CYCLIC SIEVING PHENOMENON ON DOMINANT MAXIMAL WEIGHTS OVER AFFINE KAC-MOODY ALGEBRAS

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ABSTRACT. We construct a (bi)cyclic sieving phenomenon on the union of dominant maximal weights for level ℓ highest weight modules over an affine Kac-Moody algebra with exactly one highest weight being taken for each equivalence class, in a way not depending on types, ranks and levels. In order to do that, we introduce $\boldsymbol{S}$-evaluation on the set of dominant maximal weights for each highest modules, and generalize Sagan's action in [18] by considering the datum on each affine Kac-Moody algebra. As consequences, we obtain closed and recursive formulae for cardinality of the number of dominant maximal weights for every highest weight module and observe level-rank duality on the cardinalities.

INTRODUCTION

Kac-Moody algebras were independently introduced by Kac [11] and Moody [14]. Among them, affine Kac-Moody algebras have been particularly extensively studied for their beautiful representation theory as well as for their remarkable connections to other areas such as mathematical physics, number theory, combinatorics, and so on. Nevertheless, many basic questions are still unresolved. For instance the behaviour of weight multiplicities and combinatorial features of dominant maximal weights are not fully understood (see [13, Introduction]).

Throughout this paper, $\mathfrak{g}$ denotes an affine Kac-Moody algebra and $V(\Lambda)$ the irreducible highest weight module with highest weight $\Lambda \in P^+$, where P^+ denotes the set of dominant integral weights. Due to Kac [12], all weights of $V(\Lambda)$ are given by the disjoint union of δ-strings attached to maximal weights and every maximal weight is conjugate to a unique dominant maximal weight under Weyl group action. So it would be quite natural to expect that better understanding of dominant maximal weights makes a considerable contribution towards the study of representation theory of affine Kac-Moody algebras.

In [12], Kac established lots of fundamental properties concerned with $\mathrm{wt}(\Lambda)$, the set of weights of $V(\Lambda)$, using the orthogonal projection $^{-}\colon \mathfrak{h}^* \to \mathfrak{h}_0^*$. In particular, he showed that $\max^+(\Lambda)$, the set of dominant maximal weights, is in bijection with $\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ under this projection, thus it is finite. Here ℓ denotes the level of Λ. However, in the best knowledge of the authors, approachable combinatorial models, cardinality formulae and structure on $\max^+(\Lambda)$'s have not been available up to now except for limited cases, which motivates the present paper.

In 2014, Jayne and Misra [10] published noteworthy results about $\max^+(\Lambda)$ in $A_n^{(1)}$-case. They give an explicitly parametrization of $\max^+((\ell-1)\Lambda_0 + \Lambda_i)$ in terms of paths for $0 \leqslant i \leqslant n$ and $\ell \geqslant 2$, and present the following conjecture:

$$|\max^+(\ell\Lambda_0)| = \frac{1}{(n+1)+\ell} \sum_{d|(n+1,\ell)} \varphi(d) \binom{((n+1)+\ell)/d}{\ell/d}, \tag{0.1}$$

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where φ is Euler's phi function. Notably this number gives the celebrated Catalan number when $\ell = n$. Soon after, this conjecture turned out to be affirmative in [24]. The proof therein largely depends on Sagan's congruence on q-binomial coefficients [18, Theorem 2.2].

The main purpose of this paper is to investigate $\max^+(\Lambda)$ by constructing bijections with several combinatorial models and a (bi)cyclic sieving phenomenon on the combinatorial models. As applications, we can obtain closed formulae of $\max^+(\Lambda)$ for all affine types, and observe interesting symmetries by considering $\max^+(\Lambda)$ for all ranks and levels.

Set

$$P^+_{\mathrm{cl}} := P^+/\mathbb{Z}\delta \quad \text{and} \quad P^+_{\mathrm{cl},\ell} := P^+_\ell/\mathbb{Z}\delta \quad \text{for } \ell \in \mathbb{Z}_{\geqslant 0},$$

where P^+_ℓ denotes the set of level ℓ dominant integral weights and δ denotes the canonical null root of $\mathfrak{g}$.

Given a nonnegative integer ℓ, we only consider classical dominant integral weights, that is, Λ in $P^+_{\mathrm{cl},\ell}$ because there is a natural bijection between $\max^+(\Lambda)$ and $\max^+(\Lambda + k\delta)$ for every $k \in \mathbb{Z}$. We begin with the observation that the set $\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ can be embedded into $P^+_{\mathrm{cl},\ell}$ via the map

$$\begin{aligned} \iota_\Lambda : \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) &\longrightarrow P^+_{\mathrm{cl},\ell} \\ \textstyle\sum_{i=1}^n m_i \varpi_i &\longmapsto m_0\Lambda_0 + \textstyle\sum_{i=1}^n m_i\Lambda_i, \end{aligned} \tag{0.2}$$

where Q denotes the root lattice, $\varpi_i := \overline{\Lambda}_i$ and $m_0 = \ell - \sum_{i=1}^n a_i^\vee m_i$.

We then define an equivalence relation $\sim$ on $P^+_{\mathrm{cl},\ell}$ by $\Lambda \sim \Lambda'$ if and only if $\iota_\Lambda = \iota_{\Lambda'}$, equivalently $\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) = \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda'} + \overline{Q})$ (see Lemma 2.3). By definition, if $\Lambda \sim \Lambda'$, then $|\max^+(\Lambda)| = |\max^+(\Lambda')|$. Note that this equivalence relation is defined in [3] in a slightly different form. We should remark that, in [3], the authors mainly investigated a membership condition of weights for highest weight module $V(\Lambda)$ modulo a certain lattice, while we investigate $|\max^+(\Lambda)|$ and structures on the union of $\max^+(\Lambda)$'s.

Under the relation $\sim$, it turns out that the image of ι_Λ coincides with the equivalence class of Λ. We provide a complete set of pairwise inequivalent representatives of the distinguished form $(\ell-1)\Lambda_0 + \Lambda_i$, denoted by $\mathtt{DR}(P^+_{\mathrm{cl},\ell})$. For instances, in case where $\mathfrak{g} = A_n^{(1)}$, we have $\mathtt{DR}(P^+_{\mathrm{cl},\ell}) = \{(\ell-1)\Lambda_0 + \Lambda_i \mid 0 \leqslant i \leqslant n\}$ and in case where $\mathfrak{g} = E_6^{(1)}$, we have $\mathtt{DR}(P^+_{\mathrm{cl},\ell}) = \{(\ell-1)\Lambda_0 + \Lambda_i \mid i = 0, 1, 6\}$ (see Table 2.2). It follows that

$$\bigsqcup_{\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})} P^+_{\mathrm{cl},\ell}(\Lambda) = P^+_{\mathrm{cl},\ell},$$

where $P^+_{\mathrm{cl},\ell}(\Lambda)$ denotes the equivalence class of Λ under $\sim$. It should be noticed that $|P^+_{\mathrm{cl},\ell}(\Lambda)| = |\max^+(\Lambda)|$.

From this we derive a very significant consequence that the number of all equivalence classes is given by $\mathsf{N} := [\overline{P} : \overline{Q}]$, where $\overline{P}/\overline{Q}$ is isomorphic to the *fundamental group* of the root system of $\mathfrak{g}_0$ except for $\mathfrak{g} = A_{2n}^{(2)}$ (see Table 2.1). Here $\mathfrak{g}_0$ denotes the subalgebra of $\mathfrak{g}$ which is of finite type.

Next, we introduce a new statistic $\mathsf{ev}_{\boldsymbol{s}}$, called the $\boldsymbol{S}$*-evaluation*, on $P^+_{\mathrm{cl},\ell}$. Here $\boldsymbol{S}$ is a certain set, called *a root sieving set*, which is characterized by a minimal generating set of the $\mathbb{Z}_{\mathsf{N}}$-kernel of the transpose of Cartan matrix associated $\mathfrak{g}_0$ (see Convention 2.13 for details). In more detail, for all affine Kac-Moody algebras except for $D_n^{(1)}$ ($n \equiv_2 0$), $\boldsymbol{S}$ consists of a single element $(s_1, \ldots, s_n)$ and

$$\mathsf{ev}_{\boldsymbol{s}}\left(\sum_{0 \leqslant i \leqslant n} m_i\Lambda_i\right) := \sum_{1 \leqslant i \leqslant n} s_i m_i \qquad \text{for } \Lambda = \sum_{0 \leqslant i \leqslant n} m_i\Lambda_i.$$

In case where $\mathfrak{g} = D_n^{(1)}$ ($n \equiv_2 0$), we have $\boldsymbol{S} = \{\boldsymbol{s}^{(1)} = (0, 0, \ldots, 0, 2, 2), \boldsymbol{s}^{(2)} = (2, 0, 2, 0, \ldots, 2, 0, 2, 0)\}$. For the $\boldsymbol{S}$-evaluation of this type, see (2.13). Finally, exploiting this statistic, we characterize the equivalence class of $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$ in terms of $\boldsymbol{S}$-evaluation (Theorem 2.14).

Quite interestingly, the $\boldsymbol{S}$*-evaluation* on $P^+_{\mathrm{cl},\ell}$ leads us to construct a (bi)cyclic sieving phenomenon on it. The *cyclic sieving phenomenon*, introduced by Reiner-Stanton-White in [15], are generalized and developed in various aspects including combinatorics and representation theory (see [1, 2, 6, 17, 19] for examples).

Let us briefly recall the cyclic sieving phenomenon. Let X be a finite set, with an action of a cyclic group C of order m, and $X(q)$ a polynomial in q with nonnegative integer coefficients. For $d \in \mathbb{Z}_{>0}$, let ω_d be a dth primitive root of the unity. We say that $(X, C, X(q))$ exhibits the cyclic sieving phenomenon if, for all $g \in C$, we have $|X^g| = X(\omega_{o(g)})$, where $o(g)$ is the order of g and X^g is the fixed point set under the action of g.

Let us explain our initial motivation. It was shown in [15, Theorem 1.1] that $\left(\left(\begin{smallmatrix}[0,n]\\ \ell\end{smallmatrix}\right), C_{n+1}, \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q\right)$ exhibits the cyclic sieving phenomenon. Here $\left(\begin{smallmatrix}[0,n]\\ \ell\end{smallmatrix}\right)$ denotes the set of all ℓ-multisets on $\{0,1,\dots,n\}$, C_{n+1} a fixed cyclic group of order $n+1$, and $\begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q$ the q-binomial coefficient of $\binom{n+\ell}{\ell}$. We identify $\left(\begin{smallmatrix}[0,n]\\ \ell\end{smallmatrix}\right)$ with $P^+_{\mathrm{cl},\ell}$ in $A_n^{(1)}$-type as C_{n+1}-sets and let

$$P^+_{\mathrm{cl},\ell}(q) := \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q .$$

Then we observe that the generating function of $P^+_{\mathrm{cl},\ell}(q)$ $(\ell \geqslant 0)$ can be expressed in terms of the root sieving set $\boldsymbol{S} = \{(s_1, s_2, \dots, s_n) = (1, 2, \dots, n)\}$ and the canonical center $c = h_0 + h_1 + h_2 + \dots + h_n = \sum_{i=0}^n a_i^\vee h_i$ as follows:

$$\sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q) t^\ell := \sum_{\ell \geqslant 0} \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q t^\ell = \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q^i t^1} = \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q^{s_i} t^{a_i^\vee}}, \tag{0.3}$$

where s_0 is set to be 0. From this product identity it follows that $P^+_{\mathrm{cl},\ell}(q) = \sum_{\Lambda \in P^+_{\mathrm{cl},\ell}} q^{\mathsf{ev}_{\boldsymbol{S}}(\Lambda)}$. Furthermore, since C_{n+1} is isomorphic to $\overline{P}/\overline{Q}$, we conclude that the triple $\left(P^+_{\mathrm{cl},\ell}, \overline{P}/\overline{Q}, P^+_{\mathrm{cl},\ell}(q)\right)$ also exhibits the cyclic sieving phenomenon.

Then it is natural to ask whether there exists a triple for other affine Kac-Moody algebras *exhibiting* the cyclic sieving phenomenon or not. Canonically, one can construct the triple in uniform way for all affine Kac-Moody algebras as follows: We first take $P^+_{\mathrm{cl},\ell}$ as the underlying set. Second, writing the canonical center as $c = \sum_{i=0}^n a_i^\vee h_i$, we take $P^+_{\mathrm{cl},\ell}(q)$ from the following geometric series (by mimicking the $A_n^{(1)}$-case):

$$\begin{cases} \sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q) t^\ell := \displaystyle\prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q^{s_i} t^{a_i^\vee}}, & \text{if } \mathfrak{g} \text{ is not of type } D_n^{(1)} \text{ for even } n, \\ \sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q_1, q_2) t^\ell := \displaystyle\prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q_1^{\mathfrak{s}_i^{(1)}} q_2^{\mathfrak{s}_i^{(2)}} t^{a_i^\vee}} & \text{if } \mathfrak{g} \text{ is of type } D_n^{(1)} \text{ for even } n, \end{cases}$$

where s_0 is set to be 0 (see (4.3) and (5.3)). Then we have

$$\begin{cases} P^+_{\mathrm{cl},\ell}(q) = \sum_{\Lambda \in P^+_{\mathrm{cl},\ell}} q^{\mathsf{ev}_{\boldsymbol{S}}(\Lambda)} & \text{if } \mathfrak{g} \text{ is not of type } D_n^{(1)} \text{ for even } n, \\ P^+_{\mathrm{cl},\ell}(q_1, q_2) = \sum_{\Lambda \in P^+_{\mathrm{cl},\ell}} q_1^{\mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda)} q_2^{\mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda)} & \text{if } \mathfrak{g} \text{ is of type } D_n^{(1)} \text{ for even } n. \end{cases}$$

Finally, take $\overline{P}/\overline{Q}$ as the (bi)cyclic group, which completes the triple:

$$(P^+_{\mathrm{cl},\ell}, \overline{P}/\overline{Q}, P^+_{\mathrm{cl},\ell}(q)) \qquad (\text{resp. } (P^+_{\mathrm{cl},\ell}, \overline{P}/\overline{Q}, P^+_{\mathrm{cl},\ell}(q_1, q_2))). \tag{0.4}$$

We assign an appropriate $\overline{P}/\overline{Q}$-action on $P^+_{\mathrm{cl},\ell}$ (see (4.14) and (5.1)), and prove that the triple exhibits the (bi)cyclic sieving phenomenon, which can be understood as a natural generalization of the cyclic sieving triple $\left(\left(\begin{smallmatrix}[0,n]\\ \ell\end{smallmatrix}\right), C_{n+1}, \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q\right)$ in aspect of affine Kac-Moody algebras.

For the proof, we employ the following strategy. For each divisor d of N, we introduce a set $\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')$ equipped with a C_d-action obtained by generalizing Sagan's action on $(0,1)$-words in [18]. Here, $r, \boldsymbol{\nu}, \boldsymbol{\nu}'$ are chosen so that $\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')$ can be identified with $P^+_{\mathrm{cl},\ell}$ by permuting indices properly. Then we show that

$|\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')^{C_d}| = \left| \left(P^+_{\mathrm{cl},\ell}\right)^g \right|$ for all $g \in \overline{P}/\overline{Q}$ of order d. We end the proof by showing

$$|\mathbf{M}_\ell(rd, d; \boldsymbol{\nu}, \boldsymbol{\nu}')^{C_d}| = P^+_{\mathrm{cl},\ell}(\zeta_{\mathsf{N}}^{\mathsf{N}/d}).$$

From the above sieving phenomena, we derive closed formulae for $|\max^+(\Lambda)|$ for all $\Lambda \in P^+_{\mathrm{cl},\ell}$ and for affine Kac-Moody algebras of arbitrary type. For the classical types, they are explicitly written as a sum of binomial coefficients (see Section 6.1). For instance, in case where $A_n^{(1)}$ type, we obtain

$$|\max{}^+((\ell-1)\Lambda_0 + \Lambda_i)| = \sum_{d | (n+1,\ell,i)} \frac{d}{(n+1)+\ell} \sum_{d' | (\frac{n+1}{d}, \frac{\ell}{d})} \mu(d') \binom{((n+1)+\ell)/dd'}{\ell/dd'}, \tag{0.5}$$

which is a vast generalization of (0.1) (see also Theorem 4.6).

Let us view $\{|\max^+(\Lambda)|\}_{n,\ell}$ as a sequence expressed in terms of n and ℓ. Exploiting our closed formulae, we can also derive recursive formulae for $|\max^+(\Lambda)|$ (except for type $A_n^{(1)}$) and their corresponding triangular arrays. It is quite interesting to observe that several triangular arrays are already known in different contexts. For example, when $\mathfrak{g}$ is of affine C-type, our triangular arrays are known as *Lozanić's triangle* and its Pascal complement (see Subsection 6.2.1). Also, the triangular array for twisted affine even A-type is Pascal triangle with duplicated diagonals (see Appendix A).

Going further, we observe interesting interrelations among the triangular arrays of various affine Kac-Moody algebras (see Appendix A). Surprisingly, all triangular arrays for classical affine type except for untwisted affine C-type can be constructed by *boundary conditions* and the triangular array of twisted affine even A-type. Similarly, the triangular arrays for untwisted affine C-type can be constructed by boundary conditions and Pascal triangle. Considering that the triangular array of twisted affine even A-type can be obtained from Pascal triangle, we can conclude that all triangular arrays for classical affine types can be obtained from boundary conditions and Pascal triangle only.

As another byproduct of our closed formulae, we observe a symmetry which appears as level and rank are switched in a certain way. For instance, if $(n+1, \ell, i) = (\ell, n+1, j)$ for some $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant \ell - 1$, then

$$\left|\max{}^+_{A_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_i)\right| = \left|\max{}^+_{A_{\ell-1}^{(1)}}(n\Lambda_0 + \Lambda_j)\right|.$$

This symmetry is *compatible* with the classical level-rank duality for $A_n^{(1)}$ studied by Frenkel in [7] (see Subsection 6.2.2). With the closed formulae of $\max^+(\Lambda)$ in terms of binomial coefficients, we can observe interesting symmetries for all classical affine types. For instances, we have

$$\begin{cases} \left|\max{}^+_{B_n^{(1)}}(\ell\Lambda_0)\right| = \left|\max{}^+_{B_{(\ell-1)/2}^{(1)}}((2n+1)\Lambda_0)\right|, & \text{if } \ell \text{ is odd,} \\ \left|\max{}^+_{B_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \left|\max{}^+_{B_{\ell/2-1}^{(1)}}((2n+1)\Lambda_0 + \Lambda_{\ell/2-1})\right| & \text{if } \ell \text{ is even,} \end{cases}$$

by exchanging n with $(\ell-1)/2$, and n with $\ell/2 - 1$, respectively, since

$$\left|\max{}^+_{B_n^{(1)}}(\ell\Lambda_0)\right| = \binom{n + \lfloor \frac{\ell}{2} \rfloor}{n} + \binom{n + \lfloor \frac{\ell-1}{2} \rfloor}{n}, \ \left|\max{}^+_{B_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \binom{n + \lfloor \frac{\ell-1}{2} \rfloor}{n} + \binom{n + \lfloor \frac{\ell}{2} \rfloor - 1}{n}.$$

This paper is organized as follows. In Section 1, we introduce necessary notations and backgrounds for affine Kac-Moody algebras, highest weight modules and classical results on dominant maximal weights. In Section 2, we define an equivalence relation $\sim$ on $P^+_{\mathrm{cl},\ell}$ satisfying that the equivalence class of $\Lambda \in P^+_{\mathrm{cl},\ell}$ has the same cardinality with $\max^+(\Lambda)$. Then we provide the set $\mathtt{DR}(P^+_{\mathrm{cl},\ell})$ of distinguished representatives, and characterize all equivalence classes in terms of $\boldsymbol{S}$-evaluation with our sieving set $\boldsymbol{S}$. In Section 3, we generalize Sagan's action with consideration on the result in Section 2 and prove that the generalized action gives cyclic action on $P^+_{\mathrm{cl},\ell}$ indeed. In Section 4, we prove that our triple for affine Kac-Moody algebras except $D_n^{(1)}$ for even n exhibits the cyclic sieving phenomenon. In Section 5, we prove the triple for $D_n^{(1)}$ for even n exhibits bicyclic sieving phenomenon. In Section 6, we derive closed formulae, recursive formulae, and level-rank duality for the sets of dominant maximal weights from the cyclic sieving phenomenon. In Appendix A and B, we list all triangular arrays and level-rank duality for affine Kac-Moody algebras, not dealt with in Section 6.

1. Preliminaries

Let $I = \{0, 1, ..., n\}$ be an index set. An *affine Cartan datum* $(\mathsf{A}, P, \Pi, P^\vee, \Pi^\vee)$ consists of the following quintuple:

(a) a matrix $\mathsf{A} = (\mathrm{a}_{ij})_{i,j\in I}$ of corank 1, called an *affine Cartan matrix* satisfying that, for $i, j \in I$,

$$\text{(i) } \mathrm{a}_{ii} = 2, \quad \text{(ii) } \mathrm{a}_{ij} \in \mathbb{Z}_{\leqslant 0} \text{ for } i \neq j \in I, \quad \text{(iii) } \mathrm{a}_{ij} = 0 \text{ if } \mathrm{a}_{ji} = 0,$$

(c) a free abelian group $P = \bigoplus_{i=0}^{n} \mathbb{Z}\Lambda_i \oplus \mathbb{Z}\delta$, called the *weight lattice*,

(e) a linearly independent set $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of *simple roots*,

(b) a free abelian group $P^\vee = \mathrm{Hom}(P, \mathbb{Z})$, called the *coweight lattice*,

(d) a linearly independent set $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$, called the set of *simple coroots*,

subject to the condition

$$\langle h_i, \alpha_j \rangle = \mathrm{a}_{ij} \text{ and } \langle h_j, \Lambda_i \rangle = \delta_{ij} \text{ for all } i, j \in I.$$

We call Λ_i the *ith fundamental weight* and set $\mathfrak{h} := \mathbb{Q} \otimes_{\mathbb{Z}} P^\vee$. Let

$$\delta = a_0\alpha_0 + a_1\alpha_1 + \cdots + a_n\alpha_n$$

be the *null root* and

$$c = a_0^\vee h_0 + a_1^\vee h_1 + \cdots + a_n^\vee h_n$$

be the *canonical central element*. We say that a weight $\Lambda \in P$ is of *level* ℓ if

$$\langle c, \Lambda \rangle = \ell.$$

Then we have $a_i^\vee = \langle c, \Lambda_i \rangle$.

Note that there exists a non-degenerate symmetric bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}^*$ ([12, (6.2.2)]) such that

$$(\Lambda_0|\Lambda_0) = 0, \quad (\alpha_i|\alpha_j) = a_i^\vee a_i^{-1}\mathrm{a}_{ij}, \quad (\alpha_i|\Lambda_0) = \delta_{i,0}a_0^{-1} \quad \text{for } i, j \in I, \tag{1.1}$$

and

$$(\delta|\lambda) = \langle c, \lambda \rangle \quad \text{for} \quad \lambda \in P.$$

Set $P^+ := \{\Lambda \in P \mid \langle h_i, \Lambda \rangle \in \mathbb{Z}_{\geqslant 0},\ i \in I\}$. The elements of P^+ are called the *dominant integral weights*. Also, for a nonnegative integer ℓ, we set

$$P^+_\ell := \{\Lambda \in P^+ \mid \langle c, \Lambda \rangle = \ell\}.$$

We call the free abelian group $Q := \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ the *root lattice* and set $Q_+ := \bigoplus_{i\in I} \mathbb{Z}_{\geqslant 0}\alpha_i$.

Definition 1.1. The *affine Kac-Moody algebra* $\mathfrak{g}$ associated with an affine Cartan datum $(\mathsf{A}, P, \Pi, P^\vee, \Pi^\vee)$ is the Lie algebra over $\mathbb{Q}$ generated by e_i, f_i $(i \in I)$ and $h \in P^\vee$ subject to the following defining relations:

(1) $[h, h'] = 0,\ [h, e_i] = \langle h, \alpha_i \rangle e_i,\ \ [h, f_i] = -\langle h, \alpha_i \rangle f_i$ for $h, h' \in P^\vee$,

(2) $[e_i, f_j] = \delta_{i,j} h_i$ for $i, j \in I$,

(3) $(\mathrm{ad}\ e_i)^{1-\mathrm{a}_{ij}}(e_j) = (\mathrm{ad}\ f_i)^{1-\mathrm{a}_{ij}}(f_j) = 0$ if $i \neq j$.

Let $\mathfrak{g}_0$ be the subalgebra of $\mathfrak{g}$ generated by the e_i and f_i with $i \in I_0 := I\backslash\{0\}$. Then $\mathfrak{g}_0$ is the Lie algebra associated to the Cartan matrix C obtained from A by deleting the 0th row and the 0th column. For a finite dimensional Lie algebra g, let $\mathsf{g}^\dagger$ be the Lie algebra whose Cartan matrix is the transpose of the Cartan matrix of g. The following table lists $\mathfrak{g}_0$ for each affine Kac-Moody algebra $\mathfrak{g}$:

$\mathfrak{g}$	$A_n^{(1)}$	$B_n^{(1)}, D_{n+1}^{(2)}$	$C_n^{(1)}, A_{2n-1}^{(2)}, A_{2n}^{(2)}$	$D_n^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	$F_4^{(1)}$	$E_6^{(2)}$	$G_2^{(1)}$	$D_4^{(3)}$
$\mathfrak{g}_0$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	$F_4^\dagger$	G_2	$G_2^\dagger$

Table 1.1. $\mathfrak{g}_0$ for each type

A $\mathfrak{g}$-module V is called a *weight module* if it admits a *weight space decomposition*

$$V = \bigoplus_{\mu \in P} V_\mu, \quad \text{where } V_\mu = \{v \in V \mid h \cdot v = \langle h, \mu \rangle v \text{ for all } h \in P^\vee\}.$$

If $V_\mu \neq 0$, μ is called a *weight* of V and V_μ is the *weight space* attached to μ. A weight module V over $\mathfrak{g}$ is called *integrable* if e_i and f_i $(i \in I)$ act locally nilpotent on V.

Definition 1.2. The category $\mathcal{O}_{\rm int}$ consists of integrable $\mathfrak{g}$-modules V satisfying the following conditions:

(1) V admits a weight space decomposition $V = \bigoplus_{\mu \in P} V_\mu$ with $\dim V_\mu < \infty$ for all weights μ.
(2) There exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in P$ such that
$$\mathrm{wt}(V) \subset D(\lambda_1) \cup \cdots \cup D(\lambda_s).$$
Here $\mathrm{wt}(V) := \{\mu \in P \mid V_\mu \neq 0\}$ and $D(\lambda) := \{\lambda - \alpha \mid \alpha \in Q_+\}$.

It is well-known that $\mathcal{O}_{\rm int}$ is a semisimple tensor category such that every irreducible objects is isomorphic to *the highest weight module* $V(\Lambda)$ $(\Lambda \in P^+)$.

A weight μ of $V(\Lambda)$ is *maximal* if $\mu + \delta \notin \mathrm{wt}(V(\Lambda))$ and the set of all maximal weights of $V(\Lambda)$ is denoted by $\max_{\mathfrak{g}}(\Lambda)$.

Proposition 1.3 ([12, (12.6.1)])**.** *For each $\Lambda \in P^+$, we have*

$$\mathrm{wt}(V(\Lambda)) = \bigsqcup_{\mu \in \max_{\mathfrak{g}}(\Lambda)} \{\mu - s\delta \mid s \in \mathbb{Z}_{\geqslant 0}\}.$$

Denote by $\max_{\mathfrak{g}}^+(\Lambda)$ the set of all dominant maximal weights of $V(\Lambda)$, thus,

$$\max\nolimits_{\mathfrak{g}}^+(\Lambda) = \max\nolimits_{\mathfrak{g}}(\Lambda) \cap P^+.$$

We will omit the subscript $\mathfrak{g}$ for simplicity if there is no danger of confusion. It is well-known that

$$\max(\Lambda) = W \cdot \max{}^+(\Lambda), \qquad \text{where } W \text{ is the Weyl group of } \mathfrak{g}.$$

Let $\mathfrak{h}_0$ be the vector space spanned by $\{h_i \mid i \in I_0\}$. Recall the *orthogonal projection* $^-\colon \mathfrak{h}^* \to \mathfrak{h}_0^*$, which is introduced in ([12, (6.2.7)]), by

$$\mu \longmapsto \overline{\mu} = \mu - \langle c, \mu \rangle \Lambda_0 - (\mu | \Lambda_0)\delta.$$

Let $\overline{Q}$ (resp. $\overline{P}$) be the image of Q (resp. P) under this map. We also use $\langle\ ,\ \rangle$ and $(\ |\ )$ to denote bilinear forms for $\mathfrak{g}_0$ since they can be obtained by restricting $\langle\ ,\ \rangle$ and $(\ |\ )$ to $\mathfrak{h}_0 \times \mathfrak{h}_0^*$ and $\mathfrak{h}_0^* \times \mathfrak{h}_0^*$ (via $^-$) respectively.

Define

$$\ell\mathcal{C}_{\rm af} := \{\mu \in \mathfrak{h}_0^* \mid \langle h_i, \mu \rangle \geqslant 0 \text{ for } i \in I_0,\ (\mu|\theta) \leqslant \ell\} \qquad \text{where } \theta := \delta - a_0\alpha_0. \tag{1.2}$$

Proposition 1.4 ([12, Proposition 12.6])**.** *The map $\mu \longmapsto \overline{\mu}$ defines a bijection from $\max^+(\Lambda)$ onto $\ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{Q})$ where Λ is of level ℓ. In particular, the set $\max^+(\Lambda)$ is finite and described as follows:*

$$\max{}^+(\Lambda) = \{\lambda \in P^+ \mid \lambda \leqslant \Lambda \text{ and } \Lambda - \lambda - \delta \notin Q_+\}. \tag{1.3}$$

For reader's understanding, let us collect notations required to develop our arguments.

