all $F \in \mathcal{F}$. The positive integer $\sigma(t(F))$ is the k^{th} smallest element of $\sigma(F)$, where $k = f(t(F))$.*

Example 3.6. *Let $n = 5$. Moreover let $\mathcal{F}$ and let $t : \mathcal{F} \to [n]$ be as defined in Example 3.3. Furthermore, let $F_i = [i]$ for all $1 \leq i \leq 5$ so $t(F_i) = i$. Lastly, let $f : [n] \to \mathbb{N}$ be defined by $f(1) = 1$, $f(2) = 1$, $f(3) = 2$, $f(4) = 4$, and $f(5) = 2$. The map f is a configuration of t. For instance, since $F_3 = \{1, 2, 3\}$, $t(F_3) = 3$, $f(t(F_3)) = 2$, $|F_3| = 3$, and $f(t(F_3)) \leq |F_3|$. Similarly, $f(t(F_1)) = 1 \leq 1 = |F_1|$, $f(t(F_2)) = 1 \leq 2 = |F_2|$, $f(t(F_4)) = 4 \leq 4 = |F_4|$, and $f(t(F_5)) = 2 \leq 5 = |F_5|$.*

Moreover, the permutation $\sigma : [n] \to [n]$ defined by $\sigma = 41352$ satisfies f. For example, consider $F_3 = \{1, 2, 3\}$. As before, $t(F_3) = 3$ and $f(t(F_3)) = 2$, so $k = 2$. Moreover, $\sigma(t(F_3)) = \sigma(3) = 3$. Lastly, $\sigma(F_3) = \{\sigma(1), \sigma(2), \sigma(3)\} = \{1, 3, 4\}$, and $\sigma(t(F_3)) = 3$ is the second smallest element of $\sigma(F_3)$. Similarly, for F_1, $k = 1$, $\sigma(t(F_1)) = 1$, and $\sigma(F_1) = \{4\}$; for F_2, $k = 1$, $\sigma(t(F_2)) = 1$, and $\sigma(F_2) = \{1, 4\}$; for F_4, $k = 4$, $\sigma(t(F_4)) = 5$, and $\sigma(F_4) = \{1, 3, 4, 5\}$; and for F_5, $k = 2$, $\sigma(t(F_5)) = 2$, and $\sigma(F_5) = \{1, 2, 3, 4, 5\}$.

Configurations satisfy the following property, its usefulness will become more apparent in the next section.

Lemma 3.7. *Let $n \in \mathbb{N}$, and let $\mathcal{F}$ be a family of subsets of $[n]$ that has a transversal $t : \mathcal{F} \to [n]$. Then every permutation $\sigma : [n] \to [n]$ satisfies exactly one configuration f of t.*

Proof. Let $\sigma : [n] \to [n]$ be a permutation. Then σ satisfies the configuration f of t defined by letting, for all $F \in \mathcal{F}$, $f(t(F)) = k$ where $\sigma(t(F))$ is the k^{th} smallest element of the set $\sigma(F)$. Now, suppose that σ satisfies more than one configuration of t. Then, let f_1 and f_2 be two distinct configurations of t. Because $f_1 \neq f_2$, there is an element $F \in \mathcal{F}$ such that $f_1(t(F)) \neq f_2(t(F))$. So write $k_1 = f_1(t(F))$ and write $k_2 = f_2(t(F))$. Since σ satisfies f_1, Definition 3.5 implies that $\sigma(t(F))$ is the k_1^{th} smallest element of $\sigma(F)$. Moreover, since σ satisfies f_2, Definition 3.5 implies that $\sigma(t(F))$ is the k_2^{th} smallest element of $\sigma(F)$. However, this is impossible because $k_1 = f_1(t(F)) \neq f_2(t(F)) = k_2$. $\square$

Now, we define the following stronger form of the marriage condition that was defined by Chang [4] and Hirst and Hughes in [13].

Definition 3.8. *(cf. ([13], Theorem 4)) Let $n \in \mathbb{N}$, let $\mathcal{F}$ be a finite family of subsets of $[n]$, and write $m = |\mathcal{F}|$. Then $\mathcal{F}$ is* shellable *if there exists a bijection $\sigma_{\mathcal{F}} : [m] \to \mathcal{F}$ such that for all $k \in [m]$,*

$$\left| \bigcup_{i=1}^{k} \sigma_{\mathcal{F}}(i) \right| = k. \tag{1}$$

Informally, $\sigma_{\mathcal{F}}$ maps each k to a subset, such that the union of the first k subsets has cardinality k.

Remark 3.9. Shellable families of sets are connected to Theorem 3.4. Chang ([4], Theorem 1) noted that a simple consequence of Hall Jr.'s work ([11], Theorem 2) is that a finite family $\mathcal{F}$ of subsets of $[n]$ has exactly one transversal if and only if $\mathcal{F}$ is shellable. Later on, Hirst and Hughes showed that this can be proved using a subsystem of second order arithmetic called RCA_0 [13] and proved an extension of this result involving infinite families of finite sets in the context of reverse mathematics. From the aforementioned characterization of finite families of subsets of $[n]$ that have exactly one transversal, we have, by Theorem 3.4, that all shellable families satisfy the marriage condition.

Remark 3.10. The term shellable is not used in [4], [11], and [13]. However, we use this terminology because Definition 3.8 resembles the definition of a *shellable pure d-dimensional simplicial complex* ([3], Appendix A2.4, Definition A2.4.1). The differences between Definition 3.8 and Definition A2.4.1 are as follows. The sets in Definition 3.8 do not require additional conditions on the cardinalities of the members of $\mathcal{F}$. Also, in Definition A2.4.1, the requirement of the existence of a bijection $\sigma_{\mathcal{F}} : [m] \to \mathcal{F}$ as described in Definition 3.8 is relaxed to requiring the existence of a certain bijection from a subset of $[m]$ to a subset of $\mathcal{F}$.

