LOZENGE TILINGS OF THE EQUILATERAL TRIANGLE

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Abstract. We consider incomplete tilings of the equilateral triangle of edge length n that is subdivided into n^2 regular equilateral smaller unit triangles. Pairs of the unit triangles that share a side may be converted into lozenges, leaving some subset of the unit triangles untouched. We count numerically these coverings by lozenges and unit triangles for edge lengths $n \le 6$: the total and the detailed refinement as a function of the number of lozenges.

1. Lozenge Tilings

1.1. **Basic Geometry.** An equilateral triangle of integer side length n may be divided into n^2 equilateral triangles of unit side length by regular subdivision of each side into n sections and drawing lines through these parallel to all three sides. This creates a graph with [2, A000217]

$$(n+1)+n+(n-1)+\cdots+1 = T_{n+1} = \frac{(n+1)(n+2)}{2} \tag{1}$$

vertices (corners of the unit triangles) and [2, A045943]

$$M_n = 3T_n = 3\frac{n(n+1)}{2} \tag{2}$$

edges (edges of the unit triangles). The vertices have coordinates (i,j) with $i \ge 0$, $j \ge 0$, $i+j \le n$ in a system with two axes with a pointing difference of 60°. Commensurate with Euler's Formula [4], the number of faces plus the number of vertices equals the number of edges plus 1:

$$n^2 + T_{n+1} = M_n + 1. \tag{3}$$

The number of edges on the perimeter of the big triangle is three times the number of segments, $3n$, so the number of edges internal to the big triangle is

$$M_n - 3n = M_{n-1}. \tag{4}$$

The number of vertices *inside* the big triangle is the number of vertices which are not on one of the sides of the big triangle; so subtracting $3n$, the number of vertices on the big triangle's sides, from (1) yields

$$T_{n+1} - 3n = T_{n-2} \tag{5}$$

for the number of internal vertices.

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1.2. **Lozenge Sets.** In conjunction with this work, a lozenge is created by removing one of the inner edges; this merges the two unit adjacent triangles that have that edge in common. A lozenge tiling with l non-overlapping lozenges is created by removing l of the inner edges under the constraint that no pair of removed edges must be two edges of the same triangle—which would create tilings with pieces that are larger than the lozenge. So the constraint means that once an inner edge has been removed (to become the short diagonal of a unit lozenge), none of the 4 edges of that lozenge must be removed.

If l is the number of lozenges, $n^2 - 2l$ is the number of free triangles. An obvious upper bound for l is the "capacity"

$$l \le n^2/2 \equiv c(n) \tag{6}$$

because each lozenge covers 2 triangles.

Definition 1. *$L_{n,l}$ is the number of tilings of the equilateral triangle with edges of length n with l non-overlapping lozenges and $n^2 - 2l$ unit triangles.*

Algorithm 1. *A simple strategy to count the tilings $L_{n,l}$ is to generate the set of M_{n-1} inner edges, to scan all $2^{M_{n-1}}$ subsets of removing them, and to count all the subsets that meet the criterion that no pair of removed edges is part of the same triangle. If the constraint were absent, the number of subsets follows from the usual combinatorial selection, so with (4) this constitutes an upper bound*

$$L_{n,l} \le \binom{M_{n-1}}{l}. \tag{7}$$

The lozenges have three different orientations with axes differing by angles of 120°. We classify them according to the removed edge being horizontal, falling left-to-right or rising left-to-right. If one takes a set of lozenges of a common orientation and shoves them in closest packing into a corner of the big triangle, one sees that a tiling with

$$l = (n-2) + (n-1) + \cdots 1 = T_{n-2} \tag{8}$$

lozenges (and n isolated unit triangles) is achievable.

2. Example for Side length of 3

The lozenge tilings generated from a big triangle with side length $n = 3$ are illustrated in Figures 1–4, sorted by the number of lozenges from $l = 0$ up to $l = 3$. Some of the diagrams have multiplicities larger than one if rotations by multiples of 120° or flips across one of the three lines of symmetry of the big triangle generate further diagrams of the same shape. (The isosceles triangle has a symmetry group of order 6, where the 3 flips along a diagonal have order 2 and the rotations by 120° or 240° have order 3. The multiplicity if 6 divided by the order of the symmetry group once the lozenges are inserted.) The configurations generated by these symmetry operations of the triangle are considered distinct here.