- ◇ For $(n+1)$-tuples $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ and $\gamma' = (\gamma'_0, \gamma'_1, \ldots, \gamma'_{n'})$ of integers, $0 \leqslant a \leqslant b \leqslant n$, we set
  - $\gamma_{[a,b]} := (\gamma_a, \ldots, \gamma_b)$, $\gamma_{\leqslant a} := \gamma_{[0,a]}$, and $\gamma_{\geqslant b} := \gamma_{[b,n]}$.
  - $\gamma * \gamma' := (\gamma_0, \gamma_1, \ldots, \gamma_n, \gamma'_0, \gamma'_1, \ldots, \gamma'_{n'})$.
- ◇ For words $\mathbf{w} = w_1 w_2 \cdots w_n$ and $\mathbf{w}' = w'_1 w'_2 \cdots w'_n$, we set $\mathbf{w} * \mathbf{w}' := w_1 w_2 \cdots w_n w'_1 w'_2 \cdots w'_n$.
- ◇ For a nonnegative integer m and a positive integer k, we denote by m^k the sequence $\underbrace{m, m, \ldots, m}_{k\text{-times}}$.
- ◇ Let k be a positive integer.
  - For $m, m' \in \mathbb{Z}$, we write $m \equiv_k m'$ if k divides $m - m'$, and $m \not\equiv_k m'$ otherwise.
  - For $\mathbf{m} = (m_1, m_2, \ldots, m_n), \mathbf{m}' = (m'_1, m'_2, \ldots, m'_n) \in \mathbb{Z}^n$, we write $\mathbf{m} \equiv_k \mathbf{m}'$ if $m_i \equiv_k m'_i$ for all $i = 1, 2, \ldots, n$.
- ◇ For a matrix M, we denote by $M_{(i)}$ the ith row of M and by $M^{(i)}$ the ith column of M.
- ◇ For an invertible matrix M, we denote by $\widetilde{M}$ the inverse matrix of M.

◇ For a commutative ring R with the unity and a positive integer n, the *dot product* on R^n denotes the map $\bullet : R^n \times R^n \to R$ defined by

$$(x_1, x_2, \ldots, x_n) \bullet (y_1, y_2, \ldots, y_n) = \sum_{1 \leqslant i \leqslant n} x_i y_i.$$

◇ For a statement P, $\delta(P)$ is defined to be 1 if P is true and 0 if P is false.

2. Sets in bijection with $\max^+(\Lambda)$

In this section, all affine Kac-Moody algebras will be affine Kac-Moody algebras other than $A^{(2)}_{2n}$. In fact, we exclude the case $A^{(2)}_{2n}$ for the simplicity of our statements. All the notations and terminologies in the previous section will be used without change.

Choose an arbitrary element $\Lambda \in P^+_\ell$. The purpose of this section is to understand a combinatorial structure of $\max^+(\Lambda)$ by investigating sets in bijection with $\max^+(\Lambda)$ which are induced from certain restrictions of the orthogonal projection $^- : \mathfrak{h}^* \to \mathfrak{h}^*_0$.

As seen in Proposition 1.4, the set $\ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{Q})$ plays a key role in the study of $\max^+(\Lambda)$. Hereafter we will assume that Λ is of the form $\sum_{0 \leqslant i \leqslant n} p_i \Lambda_i$ because

$$\ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{Q}) = \ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda + k\delta} + \overline{Q}) \quad \text{for all } k \in \mathbb{Z}.$$

Set

$$P^+_{\rm cl} := P^+/\mathbb{Z}\delta.$$

We identify $P^+_{\rm cl}$ with $\sum_{0 \leqslant i \leqslant n} \mathbb{Z}_{\geqslant 0}\Lambda_i$ in the obvious manner. As a set, $P^+_{\rm cl}$ coincides with the set of classical dominant integral weights arising in the context of quantum affine Lie algebra $U'_q(\mathfrak{g})$ (for details, see [8]). We also set

$$P^+_{{\rm cl},\ell} := P^+_\ell/\mathbb{Z}\delta,$$

which is identified with $P^+_\ell \cap \sum_{0 \leqslant i \leqslant n} \mathbb{Z}_{\geqslant 0}\Lambda_i$.

2.1. **Description of $\ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{Q})$.** As mentioned in the above, $\mathfrak{g}$ denotes an affine Kac-Moody algebra other than $A^{(2)}_{2n}$.

Set

$$\begin{aligned}
\Pi_0 &:= \{\overline{\alpha}_i \mid i \in I_0\} \quad \text{(the set of simple roots of } \mathfrak{g}_0),\\
\boldsymbol{\varpi} &:= \{\varpi_i \mid i \in I_0\} \quad \text{(the set of fundamental dominant weights of } \mathfrak{g}_0).
\end{aligned}$$

Both Π_0 and $\boldsymbol{\varpi}$ are bases for $\mathbb{Q}\boldsymbol{\varpi}$, and the transition matrix $[\mathrm{Id}]^{\boldsymbol{\varpi}}_{\Pi_0}$ is equal to Cartan matrix C of $\mathfrak{g}_0$. For reader's understanding, let us recall that

$$\overline{\alpha}_0 = -\sum_{1 \leqslant i \leqslant n} a_i \overline{\alpha}_i, \qquad \overline{\Lambda}_i = \begin{cases} \varpi_i & \text{if } i \neq 0,\\ 0 & \text{if } i = 0,\end{cases}$$

and

$$\overline{\alpha}_i = \sum_{1 \leqslant j \leqslant n} a_{ji}\varpi_j, \qquad \varpi_i = \sum_{1 \leqslant j \leqslant n} d_{ji}\overline{\alpha}_j \quad (i \in I_0).$$

Here $\mathsf{C} = (a_{ij})_{i,j \in I_0}$ and $\widetilde{\mathsf{C}} = (d_{ij})_{i,j \in I_0}$ is the inverse of C.

Choose any element $\Lambda = \sum_{0 \leqslant i \leqslant n} p_i \Lambda_i \in P^+_{{\rm cl},\ell}$, which will be fixed throughout this subsection. Then we have

$$\begin{aligned}
\ell\mathcal{C}_{\rm af} \cap (\overline{\Lambda} + \overline{Q}) &= \left\{ \overline{\Lambda} + \sum_{0 \leqslant j \leqslant n} k_j \overline{\alpha}_j \;\middle|\; k_j \in \mathbb{Z}, \langle h_i, \overline{\Lambda} + \sum_{0 \leqslant j \leqslant n} k_j \overline{\alpha}_j \rangle \geqslant 0 \ (i \in I_0), \ \left(\overline{\Lambda} + \sum_{0 \leqslant j \leqslant n} k_j \overline{\alpha}_j \;\middle|\; \theta \right) \leqslant \ell \right\}\\
&= \left\{ \overline{\alpha} := \overline{\Lambda} + \sum_{1 \leqslant j \leqslant n} x_j \overline{\alpha}_j \;\middle|\; \begin{array}{l} \text{(i) } \mathbf{x} := (x_1, x_2, \ldots, x_n)^t \in \mathbb{Z}^n \\ \text{(ii) } \langle h_i, \overline{\alpha} \rangle \geqslant 0 \ (i \in I_0) \\ \text{(iii) } \left(\overline{\alpha} \mid \sum_{1 \leqslant i \leqslant n} a_i \overline{\alpha}_i\right) \leqslant \ell \end{array} \right\},
\end{aligned} \tag{2.1}$$

where the second equality can be obtained by substituting x_j for $k_j - k_0 a_j$ for $j \in I_0$. Since

$$\left\langle h_i, \overline{\Lambda} + \sum_{1 \leqslant j \leqslant n} x_j \overline{\alpha}_j \right\rangle = p_i + \sum_{1 \leqslant j \leqslant n} x_j \mathrm{a}_{ij} = p_i + \mathsf{C}_{(i)}\mathbf{x},$$

one can see that the condition (ii) is satisfied if and only if $\mathsf{C}_{(i)}\mathbf{x} \geqslant -p_i$. For the condition (iii), notice that (see (1.1))

$$\left(\overline{\Lambda} + \sum_{1 \leqslant j \leqslant n} x_j \overline{\alpha}_j \,\middle|\, \sum_{1 \leqslant i \leqslant n} a_i \overline{\alpha}_i \right) = \sum_{1 \leqslant i,j \leqslant n} p_j a_i (\varpi_j | \overline{\alpha}_i) + \sum_{1 \leqslant i,j \leqslant n} x_j a_i (\overline{\alpha}_j | \overline{\alpha}_i) = \sum_{1 \leqslant i \leqslant n} (a_i^\vee p_i + a_i^\vee \mathsf{C}_{(i)}\mathbf{x}). \tag{2.2}$$

Since $\ell = \langle c, \Lambda \rangle = \sum_{0 \leqslant i \leqslant n} a_i^\vee p_i$, this computation implies that the condition (iii) in (2.1) is satisfied if and only if $\sum_{1 \leqslant i \leqslant n} a_i^\vee \mathsf{C}_{(i)}\mathbf{x} \leqslant a_0^\vee p_0$. As a consequence, $\ell \mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ can be written as

$$\left\{ \overline{\Lambda} + \sum_{1 \leqslant j \leqslant n} x_j \overline{\alpha}_j \;\middle|\; \begin{array}{l} \text{(i) } \mathbf{x} := (x_1, x_2, \dots, x_n)^t \in \mathbb{Z}^n \\ \text{(ii) } -p_i \leqslant \mathsf{C}_{(i)}\mathbf{x} \text{ for } i \in I_0 \\ \text{(iii) } \sum_{1 \leqslant i \leqslant n} a_i^\vee \mathsf{C}_{(i)}\mathbf{x} \leqslant a_0^\vee p_0. \end{array} \right\} \tag{2.3}$$

Finally, using the substitution $m_j := \mathsf{C}_{(j)}\mathbf{x} + p_j$ for $j \in I_0$, we obtain the description of $\ell \mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ in terms of the basis $\boldsymbol{\varpi}$.

Proposition 2.1. *Let $\Lambda = \sum_{0 \leqslant i \leqslant n} p_i \Lambda_i \in P_\ell^+$. Then we have*

$$\ell \mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) = \left\{ \sum_{1 \leqslant i \leqslant n} m_i \varpi_i \;\middle|\; \begin{array}{l} \text{(i) } \sum_{1 \leqslant i \leqslant n} (m_i - p_i) \widetilde{\mathsf{C}}^{(i)} \in \mathbb{Z}^n \\ \text{(ii) } (m_1, m_2, \dots, m_n)^t \in \mathbb{Z}^n_{\geqslant 0} \\ \text{(iii) } \sum_{1 \leqslant i \leqslant n} a_i^\vee m_i \leqslant \ell \end{array} \right\}. \tag{2.4}$$

Proof. Let $\mathbf{x} := (x_1, x_2, \dots, x_n)^t \in \mathbb{Z}^n$. Note that

$$\overline{\Lambda} + \sum_{1 \leqslant i \leqslant n} x_i \overline{\alpha}_i = \sum_{1 \leqslant i \leqslant n} \left(p_i + \sum_{1 \leqslant j \leqslant n} x_j \mathrm{a}_{ij} \right) \varpi_i = \sum_{1 \leqslant i \leqslant n} \left(p_i + \mathsf{C}_{(i)}\mathbf{x} \right) \varpi_i.$$

Set $m_j := \mathsf{C}_{(j)}\mathbf{x} + p_j$. Since $\mathbf{x} = \sum_{1 \leqslant i \leqslant n} \widetilde{\mathsf{C}}^{(i)}(m_i - p_i)$, (i) of (2.3) is equivalent to (i) of (2.4). By direct calculation, one can see that $\mathsf{C}_{(j)}\mathbf{x}$ is an integer. Thus, by the definition of m_i, (ii) of (2.3) is equivalent to (ii) of (2.4). For the condition (iii), observe that

$$\sum_{1 \leqslant i \leqslant n} a_i^\vee \mathsf{C}_{(i)}\mathbf{x} = \sum_{1 \leqslant i \leqslant n} a_i^\vee \mathsf{C}_{(i)} \left(\sum_{1 \leqslant j \leqslant n} \widetilde{\mathsf{C}}^{(j)} (m_j - p_j) \right) = \sum_{1 \leqslant i \leqslant n} a_i^\vee (m_i - p_i) \leqslant a_0^\vee p_0.$$

This tells us that $\sum_{1 \leqslant i \leqslant n} a_i^\vee \mathsf{C}_{(i)}\mathbf{x} \leqslant a_0^\vee p_0$ if and only if $\sum_{1 \leqslant i \leqslant n} a_i^\vee m_i \leqslant \ell$, as required. $\square$

Example 2.2. Let $\mathfrak{g}$ be the affine Kac-Moody algebra of type $A_2^{(1)}$ and $\Lambda = 2\Lambda_0 + \Lambda_1$. In this case, $a_0^\vee = a_1^\vee = a_2^\vee = 1$, and $\widetilde{\mathsf{C}} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Hence, by Proposition 2.1, we have

$$\begin{aligned} 3\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) &= \left\{ m_1 \varpi_1 + m_2 \varpi_2 \;\middle|\; \begin{array}{l} \text{(i) } (m_1 - 1) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + m_2 \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} \in \mathbb{Z}^2 \\ \text{(ii) } m_1, m_2 \in \mathbb{Z}_{\geqslant 0} \\ \text{(iii) } m_1 + m_2 \leqslant 3 \end{array} \right\} \\ &= \{ \varpi_1, 2\varpi_2, 2\varpi_1 + \varpi_2 \}. \end{aligned}$$

2.2. **Equivalence relation on $P^+_{\mathrm{cl},\ell}$.** Let $\Lambda \in P^+_{\mathrm{cl},\ell}$. Consider the map $\iota_\Lambda : \ell \mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) \to P^+_{\mathrm{cl},\ell}$ defined by

$$\iota_\Lambda \left(\sum_{1 \leqslant i \leqslant n} m_i \varpi_i \right) = m_0 \Lambda_0 + \sum_{1 \leqslant i \leqslant n} m_i \Lambda_i,$$

where

$$m_0 = \ell - \sum_{1 \leqslant i \leqslant n} a_i^\vee m_i.$$

This map is well-defined since all m_i's are nonnegative integers for all $0 \leqslant i \leqslant n$ by Proposition 2.1 and $\sum_{0 \leqslant i \leqslant n} m_i \Lambda_i$ has level ℓ. In particular, it is injective.

We now define a relation $\sim$ on $P^+_{\mathrm{cl},\ell}$, called the *sieving equivalence relation*, by

$$\Lambda \sim \Lambda' \quad \text{if and only if} \quad \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q}) = \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda'} + \overline{Q}). \tag{2.5}$$

It is easy to see that $\sim$ is indeed an equivalence relation on $P^+_{\mathrm{cl},\ell}$. The following lemma is straightforward.

Lemma 2.3. *For $\Lambda, \Lambda' \in P^+_{\mathrm{cl},\ell}$, the following are equivalent.*

(1) $\Lambda \sim \Lambda'$.
(2) $\iota_\Lambda = \iota_{\Lambda'}$.
(3) $\overline{\Lambda} + \overline{Q} = \overline{\Lambda'} + \overline{Q}$.
(4) $\Lambda' \in \mathrm{Im}(\iota_\Lambda)$.

Proof. The equivalence of (1) and (2) is straightforward from the definition, and that of (1) and (3) follows from the fact that either $(\overline{\Lambda} + \overline{Q}) \cap (\overline{\Lambda'} + \overline{Q}) = \varnothing$ or $\overline{\Lambda} + \overline{Q} = \overline{\Lambda'} + \overline{Q}$ because $(\overline{\Lambda} + \overline{Q})$ and $(\overline{\Lambda'} + \overline{Q})$ are translations of $\overline{Q}$.

Next, let us show that (2) implies (4). Suppose that $\iota_\Lambda = \iota_{\Lambda'}$. Then $\overline{\Lambda'} \in \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda'} + \overline{Q}) = \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$ and so $\iota_\Lambda(\overline{\Lambda'}) = \iota_{\Lambda'}(\overline{\Lambda'}) = \Lambda'$.

Finally, let us show that (4) implies (3). Assume that $\Lambda' = \sum_{0 \leqslant i \leqslant n} m'_i \Lambda_i \in \mathrm{Im}(\iota_\Lambda)$. Since $\Lambda' \in P^+_{\mathrm{cl},\ell}$, this gives $\overline{\Lambda'} = \sum_{1 \leqslant i \leqslant n} m'_i \varpi_i \in \ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})$. Therefore, $\overline{\Lambda} + \overline{Q} = \overline{\Lambda'} + \overline{Q}$. This completes the proof. $\square$

Let $P_0 := \mathbb{Z}\varpi$ be the weight lattice of $\mathfrak{g}_0$ and $Q_0 := \mathbb{Z}\Pi_0$ the root lattice of $\mathfrak{g}_0$. Then P_0/Q_0 is known to be a finite group, called the *fundamental group of Φ_0 (the set of roots of $\mathfrak{g}_0$).* Its structure is well-known in the literature. For instance, see [9].

$\mathfrak{g}_0$	A_n	D_n	E_6	E_7	E_8	$B_n \overset{t}{\leftrightarrow} C_n$	F_4	G_2
P_0/Q_0	$\mathbb{Z}_{n+1}$	$\mathbb{Z}_4$ if n is odd, $\mathbb{Z}_2 \times \mathbb{Z}_2$ if n is even	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{e\}$	$\mathbb{Z}_2$	$\{e\}$	$\{e\}$

TABLE 2.1. Fundamental groups

It should be noticed that, except for $A^{(2)}_{2n}$ type, it holds that $\overline{Q} = Q_0$ and $\overline{P} = P_0$. Lemma 2.3 (3) shows that there are at most $|P_0/Q_0|$ equivalence classes on $P^+_{\mathrm{cl},\ell}$. In the following, we provide a complete list of representatives of very simple form. For each type, let us define a set $\mathtt{DR}(P^+_{\mathrm{cl},\ell})$, called the set of *distinguished representatives*, as in Table 2.2. One can prove in a direct way the following lemma.

Lemma 2.4. *$\mathtt{DR}(P^+_{\mathrm{cl},\ell})$ is a complete set of pairwise inequivalent representatives of $P^+_{\mathrm{cl},\ell}/\sim$, the set of equivalence classes of $P^+_{\mathrm{cl},\ell}$ under the sieving equivalence relation. In particular, the number of equivalence classes is given by $|P_0/Q_0|$.*

Proof. Here we will deal with $A^{(1)}_n$ type only since other types can be verified in the exactly same manner. Since the number of elements in $\mathtt{DR}(P^+_{\mathrm{cl},\ell})$ is equal to $n+1$, it suffices to show that every element is pairwise inequivalent, that is, it is enough to show that

$$\overline{(\ell-1)\Lambda_0 + \Lambda_i} - \overline{(\ell-1)\Lambda_0 + \Lambda_j} = \overline{\Lambda}_i - \overline{\Lambda}_j \notin \overline{Q} \ (= Q_0).$$

Equivalently, it suffices to show that

$$[\overline{\Lambda}_i - \overline{\Lambda}_j]_{\Pi_0} \notin \mathbb{Z}^n \quad (0 \leqslant i < j \leqslant n).$$

Type	$\mathtt{DR}(P^+_{\mathrm{cl},\ell})$	$\lvert\mathtt{DR}(P^+_{\mathrm{cl},\ell})\rvert$ $(=\lvert P_0/Q_0\rvert)$
$A_n^{(1)}$	$\{(\ell-1)\Lambda_0+\Lambda_i \mid i=0,1,\ldots,n\}$	$n+1$
$B_n^{(1)}, D_{n+1}^{(2)}, E_7^{(1)}$	$\{(\ell-1)\Lambda_0+\Lambda_i \mid i=0,n\}$	2
$C_n^{(1)}, A_{2n-1}^{(2)}$	$\{(\ell-1)\Lambda_0+\Lambda_i \mid i=0,1\}$	2
$D_n^{(1)}$	$\{(\ell-1)\Lambda_0+\Lambda_i \mid i=0,1,n-1,n\}$	4
$E_6^{(1)}$	$\{(\ell-1)\Lambda_0+\Lambda_i \mid i=0,1,6\}$	3
$F_4^{(1)}$, $E_6^{(2)}$, $G_2^{(1)}$, $D_4^{(3)}$, $E_8^{(1)}$	$\{\ell\Lambda_0\}$	1

TABLE 2.2. Distinguished representatives

Note that

$$[\overline{\Lambda}_i]_{\Pi_0} = \begin{cases} 0 & \text{if } i=0,\\ \widetilde{\mathsf{C}}^{(i)} & \text{if } i>0 \end{cases}$$

and the first coordinate of $\widetilde{\mathsf{C}}^{(i)}$ is $1-i/(n+1)$. Since $0\leqslant i<j\leqslant n$, the first coordinate of $[\overline{\Lambda}_i-\overline{\Lambda}_j]_{\Pi_0}$ is not an integer, as required. ☐

For $\Lambda\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})$, let $P^+_{\mathrm{cl},\ell}(\Lambda)$ denote the equivalence class of Λ, i.e., $P^+_{\mathrm{cl},\ell}(\Lambda):=\{\Lambda'\in P^+_{\mathrm{cl},\ell}\mid \Lambda\sim\Lambda'\}$. Then

$$\iota_\Lambda : \ell\mathcal{C}_{\mathrm{af}}\cap(\overline{\Lambda}+\overline{Q})\to P^+_{\mathrm{cl},\ell}(\Lambda)$$

is bijective, and its inverse is given by $^-|_{P^+_{\mathrm{cl},\ell}(\Lambda)}$, the restriction of $^-\colon \mathfrak{h}^*\to\mathfrak{h}_0^*$ to $P^+_{\mathrm{cl},\ell}(\Lambda)$. Notice that if $\Lambda\not\sim\Lambda'$, then $\big(\ell\mathcal{C}_{\mathrm{af}}\cap(\overline{\Lambda}+\overline{Q})\big)\cap\big(\ell\mathcal{C}_{\mathrm{af}}\cap(\overline{\Lambda'}+\overline{Q})\big)=\varnothing$, so we have bijections:

$$\bigsqcup_{\Lambda\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})}\ell\mathcal{C}_{\mathrm{af}}\cap(\overline{\Lambda}+\overline{Q}) \xleftrightarrow{1-1} \bigsqcup_{\Lambda\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})}P^+_{\mathrm{cl},\ell}(\Lambda)=P^+_{\mathrm{cl},\ell} \tag{2.6}$$

2.3. **Equivalence classes.** For each $\Lambda\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we give a simple description of the equivalence class $P^+_{\mathrm{cl},\ell}(\Lambda)$. For this purpose, we recall the following elementary fact from linear algebra.

Lemma 2.5. *Let $\beta=\{\beta_1,\beta_2,\ldots,\beta_n\}$ and $\gamma=\{\gamma_1,\gamma_2,\ldots,\gamma_n\}$ be bases for $\mathbb{Q}^n$ such that $\mathbb{Z}\gamma\subseteq\mathbb{Z}\beta$. Let $M=[\mathrm{Id}]^\beta_\gamma$ be the change of coordinate matrix that change γ-coordinates into β-coordinates. Then for any $v\in\mathbb{Z}\beta$, it holds that $v\in\mathbb{Z}\gamma$ if and only if $\widetilde{M}_{(i)}[v]_\beta\in\mathbb{Z}$ for all $i=1,2,\ldots,n$.*

Choose an arbitrary element $x\in P_0$. Lemma 2.5 tells us that $x\in Q_0$ if and only if $\widetilde{\mathsf{C}}_{(i)}[x]_\varpi\in\mathbb{Z}$ for all $i=1,2,\ldots,n$. Let $\{\mathrm{e}_1,\mathrm{e}_2,\ldots,\mathrm{e}_n\}$ be the standard basis of $\mathbb{Z}^n$. Since

$$\mathrm{e}_j=\mathsf{C}_{(j)}\widetilde{\mathsf{C}}=\sum_{1\leqslant k\leqslant n}\mathsf{C}_{j,k}\widetilde{\mathsf{C}}_{(k)}\quad (j\in I_0),$$

$\mathbb{Z}^n$ is obviously a submodule of the $\mathbb{Z}$-span of $\{\widetilde{\mathsf{C}}_{(i)}\mid i\in I_0\}$, denoted by $\mathbb{Z}\{\widetilde{\mathsf{C}}_{(i)}\mid i\in I_0\}$. In the same manner, $\mathbb{Z}^n$ is a submodule of $\mathbb{Z}\{\widetilde{\mathsf{C}}^{(i)}\mid i\in I_0\}$ and

$$\mathbb{Z}\{\widetilde{\mathsf{C}}^{(i)}\mid i\in I_0\}/\mathbb{Z}^n\cong P_0/Q_0 \quad \text{(as abelian groups)}$$

since $[\varpi_i]_{\Pi_0}=\widetilde{\mathsf{C}}^{(i)}$, for all $i\in I_0$. Going further, using Table 2.1, we can deduce that

$$\mathbb{Z}\{\widetilde{\mathsf{C}}_{(i)}\mid i\in I_0\}/\mathbb{Z}^n\cong P_0/Q_0 \quad \text{(as abelian groups)}\ . \tag{2.7}$$

Recall that P_0/Q_0 is a cyclic group unless $\mathfrak{g}$ is of the type $D_n^{(1)}$ (n is even). It is not difficult to see that there is an index i_1 (resp. j_1), which may not be unique, such that

$$\widetilde{\mathsf{C}}_{(i_1)} + \mathbb{Z}^n \quad (\text{resp. } \widetilde{\mathsf{C}}^{(j_1)} + \mathbb{Z}^n\ )$$

is a generator of $\mathbb{Z}\{\widetilde{\mathsf{C}}_{(i)} \mid i \in I_0\}/\mathbb{Z}^n$ (resp. $\mathbb{Z}\{\widetilde{\mathsf{C}}^{(i)} \mid i \in I_0\}/\mathbb{Z}^n$).

In a similar way, in case where $\mathfrak{g}$ is of the type $D_n^{(1)}$ (n is even), one can see that there is a set of indices $\{i_1, i_2\}$ (resp. $\{j_1, j_2\}$), which may not be unique, such that

$$\{\widetilde{\mathsf{C}}_{(i_1)} + \mathbb{Z}^n,\ \widetilde{\mathsf{C}}_{(i_2)} + \mathbb{Z}^n\} \quad (\text{resp. } \{\widetilde{\mathsf{C}}^{(j_1)} + \mathbb{Z}^n,\ \widetilde{\mathsf{C}}^{(j_2)} + \mathbb{Z}^n\}\ )$$

is a generating set of $\mathbb{Z}\{\widetilde{\mathsf{C}}_{(i)} \mid i \in I_0\}/\mathbb{Z}^n$ (resp. $\mathbb{Z}\{\widetilde{\mathsf{C}}^{(i)} \mid i \in I_0\}/\mathbb{Z}^n$). For the convenience of computation, from now on, we fix these indices as in the table below:

$\mathfrak{g}$	$A_n^{(1)}, D_n^{(1)}(n\text{:odd}), E_7^{(1)}$	$B_n^{(1)}, D_{n+1}^{(2)}$	$C_n^{(1)}, A_{2n-1}^{(2)}$	$E_6^{(1)}$	$D_n^{(1)}(n\text{:even})$	Other types
i_k	$i_1 = n$	$i_1 = 1$	$i_1 = n$	$i_1 = 1$	$i_1 = 1,\ i_2 = n$	$i_1 = 1$
j_k	$j_1 = n$	$j_1 = n$	$j_1 = 1$	$j_1 = 1$	$j_1 = 1,\ j_2 = n$	$j_1 = 1$

Table 2.3. i_k, j_k for each type

The above discussion shows that $\widetilde{\mathsf{C}}_{(i)}[\overline{\Lambda}]_{\varpi} \in \mathbb{Z}$ for all $i \in I_0$ if and only if $\widetilde{\mathsf{C}}_{(i_k)}[\overline{\Lambda}]_{\varpi} \in \mathbb{Z}$ for $k = 1$ or $k = 1, 2$ (up to types). When $\mathfrak{g}$ is of the type $D_n^{(1)}$ ($n \equiv_2 0$), the order of the coset $\widetilde{\mathsf{C}}_{(i_k)} + \mathbb{Z}^n$ ($k = 1, 2$) in $\mathbb{Z}\{\widetilde{\mathsf{C}}_{(i)} \mid i \in I_0\}/\mathbb{Z}^n$ is given by 2. For the other types, the order of the coset $\widetilde{\mathsf{C}}_{(i_1)}$ is $|P_0/Q_0|$. For simplicity of notation, we set

$$\mathsf{N} := |P_0/Q_0| \tag{2.8}$$

With this notation, we have the following characterization.

Lemma 2.6. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra. For $x \in P_0$, we have*

$$x \in Q_0 \quad \textit{if and only if} \quad \mathrm{adj}(\mathsf{C})_{(i_k)}[x]_{\varpi} \equiv_{\mathsf{N}} 0 \tag{2.9}$$

for all $k = 1$ or $k = 1, 2$ up to types. Here, $\mathrm{adj}(\mathsf{C})$ *denotes the classical adjoint of* C.

Consider the $\mathbb{Z}$-linear map given by left multiplication by C^t

$$L_{\mathsf{C}^t} : \mathbb{Z}^n \to \mathbb{Z}^n, \quad \mathbf{x} \mapsto \mathsf{C}^t\mathbf{x}.$$

From reduction modulo N

$$\mathrm{red}_{\mathsf{N}} : \mathbb{Z} \to \mathbb{Z}_{\mathsf{N}}, \quad a \mapsto a + \mathsf{N}\mathbb{Z}$$

we can induce a $\mathbb{Z}_{\mathsf{N}}$-linear map defined by

$$L_{\overline{\mathsf{C}}^t} : (\mathbb{Z}_{\mathsf{N}})^n \to (\mathbb{Z}_{\mathsf{N}})^n, \quad \mathbf{x} \mapsto \overline{\mathsf{C}}^t\mathbf{x},$$

where $\overline{\mathsf{C}}$ is obtained from C respectively by reading entries modulo N. We simply write $\ker(\overline{\mathsf{C}}^t)$ to denote the kernel of $L_{\overline{\mathsf{C}}^t}$.