Remark 3.11. When describing the members of a shellable family, we will use a total ordering on the members of that family. Specifically, let $\mathcal{F}$ be a shellable family of subsets of $[n]$ and let $m = |\mathcal{F}|$. By Definition 3.8, there exists a bijection $\sigma_{\mathcal{F}} : [m] \to \mathcal{F}$ such that Equation 1 is satisfied for all $k \in [m]$. From this bijection $\sigma_{\mathcal{F}}$, define a total ordering $<_{\mathcal{F}}$ on the members of $\mathcal{F}$ by defining, for all members $F', F'' \in \mathcal{F}$, $F' <_{\mathcal{F}} F''$ if $\sigma_{\mathcal{F}}^{-1}(F'') <_{\mathcal{F}} \sigma_{\mathcal{F}}^{-1}(F')$. The *shelling order* of a shellable complex from ([3], Appendix A2.4, Definition A2.4.1) is a variant of this total ordering.

Example 3.12. *Let $n \in \mathbb{N}$, and define the following finite family of sets.*

$$\mathcal{F} = \{[i] : i \in [n]\}$$

Then $\mathcal{F}$ is shellable for the following reason. Firstly, $|\mathcal{F}| = n$, so the variable m in Definition 3.8 satisfies $m = n$. Next, define the bijection $\sigma_{\mathcal{F}} : [n] \to \mathcal{F}$ be letting $\sigma_{\mathcal{F}}(k) = [k]$ for all

$k \in [n]$. Then for all $k \in [n]$,

$$\left|\bigcup_{i=1}^{k} \sigma_{\mathcal{F}}(i)\right| = |[k]| = k.$$

So as $\mathcal{F}$ and $\sigma_{\mathcal{F}}$ satisfy Equation 1, $\mathcal{F}$ is shellable.

Example 3.13. *If $n \in \mathbb{N}$ and $n \geq 3$, then a family of subsets of $[n]$ that satisfies the marriage condition but is not shellable is*

$$\mathcal{F} = \{[n]\backslash\{k\} : k \in [n]\}.$$

This family satisfies the marriage condition because for any subfamily $\mathcal{F}'$ of $\mathcal{F}$ with at least one member,

$$\left| \bigcup_{F \in \mathcal{F}'} F \right| = \begin{cases} n-1 & \text{if } |\mathcal{F}'| = 1 \\ n & \text{else.} \end{cases}$$

However, if $\mathcal{F}$ is shellable, where $m = |\mathcal{F}|$, then the following holds. By Definition 3.8 and Equation 1, there exists a bijection $\sigma_{\mathcal{F}} : [m] \to \mathcal{F}$ such that $|\sigma_{\mathcal{F}}(1)| = 1$. So as $\sigma_{\mathcal{F}}(1) \in \mathcal{F}$, it follows that $\mathcal{F}$ has a member whose cardinality is one. However, for all $F \in \mathcal{F}$, $|F| = n - 1 \geq 2$. So it follows that $\mathcal{F}$ is not shellable.

Now, we prove the main result of this section. It is a partial converse of Lemma 3.7.

Theorem 3.14. *Let $n \in \mathbb{N}$. Moreover, let $\mathcal{F}$ be a family of subsets of $[n]$ such that $\mathcal{F}$ satisfies the marriage condition, let t be a transversal of $\mathcal{F}$, and assume that $|\mathcal{F}| = n$. Then $\mathcal{F}$ is shellable if and only if the following holds. For all configurations f of t, there exists a permutation $\sigma : [n] \to [n]$ that satisfies f.*

Example 3.15. *Let $n = 3$. Moreover, let $\mathcal{F} = \{\{1, 2, 3\}, \{1, 3\}\}$, and let $t : \mathcal{F} \to [n]$ be defined by $t(\{1, 2, 3\}) = 1$ and $t(\{1, 3\}) = 3$. The family $\mathcal{F}$ is not shellable since we cannot find a bijection $\sigma_{\mathcal{F}} : [m] \to \mathcal{F}$ such that $|\sigma_{\mathcal{F}}(1)| = 1$. Now, let $f : [n] \to \mathbb{N}$ be the configuration of t defined by $f(1) = 1$, $f(2) = 2$, and $f(3) = 1$. It is a configuration of t since $f(t(\{1, 2, 3\})) = f(1) = 1 \leq 3 = |\{1, 2, 3\}|$ and $f(t(\{1, 3\})) = f(3) = 1 \leq 2 = |\{1, 3\}|$. Then no permutation $\sigma : [n] \to [n]$ satisfies f as follows.*

Suppose that there is a permutation $\sigma_0 : [n] \to [n]$ that satisfies f. First, consider the element $F_1 = \{1, 2, 3\}$ of $\mathcal{F}$. Then $k = f(t(F_1)) = f(1) = 1$. Moreover, $\sigma_0(F_1) = \{1, 2, 3\}$. So as σ_0 satisfies f, $\sigma_0(t(F_1)) = \sigma_0(1)$ is the smallest element of $\{1, 2, 3\}$. Hence, $\sigma_0(1) = 1$. Next, consider the element $F_2 = \{1, 3\}$ of $\mathcal{F}$. Then $k = f(t(F_2)) = f(3) = 1$. So as σ_0 satisfies f, $\sigma_0(t(F_2)) = \sigma_0(3)$ is the smallest element of $\sigma_0(F_2) = \{\sigma_0(1), \sigma_0(3)\}$. But then, $\sigma_0(3) < \sigma_0(1)$, contradicting the fact that $\sigma_0(1) = 1$.