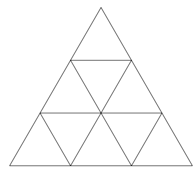

FIGURE 1. The configuration with 0 lozenges, side length 3 (multiplicity 1). $L_{3,0} = 1$. There are $n^2 = 9$ unit triangles, $T_n = 6$ point up and $T_{n-1} = 3$ point down.

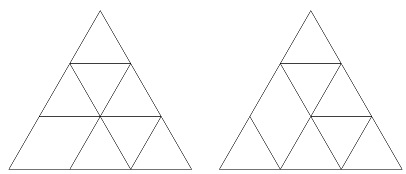

FIGURE 2. The two configurations with 1 lozenge, side length 3. It either covers a corner of the big triangle (multiplicity 3) or shares one edge with the middle part of an edge of the big triangle (multiplicity 6). $L_{3,1} = 3 + 6 = 9$

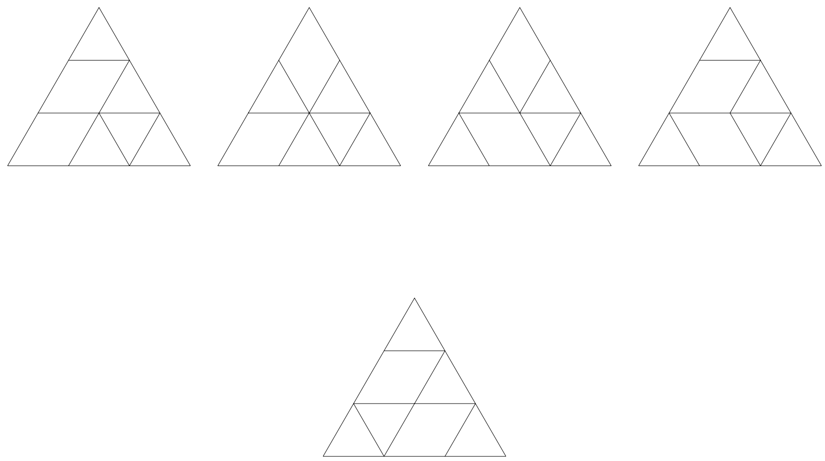

FIGURE 3. Configurations with 2 lozenges, side length 3. They may cover 2/3 of an edge of the big triangle (multiplicity 6). They may cover two different corners of the big triangle (multiplicity 3). One may cover a corner of the big triangle and the other the middle part of the opposite edge (multiplicity 6). They may share an edge and cover middle parts of two edges of the big triangle (multiplicity 6). They may touch in the middle and cover middle parts of two edges of the big triangle (multiplicity 3). $L_{3,2} = 6+3+6+6+3 = 24$.