Since $[\overline{\alpha}_i]_{\varpi} = \mathsf{C}^{(i)}$ for all $i = 1, 2, \ldots, n$, by Lemma 2.6, we deduce that $\mathrm{adj}(\overline{\mathsf{C}})_{(i_k)} \in \ker(\overline{\mathsf{C}}^t)$ for all $k = 1$ or $k = 1, 2$ up to types.

Lemma 2.7. *In the above setting,* $\{\mathrm{adj}(\overline{\mathsf{C}})_{(i_k)} : k = 1 \text{ or } k = 1, 2 \text{ up to types}\}$ *is a minimal generating set of* $\ker(\overline{\mathsf{C}}^t)$.

Proof. To prove the assertion, it suffices to show that $\ker(\overline{\mathsf{C}}^t) \cong P_0/Q_0$ as abelian groups, equivalently $\ker(\overline{\mathsf{C}}^t) \cong \mathbb{Z}\{\widetilde{\mathsf{C}}^t_{(i)} \mid i \in I_0\}/\mathbb{Z}^n$ by (2.7).

Define $f : \mathbb{Z}\{\widetilde{\mathsf{C}}^t_{(i)} \mid i \in I_0\}/\mathbb{Z}^n \to \ker(\overline{\mathsf{C}}^t)$ as follows: For $\mathbf{m} = (m_1, m_2, \dots, m_n)^t \in \mathbb{Z}^n$ (so, $\widetilde{\mathsf{C}}^t \cdot \mathbf{m} + \mathbb{Z}^n \in \mathbb{Z}\{\widetilde{\mathsf{C}}^t_{(i)} \mid i \in I_0\}/\mathbb{Z}^n$), we define

$$f\left(\widetilde{\mathsf{C}}^t \cdot \mathbf{m} + \mathbb{Z}^n\right) = \mathrm{red}_{\mathsf{N}}\left(\mathsf{N}\widetilde{\mathsf{C}}^t \cdot \mathbf{m}\right) \in (\mathbb{Z}_{\mathsf{N}})^n,$$

Since $\mathsf{C}^t\left(\mathsf{N}\widetilde{\mathsf{C}}^t \cdot \mathbf{m}\right) \in (\mathsf{N}\mathbb{Z})^n$, f is well-defined. Also, by definition, f is a group homomorphism.

Next, assume that for $\mathbf{m}, \mathbf{m}' \in \mathbb{Z}^n$,

$$f\left(\widetilde{\mathsf{C}}^t \cdot \mathbf{m} + \mathbb{Z}^n\right) = f\left(\widetilde{\mathsf{C}}^t \cdot \mathbf{m}' + \mathbb{Z}^n\right) \in (\mathbb{Z}_{\mathsf{N}})^n.$$

Then

$$\mathsf{N}\left(\widetilde{\mathsf{C}}^t \cdot (\mathbf{m} - \mathbf{m}')\right) \in (\mathsf{N}\mathbb{Z})^n,$$

which implies that $\widetilde{\mathsf{C}}^t \cdot (\mathbf{m} - \mathbf{m}') \in \mathbb{Z}^n$. Hence f is injective.

For the surjectivity, take any $\overline{\mathbf{x}} = (x_1, x_2, \dots, x_n) \in \ker(\overline{\mathsf{C}}^t)$. Then $\mathbf{m} = \dfrac{1}{\mathsf{N}}\mathsf{C}^t \cdot \overline{\mathbf{x}} \in \mathbb{Z}^n$ by the definition of $\ker(\overline{\mathsf{C}}^t)$. Thus we have

$$f\left(\widetilde{\mathsf{C}}^t \cdot \mathbf{m} + \mathbb{Z}^n\right) = \overline{\mathbf{x}}.$$

□

Convention 2.8. If there is a danger of confusion, we will use $\mathrm{red}_{\mathsf{N}}(\mathbf{x})$ and $\mathrm{red}_{\mathsf{N}}(\mathsf{C})$ instead of $\overline{\mathbf{x}}$ and $\overline{\mathsf{C}}$ to emphasize the modulo N.

Theorem 2.9. *Let $\mathfrak{g}$ be an affine Kac-Moody algebra of rank $n \in \mathbb{Z}_{>0}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$. For each $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we have*

$$P^+_{\mathrm{cl},\ell}(\Lambda) = \left\{\Lambda' \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathrm{red}_{\mathsf{N}}([\overline{\Lambda'}]_{\varpi}) \in \mathrm{red}_{\mathsf{N}}([\overline{\Lambda}]_{\varpi}) + \ker(\overline{\mathsf{C}}^t)^{\perp}\right\}, \tag{2.10}$$

where

$$\ker(\overline{\mathsf{C}}^t)^{\perp} := \left\{\mathbf{x} \in \mathbb{Z}^n_{\mathsf{N}} \;\middle|\; \mathbf{x} \bullet \mathbf{y} \equiv_{\mathsf{N}} 0 \text{ for all } \ \mathbf{y} \in \ker(\overline{\mathsf{C}}^t)\right\}.$$

Here $\bullet$ is the dot product on $\mathbb{Z}^n_{\mathsf{N}}$.

Proof. The assertion follows from Lemma 2.6 together with Lemma 2.7. □

For a subset $S \subset \mathbb{Z}^n$, set $\mathrm{red}_{\mathsf{N}}(S) := \{\overline{\mathbf{s}} \subset (\mathbb{Z}_{\mathsf{N}})^n \mid \mathbf{s} \in S\}$. Motivated by (2.9), we introduce the following definition.

Definition 2.10. Let $\mathfrak{g}$ be an affine Kac-Moody algebra. We call a subset $S \subset \mathbb{Z}^n$ a *root-sieving set* if, for all $\mathbf{x} \in P_0$,

(1) $\mathbf{x} \in Q_0$ if and only if $\mathbf{s} \bullet [\mathbf{x}]_{\varpi} \equiv_{\mathsf{N}} 0$ for all $\mathbf{s} \in S$,
(2) the set $\mathrm{red}_{\mathsf{N}}(S) \subset (\mathbb{Z}_{\mathsf{N}})^n$ is $\mathbb{Z}_{\mathsf{N}}$- linearly independent, and
(3) $|\mathrm{red}_{\mathsf{N}}(S)| = |S|$.

An element in S is called a *root-sieving vector* of S.

For instance, by (2.9), we have an example of a root-sieving set:

$$\begin{cases} \{s^{(1)}, s^{(2)}\} = \{\mathrm{adj}(\mathsf{C})_{(i_1)}, \mathrm{adj}(\mathsf{C})_{(i_2)}\} & \text{when } \mathfrak{g} = D^{(1)}_n \text{ (for even } n), \\ \{s\} = \{\mathrm{adj}(\mathsf{C})_{(i_1)}\} & \text{otherwise,} \end{cases}$$

Let $\Lambda = \sum_{0 \leqslant i \leqslant n} p_i\Lambda_i, \Lambda' = \sum_{0 \leqslant i \leqslant n} p'_i\Lambda_i \in P^+_{\mathrm{cl},\ell}$. Combining Lemma 2.3 with Definition 2.10, we can deduce the following characterization on the sieving equivalence relation $\sim$:

$$\Lambda \sim \Lambda' \quad \text{if and only if} \quad \mathbf{s} \bullet (p_1, p_2, \dots, p_n) \equiv_{\mathsf{N}} \mathbf{s} \bullet (p'_1, p'_2, \dots, p'_n) \quad \text{for all } \mathbf{s} \in S. \tag{2.11}$$

Here, $\bullet$ is the dot product on $\mathbb{Z}^n$.

Example 2.11. Let $\mathfrak{g} = A_3^{(1)}$. Then

$$\mathsf{C} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad \widetilde{\mathsf{C}} = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{bmatrix} \quad \text{and} \quad \mathrm{adj}(\mathsf{C}) = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}.$$

Let $i_1 = 3$. Note that

$$2\widetilde{\mathsf{C}}_{(3)} + \mathbb{Z}^3 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{3}{2} \end{bmatrix} + \mathbb{Z}^3 = \widetilde{\mathsf{C}}_{(2)} + \mathbb{Z}^3$$

and

$$3\widetilde{\mathsf{C}}_{(3)} + \mathbb{Z}^3 = \begin{bmatrix} \frac{3}{4} & \frac{3}{2} & \frac{9}{4} \end{bmatrix} + \mathbb{Z}^3 = \widetilde{\mathsf{C}}_{(1)} + \mathbb{Z}^3.$$

That is, for $\mathbf{x} \in P_0$, $\widetilde{\mathsf{C}}_{(3)}[\mathbf{x}]_\varpi \in \mathbb{Z}$ if and only if $\widetilde{\mathsf{C}}_{(i)}[\mathbf{x}]_\varpi \in \mathbb{Z}$ for all $i = 1, 2, 3$. It means that

$$\mathbf{x} \in Q_0 \quad \text{if and only if} \quad 4\left(\widetilde{\mathsf{C}}_{(3)}\right)^t \bullet [\mathbf{x}]_\varpi \equiv_4 0.$$

Thus $\left\{4\left(\widetilde{\mathsf{C}}_{(3)}\right)^t\right\} = \left\{\left(\mathrm{adj}(\mathsf{C})_{(3)}\right)^t\right\} = \left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\}$ is a root sieving set.

In the rest of this subsection, we classify all root sieving sets up to modulo N.

Lemma 2.12. *Let $S \subset \mathbb{Z}^n$ with $|S| = |\mathrm{red}_\mathsf{N}(S)|$. Then S is a root sieving set if and only if $\mathrm{red}_\mathsf{N}(S)$ is a $\mathbb{Z}_\mathsf{N}$-basis of $\ker(\overline{\mathsf{C}}^t)$.*

Proof. (a) Suppose that $\mathrm{red}_\mathsf{N}(S)$ is a $\mathbb{Z}_\mathsf{N}$-basis of $\ker(\overline{\mathsf{C}}^t)$. Since $\ker(\overline{\mathsf{C}}^t) \subset (\mathbb{Z}_\mathsf{N})^n$, $\mathrm{red}_\mathsf{N}(S)$ should be $\mathbb{Z}_\mathsf{N}$-linearly independent. Therefore, it suffices to show that S satisfies the condition (1) in Definition 2.10.

We first show that if $\mathbf{x} \in Q_0$, then we have $[\mathbf{x}]_\varpi \bullet \mathbf{s} \equiv_\mathsf{N} 0$ for all $s \in S$. Take $\mathbf{x} = \sum_{1 \leqslant i \leqslant n} t_i \overline{\alpha}_i \in Q_0$. Since $\mathrm{red}_\mathsf{N}(S) \subset \ker(\overline{\mathsf{C}}^t)$,

$$[\mathbf{x}]_\varpi \bullet s = \sum_{1 \leqslant i \leqslant n} t_i [\overline{\alpha}_i]_\varpi \bullet s = \sum_{1 \leqslant i \leqslant n} t_i (\mathsf{C}^t)_{(i)} \mathbf{s} \equiv_\mathsf{N} 0, \quad \text{for all } \mathbf{s} \in S.$$

Next, we assume that there is $\mathbf{x} \notin Q_0$ satisfying $\mathbf{s} \bullet [\mathbf{x}]_\varpi \equiv_\mathsf{N} 0$ for all $\mathbf{s} \in S$. Since $\mathbf{x} \notin Q_0$, we have

$$\mathrm{adj}(\mathsf{C})_{(i_{k'})} \bullet [\mathbf{x}]_\varpi \not\equiv_\mathsf{N} 0 \quad \text{for } k' = 1 \text{ or } k' = 1, 2 \text{ up to types},$$

by Lemma 2.6. However, since $\mathbf{s} \bullet [\mathbf{x}]_\varpi \equiv_\mathsf{N} 0$ for $\mathbf{s} \in S$, there are no $t_s \in \mathbb{Z}$ such that $\sum_{\mathbf{s} \in S} t_\mathbf{s} \mathbf{s} \equiv_\mathsf{N} \mathrm{adj}(\mathsf{C})_{(i_{k'})}$. Since $\mathrm{adj}(\mathsf{C})_{(i_k)} \in \ker(\overline{\mathsf{C}}^t)$, it contradicts to the assumption that $\mathrm{red}_\mathsf{N}(S)$ is a $\mathbb{Z}_\mathsf{N}$-basis of $\ker(\overline{\mathsf{C}}^t)$.

(b) Suppose that S is a root sieving set. By definition, $|S| = |\mathrm{red}_\mathsf{N}(S)|$, $\mathrm{red}_\mathsf{N}(S) \subset \ker(\overline{\mathsf{C}}^t)$, and $\mathrm{red}_\mathsf{N}(S)$ is $\mathbb{Z}_\mathsf{N}$-linearly independent. Therefore, we have to show that the $\mathbb{Z}_\mathsf{N}$-span of $\mathrm{red}_\mathsf{N}(S)$ equals $\ker(\overline{\mathsf{C}}^t)$.

Note that

$$P_0/Q_0 = \mathbb{Z}\{\varpi_{j_1} + Q_0\} \text{ or } \mathbb{Z}\{\varpi_{j_1} + Q_0, \varpi_{j_2} + Q_0\} \text{ up to types.}$$

Therefore, for any $\mathbf{y} = (y_1, y_2, \ldots, y_n)^t \in \ker(\overline{\mathsf{C}}^t)$ and $\Lambda \in P$, $(\mathbf{y} \bullet [\mathbf{x}]_\varpi)$ is determined by $\mathbf{y} \bullet [\varpi_{j_k}]_\varpi = y_{j_k}$, that is, $\mathbf{y}$ is determined by y_{j_1} (resp. y_{j_1} and y_{j_2}). Now it suffices to show that for any $\mathbf{y} \in \ker(\overline{\mathsf{C}}^t)$, there are $\mathbb{Z}_\mathsf{N}$-solutions for the following equation:

$$\mathbf{y} = x\,\mathbf{s} \quad \text{or} \quad \mathbf{y} = x^{(1)}\,\mathbf{s}^{(1)} + x^{(2)}\,\mathbf{s}^{(2)} \tag{2.12}$$

Here $\mathrm{red}_\mathsf{N}(S) = \{\mathbf{s}\}$ (resp. $\{\mathbf{s}^{(1)}, \mathbf{s}^{(2)}\}$). Since $\mathbf{y}$ is determined by y_{j_1} (resp. y_{j_1} and y_{j_2}), the linearly independence of $\mathrm{red}_\mathsf{N}(S)$ implies the existence of the solution to (2.12). □

Lemma 2.12 implies that there are finitely many root sieving sets for each type up to modulo N. Combining Lemma 2.7 with Table 2.1, we can complete the classification of root sieving sets up to modulo N, which is presented in the table below.

Type	Root sieving sets up to modulo N
$A_n^{(1)}$	$\{k(1,2,\dots,n)\}$, for $(k,n+1)=1$
$B_n^{(1)}, D_{n+1}^{(2)}$	$\{(0,0,\dots,0,1)\}$
$C_n^{(1)}, A_{2n-1}^{(2)}$	$\{(\delta(j\equiv_2 1))_{j=0,1,\dots,n}\}$
$D_n^{(1)} (n\equiv_2 0)$	$\{(0,0,\dots,0,2,2),(2,0,2,0,\dots,2,0,2,0)\}$, $\{(0,0,\dots,0,2,2),(2,0,2,0,\dots,2,0,0,2)\}$, $\{(2,0,2,0,\dots,2,0,2,0),(2,0,2,0,\dots,2,0,0,2)\}$
$D_n^{(1)} (n\equiv_2 1)$	$\{k(2,0,2,0,\dots,0,2,1,3)\}$, for $k=1,3$
$E_6^{(1)}$	$\{k(1,0,2,0,1,2)\}$, for $k=1,2$
$E_7^{(1)}$	$\{(0,1,0,0,1,0,1)\}$
Remaining types	$\{(0,0,\dots,0)\}$

TABLE 2.4. Root sieving sets up to modulo N

Convention 2.13.

(1) From now on, we choose a special root sieving set, denoted by $\boldsymbol{S}$, as follows:

$$\boldsymbol{S}=\begin{cases}\{\boldsymbol{s}=(1,2,\dots,n)\} & \text{if } \mathfrak{g}=A_n^{(1)},\\ \{\boldsymbol{s}=(1,0,2,0,1,2)\} & \text{if } \mathfrak{g}=E_6^{(1)},\\ \{\boldsymbol{s}=(2,0,2,0,\dots,0,2,1,3)\} & \text{if } \mathfrak{g}=D_n^{(1)} \text{ and } n\equiv_2 1,\\ \{\boldsymbol{s}^{(1)}=(0,0,\dots,0,2,2),\ \boldsymbol{s}^{(2)}=(2,0,2,0,\dots,2,0,2,0)\} & \text{if } \mathfrak{g}=D_n^{(1)} \text{ and } n\equiv_2 0.\end{cases}$$

For the other types, we choose $\boldsymbol{S}$ as in Table 2.4.

(2) For a root sieving vector $\boldsymbol{s}=(s_1,s_2,\dots,s_n)$, we denote $(0,s_1,s_2,\dots,s_n)$ by $\widetilde{\boldsymbol{s}}$.

With the root sieving sets $\boldsymbol{S}$ given in Convention 2.13, we define a new statistics $\mathsf{ev}_{\boldsymbol{s}}$, called the $\boldsymbol{S}$-*evaluation*,

$$\mathsf{ev}_{\boldsymbol{s}}: P^+_{\mathrm{cl},\ell}\to\mathbb{Z}^k_{\geqslant 0},\quad \sum_{0\leqslant i\leqslant n} m_i\Lambda_i\mapsto\left(\widetilde{\boldsymbol{s}}^{(k)}\bullet\mathbf{m}\right)_{k=1\text{ or }1,2}. \tag{2.13}$$

Here, $\widetilde{\boldsymbol{s}}^{(1)}=\widetilde{\boldsymbol{s}}$ in cases except for $D_n^{(1)}$ $(n\equiv_2 0)$ and $\mathbf{m}=(m_0,m_1,\dots,m_n)$. For $\Lambda\in P^+_{\mathrm{cl},\ell}$, we call $\mathsf{ev}_{\boldsymbol{s}}(\Lambda)$ the $\boldsymbol{S}$-*evaluation* of Λ. For later use, we list $\mathsf{ev}_{\boldsymbol{s}}(\Lambda)$ for all $\Lambda=(\ell-1)\Lambda_0+\Lambda_i\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})$ in Table 2.5.

	$A_n^{(1)}, C_n^{(1)}, A_{2n-1}^{(2)}$	$B_n^{(1)}, D_{n+1}^{(2)}, E_7^{(1)}$	$D_n^{(1)}$	$E_6^{(1)}$
$\mathsf{ev}_{\boldsymbol{s}}(\Lambda)$	i	δ_{in}	$(2(\delta_{i,n-1}+\delta_{i,n}),2(\delta_{i,1}+\delta_{i,n-1}))$ $(n\equiv_2 0)$, $2\delta_{i,1}+\delta_{i,n-1}+3\delta_{i,n}$ $(n\equiv_2 1)$	$\delta_{i,n}+2\delta_{i,6}$
	For the remaining types, $\mathsf{ev}_{\boldsymbol{s}}(\Lambda)=0$			

TABLE 2.5. $\mathsf{ev}_{\boldsymbol{s}}((\ell-1)\Lambda_0+\Lambda_i)$ for each type

The following theorem follows from (2.11).

Theorem 2.14. *Let $\boldsymbol{S}$ be the root sieving set given in Convention 2.13. For any $\Lambda=(\ell-1)\Lambda_0+\Lambda_i\in\mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we have*

$$P^+_{\mathrm{cl},\ell}(\Lambda)=\left\{\Lambda'\in P^+_{\mathrm{cl},\ell}\ \middle|\ \mathsf{ev}_{\boldsymbol{s}}(\Lambda')\equiv_{\mathsf{N}}\mathsf{ev}_{\boldsymbol{s}}(\Lambda)\right\}. \tag{2.14}$$

Example 2.15. Let $\mathfrak{g} = A_3^{(1)}$ and $\ell = 2$. In this case, $a_i^\vee = 1$, $\boldsymbol{s} = (1,2,3)$ and $\mathsf{ev}_{\boldsymbol{s}}((\ell-1)\Lambda_0 + \Lambda_i) = i$ for $i = 0,1,2,3$. Then

$$\begin{aligned} P^+_{\mathrm{cl},2} &= \left\{ \sum_{0 \leqslant i \leqslant 3} m_i \Lambda_i \in P^+_2 \;\middle|\; \sum_{0 \leqslant j \leqslant 3} m_j = 2 \right\} \\ &= \{2\Lambda_0, 2\Lambda_1, 2\Lambda_2, 2\Lambda_3, \Lambda_0+\Lambda_1, \Lambda_0+\Lambda_2, \Lambda_0+\Lambda_3, \Lambda_1+\Lambda_2, \Lambda_1+\Lambda_3, \Lambda_2+\Lambda_3\} \end{aligned}$$

and, for each $i = 0,1,2,3$,

$$P^+_{\mathrm{cl},2}(\Lambda_0+\Lambda_i) = \left\{ \sum_{0 \leqslant i \leqslant 3} m_i \Lambda_i \in P^+_2 \;\middle|\; \sum_{0 \leqslant j \leqslant 3} m_j = 2 \text{ and } \sum_{0 \leqslant j \leqslant 3} j m_j \equiv_4 i \right\}.$$

For instance,

$$P^+_{\mathrm{cl},2}(2\Lambda_0) = \{2\Lambda_0, 2\Lambda_2, \Lambda_1+\Lambda_3\}.$$

Remark 2.16. Even in case where $\mathfrak{g} = A_{2n}^{(2)}$, we can define the sieving equivalence relation as in (2.5). In this case, there is only one equivalence class and hence we may define the distinguished representative $\mathtt{DR}(P^+_{\mathrm{cl},\ell})$ as $\{\ell\Lambda_0\}$. Then we have the same bijection described in (2.6), which implies that for any $\Lambda \in P^+_\ell$,

$$|\ell\mathcal{C}_{\mathrm{af}} \cap (\overline{\Lambda} + \overline{Q})| = |P^+_{\mathrm{cl},\ell}|.$$

3. Sagan's action and generalization

From this section, we will investigate the structure and enumeration of $P^+_{\mathrm{cl},\ell}(\Lambda)$ for all $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$ in a viewpoint of (bi)cyclic sieving phenomena ([15]). In order to do this, we give a suitable (bi)cyclic group action on $P^+_{\mathrm{cl},\ell}$. This will be achieved by generalizing Sagan's action in [18] under consideration on our results in the previous sections.

For each positive integer m, we fix a cyclic group C_m of order m and a generator σ_m of C_m. Note that every C_m-action is completely determined by the action of σ_m.

In [18, §2], Sagan introduced an interesting cyclic group action on sets consisting of $(0,1)$-words. Here we provide a generalized version of this action, which will play a key role in our demonstration of cyclic sieving phenomena associated with dominant maximal weights. To do this, we first recall Sagan's action.

Let

$$\mathcal{W}_{n,\ell} := \left\{ \mathbf{w} = w_1 w_2 \cdots w_{n+\ell} \;\middle|\; w_i = 0,1 \quad \text{for } i = 1,2,\ldots,n+\ell, \text{ and } \sum_{1 \leqslant i \leqslant n+\ell} w_i = \ell \right\}, \tag{3.1}$$

which is in one to one correspondence with $P^+_{\mathrm{cl},\ell}$ of type $A_n^{(1)}$ via

$$\sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \mapsto \underbrace{11\cdots1}_{m_0}0\underbrace{11\cdots1}_{m_1}0\cdots\cdots0\underbrace{11\ldots1}_{m_n}. \tag{3.2}$$

For any $d \in \mathbb{Z}_{\geqslant 1}$, we define a $C_d = \langle \sigma_d \rangle$-action on $\mathcal{W}_{n,\ell}$ as follows: Given a $(0,1)$-word $\mathbf{w} = w_1 w_2 \cdots w_{n+\ell} \in \mathcal{W}_{n,\ell}$, break it into subwords of length d,

$$\begin{aligned} \mathbf{w} &= w_1 w_2 \cdots w_d \mid w_{d+1} w_{d+2} \cdots w_{2d} \mid \cdots \mid w_{(t-1)d+1} w_{(t-1)d+2} \cdots w_{td} \mid w_{td+1} \cdots w_{n+\ell} \\ &= w^1 \mid w^2 \mid \cdots \mid w^t \mid w^0, \end{aligned}$$

where $t = \left\lfloor \frac{n+\ell}{d} \right\rfloor$,

$$w^j := w_{(j-1)d+1} w_{(j-1)d+2} \cdots w_{jd} \quad \text{for } 1 \leqslant j \leqslant t, \quad \text{and} \quad w^0 := w_{td+1} \cdots w_{n+\ell}.$$

Note that C_d acts on each subword w^j by cyclic shift:

$$\sigma_d \cdot w^j := w_{jd}, w_{(j-1)d+1} w_{(j-1)d+2} \cdots w_{jd-1}.$$

Assume that j_0 is the smallest integer such that $\sigma_d \cdot w^{j_0} \neq w^{j_0}$. Then Sagan's action $\blacksquare$ is defined by

$$\sigma_d \blacksquare \mathbf{w} := w^1 \mid w^2 \mid \cdots \mid w^{j_0-1} \mid \sigma_d \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w^0.$$

If there is no such j_0 in {1,2,…,t}, set $\sigma_d \blacksquare \mathbf{w} := \mathbf{w}$.

Example 3.1. Note that

$$\mathcal{W}_{3,2} = \{11000,\ 01100,\ 00110,\ 10010,\quad 10001,\ 01001,\ 00101,\ 00011,\quad 10100,\ 01010\}.$$

Under the above C_4-action on $\mathcal{W}_{3,2}$, we have three orbits given by

$$\{1100|0,\ 0110|0,\ 0011|0,\ 1001|0\},\quad \{1000|1,\ 0100|1,\ 0010|1,\ 0001|1\},\quad \{1010|0,\ 0101|0\}.$$

Via the correspondence in (3.2), we can transport Sagan's actions on $\mathcal{W}_{n,\ell}$ to $P^+_{\mathrm{cl},\ell}$ of type $A^{(1)}_n$. In the following, we will generalize this approach to other types. Although our setting is more general, basically we construct a set in bijection with $P^+_{\mathrm{cl},\ell}$ and define cyclic group actions on it by mimicking Sagan's actions.

In this section, we assume that d, k are positive integers and ℓ is a nonnegative integer. Given a kd-tuple $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbb{Z}^{kd}_{\geqslant 0}$, we set

$$\mathbf{m}[j;d] := \sum_{0 \leqslant t \leqslant d-1} m_{jd+t} \quad (0 \leqslant j \leqslant k-1).$$

Also, given a k-tuple $\boldsymbol{\nu} = (\nu_0, \nu_1, \dots, \nu_{k-1}) \in \mathbb{Z}^k_{>0}$, we set

$$\mathbf{M}_\ell(d;\boldsymbol{\nu}) := \left\{ \mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbb{Z}^{kd}_{\geqslant 0} \;\middle|\; \sum_{0 \leqslant j \leqslant k-1} \nu_j \mathbf{m}[j;d] = \ell \right\}. \tag{3.3}$$

In particular, if $d = 1$ then $\mathbf{m}[j;1] = m_j$ and

$$\mathbf{M}_\ell(1;\boldsymbol{\nu}) = \left\{ \mathbf{m} = (m_0, m_1, \dots, m_{k-1}) \in \mathbb{Z}^{k}_{\geqslant 0} \;\middle|\; \boldsymbol{\nu} \bullet \mathbf{m} = \ell \right\}. \tag{3.4}$$

To each $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$ we associate a word

$$\mathbf{w}(\mathbf{m};d;\boldsymbol{\nu}) := w_1 w_2 \cdots w_{u_\mathbf{m}}$$

with entries in $\{0, \nu_0, \nu_1, \dots, \nu_{k-1}\}$ defined by the following algorithm:

Algorithm 3.2. (Algorithm for $\mathbf{w}(\mathbf{m};d;\boldsymbol{\nu})$) Assume we have a kd-tuple $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$.

- (**A**1) Set $\mathbf{w}$ to be the empty word and $j = 0, t = 0$. Go to (**A**2).
- (**A**2) Set $\mathbf{w}$ to be the word obtained by concatenating m_{jd+t} ν_j's at the right of $\mathbf{w}$. If $j = k-1$ and $t = d-1$, return $\mathbf{w}$ and terminate the algorithm. Otherwise, go to (**A**3).
- (**A**3) Set $\mathbf{w}$ to be the word obtained by concatenating 0 at the right. Go to (**A**4).
- (**A**4) If $t \neq d-1$ then set $t = t+1$ and go to (**A**2). If $t = d-1$ set $j = j+1$ and $t = 0$, and go to (**A**2).

As seen in Algorithm 3.2, the length $u_\mathbf{m}$ determined by $\mathbf{m}$ and the formula for $u_\mathbf{m}$ is given as follows:

$$u_\mathbf{m} = \left(\sum_{0 \leqslant j \leqslant kd-1} m_j \right) + (kd-1).$$

For $\mathbf{m}, \mathbf{m}' \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$, the lengths $u_\mathbf{m}$ and $u_{\mathbf{m}'}$ are not necessarily equal to each other. On the contrary, for all $\mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$, the number of 0's in $\mathbf{w}(\mathbf{m};d;\boldsymbol{\nu})$ is uniquely determined by $kd-1$ (see Example 3.4 below).

Set

$$\mathcal{W}_\ell(d;\boldsymbol{\nu}) := \{\mathbf{w}(\mathbf{m};d;\boldsymbol{\nu}) \mid \mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu})\},$$

which can be viewed as a generalization of $\mathcal{W}_{n,\ell}$ since $\mathcal{W}_\ell(1;\boldsymbol{\nu})$ recovers $\mathcal{W}_{n,\ell}$ when $k = n+1$ and $\boldsymbol{\nu} = (1, 1, \dots, 1) \in \mathbb{Z}^k$.

Lemma 3.3. *The map*

$$\Psi : \mathbf{M}_\ell(d;\boldsymbol{\nu}) \to \mathcal{W}_\ell(d;\boldsymbol{\nu}), \quad \mathbf{m} \mapsto \mathbf{w}(\mathbf{m};d;\boldsymbol{\nu}) \tag{3.5}$$

is injective and hence bijective.