Proof. First, assume that $\mathcal{F}$ is not shellable, and let t be a transversal of $\mathcal{F}$. Because $\mathcal{F}$ is not shellable, we will not use a total ordering to describe the members of this family. Moreover, since $\mathcal{F}$ is not shellable, Equation 1 is false for at least one element $k \in [m]$, where $m = |\mathcal{F}|$, implying, since $|\mathcal{F}| = n$, that at least one of the following holds.

(1) There are two distinct elements $k_1, k_2 \in [n]$ and a member $F \in \mathcal{F}$ such that $k_1, k_2 \in F$, no other member of $\mathcal{F}$ contains k_1, and no other member of $\mathcal{F}$ contains k_2.
(2) For all $k \in [n]$, there are at least two members of $\mathcal{F}$ that contain k.

Suppose that (1) holds, and let F be as described in (1). Because $|\mathcal{F}| = n$ and because t is an injective map from $\mathcal{F}$ to $[n]$, t is a bijection from $\mathcal{F}$ to $[n]$. But then, $k_1, k_2 \in t(\mathcal{F})$ and there exists a member $F_1 \in \mathcal{F}$ that is different from F (recall what we mean by different members in Section 3) such that $t(F_1) = k_1$ or $t(F_1) = k_2$. But, $k_1 \notin F_1$ and $k_2 \notin F_1$, implying that $t(F_1) \notin F_1$, which is impossible by Definition 3.1.

So assume that (2) holds. Let f be the configuration of t defined by $f(k) = 1$ for all $k \in [n]$, and suppose that there is a permutation $\sigma_0 : [n] \to [n]$ such that σ_0 satisfies f. Select an element $k_1 \in [n]$. Because $|\mathcal{F}| = n$, Equation 1 implies that $k_1 \in t(\mathcal{F})$. Hence, there exists a member $F_1 \in \mathcal{F}$ such that $t(F_1) = k_1$. By (2), there also exists a member F_2 of $\mathcal{F}$ that is different from F_1 and that satisfies $k_1 \in F_2$. Lastly, let $k_2 = t(F_2)$. Note that $k_2 \neq k_1$ since t is an injection. As $f(k_2) = 1$ and σ_0 satisfies f, $\sigma_0(k_2) = \sigma_0(t(F_2))$ is the smallest element of $\sigma_0(F_2)$. And as $k_1 \in F_2$, $\sigma_0(k_1) \in \sigma_0(F_2)$, so $\sigma_0(k_1) > \sigma_0(k_2)$. Repeating this argument with k_2 replacing k_1 gives an element $k_3 \in [n]$ such that $\sigma_0(k_2) > \sigma_0(k_3)$, repeating this argument with k_3 replacing k_2 gives an element $k_4 \in [n]$ such that $\sigma_0(k_3) > \sigma_0(k_4)$, and so on. Hence, there is an infinite sequence $(k_i)_{i=1,2,\ldots}$ of elements in $[n]$ such that $\sigma_0(k_i) > \sigma_0(k_{i+1})$ for all $i \in \mathbb{N}$. However, $[n]$ is finite. So there are positive integers $i, j \in \mathbb{N}$ such that $k_i = k_{i+j}$. But then,

$$\sigma_0(k_i) > \sigma_0(k_{i+1}) > \cdots > \sigma_0(k_i),$$

which is impossible. Therefore if $\mathcal{F}$ is not shellable then no permutation satisfies the configuration f.

Next, assume that $\mathcal{F}$ is shellable. Because $\mathcal{F}$ is shellable, we will use the total ordering as described in Remark 3.11 to describe the members of this family. We proceed by induction on n. If $n = 1$, then the only family of subsets of $\{1\}$ with a transversal is the family $\mathcal{F} = \{\{1\}\}$. Moreover, with t being the transversal of $\mathcal{F}$ defined by mapping $\{1\}$ to 1, the only configuration f that satisfies t is the function $f : \{1\} \to \mathbb{N}$ defined by $f(1) = 1$, and any permutation $\sigma : \{1\} \to \{1\}$ satisfies f.

So let $n \geq 2$ and assume that the induction hypothesis holds. Let t be a transversal of $\mathcal{F}$ and let f be a configuration of t. Because $\mathcal{F}$ is shellable, Definition 3.8 and Remark 3.11 imply that there is an element $n' \in [n]$ such that, for all $F \in \mathcal{F}$, $n' \notin F$ or $t(F) = n'$. So without loss of generality, assume that $n' = n$. Let $\mathcal{F}'$ be the family of sets defined by

$$\mathcal{F}' = \{F \in \mathcal{F} : t(F) \neq n\}.$$

As $n \notin F$ for all $F \in \mathcal{F}$ such that $t(F) \neq n$, $\mathcal{F}'$ is a family of subsets of $[n-1]$. Next, define $t' : \mathcal{F}' \to [n-1]$ by letting $t'(F) = t(F)$ for all $F \in \mathcal{F}'$. Moreover, because t is a transversal of $\mathcal{F}$, t' is a transversal of $\mathcal{F}'$. By Definition 3.8 and the choice of $n = n'$, $\mathcal{F}'$ is shellable for the following reason.

Define the bijection $\sigma_{\mathcal{F}'} : [n-1] \to \mathcal{F}'$ by

$$\sigma_{\mathcal{F}'}(k) = \sigma_{\mathcal{F}}(k)$$

for all $k \in [n-1]$. Because $\sigma_{\mathcal{F}}$ satisfies Equation 1 of Definition 3.8, $\sigma_{\mathcal{F}'}$ satisfies Equation 1 of Definition 3.8. Hence, $\mathcal{F}'$ is shellable. So by the induction hypothesis, there exists a permutation $\sigma' : [n-1] \to [n-1]$ that satisfies all configurations f' of t'.