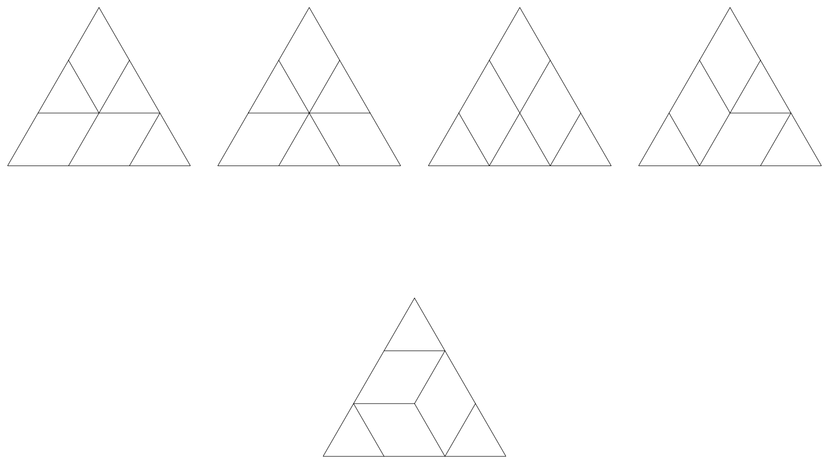

FIGURE 4. The configurations with 3 lozenges, side length 3 [7, Fig. 9.57]. They may cover 2/3 of an edge of the big triangle and one corner (multiplicity 6). They may cover the three corners of the big triangle (multiplicity 1). They may cover 1/3 of two edges of the big triangle and have the same orientation (multiplicity 3). They may cover 2/3 of one edge of the big triangle and the middle of another (multiplicity 6). They may touch in the middle and cover middle parts of all edges of the big triangle (multiplicity 2, two circulations, [3, Fig 3]). $L_{3,3} = 6 + 1 + 3 + 6 + 2 = 18$.

3. Special Cases

3.1. **No Lozenge.**
The count of

$$L_{n,0} = 1 \tag{9}$$

for every $l = 0$ means that for each side length n there is one way of not merging any triangles into lozenges.

3.2. **One Lozenge.**
The appearance of the triangular matchstick numbers

$$L_{n,1} = M_{n-1}, \tag{10}$$

$$\sum_{n\ge 0} L_{n,1}x^n = \frac{3x^2}{(1-x)^3}, \tag{11}$$

is immediately plausible: recall that deleting one of the internal edges creates a lozenge by merging the two triangles that share that edge, so (10) just restates (4).

3.3. **Two Lozenges.**
Two lozenges are created by deleting two internal edges, which can be selected in $\binom{M_{n-1}}{2}$ ways according to (7). Some of these pairs of deleted edges do not represent lozenge tilings because they are spatially correlated as considered in Section 1.2.

Definition 2. *A V subgraph is a pair of internal edges (in the full graph without lozenges) that share one common vertex, where the two edge directions differ by an angle of* $60°$.

Definition 3. *The center of a V graph is the vertex with degree 2.*

Each V subgraph reduces the number of lozenge tilings by one. There are two distinct sets of V's:

- If the common vertex of a pair of edges is an internal vertex, 6 distinct V's exist which can be created by rotating one of them by multiples of 60 degrees. There are T_{n-2} internal vertices, so there are $6T_{n-2}$ V's of that type.
- If the common vertex of a pair of edges is a vertex on a side of the big triangle, one V exists with its edges reaching into the interior of the big triangle. The exception are the 3 vertices at the corners of the big triangle, where no V exists because no *internal* edges meet there. So the eligible number of vertices for these V's is not $3n$ but only $3(n-1)$.

The total number of V subgraphs is $6T_{n-2} + 3(n-1) = 3(n-1)^2$, which must be interpreted as 0 if $n < 1$. This reduces the number of configurations with 2 lozenges to [2, A326367]

$$L_{n,2} = \binom{M_{n-1}}{2} - 3(n-1)^2 = \frac{3}{8}(n-1)(n-2)(3n^2+3n-4), \quad n \ge 1. \tag{12}$$

The ordinary generating function is

$$\sum_{n\ge 0} L_{n,2}x^n = 3x^3\frac{(2+x)(4-x)}{(1-x)^5}. \tag{13}$$

3.4. **Three Lozenges.** Three lozenges are created by deleting three internal edges, which can be selected in $V^{3,a} = \binom{M_{n-1}}{3}$ ways according to (7). Some of these triples of deleted edges do not represent lozenge tilings because they are spatially correlated as defined in Section 1.2. The types of correlations are:

- The three edges are a subgraph with two components. One component is a V as defined above (2 edges, 3 vertices) and the other component an internal edge (2 vertices) which does not have a vertex in common with the V: Set (b) in Figure 5.
- A unit triangle (3 edges, 3 vertices, connected) where all three edges are internal edges: set (c) in Figure 5. There are 3 unit triangles at the corners of the big triangle which have only 1 internal edge and $n-2$ unit triangles at each side of the big triangle which have only 2 internal edges. So $3 + 3(n-2) = 3(n-1)$ triangles are not in that class, and $V^{3,c} = n^2 - 3(n-1)$ unit triangles are in that class.
- Zigzag subgraphs (3 edges, 4 vertices, connected) constructed by an overlay of two V's such that they share an edge but have distinct centers: set (d) in Figure 5. For each shared internal edge, there are generally two zigzag subgraphs, but if the internal edge has one vertex on the side of the big triangle, only one zigzag subgraph emerges, and if both vertices are on sides of the big triangle, no zigzag subgraph emerges. The classification of the M_{n-1} internal edges of (4) according to these three vertex types is:
  - 3 (at the corners of the big triangle) have two vertices at the big triangle's sides.
  - $6(n-1)-6 = 6n-12$ have one vertex at the big triangle's sides. This is 2 for each of the $3(n-1)$ V with a center at the big triangle's side derived in Section 3.3, reduced by the overcount for the edges of the previous bullet.
  - M_{n-3} have only internal vertices. This is obtained from (2) by deleting all $3n$ vertices on the big triangle and edges that connect to them.