Proof. For each $\mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$, we have to apply (**A**3) $(kd-1)$-times to obtain $\mathbf{w}(\mathbf{m};d;\boldsymbol{\nu})$ via Algorithm 3.2. This says that every word $\mathbf{w} \in \mathcal{W}_\ell(d;\boldsymbol{\nu})$ contains exactly $(kd-1)$-zero.

Define a map $\Psi^{-1} : \mathcal{W}_\ell(d;\boldsymbol{\nu}) \to \mathbf{M}_\ell(d;\boldsymbol{\nu})$ as follows: Let $\mathbf{w} \in \mathcal{W}_\ell(d;\boldsymbol{\nu})$. For each $1 \leqslant i \leqslant kd-1$, let z_i denote the position of the ith zero when we read $\mathbf{w}$ from left to right, and we set $z_0 := 0$ and $z_{kd} := u_{\mathbf{m}} + 1$. For each $0 \leqslant j \leqslant kd-1$, let $m_i = z_{i+1} - z_i - 1$. Define

$$\Psi^{-1}(\mathbf{w}) = (m_0, m_1, \dots, m_{kd-1}).$$

Recall that, for each $\mathbf{m} = (m_0, m_1, \dots, m_{kd-1}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$, $\mathbf{w}(\mathbf{m}; d; \boldsymbol{\nu})$ is obtained by applying Algorithm 3.2 to $\mathbf{m}$, which shows that there are exactly m_i nonzero entries between the ith 0 and the $(i+1)$st 0 when we read $\mathbf{w}(\mathbf{m}; d; \boldsymbol{\nu})$ from left to right for $0 \leqslant i \leqslant kd-1$. Here the 0th and kdth 0's are set to be the empty word (see Table 3.1 for details). Obviously it holds that

$$\Psi^{-1} \circ \Psi(\mathbf{m}) = \mathbf{m} \quad \text{for each} \;\; \mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$$

and hence Ψ is injective. □

Example 3.4. Let $d = 2, k = 2, \ell = 4$, and $\boldsymbol{\nu} = (1, 2)$. Then

$$\begin{aligned}\mathbf{M}_4(2;(1,2)) &= \left\{(m_0,m_1,m_2,m_3) \in \mathbb{Z}^4_{\geqslant 0} \,\middle|\, m_0+m_1+2m_2+2m_3 = 4\right\} \\ &= \left\{\begin{array}{lll} (4,0,0,0), & (2,0,1,0), & (0,0,2,0), \\ (3,1,0,0), & (2,0,0,1), & (0,0,1,1), \\ (2,2,0,0), & (1,1,1,0), & (0,0,0,2), \\ (1,3,0,0), & (1,1,0,1), & \\ (0,4,0,0), & (0,2,1,0), & \\ & (0,2,0,1) & \end{array}\right\}.\end{aligned}$$

Using Algorithm 3.2, one can obtain $\Psi(\mathbf{m})$ for each $\mathbf{m} \in \mathbf{M}_4(2;(1,2))$ as follows:

$$\mathcal{W}_4(2;(1,2)) = \left\{\begin{array}{lll} \Psi((4,0,0,0)) = 1111000, & \Psi((2,0,1,0)) = 110020, & \Psi((0,0,2,0)) = 00220, \\ \Psi((3,1,0,0)) = 1110100, & \Psi((2,0,0,1)) = 110002, & \Psi((0,0,1,1)) = 00202, \\ \Psi((2,2,0,0)) = 1101100, & \Psi((1,1,1,0)) = 101020, & \Psi((0,0,0,2)) = 00022, \\ \Psi((1,3,0,0)) = 1011100, & \Psi((1,1,0,1)) = 101002, & \\ \Psi((0,4,0,0)) = 0111100, & \Psi((0,2,1,0)) = 011020, & \\ & \Psi((0,2,0,1)) = 011002, & \end{array}\right\}.$$

In particular, $\Psi((2,0,1,0)) = 110020$ can be computed as follows:

	(**A**1)	(**A**2)	(**A**3)	(**A**4)	(**A**2)	(**A**3)	(**A**4)	(**A**2)	(**A**3)	(**A**4)	(**A**2)
$\mathbf{w}$	$\varnothing$	11	110	110	110	1100	1100	11002	110020	110020	110020
j	0	0	0	0	0	0	1	1	1	1	1
t	0	0	0	1	1	1	0	0	0	1	1

TABLE 3.1. The process of obtaining $\Psi((2,0,1,0))$ by Algorithm 3.2

Remark 3.5.

(1) There are five words of length 7, six words of length 6, and three words of length 5 in $\mathcal{W}_4(2;(1,2))$. This shows that the lengths of $\mathbf{m}$'s may be different.

(2) The set $\mathcal{W}_\ell(d;\boldsymbol{\nu})$ may be complicated to some extent. The definition in (3.1) implies that all words of length $n+\ell$ consist of n 0's and ℓ 1's are in $\mathcal{W}_{n,\ell}$, but which fails to characterize $\mathcal{W}_\ell(d;\boldsymbol{\nu})$. For instance, although 110200, 110020 and 110002 have the same number of i's ($i = 0, 1, 2$), Example 3.4 shows that

$$110020, 110002 \in \mathcal{W}_4(2;(1,2)) \quad \text{but} \quad 110200 \notin \mathcal{W}_4(2;(1,2)).$$

Now we define a $C_d = \langle \sigma_d \rangle$-action on $\mathcal{W}_\ell(d;\boldsymbol{\nu})$. First, we break $\mathbf{w} = w_1 w_2 \dots w_u$ into subwords of length d as many as possible in order as follows:

$$\begin{aligned}\mathbf{w} &= w_1 w_2 \cdots w_d \mid w_{d+1} w_{d+2} \cdots w_{2d} \mid \cdots \mid w_{(k-1)d+1} w_{(k-1)d+2} \cdots w_{td} \mid w_{td+1} \cdots w_u \\ &= w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{td+1} \cdots w_u,\end{aligned} \tag{3.6}$$

where $t = \lfloor u/d \rfloor$ and $w^j = w_{(j-1)d+1}w_{(j-1)d+2}\cdots w_{jd}$ for $1 \leqslant j \leqslant t$. Note that σ_d acts on each subword w^j by cyclic shift, i.e.,

$$\sigma_d \cdot w^j := w_{jd}w_{(j-1)d+1}w_{(j-1)d+2}\cdots w_{jd-1}.$$

Assume that j_0 is the smallest integer such that $\sigma_d \cdot w_0^j \neq w_0^j$. Then we set

$$\sigma_d \blacksquare \mathbf{w} := w^1 \mid w^2 \mid \cdots \mid w^{j_0-1} \mid \sigma_d \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{td+1}\cdots w_u. \tag{3.7}$$

If there is no such j_0, we set $\sigma_d \blacksquare \mathbf{w} := \mathbf{w}$.

Theorem 3.6. *For any $\boldsymbol{\nu} = (\nu_0, \nu_1, \ldots, \nu_{k-1}) \in \mathbb{Z}^k_{>0}$, the action defined as above is indeed a C_d-action on $\mathcal{W}_\ell(d;\boldsymbol{\nu})$.*

Proof. From the definition in (3.7), one can see that $e \blacksquare \mathbf{w} = \underbrace{(\sigma_d \blacksquare (\sigma_d \blacksquare \cdots \blacksquare (\sigma_d \blacksquare \mathbf{w})\cdots))}_{d} = \mathbf{w}$ for all $\mathbf{w} \in \mathcal{W}_\ell(d;\nu)$. Therefore, our assertion can be justified by showing that $\mathcal{W}_\ell(d;\nu)$ is closed under the action of σ_d. To do this, for any $\mathbf{w} \in \mathcal{W}_\ell(d;\nu)$, we will find an element $\mathbf{m}' \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$ such that $\Psi(\mathbf{m}') = \sigma_d \blacksquare \mathbf{w}$.

Let $\mathbf{w} \in \mathcal{W}_\ell(d;\nu)$. We may assume that $\sigma_d \blacksquare \mathbf{w} \neq \mathbf{w}$. Break $\mathbf{w}$ into subwords

$$\begin{aligned}\mathbf{w} &= w_1w_2\cdots w_d \mid w_{d+1}w_{d+2}\cdots w_{2d} \mid \cdots \mid w_{(t-1)d+1}w_{(t-1)d+2}\cdots w_{td} \mid w_{td+1}\cdots w_u \\ &= w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{td+1}\cdots w_u\end{aligned}$$

as in (3.7). Since $\sigma_d \blacksquare \mathbf{w} \neq \mathbf{w}$, there exists an index $1 \leqslant j_0 \leqslant t$ such that

$$\sigma_d \blacksquare \mathbf{w} = w^1 \mid w^2 \mid \cdots \mid w^{j_0-1} \mid \sigma_d \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{td+1}\cdots w_u.$$

Note that for each $1 \leqslant j \leqslant j_0$, w^j consists of d 0's or d ν_r's for some $0 \leqslant r < k$. Thus, the number of zeros in $w^1w^2\cdots w^{j_0-1}$ is $s \times d$ for some $s \in \mathbb{Z}_{\geqslant 0}$. Moreover, from Algorithm 3.2 and $\sigma_d \blacksquare w^{j_0} \neq w^{j_0}$, we see that

(i) ν_{s+1} can appear in $\mathbf{w}$ after $(s+1)\times d$ zeros occurrence, and
(ii) w^{j_0} consists of z 0's and $(d-z)$ ν_s's for some $z \geqslant 1$.

By Lemma 3.3, we can write $\Psi^{-1}(\mathbf{w})$ as $\mathbf{m} = (m_0, m_1, \ldots, m_{kd-1})$. Now we shall construct a tuple $\mathbf{m}' = (m'_0, m'_1, \ldots, m'_{kd-1}) \in \mathbb{Z}^{kd}_{\geqslant 0}$ satisfying that $\mathbf{m}' = \Psi^{-1}(\sigma_d \blacksquare \mathbf{w}) \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$ in the following steps:

Step 1. Take $m'_i = m_i$ for $0 \leqslant i \leqslant sd-1$.
Step 2. Recall that z denotes the number of 0's in w^{j_0}. Take

$$\begin{cases} m'_{sd} = m_{sd}+1,\ m'_{sd+1} = m_{sd+1},\ m'_{sd+z} = m_{sd+z}-1, \quad \text{and} \quad m'_i = m_i & \text{if } w_{j_0d} = \nu_s, \\ m'_{sd} = m_{sd}-p,\ m'_{sd+1} = p,\ m'_{sd+z} = m_{sd+z-1}+m_{sd+z} \quad \text{and} \quad m'_i = m_{i-1} & \text{if } w_{j_0d} = 0, \end{cases} \tag{3.8}$$

for $sd+1 < i < sd+z$, where $w^{j_0} = \underbrace{\nu_{s+1}\nu_{s+1}\cdots\nu_{s+1}}_{p}0**\cdots*w_{j_0d}$.

Step 3. Take $m'_i = m_i$ for $sd+z+1 \leqslant i \leqslant kd-1$.

By the construction of $\mathbf{m}'$, we have

$$\sum_{0\leqslant j\leqslant k-1} \nu_j \mathbf{m}'[j;d] = \sum_{0\leqslant j\leqslant k-1} \nu_j \mathbf{m}[j;d] = \ell.$$

Thus, we have $\mathbf{m}' \in \mathbf{M}_\ell(d;\boldsymbol{\nu})$. Moreover, (3.8) implies $\Psi(\mathbf{m}') = \sigma_d \blacksquare \mathbf{w}$, by Algorithm 3.2. ☐

Now we define a C_d-action on $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ by transporting the C_d-action $\blacksquare$ on $\mathcal{W}_\ell(d;\boldsymbol{\nu})$ via the bijection Ψ, that is,

$$\sigma_d \blacksquare \mathbf{m} := \Psi^{-1}(\sigma_d \blacksquare \Psi(\mathbf{m})) \quad \text{for all } \mathbf{m} \in \mathbf{M}_\ell(d;\boldsymbol{\nu}). \tag{3.9}$$

Example 3.7. Let $d = 2, k = 2, \ell = 4$, and $\boldsymbol{\nu} = (1,2)$. For $\mathbf{m} = (3,1,0,0), \mathbf{m}' = (2,0,1,0), \mathbf{m}'' \in \mathbf{M}_4(2;(1,2))$, we have the following commutative diagrams:

$$\begin{tikzcd} \mathbf{m} = (3,1,0,0) \arrow[r, "\Psi"] \arrow[d, dotted, "\sigma_2"] & 11|10|10|0 \arrow[d, "\sigma_2"] & \mathbf{m}' = (1,1,1,0) \arrow[r, "\Psi"] \arrow[d, dotted, "\sigma_2"] & 10|10|20 \arrow[d, "\sigma_2"] & \mathbf{m}'' = (0,0,1,1) \arrow[r, "\Psi"] \arrow[d, dotted, "\sigma_2"] & 00|20|2 \arrow[d, "\sigma_2"] \\ \sigma_2 \blacksquare \mathbf{m} = (2,2,0,0) & 11|01|10|0, \arrow[l, "\Psi^{-1}"'] & \sigma_2 \blacksquare \mathbf{m}' = (0,2,1,0) & 01|10|20, \arrow[l, "\Psi^{-1}"'] & \sigma_2 \blacksquare \mathbf{m}'' = (0,0,0,2) & 00|02|2 \arrow[l, "\Psi^{-1}"'] \end{tikzcd}$$

Remark 3.8.

(1) Suppose that C_d acts on $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ as in (3.9). Then, for any $r \in \mathbb{Z}_{>0}$, $\mathbf{M}_\ell(d;\boldsymbol{\nu})$ is also equipped with a C_{rd}-action $\blacksquare_d$, which is given by

$$\sigma_{rd} \blacksquare_d \mathbf{m} := \sigma_d \blacksquare \mathbf{m}. \tag{3.10}$$

(2) In (3.10), if $d = 1$ then the C_r-action $\blacksquare_1$ on $\mathbf{M}_\ell(1;\boldsymbol{\nu})$ is trivial.

Let us generalize the above setting a little further. Let d, k, k' and r be positive integers and ℓ a nonnegative integer. For $\boldsymbol{\nu} = (\nu_0, \nu_1, \dots, \nu_{k-1}) \in \mathbb{Z}_{>0}^k$ and $\boldsymbol{\nu}' = (\nu'_0, \nu'_1, \dots, \nu'_{k'-1}) \in \mathbb{Z}_{>0}^{k'}$, set

$$\mathbf{M}_\ell(rd, d;\boldsymbol{\nu},\boldsymbol{\nu}') := \left\{ \mathbf{m} \in \mathbb{Z}_{\geqslant 0}^{krd+k'd} \;\middle|\; \sum_{0\leqslant j\leqslant k-1} \nu_j \mathbf{m}[j;rd] + \sum_{0\leqslant j\leqslant k'-1} \nu'_j \mathbf{m}[kr+j;d] = \ell \right\}. \tag{3.11}$$

Using the actions given in (3.7) and (3.10), we define a new C_{rd}-action, denoted by $\blacksquare_{rd,d}$, on $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$ as follows: Given $\mathbf{m} \in \mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$, we break it into $\mathbf{m}_{\leqslant krd-1} \in \mathbf{M}_l(rd;\boldsymbol{\nu})$ and $\mathbf{m}_{\geqslant krd} \in \mathbf{M}_{l'}(d;\boldsymbol{\nu}')$, where $\ell = l + l'$. Now, we define

$$\sigma_{rd} \blacksquare_{rd,d} \mathbf{m} := \begin{cases} (\sigma_{rd} \blacksquare \mathbf{m}_{\leqslant krd-1}) * \mathbf{m}_{\geqslant krd} & \text{if } \sigma_{rd} \blacksquare \mathbf{m}_{\leqslant krd-1} \neq \mathbf{m}_{\leqslant krd-1}, \\ \mathbf{m}_{\leqslant krd-1} * (\sigma_{rd} \blacksquare_d \mathbf{m}_{\geqslant krd}) & \text{otherwise.} \end{cases} \tag{3.12}$$

Example 3.9. Let $d = 2, k = 1, k' = 2, r = 2, \ell = 8, \boldsymbol{\nu} = (1)$, and $\boldsymbol{\nu}' = (1,2)$. Then

$$\mathbf{M}_8(4,2;(1),(1,2)) = \left\{ \mathbf{m} \in \mathbb{Z}_{\geqslant 0}^8 \;\middle|\; (m_0 + m_1 + m_2 + m_3) + (m_4 + m_5) + 2(m_6 + m_7) = 8 \right\},$$

where $\mathbf{m} = (m_0, m_1, m_2, m_3, m_4, m_5, m_6)$.

(1) For $\mathbf{m} = (6,0,0,0,1,1,0,0) \in \mathbf{M}_8(4,2;(1),(1,2))$, break $\mathbf{m}$ into

$$\mathbf{m}_{\leqslant 3} = (6,0,0,0) \in \mathbf{M}_6(4;(1)) \quad \text{and} \quad \mathbf{m}_{\geqslant 4} = (1,1,0,0) \in \mathbf{M}_2(2;(1,2)).$$

Since $\Psi((6,0,0,0)) = 1111|1100|0$, it follows that $\sigma_4 \blacksquare \Psi((6,0,0,0)) = 1111|0110|0$ and so

$$\sigma_4 \blacksquare (6,0,0,0) = (4,2,0,0).$$

Thus, (3.12) shows that

$$\sigma_4 \blacksquare_{4,2} \mathbf{m} = (\sigma_4 \blacksquare \mathbf{m}_{\leqslant 3}) * \mathbf{m}_{\geqslant 4} = (4,2,0,0 \mid 1,1,0,0).$$

(2) For $\mathbf{m} = (4,0,0,0,1,1,1,0) \in \mathbf{M}_8(4,2;(1),(1,2))$, break $\mathbf{m}$ into

$$\mathbf{m}_{\leqslant 3} = (4,0,0,0) \in \mathbf{M}_4(4;(1)) \quad \text{and} \quad \mathbf{m}_{\geqslant 4} = (1,1,1,0) \in \mathbf{M}_4(1;(1,2)).$$

Since $\Psi(\mathbf{m}_{\leqslant 3}) = 1111|000$, one can see that $\sigma_4 \blacksquare (4,0,0,0) = (4,0,0,0)$. In Example 3.7, we have already shown that $\sigma_2 \blacksquare \mathbf{m}_{\geqslant 4} = (0,2,1,0)$. Thus, by (3.12), we have

$$\sigma_4 \blacksquare_{4,2} \mathbf{m} = \mathbf{m}_{\leqslant 3} * (\sigma_2 \blacksquare \mathbf{m}_{\geqslant 4}) = (4,0,0,0 \mid 0,2,1,0).$$

Remark 3.10. The expression for $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$ may not be unique. For instance, $\mathbf{M}_\ell(4,2;(1),(2^k)) = \mathbf{M}_\ell(2,1;(1^2),(2^{2k}))$ $(k \in \mathbb{Z}_{\geqslant 1})$ as sets. They should be distinguished since the former has a C_4-action $\blacksquare_{4,2}$, while the latter has a C_2-action $\blacksquare_{2,1}$, which is different from $(\blacksquare_{4,2})^2$.

In the following, we will extend Ψ in (3.5) to $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$. The resulting map is denoted by $\widehat{\Psi}$.

Definition 3.11. Let $\mathbf{m} \in \mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$. Break $\mathbf{m}$ into $\mathbf{m}_{\leqslant krd-1} \in \mathbf{M}_l(rd;\boldsymbol{\nu})$ and $\mathbf{m}_{\geqslant krd} \in \mathbf{M}_{l'}(d;\boldsymbol{\nu}')$, where $\ell = l + l'$. Let $\mathbf{w}^{(1)} = \Psi(\mathbf{m}_{\leqslant krd-1})$ and $\mathbf{w}^{(2)} = \Psi(\mathbf{m}_{\geqslant krd})$. Define

$$\widehat{\Psi}(\mathbf{m}) = \mathbf{w}^{(1)} * 0 * \mathbf{w}^{(2)}. \tag{3.13}$$

Notice that 0 denotes the (krd)-th zero in $\widehat{\Psi}(\mathbf{m})$ when read from left to right.

Using the injectivity of Ψ, one can easily see that $\widehat{\Psi}$ is injective. Indeed, the inverse of $\widehat{\Psi}$ is defined in the following way: For an element $\mathbf{w} \in \widehat{\Psi}(\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}'))$, break it into $\mathbf{w}^{(1)} * 0 * \mathbf{w}^{(2)}$, where 0 denotes the krdth zero. Then

$$\widehat{\Psi}^{-1}(\mathbf{w}) := \Psi^{-1}(\mathbf{w}^{(1)}) * \Psi^{-1}(\mathbf{w}^{(2)}). \tag{3.14}$$

Convention 3.12. Hereafter the blue zero 0 will denote the krd-th zero in $\widehat{\Psi}(\mathbf{m})$ when we read it from left to right.

Example 3.13. Let $d = 1, k = 1, k' = 2, r = 2, \ell = 6, \boldsymbol{\nu} = (1)$, and $\boldsymbol{\nu}' = (1,2)$. Then

$$\mathbf{M}_6(2,1;(1),(1,2)) = \left\{\mathbf{m} \in \mathbb{Z}^4_{\geqslant 0} \;\middle|\; (m_0+m_1)+m_2+2m_3 = 6\right\},$$

where $\mathbf{m} = (m_0, m_1, m_2, m_3)$. For $\mathbf{m} = (1,2,1,1) \in \mathbf{M}_6(2,1;(1),(1,2))$, break $\mathbf{m}$ into

$$\mathbf{m}_{\leqslant 1} = (1,2) \in \mathbf{M}_3(2;(1)) \quad \text{and} \quad \mathbf{m}_{\geqslant 2} = (1,1) \in \mathbf{M}_3(1;(1,2)).$$

Since $\mathbf{m}_{\leqslant 1} = (1,2) \in \mathbf{M}_3(2;(1))$ (resp. $\mathbf{m}_{\geqslant 2} = (1,1) \in \mathbf{M}_3(1;(1,2))$), we have $\Psi((1,2)) = 1011$ (resp. $\Psi((1,1)) = 102$). Thus we have

$$\widehat{\Psi}((1,2,1,1)) = 10110102.$$

4. Cyclic sieving phenomena(except for $D_n^{(1)}(n \equiv_2 0)$)

The *cyclic sieving phenomenon* was introduced by Reiner-Stanton-White in [15]. Let X be a finite set, with an action of a cyclic group C of order m. Elements within a C-orbit share the same stabilizer subgroup, whose cardinality we will call the *stabilizer-order* for the orbit. Let $X(q)$ be a polynomial in q with nonnegative integer coefficients. For $d \in \mathbb{Z}_{>0}$, let ω_d be a dth primitive root of the unity. We say that $(X, C, X(q))$ exhibits the cyclic sieving phenomenon if, for all $c \in C$, we have

$$|X^c| = X(\omega_{o(c)}),$$

where $o(c)$ is the order of c and X^c is the fixed point set under the action of c. Note that this condition is equivalent to the following:

$$X(q) \equiv \sum_{0 \leqslant i \leqslant m-1} b_i q^i \pmod{q^m - 1},$$

where b_i counts the number of C-orbits on X for which the stabilizer-order divides i.

In this section, we suppose that $\mathfrak{g}$ is any affine Kac-Moody algebra of rank n except for $D_n^{(1)}(n \equiv_2 0)$ and $A_{2n}^{(2)}$. For a nonnegative integer ℓ, let $X = P^+_{\mathrm{cl},\ell}$, $C = C_{\mathsf{N}}$, with N in (2.8), and we define

$$X(q) = P^+_{\mathrm{cl},\ell}(q) := \sum_{\Lambda \in P^+_{\mathrm{cl},\ell}} q^{\mathsf{ev}_{\boldsymbol{S}}(\Lambda)}, \tag{4.1}$$

where $\boldsymbol{S}$ is the root-sieving set given in Convention 2.13. Then Theorem 2.14 tells that

$$P^+_{\mathrm{cl},\ell}(q) = \sum_{i \geqslant 0} \left| \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_{\boldsymbol{S}}(\Lambda) = i \right\} \right| q^i \equiv \sum_{\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})} \left| P^+_{\mathrm{cl},\ell}(\Lambda) \right| q^{\mathsf{ev}_{\boldsymbol{S}}(\Lambda)} \pmod{q^{\mathsf{N}} - 1}. \tag{4.2}$$

Remark 4.1. Let $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ and $s_0 := 0$. Then $P^+_{\mathrm{cl},\ell}(q)$ can be defined by the geometric series:

$$\sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q) t^\ell = \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q^{s_i} t^{a_i^\vee}}. \tag{4.3}$$

Note that the coefficient of t^ℓ of the right hand side is given by

$$\sum_{j \geqslant 0} \left| \left\{ \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \;\middle|\; \sum_{0 \leqslant i \leqslant n} a_i^\vee m_i = \ell \text{ and } \sum_{0 \leqslant i \leqslant n} s_i m_i = j \right\} \right| q^j.$$

The purpose of this section is to show that $(P^+_{\mathrm{cl},\ell}, C_{\mathsf{N}}, P^+_{\mathrm{cl},\ell}(q))$ exhibits the cyclic sieving phenomenon.

Let us introduce some necessary notations. When a finite group G acts on X, we denote by X^G the set of fixed points under the action of G. For any $g \in G$, we let $X^g := X^{\langle g \rangle}$. For $n \in \mathbb{Z}_{\geqslant 0}$ and $k \in \mathbb{Z}_{\geqslant 0} \cap \mathbb{Z}_{\leqslant n}$, we let *q-binomial coefficient* which are defined as follows:

$$[n]_q := \frac{q^n - 1}{q - 1}, \quad [n]_q! := \prod_{1 \leqslant k \leqslant n} [k]_q, \quad \text{and} \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! [n-k]_q!}.$$

Let $\mathscr{F} = \{\Lambda_0, \Lambda_1, \ldots, \Lambda_n\}$ and let

$$\phi_{\mathscr{F}} : P^+_{\mathrm{cl}} \to \mathbb{Z}^{n+1}, \quad \Lambda \mapsto [\Lambda]_{\mathscr{F}} \tag{4.4}$$

be the map given by the matrix representation in terms of $\mathscr{F}$.

4.1. $A^{(1)}_n$ **type.** To begin with, we review a result in [15]. For a positive integer N, let $[0,N] := \{0,1,2,\ldots,N\}$ and $\left(\!\binom{[0,N]}{\ell}\!\right)$ the set of all ℓ-multisubsets of $[0,N]$. Then the symmetric group $\mathfrak{S}_{[0,N]}$ on $[0,N]$ acts on $\left(\!\binom{[0,N]}{\ell}\!\right)$. We say that a cyclic group C of order m acts *nearly freely* on $[0,N]$ if it is generated by a permutation $c \in \mathfrak{S}_{[0,N]}$ whose cycle type is either

(4.5)
(1) j cycles of size m so that $N+1 = jm$, or
(2) j cycles of size m and one singleton cycle, so that $N+1 = jm+1$

for some positive integer j.

Lemma 4.2. [15, Theorem 1.1 (a)] *Let a cyclic group C of order m act nearly freely on $[0,N]$. Then the triple*

$$\left(\left(\!\binom{[0,N]}{\ell}\!\right), C, \begin{bmatrix} N+\ell \\ \ell \end{bmatrix}_q\right)$$

exhibits the cyclic sieving phenomenon.

Recall that $\mathsf{N} = n+1$ and $\langle c, \Lambda_i\rangle = 1$ for all $i \in I$ when $\mathfrak{g} = A^{(1)}_n$. Let us define a C_{n+1}-action on $P^+_{\mathrm{cl},\ell}$ by

$$\sigma_{n+1} \cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i = \sum_{0\leqslant i\leqslant n} m_{i+1}\Lambda_i, \qquad \text{where } m_{n+1} = m_0. \tag{4.6}$$

On the other hand, for the long cycle $\sigma = (0,1,2,\ldots,n) \in \mathfrak{S}_{[0,n]}$ of order $n+1$, the cyclic group $\langle\sigma\rangle$ acts freely on $\left(\!\binom{[0,n]}{\ell}\!\right)$. For simplicity, let us use $0^{m_0}1^{m_1}\cdots n^{m_n}$ to denote the multiset with m_i i's for all $0 \leqslant i \leqslant n$. There is a natural bijection, say κ, between $P^+_{\mathrm{cl},\ell}$ and $\left(\!\binom{[0,n]}{\ell}\!\right)$

$$\kappa : P^+_{\mathrm{cl},\ell} \to \left(\!\binom{[0,n]}{\ell}\!\right), \quad \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \mapsto 0^{m_0}1^{m_1}\cdots n^{m_n} \tag{4.7}$$

preserving group actions, more precisely, satisfying that

$$\kappa\left(\sigma_{n+1} \cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = \sigma \cdot 0^{m_0}1^{m_1}\cdots n^{m_n} \ (= 0^{m_n}1^{m_0}\cdots n^{m_{n-1}}). \tag{4.8}$$

Moreover, there is a bijection between $P^+_{\mathrm{cl},\ell}$ and the set $\mathrm{Par}(n,\ell)$ of partitions contained in $n \times \ell$ rectangle defined by

$$\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \mapsto (n^{m_n}(n-1)^{m_{n-1}}\cdots 1^{m_1})'. \tag{4.9}$$

Here $(n^{m_n}(n-1)^{m_{n-1}}\cdots 1^{m_1})$ denotes the partition having m_i part equal to i $(1 \leqslant i \leqslant n)$ and μ' the conjugate of μ for any partition μ. It can be easily seen that $\mathsf{ev}_s(\Lambda)$ is equal to the size of the corresponding partition, thus [21, Proposition 1.7.3] says that

$$P^+_{\mathrm{cl},\ell}(q) = \sum_{i\geqslant 0}\left|\left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) = i\right\}\right| q^i = \sum_{i\geqslant 0}\left|\{\lambda \in \mathrm{Par}(n,\ell) \mid |\lambda| = i\}\right| q^i = \begin{bmatrix} n+\ell \\ \ell \end{bmatrix}_q. \tag{4.10}$$

Theorem 4.3. *Under the C_{n+1}-action in (4.6), the triple*

$$\left(P^+_{\mathrm{cl},\ell}, C_{n+1}, P^+_{\mathrm{cl},\ell}(q)\right) \tag{4.11}$$

exhibits the cyclic sieving phenomenon.