Let $m = f(n)$, and let F_σ be the element of $\mathcal{F}$ such that $t(F_\sigma) = n$. There is an order embedding $\kappa : [n-1] \to [n]$ such that the element $k \in [n] \backslash \kappa([n-1])$ is the m^{th} smallest element of $\kappa(F_\sigma)$. With κ defined, define $\sigma : [n] \to [n]$ as follows. Let $\sigma(n)$ be the element of $[n]$ that is not in $\kappa([n-1])$, and, for all $k \in [n-1]$, let $\sigma(k) = \kappa(\sigma'(k))$. Because $n = n'$ and $n' \in F$ for exactly one element $F \in \mathcal{F}$, σ satisfies f. From this, the theorem follows. $\square$

Remark 3.16. A family $\mathcal{F}$ satisfying the condition $|\mathcal{F}| = n$ is called a *critical block* in [11]. In [11], Hall Jr. used this notion as a very important ingredient in extending Hall's Marriage Theorem to infinite families of finite sets.

As a corollary, we show the following.

Corollary 3.17. *Let $n \in \mathbb{N}$. Moreover, let $\mathcal{F}$ be a family of subsets of $[n]$ that has a transversal, let t be a transversal of $\mathcal{F}$, and assume that $|\mathcal{F}| = n$. Then every configuration f of t is satisfied by some permutation $\sigma : [n] \to [n]$ if and only if the following holds. The configuration f_0 of t defined by $f_0(t(F)) = 1$ for all $F \in \mathcal{F}$ is satisfied by some permutation $\sigma_0 : [n] \to [n]$.*

Example 3.18. *The family of sets in Example 3.15 is, as shown in that example, a family where the configuration f_0 as defined in Corollary 3.17 is not satisfied by any permutation.*

Proof. By Theorem 3.4, $\mathcal{F}$ has a transversal if and only if $\mathcal{F}$ satisfies the marriage condition. So by Theorem 3.14, it is enough to prove that $\mathcal{F}$ is shellable if and only if the configuration f_0 of t as described in the corollary is satisfied by some permutation $\sigma_0 : [n] \to [n]$. The first part of the proof of Theorem 3.14 proves that if $\mathcal{F}$ is not shellable, then the configuration f_0 is not satisfied by any permutation. So assume that $\mathcal{F}$ is shellable, and use a total order to describe the members of $\mathcal{F}$ by letting $\sigma_{\mathcal{F}} : [n] \to \mathcal{F}$ be as described in Definition 3.8. Define the permutation $\sigma_0 : [n] \to [n]$ by having

$$\sigma_0(t(\sigma_{\mathcal{F}}(k)) = n - k + 1$$

for all $k \in [n]$. This permutation satisfies f_0 because for all $k \in [n]$, $\sigma_0(t(\sigma_{\mathcal{F}}(k))) = n-k+1$ is the smallest element of $\sigma_0(\sigma_{\mathcal{F}}(k))$. This completes the proof of the corollary. $\square$

4. Applications to skew tableaux

In this section, we describe how the results in the previous section can be applied to skew shapes. Specifically, we introduce a generalization of standard skew tableaux and Edelman and Greene's balanced tableaux, then prove some existence results about these generalized structures as described in Section 1 by using the characterization of the stronger form of the marriage condition. Afterwards, we briefly indicate other ways in which we can apply the results of Section 3.

Definition 4.1. *(cf. [16], p.7 and [23], Definition 2.1.1, Definition 3.7.1) Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_{\ell'})$ be partitions of positive integers such that $\ell' \leq \ell$ and $\mu_i \leq \lambda_i$ for all $1 \leq i \leq \ell'$. Moreover, let*

$$X = \bigcup_{i=1}^{\ell} \bigcup_{j=\mu_i+1}^{\lambda_i} \{(i,j)\}.$$

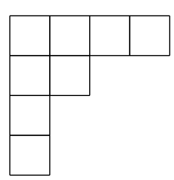
Lastly, let $X' \subset \mathbb{N}^2$ be such that $X' = X + v$ for some $v \in \mathbb{Z}^2$, $X' - (0,1) \nsubseteq \mathbb{N}^2$, and $X' - (1,0) \nsubseteq \mathbb{N}^2$. Then define the skew shape *λ/μ to be the Young diagram that is equal to X'.*

If $\mu = \emptyset$ is the empty partition, then for any partition λ of a positive integer, define the skew shape λ/μ to be the Young diagram that is equal to

$$\bigcup_{i=1}^{\ell} \bigcup_{j=1}^{\lambda_i} \{(i,j)\}$$

and call this Young diagram the Young diagram of λ. *We also call the Young diagram of λ a* normal shape. *Lastly, if $\lambda = \emptyset$ is the empty partition, then we define the Young diagram of λ to be the empty set.*

Example 4.2. *Let $\lambda = (4,2,1,1)$, and consider the Young diagram of λ. By Definition 4.1, row 1 of this diagram consists of $\lambda_1 = 4$ cells, row 2 of this diagram consists of $\lambda_2 = 2$ cells, row 3 of this diagram consists of $\lambda_3 = 1$ cell, and row 4 of diagram consists of $\lambda_4 = 1$ cell. Hence, the Young diagram is as follows.*



Remark 4.3. Let λ be a partition of a non-negative integer. Then we will refer to the Young diagram of λ as λ. In particular, we can speak of cells of λ or even rows of λ. Since we will do this, we will say things such as "let λ be a normal shape" when mentioning the Young diagram of λ.