$$M_{n-1} = 3 + (6n-12) + M_{n-3}. \tag{14}$$

  So the number of zigzag subgraphs is

$$V^{3,d} = 0 \times 3 + 1 \times (6n-12) + 2 \times M_{n-3} = 3(n-1)(n-2). \tag{15}$$

- Fork subgraphs (3 edges, 4 vertices, connected), containing an internal edge and two copies rotated around one of its vertices by 60 and again 60 degrees: set (e) in Figure 5. It could be regarded as an overlay of two V's, such that they share an edge and have the same center. This center must be an internal vertex. At each internal vertex 6 fork subgraphs exist (by rotation of one fork subgraph by multiples of 60° around the center), so according to (5) there are $V^{3,e} = 6T_{n-2}$ fork subgraphs.

According to the first bullet there are $3(n-1)^2$ V's and for each of them $M_{n-1}-2$ remaining edges (after 2 have been used for the V) eligible for the disconnected edge. The first bullet proposes a set of

$$V^{3,b} = 3(n-1)^2(M_{n-1}-2) = \frac{3}{2}(n-1)^2(3n^2-3n-4) \tag{16}$$

graphs as correction to $\binom{M_{n-1}}{3}$. However, this term does not recognize that the additional edge may have common vertices with the V, so it includes some of the

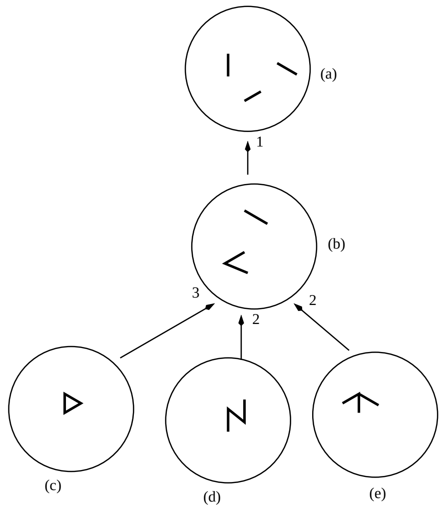

FIGURE 5. Poset diagram of the deleted edges for $l = 3$ deleted unit edges in general, and specializations where two of them are correlated in a V graph, and where three of them are correlated in a triangle, a zigzag or a fork subgraph.

correlated types of the other bullets. The term is actually too large, because some of the other correlated types can be constructed in more than one way as a V plus an edge, and the term counts them more than once although they exist only once in the full graph. The following procedure handles them with the principle of inclusion-exclusion [5]. In fact these are the correlated types that can be constructed in more than one way from two V's:

- Fork subgraphs are counted twice by (16), depending on which of the two V's is marked. (16) must be reduced by the number of fork subgraphs, $V^{3,e}$.
- Zigzag subgraphs are counted twice by (16), depending on which of the two V's is marked. (16) must be reduced by the number of zigzag subgraphs, $V^{3,d}$.
- Unit triangles are counted thrice by (16), depending on which pair of sides is considered the V. (16) must be reduced by two times the number of internal triangles, $2V^{3,c}$.