Proof. Our assertion follows from Lemma 4.2, (4.7) and (4.10). ☐

Remark 4.4. For $0 \leqslant i \leqslant n$, the image of $P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_i)$ under the correspondence in (4.9) is $\{\lambda \in \mathrm{Par}(n,\ell) \mid |\lambda| \equiv_{n+1} i\}$, which yields the identity:

$$|\max{}^+((\ell-1)\Lambda_0+\Lambda_i)| = |\,\{\lambda \in \mathrm{Par}(n,\ell) \mid |\lambda| \equiv_{n+1} i\}\,|.$$

This shows that $|\max^+((\ell-1)\Lambda_0+\Lambda_i)|$ $(0 \leqslant i \leqslant n)$ appear as the coefficient of $P^+_{\mathrm{cl},\ell}(q)$, more precisely,

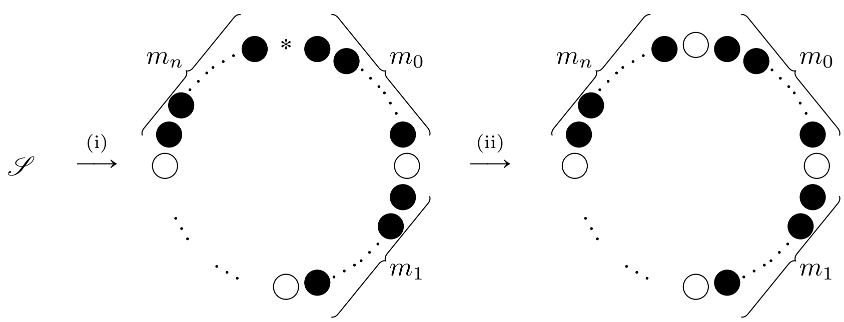
$$P^+_{\mathrm{cl},\ell}(q) \equiv \sum_{0\leqslant i\leqslant n} |\max{}^+((\ell-1)\Lambda_0+\Lambda_i)|q^i \pmod{q^{n+1}-1}. \tag{4.12}$$

The congruence (4.12) implies that $|\max^+((\ell-1)\Lambda_0+\Lambda_i)|$ counts the C_{n+1}-orbits on $P^+_{\mathrm{cl},\ell}$ for which the stabilizer-order divides i. In the following, we will give a closed formula for the number of C_{n+1}-orbits on $P^+_{\mathrm{cl},\ell}$ for which the stabilizer-order divides i.

A C_{n+1}-orbit of $P^+_{\mathrm{cl},\ell}$ can be considered as *necklace* with black beads and white beads. Fix $n,\ell \in \mathbb{Z}_{>0}$. Let $\mathsf{B}(n,\ell)$ be the set of words of n white beads and ℓ black beads. The cyclic group $C_{n+1} = \langle \sigma_{n+1}\rangle$ acts on $\mathsf{B}(n,\ell)$ by

$$\begin{aligned}&\sigma_{n+1}\cdot\Big(\underbrace{B,B,\dots,B}_{m_0},W,\underbrace{B,B,\dots,B}_{m_1},W,\dots,W,\underbrace{B,B,\dots,B}_{m_n}\Big)\\ &=\Big(\underbrace{B,B,\dots,B}_{m_n},W,\underbrace{B,B,\dots,B}_{m_0},W,\underbrace{B,B,\dots,B}_{m_1},W,\dots,W,\underbrace{B,B,\dots,B}_{m_{n-1}}\Big),\end{aligned}$$

where W denotes a white bead and B denotes a black bead. We can realize a C_{n+1}-orbit of $\mathsf{B}(n,\ell)$ as a necklace using $n+1$ white beads and ℓ black beads by (i) threading the beads into a necklace with the same order, and (ii) adding a white bead between the last B and the first B as follows: For $\mathscr{S} := \underbrace{B,B,\dots,B}_{m_0},W,\underbrace{B,B,\dots,B}_{m_1},W,\dots,W,\underbrace{B,B,\dots,B}_{m_n} \in \mathsf{B}(n,\ell)$,



A necklace $C_{n+1}\cdot\mathscr{S}$ is called *primitive* if the stabilizer subgroup of $\mathscr{S}$ is trivial.

Lemma 4.5 ([16, Theorem 7.1])**.** *The number of primitive necklaces using $n+1$ white beads and ℓ black beads is*

$$\frac{1}{(n+1)+\ell}\sum_{d|(n+1,\ell)}\mu(d)\binom{((n+1)+\ell)/d}{\ell/d},$$

where μ is the classical Möbius function.

There is a natural C_{n+1}-set isomorphism between $P^+_{\mathrm{cl},\ell}$ and $\mathsf{B}(n,\ell)$ defined by

$$(m_0,m_1,\dots,m_n)\longleftrightarrow \underbrace{B,B,\dots,B}_{m_0},W,\underbrace{B,B,\dots,B}_{m_1},W,\cdots,W,\underbrace{B,B,\dots,B}_{m_n}.$$

Combining Theorem 4.3 with (4.12) and Lemma 4.5, we derive the following closed formula.

Theorem 4.6. *For any* $(\ell-1)\Lambda_0+\Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, *we have*

$$|\max{}^+((\ell-1)\Lambda_0+\Lambda_i)| = \sum_{d|(n+1,\ell,i)} \frac{d}{(n+1)+\ell} \sum_{d'|(\frac{n+1}{d},\frac{\ell}{d})} \mu(d') \binom{((n+1)+\ell)/dd'}{\ell/dd'}. \tag{4.13}$$

Remark 4.7. In [10], Jayne-Misra conjectured that

$$|\max{}^+(\ell\Lambda_0)| = \frac{1}{(n+1)+\ell} \sum_{d|(n+1,\ell)} \varphi(d) \binom{((n+1)+\ell)/d}{\ell/d},$$

where φ is Euler's phi function. This is the case where $i=0$ in (4.13), which was proven in [24].

It should be remarked that the cardinality of $\{\lambda \in \mathrm{Par}(n,\ell) \mid |\lambda| \equiv_{n+1} i\}$ have already appeared in [5]. Hence, we can also derive Theorem 4.6 using Remark 4.4.

Corollary 4.8. *Let* $(\ell-1)\Lambda_0+\Lambda_i, (\ell-1)\Lambda_0+\Lambda_j \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$. *Then*

$$|\max{}^+((\ell-1)\Lambda_0+\Lambda_i)| = |\max{}^+((\ell-1)\Lambda_0+\Lambda_j)|$$

if and only if $(n+1,\ell,i) = (n+1,\ell,j)$.

Proof. It is a direct consequence of Theorem 4.6. □

4.2. $B^{(1)}_n, C^{(1)}_n, A^{(2)}_{2n-1}, D^{(2)}_{n+1}, E^{(1)}_6, E^{(1)}_7$ **types.** In this subsection, we assume that $\mathfrak{g}$ is an affine Kac-Moody algebra other than $A^{(1)}_n$ and $D^{(1)}_n$. In $A^{(1)}_n$-case, N is a composite unless $n+1$ is a prime. In $D^{(1)}_n$-case, $\mathsf{N}=4$ is a composite for all $n \in \mathbb{Z}_{\geqslant 4}$.

Note that $\mathfrak{S}_{[0,n]}$ acts on $P^+_{\mathrm{cl},\ell}$ by permuting indices of coefficients, that is,

$$\sigma \cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i = \sum_{0\leqslant i\leqslant n} m_{\sigma(i)}\Lambda_i \quad \text{for } \sigma \in \mathfrak{S}_{[0,n]}.$$

Let us take $\sigma \in \mathfrak{S}_{[0,n]}$ of order N as in Table 4.1. We define a $C_{\mathsf{N}} = \langle \sigma_{\mathsf{N}} \rangle$-action on $P^+_{\mathrm{cl},\ell}$ by

Types	σ	N
$B^{(1)}_n, D^{(2)}_{n+1}$	$(0,n)$	2
$C^{(1)}_n, A^{(2)}_{2n-1}$ $(n \equiv_2 1)$	$(0,1)(2,3)\cdots(n-1,n)$	2
$C^{(1)}_n, A^{(2)}_{2n-1}$ $(n \equiv_2 0)$	$(0,1)(2,3)\cdots(n-2,n-1)$	2
$E^{(1)}_6$	$(0,1,6)(2,3,5)$	3
$E^{(1)}_7$	$(0,7)(1,6)(3,5)$	2

TABLE 4.1. σ and N for other types

$$\sigma_{\mathsf{N}} \cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i := \sum_{0\leqslant i\leqslant n} m_{\sigma(i)}\Lambda_i \tag{4.14}$$

for any $\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell}$.

Theorem 4.9. *Let* $\mathfrak{g}$ *be of type* $B^{(1)}_n, C^{(1)}_n, A^{(2)}_{2n-1}, D^{(2)}_{n+1}, E^{(1)}_6$, *or* $E^{(1)}_7$. *Then, under the* C_{N}*-action given in* (4.14), *the triple*

$$\left(P^+_{\mathrm{cl},\ell}, C_{\mathsf{N}}, P^+_{\mathrm{cl},\ell}(q)\right) \tag{4.15}$$

exhibits the cyclic sieving phenomenon.

Since the method of proof for each type is essentially same, we only deal with $E_6^{(1)}$ type. Recall that

$$(a_i^\vee)_{i=0}^6 = (1,1,2,2,3,2,1), \quad \widetilde{\boldsymbol{s}} = (s_i)_{i=0}^6 = (0,1,0,2,0,1,2) \quad \text{and} \quad \sigma = (0,1,6)(2,3,5).$$

Then one can see that, for all $j = 0,1,\ldots,6$, we have

$$a_j^\vee = a_{\sigma(j)}^\vee \quad \text{and} \quad \{s_j, s_{\sigma(j)}, s_{\sigma^2(j)}\} = \begin{cases} \{0,1,2\} & \text{if } i \neq 4, \\ \{0\} & \text{if } i = 4. \end{cases} \tag{4.16}$$

By Theorem 2.14 and (4.2), we have

$$P_{\mathrm{cl},\ell}^+(q) = \sum_{i \geqslant 0} \left| \left\{ \Lambda \in P_{\mathrm{cl},\ell}^+ \;\middle|\; \mathsf{ev}_s(\Lambda) = i \right\} \right| q^i \equiv \sum_{0 \leqslant i \leqslant 2} \left| \left\{ \Lambda \in P_{\mathrm{cl},\ell}^+ \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_3 i \right\} \right| q^i \pmod{q^3 - 1}. \tag{4.17}$$

We will prove Theorem 4.9 by providing a set X with a C_3-action such that X is isomorphic to $P_{\mathrm{cl},\ell}^+$ as C_3-sets and $(X, C_3, P_{\mathrm{cl},\ell}^+(q))$ exhibits the cyclic sieving phenomenon. More precisely, we will take X as $\mathbf{M}_\ell(3,1;(1,2),(3))$.

Recall that $\mathfrak{S}_{[0,6]}$ acts on P_{cl}^+ by permuting indices of coefficients. Let $\tau = (4,3,2,6) \in \mathfrak{S}_{[0,6]}$. Since

$$\tau \cdot (a_0^\vee, a_1^\vee, a_2^\vee, a_3^\vee, a_4^\vee, a_5^\vee, a_6^\vee) = (a_{\tau(0)}^\vee, a_{\tau(1)}^\vee, a_{\tau(2)}^\vee, a_{\tau(3)}^\vee, a_{\tau(4)}^\vee, a_{\tau(5)}^\vee, a_{\tau(6)}^\vee),$$

we have

$$\tau \cdot (1,1,2,2,3,2,1) = (1,1,1 \mid 2,2,2 \mid 3).$$

Thus the image of $\tau \cdot P_{\mathrm{cl},\ell}^+$ under $\phi_{\mathscr{F}}$ in (4.4) is the same as $\mathbf{M}_\ell(3,1;(1,2),(3))$. For the definition of $\mathbf{M}_\ell(3,1;(1,2),(3))$, see (3.11).

Example 4.10. Let $\mathfrak{g} = E_6^{(1)}$ and $\ell = 3$. Then, we have

$$P_{\mathrm{cl},3}^+ = \left\{ \begin{array}{lll} 3\Lambda_0, & \Lambda_0 + \Lambda_2, & \Lambda_4, \\ 2\Lambda_0 + \Lambda_1, & \Lambda_0 + \Lambda_3, & \\ 2\Lambda_0 + \Lambda_6, & \Lambda_0 + \Lambda_5, & \\ \Lambda_0 + 2\Lambda_1, & \Lambda_1 + \Lambda_2, & \\ \Lambda_0 + \Lambda_1 + \Lambda_6, & \Lambda_1 + \Lambda_3, & \\ \Lambda_0 + 2\Lambda_6, & \Lambda_1 + \Lambda_5, & \\ 3\Lambda_1, & \Lambda_2 + \Lambda_6, & \\ 2\Lambda_1 + \Lambda_6, & \Lambda_3 + \Lambda_6, & \\ \Lambda_1 + 2\Lambda_6, & \Lambda_5 + \Lambda_6, & \\ 3\Lambda_6 & & \end{array} \right\} \quad \text{and} \quad \tau \cdot P_{\mathrm{cl},3}^+ = \left\{ \begin{array}{lll} 3\Lambda_0, & \Lambda_0 + \Lambda_3, & \Lambda_6, \\ 2\Lambda_0 + \Lambda_1, & \Lambda_0 + \Lambda_4, & \\ 2\Lambda_0 + \Lambda_2, & \Lambda_0 + \Lambda_5, & \\ \Lambda_0 + 2\Lambda_1, & \Lambda_1 + \Lambda_3, & \\ \Lambda_0 + \Lambda_1 + \Lambda_2, & \Lambda_1 + \Lambda_4, & \\ \Lambda_0 + 2\Lambda_2, & \Lambda_1 + \Lambda_5, & \\ 3\Lambda_1, & \Lambda_2 + \Lambda_3, & \\ 2\Lambda_1 + \Lambda_2, & \Lambda_2 + \Lambda_4, & \\ \Lambda_1 + 2\Lambda_2, & \Lambda_2 + \Lambda_5, & \\ 3\Lambda_2 & & \end{array} \right\}.$$

Note that the image of $\tau \cdot P_{\mathrm{cl},3}^+$ under $\phi_{\mathscr{F}}$ in (4.4) is

$$\mathbf{M}_3(3,1;(1,2),(3)) = \left\{ \begin{array}{lll} (3,0,0,0,0,0,0), & (1,0,0,1,0,0,0), & (0,0,0,0,0,0,1), \\ (2,1,0,0,0,0,0), & (1,0,0,0,1,0,0), & \\ (2,0,1,0,0,0,0), & (1,0,0,0,0,1,0), & \\ (1,2,0,0,0,0,0), & (0,1,0,1,0,0,0), & \\ (1,1,1,0,0,0,0), & (0,1,0,0,1,0,0), & \\ (1,0,2,0,0,0,0), & (0,1,0,0,0,1,0), & \\ (0,3,0,0,0,0,0), & (0,0,1,1,0,0,0), & \\ (0,2,1,0,0,0,0), & (0,0,1,0,1,0,0), & \\ (0,1,2,0,0,0,0), & (0,0,1,0,0,1,0), & \\ (0,0,3,0,0,0,0) & & \end{array} \right\}.$$

In Table 4.2, we list τ and $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$ for all types (except for $A_n^{(1)}$ and $D_n^{(1)}$).

Remark 4.11.

(1) All $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$ in Table 4.2 are contained in $\mathbb{Z}_{\geqslant 0}^{n+1}$ as subsets.
(2) In Table 4.2, we choose $\tau, r, d, \boldsymbol{\nu}, \boldsymbol{\nu}'$ to be satisfied that the image of $\tau \cdot P_{\mathrm{cl},\ell}^+$ under $\phi_{\mathscr{F}}$ in (4.4) is the same as $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$.

Types	τ	$\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$
$B_n^{(1)}$	$(n,n-1,\dots,1)$	$\mathbf{M}_\ell(2,1;(1),(1,2^{n-2}))$
$C_n^{(1)}$ $(n\equiv_2 1)$	id	$\mathbf{M}_\ell(2;(1^{(n+1)/2}))$
$C_n^{(1)}$ $(n\equiv_2 0)$	id	$\mathbf{M}_\ell(2,1;(1^{n/2}),(1))$
$A_{2n-1}^{(2)}$ $(n\equiv_2 1)$	id	$\mathbf{M}_\ell(2;(1,2^{(n-1)/2}))$
$A_{2n-1}^{(2)}$ $(n\equiv_2 0)$	id	$\mathbf{M}_\ell(2,1;(1,2^{(n-2)/2}),(2))$
$D_{n+1}^{(2)}$	$(n,n-1,\dots,1)$	$\mathbf{M}_\ell(2,1;(1),(2^{n-1}))$
$E_6^{(1)}$	$(4,3,2,6)$	$\mathbf{M}_\ell(3,1;(1,2),(3))$
$E_7^{(1)}$	$(1,7,4,3,2,6)$	$\mathbf{M}_\ell(2,1;(1,2,3),(2,4))$

TABLE 4.2. τ and $\mathbf{M}_\ell(rd,d;\boldsymbol{\nu},\boldsymbol{\nu}')$ for all types

Now, we have a C_3-action $\blacksquare_{3,1}$ on $\tau\cdot P^+_{\mathrm{cl},\ell}$ defined as follows: For $\Lambda\in\tau\cdot P^+_{\mathrm{cl},\ell}$,

$$\sigma_3 \blacksquare_{3,1} \Lambda = \phi_{\mathscr{F}}^{-1}(\sigma_3 \blacksquare_{3,1} \phi_{\mathscr{F}}(\Lambda)) = (\widehat{\Psi}\circ\phi_{\mathscr{F}})^{-1}\left(\sigma_3 \blacksquare_{3,1} (\widehat{\Psi}\circ\phi_{\mathscr{F}})(\Lambda)\right). \tag{4.18}$$

Example 4.12. Let $\mathfrak{g}=E_6^{(1)}$. For $2\Lambda_0+\Lambda_1\in\tau\cdot P^+_{\mathrm{cl},3}$, we have the following commutative diagram:

$$\begin{tikzcd}
2\Lambda_0+\Lambda_1 \arrow[r, "\phi_{\mathscr{F}}"] \arrow[d, dotted, "\sigma_3"] & (2,1,0,0,0,0) \arrow[r, "\widehat{\Psi}"] \arrow[d, "\sigma_3"] & 110|100|000 \arrow[d, "\sigma_3"] \\
3\Lambda_1 \arrow[d, dotted, "\sigma_3"] & (0,3,0,0,0,0) \arrow[l, "\phi_{\mathscr{F}}^{-1}"] \arrow[d, "\sigma_3"] & 011|100|000 \arrow[l, "\widehat{\Psi}^{-1}"] \arrow[d, "\sigma_3"] \\
\Lambda_0+2\Lambda_1 \arrow[uu, dotted, bend left, "\sigma_3"] & (1,2,0,0,0,0) \arrow[l, "\phi_{\mathscr{F}}^{-1}"] & 101|100|000 \arrow[l, "\widehat{\Psi}^{-1}"] \arrow[uu, bend right, "\sigma_3"']
\end{tikzcd}$$

Lemma 4.13. *Under the C_3-action $\blacksquare_{3,1}$ on $\mathbf{M}_\ell(3,1;(1,2),(3))$ given in (3.12),*

$$\left(\mathbf{M}_\ell(3,1;(1,2),(3)),C_3,P^+_{\mathrm{cl},\ell}(q)\right)$$

exhibits the cyclic sieving phenomenon.

Proof. Note the following:

- $|\mathbf{M}_\ell(3,1;(1,2),(3))| = \left|P^+_{\mathrm{cl},\ell}\right|$.
- C_3-orbits are of length 1 or 3.
- For any $\Lambda\in P^+_{\mathrm{cl},\ell}$, $\mathsf{ev}_{\boldsymbol{s}}(\Lambda)=\widehat{\boldsymbol{s}}\bullet\phi_{\mathscr{F}}(\tau\cdot\Lambda)$, where $\widehat{\boldsymbol{s}}:=\tau\cdot\widetilde{\boldsymbol{s}}=(0,1,2,0,1,2,0)$.

For simplicity, we set $X:=\tau\cdot P^+_{\mathrm{cl},\ell}$ and $X(i):=\tau\cdot\left\{\Lambda\in P^+_{\mathrm{cl},\ell}\;\middle|\;\mathsf{ev}_{\boldsymbol{s}}(\Lambda)\equiv_3 i\right\}$. Suppose that the following claims hold (which will be proven in the below):

Claim 1. Let $\mathbf{m}=(m_0,m_1,\dots,m_5,m_6)\in\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3}$. Then

$$\widehat{\boldsymbol{s}}\bullet\mathbf{m}=\widehat{\boldsymbol{s}}\bullet\phi_{\mathscr{F}}(\tau\cdot\Lambda)=0$$

(see Remark 4.11).

Claim 2. Let $O=\{\mathbf{m}^{(1)},\mathbf{m}^{(2)},\mathbf{m}^{(3)}\}$ be a free C_3-orbit of $\mathbf{M}_\ell(3,1;(1,2),(3))$. For each $\mathbf{m}^{(j)}$ $(j=1,2,3)$, $\widehat{\boldsymbol{s}}\bullet\mathbf{m}^{(j)}$ are distinct up to modulo 3.

By *Claim 1*, we have

$$X^{C_3} \subset X(0), \tag{4.19}$$

under the C_3-action on X given in (4.18).

By *Claim 2*, we have

$$\left|\left(X\backslash X^{C_3}\right) \cap X(0)\right| = \left|\left(X\backslash X^{C_3}\right) \cap X(1)\right| = \left|\left(X\backslash X^{C_3}\right) \cap X(2)\right|. \tag{4.20}$$

By (4.19), for $i = 1, 2$, we have

$$\left|\left(X\backslash X^{C_3}\right) \cap X(i)\right| = |X(i)| = \left|\left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_3 i\right\}\right|,$$

which is equal to the number of free C_3-orbits, by (4.20). Moreover, since

$$\begin{aligned}\left|\left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_3 0\right\}\right| &= |X(0)| = \left|X^{C_3}\right| + \left|\left(X\backslash X^{C_3}\right) \cap X(0)\right| \\ &= (\text{the number of fixed points}) + (\text{the number of free orbits}) \\ &= (\text{the number of all orbits}),\end{aligned}$$

our assertion holds.

To complete the proof, we have only to verify *Claim 1* and *Claim 2*.

(a) For *Claim 1*, suppose that $\mathbf{m} \in \mathbf{M}_\ell(3, 1; (1, 2), (3))^{C_3}$. Recall the function Ψ from (3.5). Let $\mathbf{w} = w_1 w_2 \cdots w_u := \Psi(\mathbf{m}_{\leqslant 5})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{3t+1} \cdots w_u, \qquad (t = \lfloor u/3 \rfloor)$$

of length 3 as (3.6). Since $\mathbf{m}$ is a fixed point, Algorithm 3.2 and the definition of $\sigma_3 \,\blacksquare$ in (3.9) say that

$$\mathbf{w} = \underbrace{11\cdots 1}_{3k_1} \mid 000 \mid \underbrace{22\ldots 2}_{3k_2} \mid 00, \tag{4.21}$$

for $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$ such that $\ell - 3k_1 - 6k_2 = 3k_3$ for some $k_3 \in \mathbb{Z}_{\geqslant 0}$. Therefore,

$$\mathbf{m} = (3k_1, 0, 0, 3k_2, 0, 0, k_3). \tag{4.22}$$

Thus,

$$\widehat{\boldsymbol{s}} \bullet \mathbf{m} = (0, 1, 2, 0, 1, 2, 0) \bullet (3k_1, 0, 0, 3k_2, 0, 0, k_3) = 0.$$

(b) For *Claim 2*, choose any element $\mathbf{m} \in O$. Then we have

$$\mathbf{m}_{\leqslant 5} \in \mathbf{M}_l(3; (1, 2)) \text{ and } \mathbf{m}_6 \in \mathbf{M}_{l'}(1; (3)) \text{ for some } l \text{ and } l' \text{ with } l + l' = \ell.$$

Since C_3 acts on $\mathbf{m}_6$ in a trivial way (see Remark 3.8 (2)), it suffices to consider the C_3-action on $\mathbf{m}_{\leqslant 5}$.

Let $\mathbf{w} = w_1 w_2 \cdots w_u := \Psi(\mathbf{m}_{\leqslant 5})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{3t+1} \cdots w_u$$

of length 3 as (3.6). Since O is a free orbit, there exists the smallest $1 \leqslant j_0 \leqslant t$ such that $\sigma_3 \cdot w^{j_0} \neq w^{j_0}$. Note that the definition of the C_3-action in (3.12) says that, for all $1 \leqslant j < j_0$, w^j's are $000, 111$, or 222.

Now w^{j_0} should be one of

$$100,\ 010,\ 001, \qquad 110,\ 011,\ 101, \qquad 200,\ 020,\ 002, \qquad 220,\ 022,\ 202.$$

We shall only give a proof for the case $w^{j_0} = 100$ since the other cases can be proved by the same argument.

Note that for all $1 \leqslant j < j_0$, w^j is 000 or 111 under our assumption. Assume that there is $1 \leqslant j < j_0$ such that $w^j = 000$. Then, by Algorithm 3.2, w^{j_0} is not able to contain 1 since $\mathbf{m}_{\leqslant 5} \in \mathbf{M}_l(3; (1, 2))$. Now we have $w^j = 111$, for all $1 \leqslant j < j_0$. Then

$$\begin{aligned}\sigma_3^i \,\blacksquare\, \mathbf{w} &= w^1 \mid w^2 \mid \cdots \mid w^{j_0-1} \mid \sigma_3^i \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{3t+1} \cdots w_u \\ &= 111 \mid 111 \mid \cdots \mid 111 \mid \begin{matrix} 010 \\ 001 \end{matrix} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{3t+1} \cdots w_u \quad \begin{matrix} \text{if } i = 1, \\ \text{if } i = 2. \end{matrix}\end{aligned}$$

Therefore, by the construction of Ψ^{-1}, we have

$$\sigma_3^i \blacksquare_{3,1} \mathbf{m} = \Psi^{-1}(\sigma_3^i \blacksquare \mathbf{w}) * \mathbf{m}_6 = \begin{cases} (m_0-1, m_1+1, m_2 \quad , m_3, m_4, m_5, m_6) & \text{if } i=1, \\ (m_0-1, m_1 \quad , m_2+1, m_3, m_4, m_5, m_6) & \text{if } i=2. \end{cases}$$

Hence, we have

$$\widehat{\boldsymbol{s}} \bullet (\sigma_3^i \blacksquare_{3,1} \mathbf{m} - \mathbf{m}) = \begin{cases} (0,1,2,0,1,2,0) \bullet (-1,1,0,0,0,0,0) \equiv_3 1 & \text{if } i=1, \\ (0,1,2,0,1,2,0) \bullet (-1,0,1,0,0,0,0) \equiv_3 2 & \text{if } i=2. \end{cases} \qquad \square$$

Now we are ready to prove Theorem 4.9.

Proof for Theorem 4.9. Since $|P^+_{\mathrm{cl},\ell}| = |\mathbf{M}_\ell(3,1;(1,2),(3))|$, by Lemma 4.13, we have only to see that

$$|(P^+_{\mathrm{cl},\ell})^{C_3}| = |\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3}|.$$

Note that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{C_3} = \left\{(k_1,k_1,k_2,k_2,k_3,k_2,k_1) \in \mathbb{Z}^7_{\geqslant 0} \;\middle|\; 3k_1+6k_2+3k_3=\ell\right\}. \tag{4.23}$$

We claim that

$$\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3} = \left\{(3k_1,0,0,3k_2,0,0,k_3) \in \mathbb{Z}^7_{\geqslant 0} \;\middle|\; 3k_1+6k_2+3k_3=\ell\right\} =: Y. \tag{4.24}$$

By (4.21) and (4.22), $\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3}$ is contained in Y.

For the reverse inclusion, recall the function $\widehat{\Psi}$ from Definition 3.11. For $\mathbf{m} \in Y$, we have

$$\widehat{\Psi}(\mathbf{m}) = \underbrace{1\cdots1}_{3k_1} \mid 000 \mid \underbrace{2\cdots2}_{3k_2} \mid 000 \mid \underbrace{3\cdots3}_{k_3}.$$

Thus, by (3.12) and (3.13), $\sigma_3 \blacksquare_{3,1} \mathbf{m} = \mathbf{m}$ and hence Y is contained in $\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3}$.

By (4.23) and (4.24), $\left(P^+_{\mathrm{cl},\ell}\right)^{C_3}$ and $\mathbf{M}_\ell(3,1;(1,2),(3))^{C_3}$ have the same cardinality, as required. $\square$

4.3. $D_n^{(1)}$ type ($n \equiv_2 1$).

Throughout this subsection, we set

$$\eta := \frac{n-3}{2},$$

which is an integer since n is odd. Recall $\mathsf{N}=4$,

$$(a_i^\vee)_{i=0}^n = (1,1,2,2,\dots,2,1,1) \quad \text{and} \quad \widetilde{\boldsymbol{s}} = (s_i)_{i=0}^n = (0,2,0,2,0,\dots,0,2,1,3).$$

Let

$$\upsigma = (0,n,1,n-1)(2,3)(3,4)\cdots(n-3,n-2) \in \mathfrak{S}_{[0,n]},$$

which is of order 4. Since $a_j^\vee = a_{\upsigma(j)}^\vee$ for any $j \in I$, we can define a $C_4 = \langle\sigma_4\rangle$-action on $P^+_{\mathrm{cl},\ell}$ as follows:

$$\sigma_4 \cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i := \sum_{0\leqslant i\leqslant n} m_{\upsigma(i)}\Lambda_i \quad \text{for} \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell}. \tag{4.25}$$

Theorem 4.14. *Under the C_4-action given in (4.25),*

$$\left(P^+_{\mathrm{cl},\ell}, C_4, P^+_{\mathrm{cl},\ell}(q)\right) \tag{4.26}$$

exhibits the cyclic sieving phenomenon.