Example 4.4. *Let $\lambda = (4,3,2,2)$ and $\mu = (2,2,1)$. Then $\ell = 4$, $\ell' = 3$, and for all $1 \leq i \leq \ell'$, $\mu_i \leq \lambda_i$. Hence, the skew shape λ/μ is well-defined. The set X as described in Definition 4.1 is obtained from the Young diagram of λ by deleting the $\mu_1 = 2$ left-most cells of row 1 of λ, the $\mu_2 = 2$ left-most cells of row 2 of λ, and, as $\mu_3 = 1$, the left-most cell of* *row 3 of λ. Because this set X satisfies $X - (0,1) \not\subseteq \mathbb{N}^2$ and $X - (1,0) \not\subseteq \mathbb{N}^2$, it follows that $X' = X$. Hence, by Definition 4.1, the skew shape λ/μ is the following Young diagram.*

$$\begin{ytableau} \none & \none & \; & \; \\ \none & \none & \; \\ \none & \; \\ \; & \; \end{ytableau}$$

Remark 4.5. When mentioning skew shapes λ/μ, we simply say "let λ/μ be a skew shape" without explicitly mentioning that λ and μ are partitions that satisfy the conditions described in Definition 4.1.

Definition 4.6. *(Folklore, cf.* [16, 23, 26]*) Let λ/μ be a skew shape consisting of n cells. Then a* standard skew tableau of shape λ/μ *is a bijective filling of the cells of λ/μ with numbers from $[n]$ such that entries increase along every row from left to right and entries increase along every column from top to bottom. Moreover, a* reverse standard skew tableau of shape λ/μ *is a bijective filling of the cells of λ/μ such that the entries decrease along every row from left to right and entries decrease along every column from top to bottom. If $\mu = \emptyset$, then a standard skew tableau of shape λ/μ is a* standard Young tableau of shape λ *and a reverse standard skew tableau of shape λ/μ is a* standard reverse tableau of shape λ.

Example 4.7. *Consider the skew shape λ/μ from Example 4.4. An example of a standard skew tableau of shape λ/μ is the following.*

$$\begin{ytableau} \none & \none & 1 & 4 \\ \none & \none & 2 \\ \none & 5 \\ 3 & 6 \end{ytableau}$$

An example of a reverse standard skew tableau of shape λ/μ is the following.

$$\begin{ytableau} \none & \none & 6 & 1 \\ \none & \none & 4 \\ \none & 5 \\ 3 & 2 \end{ytableau}$$

When describing families of sets, we will replace $[n]$ in the last section with the set of cells of λ/μ. Moreover, in place of the permutations $\sigma : [n] \to [n]$, we define *generalized standard skew tableaux.*

Definition 4.8. *(cf.* [23]*, Definition 2.1.3) Let λ/μ be a skew shape with n cells. Then a* generalized standard skew tableau of shape λ/μ *is a bijective filling of the cells of λ/μ with numbers from $[n]$.*

Example 4.9. *If $\lambda = (4,3,1)$ and $\mu = (2)$, then an example of a generalized skew tableau of shape λ/μ is as follows.*

$$\begin{array}{cccc} & & 3 & 5 \\ 6 & 1 & 2 & \\ 4 & & & \end{array}$$

Definition 4.10. *([23]) Let λ/μ be a skew shape. For any cell (i,j) in λ/μ, define the corresponding* hook $H_{(i,j)}$ *and* hook-length $h_{(i,j)}$ *in the same way that hooks and hook-lengths are defined for Young diagrams of partitions in:*

- $H_{(i,j)} = \{(i',j') \in \lambda/\mu : i' \geq i \text{ and } j' \geq j\}$,
- $h_{(i,j)} = |H_{(i,j)}|$.

Example 4.11. *Consider the following skew shape λ/μ, where $\lambda = (5,4,3,3)$ and $\mu = (2,2,1)$. Moreover, let r be the cell of λ/μ depicted below that is filled with a bullet. Then H_r consists of the cells that are filled with asterisks or bullets, and $h_r = 4$.*

$$\begin{array}{|c|c|c|c|c|} \hline & & \Box & \Box & \Box \\ & & \bullet & * & \\ & \Box & * & & \\ \Box & \Box & * & & \\ \hline \end{array}$$

Let λ be a normal shape. Then an *inner corner of* λ ([23], Definition 2.8.1) is a cell $r \in \lambda$ such that deleting r from λ results in another normal shape. With this definition in mind, let λ/μ be a skew shape with n cells, and consider the family of sets defined by $\mathcal{F} = \{H_r : r \in \lambda/\mu\}$. Then $\mathcal{F}$ is shellable. To see this, let $r_1, r_2, \ldots, r_n$ be a sequence of cells in λ/μ that is obtained as follows.

- Let r_1 be an inner corner of λ.
- If $1 \leq k < n$ and if r_1, r_2, $\ldots$, r_k have already been defined, then let $\lambda^{(k)}$ be the Young diagram that results from deleting cells r_1, r_2, $\ldots$, and r_k from λ, and let r_{k+1} be an inner corner of $\lambda^{(k)}$.

Lastly, let $\lambda^{(n)} = \mu$. Define $\sigma_{\mathcal{F}} : [n] \to \mathcal{F}$ by letting $\sigma_{\mathcal{F}}(k) = H_{r_k}$ for all $k \in [n]$. The bijection $\sigma_{\mathcal{F}}$ satisfies Equation 1 because, for all $k \in [n]$, $\lambda^{(k)}$ has $n-k$ cells,

$$\lambda^{(k)} \cup \bigcup_{i=1}^{k} H_{r_i} = \lambda/\mu, \tag{2}$$

and

$$\lambda^{(k)} \cap \bigcup_{i=1}^{k} H_{r_i} = \emptyset. \tag{3}$$

Hence, $\mathcal{F}$ is shellable by Definition 3.8. In particular, by Remark 3.9, $\mathcal{F}$ has a unique transversal. The unique transversal $t : \mathcal{F} \to \lambda/\mu$ of $\mathcal{F}$ is given by $t(H_r) = r$ for all $r \in \lambda/\mu$.