Alternatively, in the formal setting of sets in the inclusion-exclusion principle consider the simple graphs on one hand and the graphs with a marked vertex of the center of the V. Add an overbar to the symbols for marked graphs, and a prime to the label if a set does not include graphs of "higher correlated" type, i.e., not graphs of lower rank in the poset diagram:

- Simple graphs: we wish to enumerate the graphs of type (b), (c), (d) ane (e) counting each graph with at least one 60° angle exactly once. The 4 non-intersecting sets of graphs are
  - The set $\mathbb{V}^{3,b'}$ containing a V and a separate edge,
  - The set $\mathbb{V}^{3,c}$ containing a triangle,
  - The set $\mathbb{V}^{3,d}$ containing a zigzag graph,

– The set $\mathbb{V}^{3,e}$ containing a fork graph.

We wish to compute the cardinality

(17)
$$|\mathbb{V}^{3,b'} \cup \mathbb{V}^{3,c} \cup \mathbb{V}^{3,d} \cup \mathbb{V}^{3,e}| = |\mathbb{V}^{3,b'}| + |\mathbb{V}^{3,c}| + |\mathbb{V}^{3,d}| + |\mathbb{V}^{3,e}| = |\mathbb{V}^{3,b'}| + V^{3,c} + V^{3,d} + V^{3,e}$$

- Marked graphs: If the V center is marked, each graph of type (c) splits into 3 marked graphs and each graph of type (d) or (e) into 2 marked graphs: $|\bar{\mathbb{V}}^{3,c}| = 3|\mathbb{V}^{3,c}| = 3V^{3,c}$ and so on. Now consider the 3 non-intersecting sets of graphs of type (c) to (e), plus the super-set $\mathbb{V}^{3,b}$ of marked graphs with a V and a (not necessarily separated) edge. Then

$$|\bar{\mathbb{V}}^{3,b}| = |\bar{\mathbb{V}}^{3,b'} \cup \bar{\mathbb{V}}^{3,c} \cup \bar{\mathbb{V}}^{3,d} \cup \bar{\mathbb{V}}^{3,e}| = V^{3,b} = |\bar{\mathbb{V}}^{3b'}| + 3V^{3,c} + 2V^{3,d} + 2V^{3,e} \quad (18)$$

Eliminating $|\mathbb{V}^{3,b'}| = |\bar{\mathbb{V}}^{3,b'}|$ from the previous two equations gives

$$|\mathbb{V}^{3,b'} \cup \mathbb{V}^{3,c} \cup \mathbb{V}^{3,d} \cup \mathbb{V}^{3,e}| = V^{3,b} - 2V^{3,c} - V^{3,d} - V^{3,e}. \quad (19)$$

Considering this correction to (16) we conclude [1][2, A326368]

$$\begin{aligned} L_{n,3} &= V^{3,a} - \left[V^{3,b} - 2V^{3,c} - V^{3,d} - V^{3,e}\right] \\ &= \frac{1}{16}(n-2)(9n^5 - 9n^4 - 81n^3 + 81n^2 + 160n - 192), \quad n \geq 2, \end{aligned} \quad (20)$$

with generating function

$$\sum_{n\geq 0} L_{n,3}x^n = x^3 \frac{18 + 308x + 154x^2 - 87x^3 + 10x^4 + 2x^5}{(1-x)^7}. \quad (21)$$

3.5. **Four Lozenges.**

Conjecture 1. [1][2, A326369]

(22)
$$\begin{aligned} L_{n,4} &= \binom{M_{n-1}}{4} - \left[3(n-1)^2\binom{M_{n-1}-2}{2} - \frac{3}{2}(n-2)(11n^3 - 22n^2 - 23n + 54)\right] \\ &= \frac{3}{128}(n-2)(n-3)(9n^6 + 9n^5 - 135n^4 - 81n^3 + 670n^2 + 104n - 1216), \quad n \geq 3. \end{aligned}$$

3.6. **Five or Six Lozenges.**

Conjecture 2.

$$\begin{aligned} L_{n,5} &= \binom{M_{n-1}}{5} - \left[3(n-1)^2\binom{M_{n-1}-2}{3} \right. \\ &\quad \left. - \frac{1}{4}(4704 - 3102n + 1845n^3 - 2031n^2 + 60n^4 - 315n^5 + 63n^6)\right] \\ &= \frac{3}{1280}(n-3)(n+3)(27n^8 - 135n^7 - 387n^6 + 2835n^5 - 168n^4 - 18732n^3 + 19568n^2 + 36992n - 56320), \quad n \geq 3. \end{aligned} \quad (23)$$