For reader's understanding, let us briefly explain our strategy for proving Theorem 4.14. First, as in Table 4.2, we take

$$\tau = (n-1,n-3,\dots,4,2,1)(n,n-2,\dots,5,3) \in \mathfrak{S}_{[0,n]} \tag{4.27}$$

so that

$$\tau \cdot (a_0^\vee, a_1^\vee, \dots, a_n^\vee) = \tau \cdot (1,1,2,2,\dots,2,2,1,1) = (1,1,1,1,2,2,\dots,2).$$

By breaking $(1,1,1,1,2,2,\dots,2)$ into $(1,1,1,1 \mid 2,2 \mid 2,2 \mid \cdots \mid 2,2)$, we can identify the image of $\tau \cdot P^+_{\mathrm{cl},\ell}$ under $\phi_{\mathscr{F}}$ with $\mathbf{M}_\ell(4,2;(1),(2^\eta))$. Thus, we can define the C_4-action $\blacksquare_{4,2}$ on $\tau \cdot P^+_{\mathrm{cl},\ell}$ by

$$\sigma_4 \blacksquare_{4,2} \Lambda = \phi_{\mathscr{F}}^{-1}(\sigma_4 \blacksquare_{4,2} \phi_{\mathscr{F}}(\Lambda)) \quad \text{for } \Lambda \in \tau \cdot P^+_{\mathrm{cl},\ell}. \tag{4.28}$$

On the other hand, by breaking $(1,1,1,1,2,2,\dots,2)$ into $(1,1 \mid 1,1 \mid 2 \mid 2 \mid \cdots \mid 2)$, we can also identify the image of $\tau \cdot P^+_{\mathrm{cl},\ell}$ under $\phi_{\mathscr{F}}$ with $\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))$. Thus, we can define the C_2-action $\blacksquare_{2,1}$ on $\tau \cdot P^+_{\mathrm{cl},\ell}$ by

$$\sigma_2 \blacksquare_{2,1} \Lambda = \phi_{\mathscr{F}}^{-1}(\sigma_2 \blacksquare_{2,1} \phi_{\mathscr{F}}(\Lambda)) \quad \text{for } \Lambda \in \tau \cdot P^+_{\mathrm{cl},\ell}. \tag{4.29}$$

Then, we will show

- $\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_4}\right| = P^+_{\mathrm{cl},\ell}(\zeta_4^j)$ $\quad (j=1,3)$ using the C_4-action defined in (4.28),
- $\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = P^+_{\mathrm{cl},\ell}(-1)$ using the C_2-action defined in (4.29).

By (4.2), we have

$$P^+_{\mathrm{cl},\ell}(q) = \sum_{i \geqslant 0} \left|\left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) = i\right\}\right| q^i \equiv \sum_{0 \leqslant i \leqslant 3} \left|\left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_4 i\right\}\right| q^i \pmod{q^4-1}. \tag{4.30}$$

For simplicity, we let

$$P^+_{\mathrm{cl},\ell}(q) \equiv \sum_{0 \leqslant i \leqslant 3} b_i q^i \pmod{q^4-1}.$$

Before proving Theorem 4.14, let us introduce four key lemmas.

Lemma 4.15. *Under the C_4-action $\blacksquare_{4,2}$ on $\mathbf{M}_\ell(4,2;(1),(2^\eta))$ given in (3.12), we have*

$$\left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| = b_0 - b_2 \quad \text{and} \quad b_1 = b_3.$$

Proof. For simplicity, we set $X = \tau \cdot P^+_{\mathrm{cl},\ell}$ and $X(i) := \tau \cdot \left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_4 i\right\}$. Note the following:

- C_4-orbits are of length 1, 2 or 4.
- Under the C_4-action on X given in (4.28), $\left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| = \left|X^{C_4}\right|$.
- For any $\Lambda \in P^+_{\mathrm{cl},\ell}$, $\mathsf{ev}_s(\Lambda) = \widehat{\boldsymbol{s}} \bullet \phi_{\mathscr{F}}(\tau \cdot \Lambda)$, where $\widehat{\boldsymbol{s}} := \tau \cdot \widetilde{\boldsymbol{s}} = (0,1,2,3,0,2,0,2,\dots,0,2)$.

Suppose that the following claims hold (which will be proven below):

Claim 1. Let $\Lambda \in X^{C_4}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then $\widehat{\boldsymbol{s}} \bullet \mathbf{m} = 0$ and hence $X^{C_4} \subset X(0)$.

Claim 2. Let $\Lambda \in X \backslash X^{C_4}, \mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$, and $i,j \in \{0,1,2,3\}$. Then

$$\widehat{\boldsymbol{s}} \bullet (\sigma_4^i \blacksquare_{4,2} \mathbf{m} - \sigma_4^{i+1} \blacksquare_{4,2} \mathbf{m}) \equiv_4 \widehat{\boldsymbol{s}} \bullet (\sigma_4^j \blacksquare_{4,2} \mathbf{m} - \sigma_4^{j+1} \blacksquare_{4,2} \mathbf{m}) \not\equiv_4 0,$$

By *Claim 1*, we have

$$X(2) \subset X \backslash X^{C_4}. \tag{4.31}$$

Let $\Lambda \in X(0) \cap \left(X \backslash X^{C_4}\right)$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then we have the following cases:

Case 1. $\sigma_4^2 \blacksquare_{4,2} \Lambda = \Lambda$,
Case 2. $\sigma_4^2 \blacksquare_{4,2} \Lambda \neq \Lambda$ and $\widehat{\boldsymbol{s}} \bullet \phi_{\mathscr{F}}(\Lambda - \sigma_4 \blacksquare_{4,2} \Lambda) \equiv_4 1$ or 3,
Case 3. $\sigma_4^2 \blacksquare_{4,2} \Lambda \neq \Lambda$ and $\widehat{\boldsymbol{s}} \bullet \phi_{\mathscr{F}}(\Lambda - \sigma_4 \blacksquare_{4,2} \Lambda) \equiv_4 2$.

- In *Case 1*, we have $\widehat{\boldsymbol{s}} \bullet (\sigma_4 \blacksquare_{4,2} \mathbf{m}) = 2$ by *Claim 2* and thus $\sigma_4 \blacksquare_{4,2} \Lambda \in X(2)$. This shows that one can correspond Λ to $\sigma_4 \blacksquare_{4,2} \Lambda$ in a bijective way.
- In *Case 2*, we have $\widehat{\boldsymbol{s}} \bullet (\sigma_4^2 \blacksquare_{4,2} \mathbf{m}) = 2$ by *Claim 2* and thus $\sigma_4^2 \blacksquare_{4,2} \Lambda \in X(2)$. This shows that one can correspond Λ to $\sigma_4^2 \blacksquare_{4,2} \Lambda$ in a bijective way.
- In *Case 3*, we have $\widehat{\boldsymbol{s}} \bullet (\sigma_4^i \blacksquare_{4,2} \mathbf{m}) \equiv_4 2i$ for $i = 1,2,3$. Therefore $\sigma_4 \blacksquare_{4,2} \Lambda, \sigma_4^3 \blacksquare_{4,2} \Lambda \in X(2)$ and $\sigma_4^2 \blacksquare_{4,2} \Lambda \in X(0)$. This shows that one can correspond $\Lambda, \sigma_4^2 \blacksquare_{4,2} \Lambda$ to $\sigma_4 \blacksquare_{4,2} \Lambda, \sigma_4^3 \blacksquare_{4,2} \Lambda$ in a bijective way.

In this way, we obtain a bijection between $X(0) \cap \left(X \backslash X^{C_4}\right)$ and $X(2) \cap \left(X \backslash X^{C_4}\right)$ and thus, by (4.31),

$$\left|X(0) \cap \left(X \backslash X^{C_4}\right)\right| = \left|X(2) \cap \left(X \backslash X^{C_4}\right)\right| = |X(2)|.$$

Consequently we have

$$\begin{aligned}\left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| &= \left|X^{C_4}\right| = |X(0)| - \left|X(0) \cap \left(X \backslash X^{C_4}\right)\right| \\ &= |X(0)| - |X(2)| = b_0 - b_2.\end{aligned}$$

In the same manner, by taking $\Lambda \in X(1)$ or $X(3) \subset X \backslash X^{C_4}$, one can see that

$$|X(1)| = |X(3)| \iff b_1 = b_3.$$

To complete the proof, we have only to verify *Claim 1* and *Claim 2*.
(a) For *Claim 1*, suppose that $\Lambda \in X^{C_4}$. Recall $\widehat{\Psi}$ in (3.13) and $\phi_{\mathscr{F}}$ in (4.4). Let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$ and $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^{t_1} \mid w^{t_1+1} \mid \cdots \mid w^{t_1+t_2} \mid w^0, \tag{4.32}$$

where

- w^j is of length 4 for $1 \leqslant j \leqslant t_1$,
- w^{t_1} contains the 4th zero when we read $\mathbf{w}$ from left to right,
- w^j is of length 2 for $t_1+1 \leqslant j \leqslant t_1+t_2$,
- w^0 is the empty word or of length 1.

Here such w^{t_1} exists since the number of 0 in $\mathbf{w}$ is $n \geqslant 4$ by Algorithm 3.2 and (3.13).

Since $\Lambda \in X^{C_4}$, we have $\mathbf{m} \in \mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}$. Then $\sigma_4 \blacksquare_{4,2}$ in (3.12) and (3.13) say that

$$\underbrace{1\cdots1}_{4k_1} \mid 0000 \mid \underbrace{2\cdots2}_{2k_2} \mid 00 \mid \underbrace{2\cdots2}_{2k_3} \mid 00 \mid \cdots \mid \underbrace{2\cdots2}_{2k_{\eta+1}} \mid 0 \tag{4.33}$$

for some $k_1, k_2, \dots, k_{\eta+1} \in \mathbb{Z}_{\geqslant 0}$ such that $\sum_{1 \leqslant j \leqslant \eta+1} 4k_j = \ell$. Therefore,

$$\mathbf{m} = (4k_1, 0, 0, 0, 2k_2, 0, 2k_3, 0, \dots, 2k_{\eta+1}, 0) \tag{4.34}$$

and hence

$$\widehat{\boldsymbol{s}} \bullet \mathbf{m} = (0,1,2,3,0,2,\dots,0,2) \bullet (4k_1,0,0,0,2k_2,0,2k_3,0,\dots,2k_{\eta+1},0) = 0.$$

(b) For *Claim 2*, suppose that $\Lambda \in X \backslash X^{C_4}$. Let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$ and $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^{t_1} \mid w^{t_1+1} \mid \cdots \mid w^{t_1+t_2} \mid w^0,$$

as (4.32). Since $\Lambda \in X \backslash X^{C_4}$, there exists the smallest $1 \leqslant j_0 \leqslant t_1 + t_2$ such that $\sigma_4 \cdot w^{j_0} \neq w^{j_0}$. Note that w^{i_0} should be one of

$$\begin{array}{lll} 1000,\ 0100,\ 0010,\ 0001, & 1100,\ 0110,\ 0011,\ 1001, & \\ 1110,\ 0111,\ 1011,\ 1101, & 1010,\ 0101, & 20,\ 02. \end{array}$$

We shall only give a proof for the case where $w^{j_0} = 1000$ since the other cases can be proven by the same argument. Since $\mathbf{m} \in \mathbf{M}_\ell(4,2;(1),(2^\eta))$ and w^{j_0} contains 1, by (3.13), w^j is 1111 for all $1 \leqslant j < j_0$. Thus we have

$$\begin{aligned}\sigma_4^i \blacksquare_{4,2} \mathbf{w} &= \quad w^1 \quad \mid \quad w^2 \quad \mid \cdots \mid w^{j_0-1} \mid \sigma \cdot w^{j_0} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{4t+1} \cdots w_u \\ &= 1111 \mid 1111 \mid \cdots \mid 1111 \mid \begin{matrix} 0100 \\ 0010 \\ 0001 \end{matrix} \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{4t+1}\cdots w_u \quad \begin{matrix} \text{if } i=1, \\ \text{if } i=2, \\ \text{if } i=3, \end{matrix}\end{aligned}$$

which implies

$$\sigma_4^i \blacksquare_{4,2} \mathbf{m} = (m_0 - 1, m_1 + \delta_{i,1}, m_2 + \delta_{i,2}, m_3 + \delta_{i,3}, m_4 \dots, m_n) \quad \text{for } i = 1,2,3,$$

by the construction of $\widehat{\Psi}^{-1}$ in (3.14). Hence, we have

$$\widehat{\boldsymbol{s}} \bullet (\mathbf{m} - \sigma_4 \blacksquare_{4,2} \mathbf{m}) = (0,1,2,3,0,2,\dots,0,2) \bullet (1,-1,0,0,0,\dots,0) \equiv_4 3,$$

$$\begin{aligned}
\widehat{\boldsymbol{s}} \bullet (\sigma_4 \blacksquare_{4,2} \mathbf{m} - \sigma_4^2 \blacksquare_{4,2} \mathbf{m}) &= (0,1,2,3,0,2,\dots,0,2) \bullet (0,1,-1,0,0,\dots,0) \equiv_4 3, \\
\widehat{\boldsymbol{s}} \bullet (\sigma_4^2 \blacksquare_{4,2} \mathbf{m} - \sigma_4^3 \blacksquare_{4,2} \mathbf{m}) &= (0,1,2,3,0,2,\dots,0,2) \bullet (0,0,1,-1,0,\dots,0) \equiv_4 3, \\
\widehat{\boldsymbol{s}} \bullet (\sigma_4^3 \blacksquare_{4,2} \mathbf{m} - \mathbf{m}) &= (0,1,2,3,0,2,\dots,0,2) \bullet (-1,0,0,1,0,\dots,0) \equiv_4 3.
\end{aligned}$$

This completes the proof. $\square$

Lemma 4.16. *Under the C_4-action on $P^+_{\mathrm{cl},\ell}$ given in (4.25) and the C_4-action on $\mathbf{M}_\ell(4,2;(1),(2^\eta))$ given in (3.12), we have*

$$\left| \left(P^+_{\mathrm{cl},\ell}\right)^{C_4} \right| = \left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right|.$$

Proof. By (4.25), one can see that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{C_4} = \left\{ \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; \begin{array}{l} m_0 = m_1 = m_{n-1} = m_n, \\ m_{2j} = m_{2j+1} \quad \text{for } 1 \leqslant j \leqslant \eta, \\ m_0 + m_1 + \sum_{2\leqslant j\leqslant n-2} 2m_j + m_{n-1} + m_n = \ell \end{array} \right\}. \tag{4.35}$$

We claim that

$$\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4} = \left\{ (4k_1,0,0,0,2k_2,0,2k_3,0,\dots,2k_{\eta+1},0) \;\middle|\; k_i \in \mathbb{Z}_{\geqslant 0}, \sum_{1\leqslant i\leqslant \eta+1} 4k_i = \ell \right\} =: Y.$$

By (4.33) and (4.34), $\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}$ is contained in Y.

On the contrary, for an element $\mathbf{m} \in Y$, we have

$$\widehat{\Psi}(\mathbf{m}) = \underbrace{1\cdots1}_{4k_1} \mid 0000 \mid \underbrace{2\cdots2}_{2k_2} \mid 00 \mid \underbrace{2\cdots2}_{2k_3} \mid 00 \mid \cdots \mid \underbrace{2\cdots2}_{2k_{\eta+1}} \mid 0.$$

Thus, by (3.12) and (3.13), $\sigma_4 \blacksquare_{4,2} \mathbf{m} = \mathbf{m}$ and hence our claim follows.

Next, we have an obvious bijection $\Theta : \left(P^+_{\mathrm{cl},\ell}\right)^{C_4} \to (\mathbf{M}_\ell(4,2;(1),(2^\eta)))^{C_4}$ defined by

$$\Theta\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = (4m_0,0,0,0,2m_2,0,2m_4,0,\dots,2m_{n-3},0).$$

This completes the proof. $\square$

Lemma 4.17. *Under the C_2-action given in (3.12), we have*

$$\left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| = b_0 - b_1 + b_2 - b_3.$$

Proof. As we did in the proof of Lemma 4.15, we set $X = \tau \cdot P^+_{\mathrm{cl},\ell}$ and $X(i) := \tau \cdot \left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_s(\Lambda) \equiv_4 i\right\}$. Note the following:

- Under the C_2-action on X given in (4.29), $\left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| = \left|X^{C_2}\right|$.
- For any $\Lambda \in P^+_{\mathrm{cl},\ell}$, $\mathsf{ev}_s(\Lambda) = \widehat{\boldsymbol{s}} \bullet \phi_{\mathscr{F}}(\tau\cdot\Lambda)$, where $\widehat{\boldsymbol{s}} := \tau \cdot \widetilde{\boldsymbol{s}} = (0,1,2,3,0,2,0,2,\dots,0,2)$.

Suppose that the following claims hold:

Claim 1. Let $\Lambda \in X^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then $\widehat{\boldsymbol{s}} \bullet \mathbf{m} \equiv_2 0$ and hence $X^{C_2} \subset X(0) \cup X(2)$.

Claim 2. Let $\Lambda \in X\backslash X^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then

$$\widehat{\boldsymbol{s}} \bullet (\mathbf{m} - \sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1.$$

By *Claim1*, we have

$$X(1) \sqcup X(3) \subset X\backslash X^{C_2}. \tag{4.36}$$

Let $\Lambda \in (X(0) \sqcup X(2)) \cap X\backslash X^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. By *Claim2*, we have $\widehat{\boldsymbol{s}} \bullet (\sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1$ and thus $\sigma_2 \blacksquare_{2,1} \Lambda \in (X(1) \sqcup X(3)) \cap X\backslash X^{C_2}$. So, we obtain a bijection from $(X(0) \sqcup X(2)) \cap X\backslash X^{C_2}$ to $(X(1) \sqcup X(3)) \cap X\backslash X^{C_2}$ by mapping Λ to $\sigma_2 \blacksquare_{2,1} \Lambda$. By (4.36), we have

$$\left|(X(0) \sqcup X(2)) \cap X\backslash X^{C_2}\right| = \left|(X(1) \sqcup X(3)) \cap X\backslash X^{C_2}\right| = |X(1) \sqcup X(3)|$$

$$= |X(1)| + |X(3)|.$$

Finally we have

$$\begin{aligned}\left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| &= \left|X^{C_2}\right| = |X(0) \sqcup X(2)| - \left|(X(0) \sqcup X(2)) \cap X \backslash X^{C_2}\right| \\ &= (|X(0)| + |X(2)|) - (|X(1)| + |X(3)|) = b_0 - b_1 + b_2 - b_3.\end{aligned}$$

We omit the proof of *Claim1* and *Claim 2* since they can be proven in the same manner as those in the proof of Lemma 4.15. □

Lemma 4.18. *Under the $C_4 = \langle \sigma_4 \rangle$-action on $P^+_{\mathrm{cl},\ell}$ given in (4.25) and the C_2-action on $\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))$ given in (3.12), we have*

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = \left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right|.$$

Proof. By (4.25), one can see that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2} = \left\{ \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_{n-1}, m_1 = m_n \text{ and } 2m_0 + 2m_1 + \sum_{2 \leqslant j \leqslant n-2} m_j = \ell \right\}. \tag{4.37}$$

We claim that

$$\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2} = \left\{ (2k_1, 0, 2k_2, 0, m_4, m_5, \dots, m_n) \;\middle|\; \begin{array}{l} k_1, k_2, m_4, m_5, \dots, m_n \in \mathbb{Z}_{\geqslant 0}, \\ 2k_1 + 2k_2 + \sum_{4 \leqslant j \leqslant n} m_j = \ell \end{array} \right\} =: Y.$$

Suppose $\mathbf{m} \in \mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}$. Recall the function $\widehat{\Psi}$ from (3.13). Let $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^{t_1} \mid w_{2t_1+1} w_{2t_1+2} \cdots w_u, \tag{4.38}$$

where

- w^j is of length 2 for $1 \leqslant j \leqslant t_1$,
- w^{t_1} contains the 4th zero when we read $\mathbf{w}$ from left to right.

By (3.12) and (3.13), we have

$$\mathbf{w} = \underbrace{1 \cdots 1}_{2k_1} \mid 00 \mid \underbrace{1 \cdots 1}_{2k_2} \mid 00 \mid w_{2t_1+1} w_{2t_1+2} \cdots w_u$$

for some $k_1, k_2, \in \mathbb{Z}_{\geqslant 0}$; i.e., $w^{t_1} = 00$. Therefore, $\mathbf{m} \in Y$.

On the contrary, for $\mathbf{m}' \in Y$, we have

$$\widehat{\Psi}(\mathbf{m}') = \underbrace{1 \cdots 1}_{2k_1} \mid 00 \mid \underbrace{1 \cdots 1}_{2k_2} \mid 00 \mid \underbrace{2 \cdots 2}_{m'_4} 0 \underbrace{2 \cdots 2}_{m'_5} 0 \cdots\cdots 0 \underbrace{2 \cdots 2}_{m'_n}.$$

Therefore, $\sigma_2 \blacksquare_{2,1} \mathbf{m}' = \mathbf{m}'$ and hence our assertion follows.

Hence, we have an obvious bijection $\Theta : \left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2} \to \mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}$ defined by

$$\Theta\left(\sum_{0 \leqslant i \leqslant n} m_i \Lambda_i\right) = (2m_0, 0, 2m_1, 0, m_2, m_3, \dots, m_{n-2}).$$

This completes the proof. □

Proof of Theorem 4.14. Let ζ_4 be a 4th primitive root of unity. We will see that

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^j}\right| = P^+_{\mathrm{cl},\ell}(\zeta_4^j) \quad \text{for } j = 0, 1, 2, 3.$$

- When $j = 0$, since $\left|\left(P^+_{\mathrm{cl},\ell}\right)\right| = P^+_{\mathrm{cl},\ell}(1)$, it is trivial.
- For the case $j \in \{1, 3\}$, note that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{C_4} = \left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^j} \quad \text{and} \quad P^+_{\mathrm{cl},\ell}(\zeta_4^j) = b_0 + b_1\zeta_4^j - b_2 - b_3\zeta_4^j.$$

Lemma 4.15 and Lemma 4.16 say that

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_4}\right| = \left|\mathbf{M}_\ell(4,2;(1),(2^\eta))^{C_4}\right| = b_0 - b_2 = P^+_{\mathrm{cl},\ell}(\zeta_4^j).$$

- For the case $j=2$, note that

$$P^+_{\mathrm{cl},\ell}(-1) = b_0 - b_1 + b_2 - b_3.$$

Lemma 4.17 and Lemma 4.18 say that

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = \left|\mathbf{M}_\ell(2,1;(1^2),(2^{2\eta}))^{C_2}\right| = b_0 - b_1 + b_2 - b_3 = P^+_{\mathrm{cl},\ell}(-1).$$

Thus our assertion holds. □

5. Bicyclic sieving phenomenon for $D_n^{(1)}$

We start with reviewing the notion of bicyclic sieving phenomenon. For details, see [2, Section 3] or [19, Section 9].

Let X be a finite set with a permutation action of a finite *bicyclic group*, that is, a product $C_k \times C_{k'}$ for some $k,k' \in \mathbb{Z}_{>0}$. Fix embeddings $\omega : C_k \to \mathbb{C}^\times$ and $\omega' : C_{k'} \to \mathbb{C}^\times$ into the complex roots of unity. Let $X(q_1,q_2) \in \mathbb{Z}_{\geqslant 0}[q_1,q_2]$.

Proposition 5.1 ([2], Proposition 3.1)**.** *In the above situation, the following two conditions on the triple $(X, C_k \times C_{k'}, X(q_1,q_2))$ are equivalent:*

(1) *For any $(c,c') \in C_k \times C_{k'}$,*

$$X(\omega(c),\omega(c')) = \left|\left\{x \in X \mid (c,c')\,x = x\right\}\right|.$$

(2) *The coefficients $a(j_1,j_2)$ uniquely defined by the expansion*

$$X(q_1,q_2) \equiv \sum_{\substack{0\leqslant j_1<k\\ 0\leqslant j_2<k'}} a(j_1,j_2) q_1^{j_1} q_2^{j_2} \pmod{q_1^k - 1, q_2^{k'} - 1}$$

have the following interpretation: $a(j_1,j_2)$ is the number of orbits of $C_k \times C_{k'}$ on X for which the $C_k \times C_{k'}$-stabilizer subgroup of any element in the orbit lies in the kernel of the $C_k \times C_{k'}$-character $\rho^{(j_1,j_2)}$ defined by

$$\rho^{(j_1,j_2)}(c,c') = \omega(c)^{j_1}\omega'(c')^{j_2}.$$

Definition 5.2. When either of the above two conditions holds, we say that the triple $(X, C_k \times C_{k'}, X(q_1,q_2))$ exhibits the *bicyclic sieving phenomenon*.

In this section, we let $\mathfrak{g} = D_n^{(1)}$ ($n \equiv_2 0$). In this case, $(a_i^\vee)_{i=0}^n = (1,1,2,2,\ldots,2,1,1)$,

$$\widetilde{\mathfrak{s}}^{(1)} = (s_i^{(1)})_{i=0}^n = (0,0,\ldots,0,2,2) \quad \text{and} \quad \widetilde{\mathfrak{s}}^{(2)} = (s_i^{(2)})_{i=0}^n = (0,2,0,2,0,\ldots,2,0).$$

We set

$$\mathfrak{s}^{(1)} := \frac{1}{2}\widetilde{\mathfrak{s}}^{(1)} = (0,0,\ldots,0,1,1) \quad \text{and} \quad \mathfrak{s}^{(2)} := \frac{1}{2}\widetilde{\mathfrak{s}}^{(2)} = (0,1,0,1,0,\ldots,1,0).$$

Let

$$\sigma_1 = (0,n)(1,n-1) \in \mathfrak{S}_{[0,n]} \quad \text{and} \quad \sigma_2 = (0,1)(2,3)\cdots(n-4,n-3)(n-1,n) \in \mathfrak{S}_{[0,n]}.$$

Note that σ_1 and σ_2 commute to each other in $\mathfrak{S}_{[0,n]}$, so $\langle\sigma_1,\sigma_2\rangle \simeq C_2 \times C_2$. Thus, we can define a $C_2 \times C_2 = \langle\sigma_2\rangle \times \langle\sigma_2\rangle$-action on $P^+_{\mathrm{cl},\ell}$ by

$$(\sigma_2,e)\cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i := \sum_{0\leqslant i\leqslant n} m_{\sigma_1(i)}\Lambda_i \quad \text{and} \quad (e,\sigma_2)\cdot \sum_{0\leqslant i\leqslant n} m_i\Lambda_i := \sum_{0\leqslant i\leqslant n} m_{\sigma_2(i)}\Lambda_i. \tag{5.1}$$

Here e denotes the identity of C_2. Note that

$$a_j^\vee = a_{\sigma_k(j)}^\vee \quad \text{and} \quad \begin{cases} \left\{\mathfrak{s}_j^{(k)}, \mathfrak{s}_{\sigma_k(j)}^{(k)}\right\} = \{0,1\} & \text{if } \sigma_k(j) \neq j, \\ \mathfrak{s}_j^{(k)} = 0 & \text{if } \sigma_k(j) = j, \end{cases} \tag{5.2}$$

for any $k = 1, 2$ and $j = 0, 1, \ldots, n$.

Let

$$P^+_{\mathrm{cl},\ell}(q_1, q_2) := \sum_{\Lambda \in P^+_{\mathrm{cl},\ell}} q_1^{\mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda)} q_2^{\mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda)},$$

where $\mathsf{ev}_{\mathfrak{s}^{(t)}}(\Lambda) : P^+_{\mathrm{cl},\ell} \to \mathbb{Z}_{\geqslant 0}$ $(t = 1, 2)$ is defined as follow:

$$\sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \mapsto \mathfrak{s}^{(t)} \bullet \mathbf{m}.$$

Alternatively, $P^+_{\mathrm{cl},\ell}(q_1, q_2)$ can be defined by the geometric series as in the other affine types:

$$\sum_{\ell \geqslant 0} P^+_{\mathrm{cl},\ell}(q_1, q_2) t^\ell := \prod_{0 \leqslant i \leqslant n} \frac{1}{1 - q_1^{\mathfrak{s}_i^{(1)}} q_2^{\mathfrak{s}_i^{(2)}} t^{a_i^\vee}}. \tag{5.3}$$

We set

$$\mathsf{ev}_{\mathfrak{s}}(\Lambda) := (\mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda), \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda)).$$

Note that all components of $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$ are even. Therefore, by Theorem 2.14, we have

$$\begin{aligned} P^+_{\mathrm{cl},\ell}(q_1, q_2) &= \sum_{i_1, i_2 \geqslant 0} \left| \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) = i_1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) = i_2 \right\} \right| q_1^{i_1} q_2^{i_2} \\ &\equiv \sum_{0 \leqslant i_1, i_2 \leqslant 1} \left| \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) \equiv_2 i_1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) \equiv_2 i_2 \right\} \right| q_1^{i_1} q_2^{i_2} \pmod{q_1^2 - 1, q_2^2 - 1}. \end{aligned} \tag{5.4}$$

For simplicity, we let

$$P^+_{\mathrm{cl},\ell}(q_1, q_2) \equiv \sum_{0 \leqslant i_1, i_2 \leqslant 1} b_{(i_1, i_2)} q^{i_1} q^{i_2} \pmod{q_1^2 - 1, q_2^2 - 1}.$$

Throughout this subsection, we set

$$\eta'_1 = n - 3 \quad \text{and} \quad \eta'_2 = \frac{n-4}{2}.$$

Theorem 5.3. *Under the $C_2 \times C_2$-action given in (5.1),*

$$\left(P^+_{\mathrm{cl},\ell}, C_2 \times C_2, P^+_{\mathrm{cl},\ell}(q) \right)$$

exhibits the bicyclic sieving phenomenon.