Example 4.12. *Let $\lambda = (4,2,2)$ and let $\mu = (1)$. Next, let $\mathcal{F} = \{H_r : r \in \lambda/\mu\}$. We illustrate why $\mathcal{F}$ is shellable. The normal shape λ is depicted below and all inner corners of λ are filled with bullets.*

			$\bullet$
	$\bullet$		

Pick the inner corner $r_1 = (3,2)$ of λ. Then the normal shape $\lambda^{(1)}$ is depicted below and all inner corners of $\lambda^{(1)}$ are filled with bullets

			$\bullet$
	$\bullet$		
$\bullet$			

and we can pick the inner corner $r_2 = (1,4)$ of $\lambda^{(1)}$. Continuing in this way, one possibility is the following sequence of cells in λ/μ depicted below.

	r_6	r_5	r_2
r_7	r_3		
r_4	r_1		

In particular, $r_3 = (2,2)$, $r_4 = (3,1)$, $r_5 = (1,3)$, $r_6 = (1,2)$, and $r_7 = (2,1)$. Now, define $\sigma_{\mathcal{F}} : \{1,2,\ldots,7\} \to \mathcal{F}$ so that $\sigma_{\mathcal{F}}(1) = H_{(3,2)}$, $\sigma_{\mathcal{F}}(2) = H_{(1,4)}$, $\sigma_{\mathcal{F}}(3) = H_{(2,2)}$, $\sigma_{\mathcal{F}}(4) = H_{(3,1)}$, $\sigma_{\mathcal{F}}(5) = H_{(1,3)}$, $\sigma_{\mathcal{F}}(6) = H_{(1,2)}$, and $\sigma_{\mathcal{F}}(7) = H_{(2,1)}$. This bijection satisfies Equation 2 and Equation 3. Hence, $\mathcal{F}$ is shellable.

Edelman and Greene introduced the following variant of standard Young tableaux.

Definition 4.13. *(Edelman and Greene, [5]) Let λ be a normal shape containing n cells. Then a* balanced tableau *of shape λ is a bijective filling of the cells of λ with numbers from $[n]$ such that if $(i,j) \in \lambda$ and if i' is the largest positive integer such that $(i',j) \in \lambda$, if $k = i' - i + 1$, and if $S_{i,j}$ is the set of entries m such that m is contained in a cell in $H_{(i,j)}$, then the entry in cell (i,j) of λ is the k^{th} smallest entry of $S_{i,j}$.*

Example 4.14. *Let $\lambda = (4,3,2)$. Then a balanced tableau of shape λ is as follows.*

4	5	8	3
6	7	9	
1	2		

For instance, let $i = 2$ and $j = 1$. Then the entry contained in cell (i,j) of λ is 6. Moreover, the largest integer i' such that $(i',j) \in \lambda$ is 3, $k = i' - i + 1 = 3 - 2 + 1 = 2$, $H_{(i,j)} = \{(2,1),(2,2),(2,3),(3,1)\}$ and $S_{i,j}$, the set of entries m of this tableau such that m is contained in a cell in $H_{(i,j)}$, equals $\{1,6,7,9\}$. Hence, the k^{th} smallest entry of $S_{i,j}$ is 6, which is the entry in cell $(2,1)$ of the above tableau.

In order to generalize standard skew tableaux, reverse standard skew tableaux, and balanced tableaux, we introduce the following special case of configurations from Definition 3.5.

Definition 4.15. *Let λ/μ be a skew shape. A* configuration *of λ/μ is a function $f : \lambda/\mu \to \mathbb{N}$ from the cells of λ/μ to the positive integers so that if $r \in \lambda/\mu$, then $f(r) \in \mathbb{N}$ and $f(r) \leq h_r$.*

Remark 4.16. We say that f is a configuration of λ/μ rather than say that f is a configuration of the transversal t of the set $\mathcal{F} = \{H_r : r \in \lambda/\mu\}$ defined by $t(H_r) = r$ for all $r \in \lambda/\mu$.

Example 4.17. *Consider the skew shape λ/μ where $\lambda = (3, 2, 1)$ and $\mu = (1)$. We denote configurations f of λ/μ by filling, for all $r \in \lambda/\mu$, cell r with the number $f(r)$. For instance, three configurations of λ/μ are the following.*

$$\begin{array}{ccc} & 1 & 1 \\ 1 & 1 & \\ 1 & & \end{array} \qquad \begin{array}{ccc} & 2 & 1 \\ 2 & 1 & \\ 1 & & \end{array} \qquad \begin{array}{ccc} & 3 & 1 \\ 3 & 1 & \\ 1 & & \end{array}$$

Now, we define the special case of the notion of satisfaction from Definition 3.5.

Definition 4.18. *Let T be a generalized standard skew tableau of shape λ/μ and let f be a configuration of λ/μ. Then T* satisfies *f if for all cells $r \in \lambda/\mu$, the entry in cell r of T is the k^{th} smallest, where $k = f(r)$, entry in the set of entries of T that are located in the hook H_r.*

In particular, a standard skew tableau of shape λ/μ is precisely a generalized standard skew tableau of shape λ/μ that satisfies the configuration f_0 of λ/μ defined by $f_0(r) = 1$ for all $r \in \lambda/\mu$, and a reverse standard skew tableau of shape λ/μ is precisely a generalized standard skew tableau of shape λ/μ that satisfies the configuration f_1 of λ/μ defined by $f_1(r) = h_r$ for all $r \in \lambda/\mu$. We will see examples of this in Example 4.19.

Moreover, if λ is a normal shape, then let f be the configuration of λ such that, for all $(i, j) \in \lambda$, if i' is the largest positive integer such that $(i', j) \in \lambda$, then $f((i, j)) = i' - i + 1$. So any tableau T of shape λ is balanced if and only if T satisfies f. This characterization of balanced tableaux was used in [5] as the definition of balanced tableaux; the special case of Definition 4.15 for normal shapes also appears in [5] under a different name. Namely, Edelman and Greene call $f(r)$ the *hook rank* of r. However, they only use hook ranks to define balanced tableaux. In this paper, we have a very different emphasis as we focus on properties of the configurations themselves.