Conjecture 3.

$$L_{n,6} = \binom{M_{n-1}}{6} - \Big[3(n-1)^2\binom{M_{n-1}-2}{4} \tag{24}$$
$$-\frac{1}{16}(-131088+61472n-41206n^3+69420n^2-90n^6-918n^7+153n^8-10851n^4+9828n^5)\Big]$$
$$=\frac{1}{5120}(81n^{12}-486n^{11}-2835n^{10}+21870n^9+26775n^8-384786n^7+131751n^6+3275730n^5$$
$$-3798716n^4-13254088n^3+22481984n^2 19678080n-42024960), \quad n \ge 4.$$

The common shape of $L_{n,l}$ is that M_{n-1} is a polynomial of degree 2, and that $\binom{M_{n-1}}{l}$, the upper bound (7), is a polynomial of degree $2l$. The first-order corrections of the leading term $3(n-1)^2\binom{M_{n-1}-2}{l-2}$, counting uncorrelated V subgraphs, are of lesser degree $2l-2$, because at larger n the spatial correlations of the (deleted) internal edges play a lesser role. Conjectures for polynomials ensue assuming that the second-order corrections are of degree $2l-4$, once a sufficiently large set of $L_{n,l}$ for small n is known.

4. Summary

Table 1 summarizes the numerical results which were calculated by the program listed in the Appendix.

Row sums $\sum_{l\ge 0} L_{n,l}$ are 1, 4, 52, 2158, 286242, 121479420,... Following the conjectured (8), the maximum l for nonzero entries is T_{n-2}.

The values $L_{n,T_{n-2}} = 1, 3, 18, 187, 3135, 81462, 3198404, 186498819, 15952438877, 1983341709785$,... of these configurations with the maximum number of lozenges have already been computed by Santos [6, Table 1][7, Table 9.2].

Appendix A. JAVA Program

A.1. **Algorithm.** The ancillary directory contains a Java program that generates Table 1. The main function in `LozeTil2.java` uses an edge-growing recursive algorithm which computes a lozenge statistics $L_{n,l}$ refined by the set of lozenges that have one of their 4 sides on one of the three sides of the big triangle.

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$n\backslash l$	0	1	2	3	4	5	6	7
1	1							
2	1	3						
3	1	9	24	18				
4	1	18	126	434	762	630	187	
5	1	30	387	2814	12699	36894	69242	81936
6	1	45	915	11127	90270	515970	2139120	6523428
7	1	63	1845	33365	417435	3836439	26841853	146208393
8	1	84	3339	83568	1478160	19662060	204334715	1701554868
9	1	108	5586	184254	4354497	78536358	1124301411	13119112488
10	1	135	8802	369254	11203269	261985815	4914087052	75970268748
11	1	165	13230	686952	25970895	762098799	18070041680	355864850838
12	1	198	19140	1203930	55414395	1990014156	58055896449	1414611219018
13	1	234	26829	2009018	110505120	4761037260	167316709165	4931688363498
14	1	273	36621	3217749	208300257	10594451901	440911546295	15439933756251
15	1	315	48867	4977219	374375664	22178743326	1077784772922	44182928710470

$n\backslash l$	8	9	10	11
5	57672	21432	3135	
6	14683401	24256853	28975770	24383838
7	628823088	2153224090	5892984618	12892017948
8	11554013295	64766667704	302315092020	1181998895448
9	127156871457	1038068322606	7212713283360	42993319234518
10	987147811836	10940096605816	104581114754595	869988063985737
11	5938169156829	85230974965513	1064629166358066	11681266282861098
12	29375579984238	527873999198830	8307168403048731	115585010198220444
13	124419130905960	2728420121843584	52640100670770348	902231390539173210
14	464317587238419	12178604171344167	282021772415608164	5822744874311864316
15	1559497806005040	48137813623437500	1315457502665712336	32139701729335767774

TABLE 1. Number $L_{n,l}$ of lozenge tilings for sides $n \geq 1$ with $l \geq 0$ lozenges [2, A273464].

URL: `https://www.mpia-hd.mpg.de/~mathar`

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