Take

$$\begin{aligned} \tau_1 &= (n, n-2, \ldots, 2, 1)(n-1, n-3, \ldots, 3) \in \mathfrak{S}_{[0,n]} \text{ and} \\ \tau_2 &= (n-1, n-3, \ldots, 3, n, n-2, \ldots, 2) \in \mathfrak{S}_{[0,n]}. \end{aligned}$$

Then we have

$$\tau_i \cdot (a_0^\vee, a_1^\vee, \ldots, a_n^\vee) = \tau_i \cdot (1, 1, 2, 2, \ldots, 2, 1, 1) = (1, 1, 1, 1, 2, 2, \ldots, 2) \qquad \text{for } i = 1, 2.$$

(a) By breaking $(1, 1, 1, 1, 2, 2, \ldots, 2)$ into $(1, 1 \mid 1, 1 \mid 2, 2, \ldots, 2)$, the image of $\tau_1 \cdot P^+_{\mathrm{cl},\ell}$ under $\phi_{\mathscr{F}}$ can be identified with $\mathbf{M}_\ell(2, 1; (1^2), (2^{\eta'_1})) \subset \mathbb{Z}^{n+1}_{\geqslant 0}$ and we can define the C_2-action on $\tau_1 \cdot P^+_{\mathrm{cl},\ell}$ by

$$\sigma_2 \blacksquare_{2,1} \Lambda = \phi_{\mathscr{F}}^{-1}(\sigma_2 \blacksquare_{2,1} \phi_{\mathscr{F}}(\Lambda)) \quad \text{for } \Lambda \in \tau_1 \cdot P^+_{\mathrm{cl},\ell}. \tag{5.5}$$

(b) By breaking $(1, 1, 1, 1, 2, 2, \ldots, 2)$ into $(1, 1 \mid 1, 1 \mid 2, 2 \mid 2, 2 \mid \cdots \mid 2, 2 \mid 2)$, the image of $\tau_2 \cdot P^+_{\mathrm{cl},\ell}$ under $\phi_{\mathscr{F}}$ can be identified with $\mathbf{M}_\ell(2, 1; (1^2, 2^{\eta'_2}), (2)) \subset \mathbb{Z}^{n+1}_{\geqslant 0}$ and we can define the C_2-action on $\tau_2 \cdot P^+_{\mathrm{cl},\ell}$ by

$$\sigma_2 \blacksquare_{2,1} \Lambda = \phi_{\mathscr{F}}^{-1}(\sigma_2 \blacksquare_{2,1} \phi_{\mathscr{F}}(\Lambda)) \quad \text{for } \Lambda \in \tau_2 \cdot P^+_{\mathrm{cl},\ell}. \tag{5.6}$$

Lemma 5.4. *Under the C_2-action on $\mathbf{M}_\ell(2, 1; (1^2), (2^{\eta'_1}))$ given in (3.12),*

$$\left| \mathbf{M}_\ell(2, 1; (1^2), (2^{\eta'_1}))^{C_2} \right| = b_{(0,0)} - b_{(1,0)} + b_{(0,1)} - b_{(1,1)} = P^+_{\mathrm{cl},\ell}(-1, 1).$$

Proof. For simplicity, we set $X_1 := \tau_1 \cdot P^+_{\mathrm{cl},\ell}$ and $X_1(i_1, i_2) := \tau_1 \cdot \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \,\middle|\, \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) \equiv_2 i_1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) \equiv_2 i_2 \right\}$. Note the following:

- C_2-orbits are of length 1 or 2.
- Under the C_2-action on X_1 given in (5.5), $\left|\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2}\right| = \left|X_1^{C_2}\right|$.
- For any $\Lambda \in P^+_{\mathrm{cl},\ell}$, $\mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) = \widehat{\mathfrak{s}}^{(1)} \bullet \phi_{\mathscr{F}}(\tau \cdot \Lambda)$, where $\widehat{\mathfrak{s}}^{(1)} := \tau \cdot \mathfrak{s}^{(1)} = (0,1,0,1,0,0,\dots,0)$.

Suppose that the following claims hold (which will be proven below):

Claim 1. Let $\Lambda \in X_1^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then $\widehat{\mathfrak{s}}^{(1)} \bullet \mathbf{m} = 0$ and hence $X_1^{C_2} \subset X_1(0,0) \cup X_1(0,1)$.

Claim 2. Let $\Lambda \in X_1 \backslash X_1^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then

$$\widehat{\mathfrak{s}}^{(1)} \bullet (\mathbf{m} - \sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1.$$

By *Claim 1*, we have

$$X_1(1,1) \cup X_1(1,0) \subset X_1 \backslash X_1^{C_2}. \tag{5.7}$$

Let $\Lambda \in (X_1(0,0) \cup X_1(0,1)) \cap X_1 \backslash X_1^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Note that

$$\widehat{\mathfrak{s}}^{(1)} \bullet \phi_{\mathscr{F}}(\tau \cdot \Lambda) \equiv_2 \begin{cases} 0 & \text{if } \Lambda \in X_1(0,0) \cup X_1(0,1), \\ 1 & \text{if } \Lambda \in X_1(1,0) \cup X_1(1,1). \end{cases}$$

By *Claim 2*, we have $\widehat{\mathfrak{s}}^{(1)} \bullet (\sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1$ and thus $\sigma_2 \blacksquare_{2,1} \Lambda \in X_1(1,0) \cup X_1(1,1)$. So, we obtain a bijection from $(X_1(0,0) \cup X_1(0,1)) \cap X_1 \backslash X_1^{C_2}$ to $(X_1(1,0) \cup X_1(1,1)) \cap X_1 \backslash X_1^{C_2}$ by mapping Λ to $\sigma_2 \blacksquare_{2,1} \Lambda$. By (5.7), we have

$$\begin{aligned}\left|(X_1(0,0) \cup X_1(0,1)) \cap X_1 \backslash X_1^{C_2}\right| &= \left|(X_1(1,0) \cup X_1(1,1)) \cap X_1 \backslash X_1^{C_2}\right| \\ &= |X_1(1,0) \cup X_1(1,1)| = \left|P^+_{\mathrm{cl},\ell}(1,0)\right| + \left|P^+_{\mathrm{cl},\ell}(1,1)\right|.\end{aligned}$$

Finally we have

$$\left|\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2}\right| = \left|X_1^{C_2}\right| = b_{(0,0)} - b_{(1,0)} + b_{(0,1)} - b_{(1,1)}.$$

To complete the proof, we have only to verify *Claim 1* and *Claim 2*.

(a) For *Claim 1,* suppose $\Lambda \in X_1^{C_2}$. Let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$ and $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{2t+1} w_{2t+2} \cdots w_u, \tag{5.8}$$

where

- w^j is of length 2 for $1 \leqslant j \leqslant t$,
- w^t contains the 4th zero when we read $\mathbf{w}$ from left to right,
- $w_j = 0$ or 2 for $2t+1 \leqslant j \leqslant u$.

Since $\Lambda \in X_1^{C_2}$, $\mathbf{w}$ should be of the form

$$\underbrace{1 \cdots 1}_{2k_1} \mid 00 \mid \underbrace{1 \cdots 1}_{2k_2} \mid 00 \mid w_{2t+1} w_{2t+2} \cdots w_u$$

for $k_1, k_2 \in \mathbb{Z}_{\geqslant 0}$. Therefore,

$$\mathbf{m} = (2k_1, 0, 2k_2, 0, m_4, m_5, \dots, m_n) \tag{5.9}$$

for some $m_4, m_5, \dots, m_n \in \mathbb{Z}_{\geqslant 0}$ such that $2k_1 + 2k_2 + \sum_{4 \leqslant j \leqslant n} 2m_j = \ell$. Thus, we have

$$\widehat{\mathfrak{s}}^{(1)} \bullet \mathbf{m} = (0,1,0,1,0,0,\dots,0) \bullet (2k_1, 0, 2k_2, 0, m_4, m_5, \dots, m_n) = 0.$$

(b) For *Claim 2*, suppose $\Lambda \in X \backslash X^{C_2}$. Let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$ and $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{2t+1} w_{2t+2} \cdots w_u,$$

as (5.8). Since $\Lambda \in X \backslash X^{C_2}$, there exists the smallest integer $1 \leqslant j_0 \leqslant t$ such that $\sigma_2 \cdot w^{j_0} \neq w^{j_0}$. Note that w^{j_0} should be one of

$$10,\ 01.$$

Note that if there are $1 \leqslant j_1 < j_2 < j_0$ such that $w^{j_1} = w^{j_2} = 00$ then, by (3.13), 1 can not appear in w^{j_0} since $\mathbf{m} \in \mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))$. Therefore, there is at most one $j \in \{1,2,\ldots,j_0-1\}$ such that $w^j = 00$. Thus, we have four cases as follows:

$$\begin{aligned}
&w^{j_0} = 10 \text{ and there is no } j \in \{1,2,\ldots,j_0-1\} \text{ such that } w^j = 00,\\
&w^{j_0} = 10 \text{ and there is one } j \in \{1,2,\ldots,j_0-1\} \text{ such that } w^j = 00,\\
&w^{j_0} = 01 \text{ and there is no } j \in \{1,2,\ldots,j_0-1\} \text{ such that } w^j = 00, \text{ and}\\
&w^{j_0} = 01 \text{ and there is one } j \in \{1,2,\ldots,j_0-1\} \text{ such that } w^j = 00.
\end{aligned}$$

We shall only give a proof for the case that $w^{j_0} = 10$ and there is no $j \in \{1,2,\ldots,j_0-1\}$ such that $w^j = 00$ since the other cases can be proved by the same argument. In this case, $\mathbf{w}$ is of the form

$$\underbrace{1\cdots1}_{2k} \mid 10 \mid w^{j_0+1} \mid \cdots \mid w^t \mid w_{2t+1}w_{2t+2}\cdots w_u$$

for some $k \in \mathbb{Z}_{\geqslant 0}$. Thus,

$$\sigma_2 \blacksquare_{2,1} \mathbf{w} = \underbrace{1\cdots1}_{2k} \mid 01 \mid w^{j_0+1} \mid \cdots \mid w^{t_1} \mid w_{2t_1+1}w_{2t_1+2}\cdots w_u$$

and hence

$$\sigma_2 \blacksquare_{2,1} \mathbf{m} = (m_0-1, m_1+1, m_2, m_3, \ldots, m_n).$$

Thus we have

$$\widehat{\mathfrak{s}}^{(1)} \bullet (\mathbf{m} - \sigma_2 \blacksquare_{2,1} \mathbf{m}) = (0,1,0,1,0,0,\ldots,0) \bullet (1,-1,0,0,0,0,\ldots,0,0) \equiv_2 1.$$

This completes the proof. ☐

Lemma 5.5. *Under the C_2-action on $\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))$ given in (3.12) and the $\langle(\sigma_2,e)\rangle$-action on $P^+_{\mathrm{cl},\ell}$ given in (5.1), we have*

$$\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2} = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)}\right|.$$

Proof. By (5.1), one can see that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)} = \left\{(m_0,m_1,\ldots,m_n) \in \mathbb{Z}^{n+1}_{\geqslant 0} \;\middle|\; \begin{array}{l} m_0 = m_n,\ m_1 = m_{n-1},\ \text{and} \\ m_0 + m_1 + \sum_{2\leqslant j\leqslant n-2} 2m_j + m_{n-1} + m_n = \ell \end{array}\right\}. \tag{5.10}$$

We claim that

$$\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2} = \left\{(2k_1,0,2k_2,0,m_4,\ldots,m_n) \;\middle|\; \begin{array}{l} k_1,k_2,m_4\ldots,m_n \in \mathbb{Z}_{\geqslant 0} \\ 2k_1 + 2k_2 + \sum_{4\leqslant j\leqslant n} 2m_j = \ell \end{array}\right\} =: Y.$$

By (5.9), $\mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2}$ is contained in Y.

On the contrary, for $\mathbf{m}' = (2k_1',0,2k_2',0,m_4',m_5',\ldots,m_n') \in Y$, we have

$$\widehat{\Psi}(\mathbf{m}') = \underbrace{1\cdots1}_{2k_1'} \mid 00 \mid \underbrace{1\cdots1}_{2k_2'} \mid 00 \mid \underbrace{2\cdots2}_{2m_4'}0\underbrace{2\cdots2}_{2m_5'}0\cdots\cdots0\underbrace{2\cdots2}_{2m_n'}.$$

Thus, $\sigma_2 \blacksquare_{2,1} \mathbf{m}' = \mathbf{m}'$ and hence the claim follows.

Thus, we have an obvious bijection $\Theta : \left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)} \to \mathbf{M}_\ell(2,1;(1^2),(2^{\eta_1'}))^{C_2}$ defined by

$$\Theta\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = (2m_0,0,2m_1,0,m_2,m_3,\ldots,m_{n-2}).$$

This completes the proof. ☐

Lemma 5.6. *Under the C_2-action on $\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))$ given in (3.12),*

$$\left|\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}\right| = b_{(0,0)} + b_{(1,0)} - b_{(0,1)} - b_{(1,1)} = P^+_{\mathrm{cl},\ell}(1,-1).$$

Proof. For simplicity, we write $X_2 := \tau_2 \cdot P^+_{\mathrm{cl},\ell}$ and $X_2(i_1,i_2) := \tau_2 \cdot \left\{ \Lambda \in P^+_{\mathrm{cl},\ell} \,\middle|\, \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) \equiv_2 i_1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) \equiv_2 i_2 \right\}$. Note the following:

- C_2-orbits are of length 1 or 2.
- Under the C_2-action on X_2 given in (5.6), $\left|\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}\right| = \left|X_2^{C_2}\right|$.
- For any $\Lambda \in P^+_{\mathrm{cl},\ell}$, $\mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) = \widehat{\mathfrak{s}}^{(2)} \bullet \phi_{\mathscr{F}}(\tau \cdot \Lambda)$, where $\widehat{\mathfrak{s}}^{(2)} := \tau \cdot \mathfrak{s}^{(2)} = (0,1,0,1,\dots,0,1,0)$.

Suppose that the following claims hold:

Claim 1. Let $\Lambda \in X_2^{C_2}$ and let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then $\widehat{\mathfrak{s}}^{(2)} \bullet \mathbf{m} = 0$ and hence $X_2^{C_2} \subset X_2(0,0) \cup X_2(1,0)$.

Claim 2. Let $\Lambda \in X_2 \backslash X_2^{C_2}$ and let $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Then

$$\widehat{\mathfrak{s}}^{(2)} \bullet (\mathbf{m} - \sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1.$$

By *Claim 1*, we have

$$X_2(0,1) \cup X_2(1,1) \subset X_2 \backslash X_2^{C_2}. \tag{5.11}$$

Let $\Lambda \in (X_2(0,0) \cup X_2(1,0)) \cap X_2 \backslash X_2^{C_2}$ and $\mathbf{m} = \phi_{\mathscr{F}}(\Lambda)$. Note that

$$\widehat{\mathfrak{s}}^{(2)} \bullet \phi_{\mathscr{F}}(\tau \cdot \Lambda) \equiv_2 \begin{cases} 0 & \text{if } \Lambda \in X_2(0,0) \cup X_2(1,0), \\ 1 & \text{if } \Lambda \in X_2(0,1) \cup X_2(1,1). \end{cases}$$

Therefore, by *Claim 2*, we have $\widehat{\mathfrak{s}}^{(2)} \bullet (\sigma_2 \blacksquare_{2,1} \mathbf{m}) \equiv_2 1$ and thus $\sigma_2 \blacksquare_{2,1} \Lambda \in X_2(0,1) \cup X_2(1,1)$. So, we obtain a bijection from $(X_2(0,0) \cup X_2(1,0)) \cap X_2 \backslash X_2^{C_2}$ to $(X_2(0,1) \cup X_2(1,1)) \cap X_2 \backslash X_2^{C_2}$ by mapping Λ to $\sigma_2 \blacksquare_{2,1} \Lambda$. By (5.11), we have

$$\begin{aligned} \left|(X_2(0,0) \cup X_2(1,0)) \cap X_2 \backslash X_2^{C_2}\right| &= \left|(X_2(0,1) \cup X_2(1,1)) \cap X_2 \backslash X_2^{C_2}\right| \\ &= |X_2(0,1) \cup X_2(1,1)| = \left|P^+_{\mathrm{cl},\ell}(0,1)\right| + \left|P^+_{\mathrm{cl},\ell}(1,1)\right|. \end{aligned}$$

Finally we have

$$\left|\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}\right| = \left|X_2^{C_2}\right| = b_{(0,0)} + b_{(1,0)} - b_{(0,1)} - b_{(1,1)}.$$

We omit the proof of *Claim1* and *Claim 2* since they can be proven in the same manner as those in the proof of Lemma 5.4. □

Lemma 5.7. *Under the C_2-action on $\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))$ given in (3.12) and the $\langle(e,\sigma_2)\rangle$-action on $P^+_{\mathrm{cl},\ell}$ given in (5.1), we have*

$$\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2} = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right|.$$

Proof. By (5.1), one can see that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)} = \left\{ \sum_{0 \leqslant i \leqslant n} m_i \Lambda_i \in P^+_{\mathrm{cl},\ell} \,\middle|\, \begin{array}{l} m_{2j} = m_{2j+1}, \text{ for } j = 0,1,\dots,\frac{n-4}{2},\ m_{n-1} = m_n, \\ m_0 + m_1 + \sum_{2 \leqslant j \leqslant n-2} 2m_j + m_{n-1} + m_n = \ell \end{array} \right\}. \tag{5.12}$$

We claim that

$$\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2} = \left\{ \left(2k_1,0,2k_2,0,\dots,2k_{\frac{n}{2}},0,k_0\right) \,\middle|\, \begin{array}{l} k_0,\dots,k_{\frac{n}{2}} \in \mathbb{Z}_{\geqslant 0} \\ 2k_0 + 2k_1 + 2k_2 + \sum_{3 \leqslant j \leqslant \frac{n}{2}} 4k_j = \ell \end{array} \right\} =: Y.$$

Let $\mathbf{m} = \mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}$ and $\mathbf{w} = \widehat{\Psi}(\mathbf{m})$. Break $\mathbf{w}$ into subwords

$$w^1 \mid w^2 \mid \cdots \mid w^t \mid w_{2t+1} w_{2t+2} \cdots w_u, \tag{5.13}$$

where

- w^j is of length 2 for $1 \leqslant j \leqslant t$,
- w^t contains the nth zero when we read $\mathbf{w}$ from left to right.

Since $\mathbf{m} \in \mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}$, $\mathbf{w}$ should be of the form

$$\underbrace{1\cdots1}_{2k_1} \mid 00 \mid \underbrace{1\cdots1}_{2k_2} \mid 00 \mid \underbrace{2\cdots2}_{2k_3} \mid 00 \mid \underbrace{2\cdots2}_{2k_4} \mid \cdots \mid \underbrace{2\cdots2}_{2k_{\frac{n}{2}}} \mid 00 \mid \underbrace{22\cdots2}_{k_0},$$

where for $j=0,1,\ldots\frac{n}{2}$, $k_j \in \mathbb{Z}_{\geqslant 0}$ and $2k_0+2k_1+2k_2+\sum_{3\leqslant j\leqslant \frac{n}{2}} 4k_j = \ell$. Therefore,

$$\mathbf{m} = (2k_1,0,2k_2,0,2k_3,0,2k_4,0,\ldots,2k_{\frac{n}{2}},0,k_0). \tag{5.14}$$

Hence, $\mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}$ is contained in Y.

On the contrary, for $\mathbf{m} = (2k_1,0,2k_2,0,\ldots,2k_{\frac{n}{2}},0,k_0) \in Y$, we have

$$\widehat{\Psi}(\mathbf{m}) = \underbrace{1\cdots1}_{2k_1} \mid 00 \mid \underbrace{1\cdots1}_{2k_2} \mid 00 \mid \underbrace{2\cdots2}_{2k_3} \mid 00 \mid \underbrace{2\cdots2}_{2k_4} \mid 00 \mid \cdots\cdots \mid 00 \mid \underbrace{2\cdots2}_{2k_{\frac{n}{2}}} \mid 00 \mid \underbrace{2\cdots2}_{k_0}.$$

Thus, $\sigma_2 \blacksquare_{2,1} \mathbf{m} = \mathbf{m}$ and hence our claim follows.

We have an obvious bijection $\Theta : \left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)} \to \mathbf{M}_\ell(2,1;(1^2,2^{\eta_2'}),(2))^{C_2}$ defined by

$$\Theta\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = (2m_0,0,2m_2,0,\ldots,2m_{n-4},0,2m_n,0,m_{n-2}).$$

This completes the proof. □

Remark 5.8. Note that $\sigma_1\sigma_2 = (0,n-1)(1,n)(2,3)(4,5)\cdots(n-4,n-3) \in \mathfrak{S}_{[0,n]}$. Therefore,

$$\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,\sigma_2)} = \left\{ (m_0,m_1,\ldots,m_n) \in \mathbb{Z}^{n+1}_{\geqslant 0} \;\middle|\; \begin{array}{l} m_0 = m_{n-1},\ m_1 = m_n, \\ m_{2j} = m_{2j+1},\ \text{for } j=1,\ldots,\frac{n-4}{2},\ \text{and} \\ m_0+m_1+\sum_{2\leqslant j\leqslant n-2} 2m_j + m_{n-1}+m_n = \ell \end{array} \right\}. \tag{5.15}$$

Combining (5.12) with (5.15), we have $\left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right| = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,\sigma_2)}\right|$.

Lemma 5.9. *For any even integer $n \geqslant 4$ and $\ell \in \mathbb{Z}_{>0}$, there is a bijection between $P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1})$ and $P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_n)$, that is, $b_{(1,0)} = b_{(1,1)}$.*

Proof. Recall that

$$\begin{aligned} P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1}) &= \left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) \equiv_2 1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) \equiv_2 1\right\} \\ P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n}) &= \left\{\Lambda \in P^+_{\mathrm{cl},\ell} \;\middle|\; \mathsf{ev}_{\mathfrak{s}^{(1)}}(\Lambda) \equiv_2 1 \text{ and } \mathsf{ev}_{\mathfrak{s}^{(2)}}(\Lambda) \equiv_2 0\right\}. \end{aligned}$$

Note that, for any $\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1}) \cup P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_n)$, we have

$$\mathsf{ev}_{\mathfrak{s}^{(1)}}\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = m_{n-1}+m_n \equiv_2 1.$$

Note also that, for any $\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1})$, we have

$$\mathsf{ev}_{\mathfrak{s}^{(2)}}\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = m_1+m_3+\cdots+m_{n-1} \equiv_2 1.$$

Therefore, for $\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1})$, if $m_1+m_3+\cdots+m_{n-3}$ is even then m_{n-1} should be odd and so $m_{n-1}>0$. If $m_1+m_3+\cdots+m_{n-3}$ is odd, m_{n-1} should be even and so m_n should be odd and so $m_n>0$.

Let $\psi : P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_{n-1}) \to P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0+\Lambda_n)$ be a function defined by

$$\psi\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = \begin{cases} \displaystyle\sum_{0\leqslant i\leqslant n-2} m_i\Lambda_i + (m_{n-1}-1)\Lambda_{n-1} + (m_n+1)\Lambda_n & \text{if } m_1+m_3+\cdots+m_{n-3} \text{ is even,} \\ \displaystyle\sum_{0\leqslant i\leqslant n-2} m_i\Lambda_i + (m_{n-1}+1)\Lambda_{n-1} + (m_n-1)\Lambda_n & \text{if } m_1+m_3+\cdots+m_{n-3} \text{ is odd.} \end{cases}$$

By the above observations, one can easily see that ψ is well-defined.

One can easily see that the function $\psi^{-1} : P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0 + \Lambda_n) \to P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0 + \Lambda_{n-1})$ defined by

$$\psi^{-1}\left(\sum_{0\leqslant i\leqslant n} m_i\Lambda_i\right) = \begin{cases} \displaystyle\sum_{0\leqslant i\leqslant n-2} m_i\Lambda_i + (m_{n-1}+1)\Lambda_{n-1} + (m_n-1)\Lambda_n & \text{if } m_1+m_3+\cdots+m_{n-3} \text{ is even,} \\ \displaystyle\sum_{0\leqslant i\leqslant n-2} m_i\Lambda_i + (m_{n-1}-1)\Lambda_{n-1} + (m_n+1)\Lambda_n & \text{if } m_1+m_3+\cdots+m_{n-3} \text{ is odd.} \end{cases}$$

is the inverse function of ψ. Thus ψ is a bijection and hence our assertion follows. □

Proof of Theorem 5.3. By (1) of Proposition 5.1, it suffices to show that

$$P^+_{\mathrm{cl},\ell}(1,1) = \left|P^+_{\mathrm{cl},\ell}\right|,\ P^+_{\mathrm{cl},\ell}(-1,1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)}\right|,\ P^+_{\mathrm{cl},\ell}(1,-1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right|,\ P^+_{\mathrm{cl},\ell}(-1,-1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,\sigma_2)}\right|.$$

We have that

- $P^+_{\mathrm{cl},\ell}(1,1) = \left|P^+_{\mathrm{cl},\ell}\right|$,
- by Lemma 5.4 and Lemma 5.5, $P^+_{\mathrm{cl},\ell}(-1,1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)}\right|$,
- by Lemma 5.6 and Lemma 5.7, $P^+_{\mathrm{cl},\ell}(1,-1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right|$ and
- by Remark 5.8 and Lemma 5.9,
$$P^+_{\mathrm{cl},\ell}(-1,-1) = P^+_{\mathrm{cl},\ell}(1,-1) = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right| = \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,\sigma_2)}\right|.$$

Hence our assertion follows. □

6. Formulae on the number of maximal dominant weights

In this section, exploiting the sieving phenomenon on $P^+_{\mathrm{cl},\ell}$, we derive a closed formula for $|\max^+(\Lambda)|$. Based on this formula, we also derive a recursive formula for $|\max^+(\Lambda)|$. Finally, we observe a remarkable symmetry, called *level-rank duality*, on dominant maximal weights.

6.1. Closed formulae on the number of dominant maximal weights.

In case of $A^{(1)}_n$ type, we have already given a closed formula for $|\max^+(\Lambda)|$ for all $\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$ (see Theorem 4.6). We here derive such a formula for an affine Kac-Moody algebra of arbitrary type. Let us start with an example for reader's understanding.

Example 6.1. Let $\mathfrak{g} = E^{(1)}_6$ and $\ell \in \mathbb{Z}_{>0}$. Then $\mathtt{DR}(P^+_{\mathrm{cl},\ell}) = \{\ell\Lambda_0, (\ell-1)\Lambda_0 + \Lambda_1, (\ell-1)\Lambda_0 + \Lambda_6\}$. Under the $C_3 = \langle\sigma_3\rangle$ action on $P^+_{\mathrm{cl},\ell}$ as in (4.14), let N_T be the number of all orbits and N_F be the number of free orbits. By Theorem 4.9 together with (4.17), we have $N_T = |\max^+(\ell\Lambda_0)|$ and $N_F = |\max^+((\ell-1)\Lambda_0 + \Lambda_1)| = |\max^+((\ell-1)\Lambda_0 + \Lambda_6)|$. Since

$$\left|P^+_{\mathrm{cl},\ell}\right| = 3N_F + \left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_3}\right| = N_T + 2N_F,$$

there follows

$$N_F = \frac{1}{3}\left(\left|P^+_{\mathrm{cl},\ell}\right| - \left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_3}\right|\right) \quad \text{and} \quad N_T = \left|P^+_{\mathrm{cl},\ell}\right| - 2N_F.$$

Notice that for any type of Kac-Moody algebra $\mathfrak{g}$,

$$|P^+_{\mathrm{cl},\ell}| = |\mathbf{M}_\ell(1;\mathbf{a}^\vee)| = |\mathbf{M}_\ell(1;\tau\cdot\mathbf{a}^\vee)| \qquad \text{for } \tau \in \mathfrak{S}_{[0,n]},$$

where $\mathbf{a}^\vee := (a^\vee_0, a^\vee_1, \ldots, a^\vee_n) = [c]_{\Pi^\vee}$. For the definition of $\mathbf{M}_\ell(1;\boldsymbol{\nu})$, see (3.4).

Let ℓ be a nonnegative integer, $t_1 \in \mathbb{Z}_{>0}$ and $t_2, t_3, \ldots, t_k \in \mathbb{Z}_{\geqslant 0}$. Define

$$\xi_\ell(t_1;\varnothing) := \begin{cases} \ell & \text{if } t_1 > 1, \\ 0 & \text{if } t_1 = 1. \end{cases}$$

Choose a nonnegative integer $i_1 \in [0, \xi_\ell(t_1;\varnothing)]$ and define

$$\xi_\ell(t_2; i_1) := \begin{cases} \left\lfloor \frac{\ell - i_1}{2} \right\rfloor & \text{if } t_2 > 0, \\ 0 & \text{if } t_2 = 0. \end{cases}$$

For $1 < r < k$, suppose that $i_1, i_2, \ldots, i_{r-1}$ are chosen and $\xi_\ell(t_r; i_1, i_2, \ldots, i_{r-1})$ is defined. Now, choose a nonnegative integer $i_r \in [0, \xi_\ell(t_r; i_1, i_2, \ldots, i_{r-1})]$ and define

$$\xi_\ell(t_{r+1}; i_1, i_2, \ldots, i_r) := \begin{cases} \left\lfloor \frac{\ell - \sum_{1 \leqslant s \leqslant r} s i_s}{r+1} \right\rfloor & \text{if } t_{r+1} > 0, \\ 0 & \text{if } t_{r+1} = 0. \end{cases}$$

Since

$$i_r \leqslant \xi_\ell(t_r; i_1, i_2, \ldots, i_{r-1}) \leqslant \frac{\ell - \sum_{1 \leqslant s \leqslant r-1} s i_s}{r} \quad \text{for } 1 < r < k,$$

we have $\sum_{1 \leqslant s \leqslant r} s i_s \leqslant \ell$. This implies that $\xi_\ell(t_{r+1}; i_1, i_2, \ldots, i_r)$ is a nonnegative integer. For $1 \leqslant r \leqslant k$, if $i_1, i_2, \ldots, i_{r-1}$ and t_r are clear in the context, we simply write $\xi_\ell[r]$ for $\xi_\ell(t_r; i_1, \ldots, i_{r-1})$. With this notation, we have the following lemma.