Example 4.19. *Consider the skew shape λ/μ from and the three configurations of λ/μ from Example 4.17. The generalized standard skew tableaux that satisfy the leftmost configuration depicted in Example 4.17 are precisely the standard skew tableaux of shape λ/μ. Moreover, the generalized standard skew tableaux that satisfy the rightmost configuration depicted in Example 4.17 are precisely the reverse standard skew tableaux of shape λ/μ. Furthermore, four examples of generalized standard skew tableaux that satisfy the middle configuration depicted in Example 4.17 are displayed below.*

$$\begin{array}{ccc} & 4 & 5 \\ 2 & 3 & \\ 1 & & \end{array} \qquad \begin{array}{ccc} & 2 & 1 \\ 4 & 3 & \\ 5 & & \end{array} \qquad \begin{array}{ccc} & 4 & 2 \\ 3 & 5 & \\ 1 & & \end{array} \qquad \begin{array}{ccc} & 3 & 2 \\ 4 & 5 & \\ 1 & & \end{array}$$

Definition 4.20. *Let λ/μ be a skew shape and h be a configuration of λ/μ. Then we write $N(h)$ to denote the number of generalized standard skew tableaux of shape λ/μ that satisfy h.*

Corollary 4.21. *Let λ/μ be a skew shape. Then the number of configurations f of λ/μ such that $N(f) > 0$ is*

$$\prod_{r\in\lambda/\mu} h_r \ .$$

Proof. There are $\prod_{r\in\lambda/\mu} h_r$ configurations f of λ/μ since $f(r) \leq h_r$ for every $r \in \lambda/\mu$. So, since $\{H_r : r \in \lambda/\mu\}$ is a shellable family of subsets of the set of cells of λ/μ by the discussion after Example 4.11, Theorem 3.14 implies that $N(f) > 0$ for all configurations f of λ/μ. From this, the corollary follows. □

A well-known formula is the *hook-length formula*, first proved by Frame, Robinson, and Thrall [8]. It is as follows. If λ is a normal shape with n cells, then the number of standard Young tableaux of shape λ equals

$$\frac{n!}{\prod_{r\in\lambda} h_r}.$$

Moreover, the above formula was also proved by Edelman and Greene to equal the number of balanced tableaux of shape λ [5]. In our context, we will show that the above formula also has interesting connections to the configurations that we are investigating.

Corollary 4.21 has the following consequence.

Theorem 4.22. *Let λ/μ be a skew shape with n cells, and let $X(\lambda/\mu)$ denote the set of configurations of λ/μ. Moreover, let N be the number of configurations f of λ/μ such that $N(f) > 0$. Then,*

$$\frac{1}{N} \sum_{f\in X(\lambda/\mu)} N(f) \quad = \quad \frac{n!}{\prod_{r\in\lambda/\mu} h_r} \ .$$

Example 4.23. *Let $\lambda/\mu = (4,3,2)/(2,1)$. Then*

$$\frac{1}{N} \sum_{f\in X(\lambda/\mu)} N(f) = \frac{n!}{\prod_{r\in\lambda/\mu} h_r} = \frac{6!}{1\cdot 3\cdot 1\cdot 3\cdot 1\cdot 2} = 40.$$

The hook-lengths are represented with the following diagram.

$$\begin{array}{cccc} & & \boxed{3} & \boxed{1} \\ & \boxed{3} & \boxed{1} & \\ \boxed{2} & \boxed{1} & & \end{array}$$

Proof. Every generalized standard skew tableau of shape λ/μ satisfies exactly one configuration of λ/μ by Lemma 3.7, so by Definition 4.20 and the fact that there are $n!$ generalized standard skew tableaux of shape λ/μ,

$$\sum_{f \in X(\lambda/\mu)} N(f) = n!.$$

Moreover, by Corollary 4.21,

$$N = \prod_{r \in \lambda/\mu} h_r.$$

From this, the theorem follows. $\square$

Remark 4.24. Theorem 4.22 is versatile. For instance, there is a formula for the number of standard skew tableaux of shape λ/μ, known as Naruse's formula. Asymptotic properties of Naruse's formula were analysed by Morales, Pak, and Panova in [18]. In particular [18], it turns out that in general, the number of standard skew tableaux of shape λ/μ divided by

$$\frac{n!}{\prod_{r \in \lambda/\mu} h_r},$$

where n is the number of cells of λ/μ, can be arbitrarily large. Hence, we can apply Theorem 4.22 to Naruse's formula and, using the work of Morales, Pak, and Panova in [18], analyse lower bounds on the number of configurations f of λ/μ such that $N(f) > 0$ and $N(f)$ is strictly less than

$$\frac{n!}{\prod_{r \in \lambda/\mu} h_r}.$$

Remark 4.25. There are variants and generalizations of Naruse's formula, the formula described in Remark 4.24, for shapes known as *skew shifted shapes* that are known [9, 20]. What we observe about these shapes is that the "hook-sets" for skew shifted shapes as defined in [9, 20] also form examples of shellable families. Hence, the results in this section can be replicated verbatim to include *skew shifted shapes*. Moreover, it is claimed by Morales, Pak, and Panova in [18] that their analysis of Naruse's formula can be extended to skew shifted shapes. It also appears that we can even extend the above to involve posets known as *d-complete posets* [20], as there is a generalization of Naruse's formula for such posets and the "hook-sets" in these formulas are a generalization of the "hook-sets" for the skew shifted shapes [20].

Lastly, we note that a special case of our work has also been considered in the literature by Viard. We derived our work independently of Viard.

Remark 4.26. Consider a finite subset S of $\mathbb{N}^2$. Next, for all $r = (i, j) \in S$, define $F_r = \{(i_1, j) \in S : i_1 \geq i\} \cup \{(i, j_1) : j_1 \geq j\}$, and define $\mathcal{F} = \{F_r : r \in S\}$. This construction is related to the tools we used in Section 3 for the following reason. By using the same explanation as the one we gave for why $\{H_r : r \in \lambda/\mu\}$ is shellable, we observe that $\mathcal{F}$ is shellable and that its unique transversal is defined by $t(F_r) = r$ for all $r \in S$.

Let $\mathcal{F}$ and t be as described in the above paragraph. Viard [28, 29] considered objects that are equivalent to configurations of t and permutations that satisfy such configurations. Viard [28, 29] asserted that he has established one direction of a special case of Theorem 3.14 by claiming to have proved a statement equivalent to asserting that all configurations f of t are satisfied by at least one permutation $\sigma : S \to S$. In particular, using his claim, he derives two consequences that imply Corollary 4.21 and Theorem 4.22. However, his arguments for that claim are complex, there are two versions of his arguments (a less general version in [28] and a more general version in [29]), both versions are different from our proof of Theorem 3.14, and it appears that they are also incomplete.

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References

1. R. Adin and Y. Roichman, *Standard Young Tableaux* Chapter 14 of *Handbook of Enumerative Combinatorics* 1^{st} *Edition*, Edited by Miklós Bóna, Chapman and Hall/CRC (2015).
2. O. Angel, A. Holroyd, D. Romik, B. Virág, *Random sorting networks,* Adv. Math. 215, 839 - 868 (2007).
3. A. Björner and F. Brenti, *Combinatorics of Coxeter Groups,* Grad. Texts in Math. 231, Springer, (2005).
4. G. Chang, *On the Number of SDR of a (t, n)-family* European J. Combin.10, 231 - 234 (1989).
5. P. Edelman and C. Greene, *Balanced Tableaux* Adv. Math. 63, 42 - 99 (1987).
6. P. Edelman and C. Greene, *Combinatorial correspondences for Young tableaux, balanced tableaux, and maximal chains in the weak Bruhat order of S_n*, Contemp. Math. 34, (1984).
7. S. Fomin, C. Greene, V. Reiner, M. Shimozono, *Balanced Labellings and Schubert Polynomials* European J. Combin. 18, 373 - 389 (1997).
8. J. Frame, G. Robinson, R. Thrall, *The hook graphs of the symmetric group* Canad. J. Math. 6, 316 - 325 (1954).
9. W. Graham and V. Kreiman, *Excited Young diagrams, equivariant K-theory, and Schubert varieties,* Trans. Amer. Math. Soc. 367 (9), 6597 - 6645 (2015)
10. P. Hall, *On Representatives of Subsets*, J. London Math. Soc, 10 (1), 26 - 30 (1935).
11. M. Hall Jr, *Distinct Representatives of Subsets*, Bull. Amer. Math. Soc. 54 (10), 922 - 926 (1948).
12. J. Hirst and N. Hughes, *Reverse mathematics and marriage problems with finitely many solutions* Arch. Math. Logic 55, 1015 - 1024 (2016).
13. J. Hirst and N. Hughes, *Reverse mathematics and marriage problems with unique solutions* Arch. Math. Logic 54, 49 - 57 (2015).

14. M. Konvalinka, *A bijective proof of the hook-length formula for skew shapes*, arXiv:1703.08414v2 (2018).
15. M. Lewin, *On the coefficients of the characteristic polynomial of a matrix* Discrete Math. 125, 255-262 (1994).
16. K. Luoto, S. Mykytiuk, S. van Willigenburg, *An introduction to quasisymmetric Schur functions - Hopf algebras, quasisymmetric functions, and Young composition tableaux* - Springer (2013).
17. D. Little, *Combinatorial aspects of the Lascoux - Schützenberger tree,* Adv. Math. 174, 236 - 253 (2003).
18. A. Morales, I. Pak, G. Panova, *Asymptotics of the number of standard Young tableaux of skew shape,* Electron. J. Combin. 70, 26 - 49 (2018).
19. A. Morales, I. Pak, G. Panova, *Hook formulas for skew shapes I. q-analogues and bijections* J. Combin. Theory Ser. A 154, 350 - 405 (2018).
20. H. Naruse and S. Okada, *Skew hook formula for d-complete posets via equivariant K-theory* Algebr. Comb. 2 (4), 541 - 571 (2019).
21. I. Pak, F. Petrov, and V. Sokolov, *Hook Inequalities* arXiv:1903.11828v2 (2019).
22. P. Reichmeider, *The Equivalence of Some Combinatorial Matching Theorems,* Polygonal Pub House (1985).
23. B. E. Sagan, *The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions (2nd Edition)*, Graduate Texts in Mathematics 203, Springer-Verlag New York Inc., 2001
24. N. Sloane, *The on-line encyclopedia of integer sequences* `http://oeis.org`.
25. R. Stanley, *Enumerative Combinatorics, Volume 1, Second Edition* Cambridge Stud. in Adv. Math. 49, Cambridge Univ. Press (2012).
26. R. Stanley, *Enumerative Combinatorics, Volume 2, First Edition* Cambridge Stud. in Adv. Math. 49, Cambridge Univ. Press (1999).
27. J. Swanson, *On the Existence of Tableaux with Given Modular Major Index*, Algebr. Comb. 1 (1), 3 - 21 (2018).
28. F. Viard, *A natural generalization of balanced tableaux*, arxiv 1407.6217v2 (2016).
29. F. Viard, *Des graphes orientés aux treillis complets : une nouvelle approche de l'ordre faible sur les groupes de Coxeter.* Université Claude Bernard Lyon 1 École doctorale InfoMath , ED 512 Spécialité : Mathématiques N. dórdre 232-2015.

Department of Mathematics, University of British Columbia, Vancouver BC V6T 1Z2, Canada

E-mail address: `bchan@math.ubc.ca`