Lemma 6.2. *Let $a, P \in \mathbb{Z}_{>0}, \ell \in \mathbb{Z}_{\geqslant 0}$ and $\boldsymbol{\nu} = (a^{t_1}, (2a)^{t_2}, \ldots, (ka)^{t_k}) \in \mathbb{Z}_{\geqslant 0}^{P+1}$ with $t_1, t_k > 0$ and $t_2, t_3, \ldots, t_{k-1} \geqslant 0$. With the above notation, we have*

$$|\mathbf{M}_\ell(1;\boldsymbol{\nu})| = \begin{cases} \displaystyle\sum_{i_1=0}^{\xi_{\ell/a}(t_1;\varnothing)} \sum_{i_2=0}^{\xi_{\ell/a}(t_2;i_1)} \cdots \sum_{i_k=0}^{\xi_{\ell/a}(t_k;i_1,i_2,\ldots,i_{k-1})} \prod_{r=1}^{k} \binom{i_r + t_r - 1 - \delta_{1,r}}{t_r - 1 - \delta_{1,r}} & \text{if } a \text{ divides } \ell, \\ 0 & \text{otherwise}, \end{cases} \tag{6.1}$$

where $\binom{-1}{-1}$ is set to be 1.

Proof. In case where a does not divide ℓ, in view of (3.4), one has that $\mathbf{M}_\ell(1;\boldsymbol{\nu}) = \varnothing$. Therefore we assume that a divides ℓ. We will prove our assertion by induction on k. Let $k = 1$. Then $\boldsymbol{\nu} = (a^{P+1})$ and

$$\mathbf{M}_\ell(1;\boldsymbol{\nu}) = \left\{ (m_0, m_1, \ldots, m_P) \in \mathbb{Z}_{\geqslant 0}^{P+1} \;\middle|\; \sum_{0 \leqslant j \leqslant P} a \cdot m_j = \ell \right\}.$$

It follows that $|\mathbf{M}_\ell(1;\boldsymbol{\nu})|$ is equal to $\binom{P + \frac{\ell}{a}}{\frac{\ell}{a}}$. On the other hand, since $P + 1 \geqslant 2$, the right hand side of (6.1) is given by

$$\sum_{0 \leqslant i_1 \leqslant \frac{\ell}{a}} \binom{i_1 + (P+1) - 2}{P + 1 - 2}.$$

Using Pascal's triangle, one can easily see that it is equal to $\binom{P + \frac{\ell}{a}}{\frac{\ell}{a}}$. Thus we can start the induction.

Let $k > 1$ and assume that our assertion holds for all positive integers less than k. Set $p_0 := \sum_{1 \leqslant j \leqslant k-1} t_j$ and $\boldsymbol{\nu}' = (a^{t_1}, (2a)^{t_2}, \ldots, ((k-1)a)^{t_{k-1}})$. Then, by (3.4),

$$\mathbf{M}_\ell(1;\boldsymbol{\nu}) = \left\{ (m_0, m_1, \ldots, m_P) \;\middle|\; \begin{array}{l} (m_0, m_1, \ldots, m_{p_0 - 1}) \in \mathbf{M}_{\ell - kai}(1;\boldsymbol{\nu}'), \\ (m_{p_0}, m_{p_0+1}, \ldots, m_P) \in \mathbf{M}_{kai}(1; ((ka)^{t_k})) \end{array} \text{ for some } 0 \leqslant i \leqslant \left\lfloor \frac{\ell}{ka} \right\rfloor \right\}.$$

It follows that

$$|\mathbf{M}_\ell(1;\boldsymbol{\nu})| = \sum_{i=0}^{\lfloor \ell/ka \rfloor} \left|\mathbf{M}_{\ell - kai}(1;\boldsymbol{\nu}')\right| \times \left|\mathbf{M}_{kai}(1; ((ka)^{t_k}))\right|.$$

Note that

$$\left|\mathbf{M}_{kai}(1; ((ka)^{t_k}))\right| = \binom{i + t_k - 1}{t_k - 1}.$$

Thus, by the induction hypothesis, we have

$$\begin{aligned}|\mathbf{M}_\ell(1;\boldsymbol{\nu})| &= \sum_{i=0}^{\lfloor \ell/ka \rfloor} \left(\sum_{i_1=0}^{\xi_{\ell/a-ki}[1]} \sum_{i_2=0}^{\xi_{\ell/a-ki}[2]} \cdots \sum_{i_{k-1}=0}^{\xi_{\ell/a-ki}[k-1]} \prod_{r=1}^{k-1} \binom{i_r + t_r - 1 - \delta_{1,r}}{t_r - 1 - \delta_{1,r}} \right) \cdot \binom{i + t_k - 1}{t_k - 1} \\ &= \sum_{(i_1,i_2,\ldots,i_{k-1},i) \in R} \left(\prod_{r=1}^{k-1} \binom{i_r + t_r - 1 - \delta_{1,r}}{t_r - 1 - \delta_{1,r}} \cdot \binom{i + t_k - 1}{t_k - 1} \right),\end{aligned} \tag{6.2}$$

where

$$R := \left\{ (i_1, i_2, \ldots, i_{k-1}, i) \in \mathbb{Z}^k \;\middle|\; 0 \leqslant i \leqslant \left\lfloor \frac{\ell}{ka} \right\rfloor \text{ and } 0 \leqslant i_r \leqslant \xi_{\ell/a-ki}[r] \text{ for } 1 \leqslant r \leqslant k-1 \right\}.$$

We claim that R is equal to

$$R' := \left\{ (i_1, i_2, \ldots, i_{k-1}, i_k) \in \mathbb{Z}^k \;\middle|\; 0 \leqslant i_r \leqslant \xi_{\ell/a}[r] \text{ for } 1 \leqslant r \leqslant k \right\}.$$

To prove $R \subseteq R'$, take $(i_1, i_2, \ldots, i_{k-1}, i) \in R$. Note that $\xi_{\ell/a-ki}[r] \leqslant \xi_{\ell/a}[r]$ for $1 \leqslant r \leqslant k-1$. Therefore it suffices to show that $i \leqslant \xi_{\ell/a}[k]$. Since $t_k > 1$, we have

$$\xi_{\ell/a}[k] = \left\lfloor \frac{\ell/a - \sum_{1 \leqslant s \leqslant k-1} s i_s}{k} \right\rfloor.$$

since

$$i_{k-1} \leqslant \xi_{\ell/a-ki}[k-1] \leqslant \frac{\ell/a - ki - \sum_{1 \leqslant s \leqslant k-2} s i_s}{k-1},$$

we have

$$i \leqslant \frac{\ell/a - \sum_{1 \leqslant s \leqslant k-1} s i_s}{k}$$

and hence $i \leqslant \xi_{\ell/a}[k]$.

For the reverse inclusion, take $(i_1, i_2, \ldots, i_k) \in R'$. The condition $i_k \leqslant \xi_{\ell/a}[k]$ implies that

$$i_k \leqslant \left\lfloor \frac{\ell/a - \sum_{1 \leqslant s \leqslant k-1} s i_s}{k} \right\rfloor \leqslant \left\lfloor \frac{\ell}{ka} \right\rfloor. \tag{6.3}$$

Therefore it suffices to show that $i_r \leqslant \xi_{\ell/a-ki_k}[r]$ for all $1 \leqslant r \leqslant k-1$. Let $r \in \{1, 2, \ldots, k-1\}$. If $t_r = 0$ then we have $i_r \leqslant \xi_{\ell/a}[r] = 0 = \xi_{\ell/a-ki_k}[r]$. Therefore, our claim holds in this case. Assume that $t_r > 0$. The first inequality in (6.3) implies that for each $1 \leqslant r \leqslant k-1$,

$$\sum_{r \leqslant s \leqslant k-1} s i_s \leqslant \ell/a - k i_k - \sum_{1 \leqslant s \leqslant r-1} s i_s.$$

Since i_s's are nonnegative for all $1 \leqslant s \leqslant k$, this inequality gives

$$i_r \leqslant \left\lfloor \frac{\ell/a - k i_k - \sum_{1 \leqslant s \leqslant r-1} s i_s}{r} \right\rfloor = \xi_{\ell/a-ki_k}[r],$$

as required.

Applying $R = R'$ to (6.2), we finally have

$$|\mathbf{M}_\ell(1;\boldsymbol{\nu})| = \sum_{i_1=0}^{\xi_{\ell/a}[1]} \sum_{i_2=0}^{\xi_{\ell/a}[2]} \cdots \sum_{i_k=0}^{\xi_{\ell/a}[k]} \prod_{r=1}^{k} \binom{i_r + t_r - 1 - \delta_{1,r}}{t_r - 1 - \delta_{1,r}}. \qquad \square$$

If there is no danger of confusion on ℓ, we write

$$\mathsf{m}(\boldsymbol{\nu}) = |\mathbf{M}_\ell(1;\boldsymbol{\nu})|.$$

From now on, we will compute the number of $\left(P^+_{\mathrm{cl},\ell}\right)^H$ for all subgroups H of C_{N} for the H-action induced by (4.14). For instance, in case where $\mathfrak{g} = E_6^{(1)}$, we showed in (4.23) that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{C_3} = \left\{ \sum_{0\leqslant i\leqslant 6} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_1 = m_6,\ m_2 = m_3 = m_5 \right\}. \tag{6.4}$$

Since $a_0^\vee = a_1^\vee = a_6^\vee = 1$, $a_2^\vee = a_3^\vee = a_5^\vee = 2$ and $a_4^\vee = 3$, by (6.4), we have

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_3}\right| = \left|\left\{(m_0,m_1,m_2)\in\mathbb{Z}^3_{\geqslant 0} \mid 3m_0+6m_1+3m_2=\ell\right\}\right|.$$

Thus $\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_3}\right|$ is equal to $\mathsf{m}(3^2,6^1)$. Similarly, for $B_n^{(1)}, C_n^{(1)}, A_{2n-1}^{(2)}, E_6^{(1)}$, and $E_7^{(1)}$, there exists a unique $\boldsymbol{\nu}$ such that $\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_3}\right|$ is equal to $\mathsf{m}(\boldsymbol{\nu})$. We list all $\boldsymbol{\nu}$'s in Table 6.1:

Types	$\left(P^+_{\mathrm{cl},\ell}\right)^{C_{\mathsf{N}}}$	$\boldsymbol{\nu}$
$B_n^{(1)}$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_n\right\}$	$(1^1, 2^{n-1})$
$C_n^{(1)}$ $(n \equiv_2 1)$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_{2j} = m_{2j+1}\ (0\leqslant j\leqslant \frac{n-1}{2})\right\}$	$(2^{(n+1)/2})$
$C_n^{(1)}$ $(n \equiv_2 0)$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_{2j} = m_{2j+1}\ (0\leqslant j\leqslant \frac{n-2}{2})\right\}$	$(1^1, 2^{n/2})$
$A_{2n-1}^{(2)}$ $(n \equiv_2 1)$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_{2j} = m_{2j+1}\ (0\leqslant j\leqslant \frac{n-1}{2})\right\}$	$(2^1, 4^{(n-1)/2})$
$A_{2n-1}^{(2)}$ $(n \equiv_2 0)$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_{2j} = m_{2j+1}\ (0\leqslant j\leqslant \frac{n-2}{2})\right\}$	$(2^2, 4^{(n-2)/2})$
$D_{n+1}^{(2)}$	$\left\{\sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_n\right\}$	(2^n)
$E_6^{(1)}$	$\left\{\sum_{0\leqslant i\leqslant 6} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_1 = m_6,\ m_2 = m_3 = m_5\right\}$	$(3^2, 6^1)$
$E_7^{(1)}$	$\left\{\sum_{0\leqslant i\leqslant 7} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_7,\ m_1 = m_6,\ m_3 = m_5\right\}$	$(2^2, 4^2, 6^1)$

TABLE 6.1. $(P^+_\ell)^{C_{\mathsf{N}}}$ and the corresponding $\boldsymbol{\nu}$ for other types

For $D_n^{(1)}(n \equiv_2 1)$ type, recall the C_4-action on $P^+_{\mathrm{cl},\ell}$ given in (4.25). In (4.35) and (4.37), we showed that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{C_4} = \left\{ \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_1 = m_{n-1} = m_n,\ m_{2j} = m_{2j+1} \quad \text{for } 1\leqslant j\leqslant \frac{n-3}{2} \right\}$$

and

$$\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2} = \left\{ \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_{n-1},\ m_1 = m_n \right\}.$$

Therefore we have

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{C_4}\right| = \mathsf{m}(4^{(n-1)/2}) \quad \text{and} \quad \left|\left(P^+_{\mathrm{cl},\ell}\right)^{\sigma_4^2}\right| = \mathsf{m}(2^{n-1}).$$

For $D_n^{(1)}(n \equiv_2 0)$ type, recall the $C_2\times C_2$-action on $P^+_{\mathrm{cl},\ell}$ given in (5.1). In (5.10) and (5.12), we showed that

$$\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)} = \left\{ \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_0 = m_n,\ m_1 = m_{n-1} \right\}$$

and

$$\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)} = \left\{ \sum_{0\leqslant i\leqslant n} m_i\Lambda_i \in P^+_{\mathrm{cl},\ell} \;\middle|\; m_{2j} = m_{2j+1},\ \text{for } j = 0,1,\dots,\frac{n-4}{2},\ m_{n-1} = m_n \right\}.$$

Therefore we have

$$\left|\left(P^+_{\mathrm{cl},\ell}\right)^{(\sigma_2,e)}\right| = \mathsf{m}(2^{n-1}) \quad \text{and} \quad \left|\left(P^+_{\mathrm{cl},\ell}\right)^{(e,\sigma_2)}\right| = \mathsf{m}(2^3, 4^{(n-4)/2}).$$

To summarize, we have the following closed formula for $|\max^+(\Lambda)|$ for each $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$.

Theorem 6.3. *For each $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, $|\max^+(\Lambda)|$ is given as in Table 6.2.*

Types	$|\max^+(\Lambda)| \quad \left(\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})\right)$
$A_n^{(1)}$	$\displaystyle\sum_{d\mid(n+1,\ell,i)} \frac{d}{(n+1)+\ell} \sum_{d'\mid(\frac{n+1}{d},\frac{\ell}{d})} \mu(d') \binom{((n+1)+\ell)/dd'}{\ell/dd'}$
$B_n^{(1)}$	$\frac{1}{2}\left(\mathsf{m}(1^3,2^{n-2}) - \mathsf{m}(1^1,2^{n-1})\right) + \delta_{i,0}\mathsf{m}(1^1,2^{n-1})$
$C_n^{(1)}$ $(n \equiv_2 1)$	$\frac{1}{2}\left(\mathsf{m}(1^{n+1}) - \mathsf{m}(2^{(n+1)/2})\right) + \delta_{i,0}\mathsf{m}(2^{(n+1)/2})$
$C_n^{(1)}$ $(n \equiv_2 0)$	$\frac{1}{2}\left(\mathsf{m}(1^{n+1}) - \mathsf{m}(1^1,2^{n/2})\right) + \delta_{i,0}\mathsf{m}(1^1,2^{n/2})$
$D_n^{(1)}$ $(n \equiv_2 1)$	$\frac{1}{4}\left(\mathsf{m}(1^4,2^{n-3}) - \mathsf{m}(2^{n-1})\right) + \frac{\delta(i=0,1)}{2}\left(\mathsf{m}(2^{n-1}) - \mathsf{m}(4^{(n-1)/2})\right) + \delta_{i,0}\mathsf{m}(4^{(n-1)/2})$
$D_n^{(1)}$ $(n \equiv_2 0)$	$\frac{1}{4}\left(\mathsf{m}(1^4,2^{n-3}) - \mathsf{m}(2^{n-1})\right) + \frac{\delta(i=0,1)}{2}\left(\mathsf{m}(2^{n-1}) - \mathsf{m}(2^3,4^{(n-4)/2})\right) + \delta_{i,0}\mathsf{m}(2^3,4^{(n-4)/2})$
$A_{2n-1}^{(2)}$ $(n \equiv_2 1)$	$\frac{1}{2}\left(\mathsf{m}(1^2,2^{n-1}) - \mathsf{m}(2^1,4^{(n-1)/2})\right) + \delta_{i,0}\mathsf{m}(2^1,4^{(n-1)/2})$
$A_{2n-1}^{(2)}$ $(n \equiv_2 0)$	$\frac{1}{2}\left(\mathsf{m}(1^2,2^{n-1}) - \mathsf{m}(2^2,4^{(n-2)/2})\right) + \delta_{i,0}\mathsf{m}(2^2,4^{(n-2)/2})$
$A_{2n}^{(2)}$	$\mathsf{m}(1^1,2^n)$
$D_{n+1}^{(2)}$	$\frac{1}{2}\left(\mathsf{m}(1^2,2^{n-1}) - \mathsf{m}(2^n)\right) + \delta_{i,0}\mathsf{m}(2^n)$
$F_4^{(1)}$	$\mathsf{m}(1^2,2^2,3)$
$E_6^{(2)}$	$\mathsf{m}(1^1,2^2,3^1,4^1)$
$G_2^{(1)}$	$\mathsf{m}(1^2,2^1)$
$D_4^{(3)}$	$\mathsf{m}(1^1,2^1,3^1)$
$E_6^{(1)}$	$\frac{1}{3}\left(\mathsf{m}(1^3,2^3,3^1) - \mathsf{m}(3^2,6^1)\right) + \delta_{i,0}\mathsf{m}(3^2,6^1)$
$E_7^{(1)}$	$\frac{1}{2}\left(\mathsf{m}(1^2,2^3,3^2,4^1) - \mathsf{m}(2^2,4^2,6)\right) + \delta_{i,0}\mathsf{m}(2^2,4^2,6)$
$E_8^{(1)}$	$\mathsf{m}(1^1,2^2,3^2,4^2,5^1,6^1)$

TABLE 6.2. Closed formulae for $|\max^+(\Lambda)|$

In Table 6.2, $\mathsf{m}(\boldsymbol{\nu})$'s appearing in the closed formulae for $|\max^+(\Lambda)|$ for the classical types can be expressed in terms of binomial coefficients as follows:

$$\mathsf{m}(1^n) = \binom{\ell+n-1}{\ell}, \ \mathsf{m}(1^1,2^n) = \binom{\lfloor\frac{\ell}{2}\rfloor+n}{\lfloor\frac{\ell}{2}\rfloor}, \ \mathsf{m}(1^2,2^n) = \binom{\lfloor\frac{\ell}{2}\rfloor+n+1}{\lfloor\frac{\ell}{2}\rfloor} + \binom{\lfloor\frac{\ell-1}{2}\rfloor+n+1}{\lfloor\frac{\ell-1}{2}\rfloor},$$

$$\mathsf{m}(1^3,2^n) = 2\binom{\lfloor\frac{\ell}{2}\rfloor+n+1}{\lfloor\frac{\ell}{2}\rfloor-1} + 2\binom{\lfloor\frac{\ell-1}{2}\rfloor+n+2}{\lfloor\frac{\ell-1}{2}\rfloor} + \binom{\lfloor\frac{\ell}{2}\rfloor+n+1}{\lfloor\frac{\ell}{2}\rfloor},$$

$$\mathsf{m}(1^4,2^{n-3}) = \begin{cases} 8\dbinom{\frac{\ell}{2}+n-1}{\frac{\ell}{2}-1} + \dbinom{\frac{\ell}{2}+n-2}{\frac{\ell}{2}} & \text{if } \ell \equiv_2 0, \\ 4\dbinom{\frac{\ell-1}{2}+n}{\frac{\ell-1}{2}} + 4\dbinom{\frac{\ell-1}{2}+n-1}{\frac{\ell-1}{2}-1} & \text{if } \ell \equiv_2 1, \end{cases}$$

$$\begin{aligned}
&\mathsf{m}(2^n) = \delta(\ell \equiv_2 0)\binom{\frac{\ell}{2}+n-1}{\frac{\ell}{2}}, \quad \mathsf{m}(2^1,4^n) = \delta(\ell \equiv_2 0)\binom{\lfloor\frac{\ell}{4}\rfloor+n}{\lfloor\frac{\ell}{4}\rfloor},\\
&\mathsf{m}(2^2,4^n) = \delta(\ell \equiv_2 0)\left(\binom{\lfloor\frac{\ell-2}{4}\rfloor+n+1}{\lfloor\frac{\ell-2}{4}\rfloor} + \binom{\lfloor\frac{\ell}{4}\rfloor+n+1}{\lfloor\frac{\ell}{4}\rfloor}\right),\\
&\mathsf{m}(2^3,4^n) = \delta(\ell \equiv_2 0)\left(2\binom{\lfloor\frac{\ell}{2}\rfloor+n+1}{\lfloor\frac{\ell}{2}\rfloor-1} + 2\binom{\lfloor\frac{\ell-1}{2}\rfloor+n+2}{\lfloor\frac{\ell-1}{2}\rfloor} + \binom{\lfloor\frac{\ell}{2}\rfloor+n+1}{\lfloor\frac{\ell}{2}\rfloor}\right),\\
&\mathsf{m}(4^n) = \delta(\ell \equiv_4 0)\binom{\frac{\ell}{4}+n-1}{\frac{\ell}{4}}.
\end{aligned}$$

For $B_n^{(1)}, C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$, the formulae in Table 6.2 can be expressed in terms of binomial coefficients. In particular, for $C_n^{(1)}$ type, the formula in even case and that in odd case can be merged regardless of rank.

Corollary 6.4. *Let $\mathfrak{g} = B_n^{(1)}, C_n^{(1)}, A_{2n}^{(2)}, D_{n+1}^{(2)}$. Then, for each $\Lambda \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$, we have a closed formula for $|\max^+(\Lambda)|$ in terms of binomial coefficients as in Table 6.3.*

Types	$\lvert\max^+(\Lambda)\rvert \quad \left(\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})\right)$	*Types*	$\lvert\max^+(\Lambda)\rvert \quad \left(\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})\right)$
$B_n^{(1)}$	$\binom{n+\lfloor\frac{\ell-\delta_{in}}{2}\rfloor}{n} + \binom{n+\lfloor\frac{\ell-1-\delta_{in}}{2}\rfloor}{n}$	$C_n^{(1)}$	$\frac{1}{2}\left(\binom{n+\ell}{n} + (-1)^i\delta(n\ell \equiv_2 0)\binom{\lfloor\frac{n+\ell}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}\right)$
$A_{2n}^{(2)}$	$\binom{n+\lfloor\frac{\ell}{2}\rfloor}{n}$	$D_{n+1}^{(2)}$	$\binom{n+\lfloor\frac{\ell-\delta_{in}}{2}\rfloor}{n}$

TABLE 6.3. Closed formulae for $|\max^+(\Lambda)|$ in terms of binomial coefficients

6.2. **Consequences of our closed formulae.** In this subsection, we introduce two consequences coming from our closed formulae, recursive formulae and level-rank duality for $|\max^+(\Lambda)|$.

6.2.1. *Recursive formulae(Triangular arrays).* The formulae in Table 6.2 and Table 6.3 enable us to compute $|\max^+(\Lambda)|$ recursively. We list recursive formulae for $|\max^+(\Lambda)|$ for all types except for $A_n^{(1)}$ and exceptional types. For clarity, we use $\max_{\mathfrak{g}}(\Lambda)$ to denote the set of dominant maximal weights to emphasize the rank of affine Kac-Moody algebras in consideration. Here we deal with only the $C_n^{(1)}$ type. For other types, see Appendix A.

Define $T_0^{C^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_0^{C^{(1)}}(n,0) = 1 \ (n \geqslant 0), \quad T_0^{C^{(1)}}(0,\ell) = 1 \ (\ell \geqslant 1), \text{ and}$$

$$T_0^{C^{(1)}}(n,\ell) = T_0^{C^{(1)}}(n,\ell-1) + T_0^{C^{(1)}}(n-1,\ell) - \delta(n\ell \equiv_2 1)\binom{\frac{n+\ell}{2}-1}{\frac{\ell-1}{2}} \ (n \geqslant 1,\ \ell \geqslant 1). \tag{6.5}$$

Using the formula in Table 6.3, one can see that $|\max^+_{C_n^{(1)}}(0)| = T_0^{C^{(1)}}(n,0)$ $(n \geqslant 2)$, $|\max^+_{C_2^{(1)}}(\ell\Lambda_0)| = T_0^{C^{(1)}}(2,\ell)$ $(\ell \geqslant 0)$, and $|\max^+_{C_n^{(1)}}(\ell\Lambda_0)|$ $(n \geqslant 3, \ell \geqslant 1)$ satisfies the same recursive relation as (6.5). Therefore we can conclude that $T_0^{C^{(1)}}(n,\ell) = |\max^+_{C_n^{(1)}}(\ell\Lambda_0)|$ for all $n \in \mathbb{Z}_{\geqslant 2},\ \ell \in \mathbb{Z}_{\geqslant 0}$. In particular, as a triangular array, $T_0^{C^{(1)}}$ can be described as follows:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & {\scriptstyle n=0} & & {\scriptstyle \ell=0} & & & & & & & \\
 & & & & & & {\scriptstyle n=1} & & 1 & & {\scriptstyle \ell=1} & & & & & & \\
 & & & & & {\scriptstyle n=2} & & 1 & & 1 & & {\scriptstyle \ell=2} & & & & & \\
 & & & & {\scriptstyle n=3} & & 1 & & 1 & & 1 & & {\scriptstyle \ell=3} & & & & \\
 & & & {\scriptstyle n=4} & & 1 & & 2 & & 2 & & 1 & & {\scriptstyle \ell=4} & & & \\
 & & {\scriptstyle n=5} & & 1 & & 2 & & 4 & & 2 & & 1 & & {\scriptstyle \ell=5} & & \\
 & {\scriptstyle n=6} & & 1 & & 3 & & 6 & & 6 & & 3 & & 1 & & {\scriptstyle \ell=6} & \\
{\scriptstyle n=7} & & 1 & & 3 & & 9 & & 10 & & 9 & & 3 & & 1 & & {\scriptstyle \ell=7} \\
 & 1 & & 4 & & 12 & & 19 & & 19 & & 12 & & 4 & & 1 & \\
\iddots & & \iddots & & \iddots & & \iddots & & \vdots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array},$$

which is known to be *Lozanić's triangle* [20, A034851]. To compare $T_0^{C^{(1)}}(n,\ell)$ with $|\max^+_{C_n^{(1)}}(\ell\Lambda_0)|$, we write the triangular array $T_0^{C^{(1)}}$ in a different direction from [20]. In the rest of this paper, we use this convention for triangular arrays.

Similarly, define $T_1^{C^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$\begin{aligned} &T_1^{C^{(1)}}(n,0) = 0 \ (n \geqslant 0), \quad T_1^{C^{(1)}}(0,\ell) = 0 \ (\ell \geqslant 1), \text{ and} \\ &T_1^{C^{(1)}}(n,\ell) = T_1^{C^{(1)}}(n,\ell-1) + T_1^{C^{(1)}}(n-1,\ell) + \delta(n,\ell \equiv_2 1)\binom{\frac{n+\ell}{2}-1}{\frac{\ell-1}{2}} \ (n \geqslant 1, \ \ell \geqslant 1). \end{aligned}$$

Then for all $n \in \mathbb{Z}_{\geqslant 2}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_1^{C^{(1)}}(n,\ell) = |\max^+_{C^{(1)}}((\ell-1)\Lambda_0 + \Lambda_1)|$. In particular, as a triangular array, $T_1^{C^{(1)}}$ can be described as follows:

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & {\scriptstyle n=0} & 0 & {\scriptstyle \ell=0} & & & & & & & \\
 & & & & & & {\scriptstyle n=1} & 0 & & 0 & {\scriptstyle \ell=1} & & & & & & \\
 & & & & & {\scriptstyle n=2} & 0 & & 1 & & 0 & {\scriptstyle \ell=2} & & & & & \\
 & & & & {\scriptstyle n=3} & 0 & & 1 & & 1 & & 0 & {\scriptstyle \ell=3} & & & & \\
 & & & {\scriptstyle n=4} & 0 & & 2 & & 2 & & 2 & & 0 & {\scriptstyle \ell=4} & & & \\
 & & {\scriptstyle n=5} & 0 & & 2 & & 4 & & 4 & & 2 & & 0 & {\scriptstyle \ell=5} & & \\
 & {\scriptstyle n=6} & 0 & & 3 & & 6 & & 10 & & 6 & & 3 & & 0 & {\scriptstyle \ell=6} & \\
{\scriptstyle n=7} & 0 & & 3 & & 9 & & 16 & & 16 & & 9 & & 3 & & 0 & {\scriptstyle \ell=7} \\
\iddots & & \iddots & & \iddots & & \iddots & & \vdots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array},$$

which is known in [20, A034852] and $T_1^{C^{(1)}} = (\text{Pascal triangle}) - T_0^{C^{(1)}}$.

6.2.2. *Level-rank duality.* From closed formulae in the subsection 6.1 and triangular arrays in the subsection 6.2.1, we observe a very noteworthy symmetry, called *level-rank duality*, between certain sets of dominant maximal weights except for $D_n^{(1)}$ and $A_{2n-1}^{(2)}$. Here we deal with only the $A_n^{(1)}$ type. For other types, see Appendix B.

Let $\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$. From Table 6.2 it follows that

$$\left|\max^+_{A_n^{(1)}}(\Lambda)\right| = \sum_{d|(n+1,\ell,i)} \frac{d}{(n+1)+\ell} \sum_{d'|(\frac{n+1}{d},\frac{\ell}{d})} \mu(d') \binom{((n+1)+\ell)/dd'}{\ell/dd'}.$$

Hence, for $n \geqslant 1$ and $\ell > 1$, if $(n+1,\ell,i) = (\ell,n+1,j)$ for some $0 \leqslant i,j \leqslant \min(n,\ell)$, then

$$\left|\max^+_{A_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_i)\right| = \left|\max^+_{A_{\ell-1}^{(1)}}(n\Lambda_0 + \Lambda_j)\right|, \tag{6.6}$$

i.e., when we exchange $n+1$ with ℓ, the number of dominant maximal weights remains same.

Let us deal with the relation between our duality and Frenkel's duality in [7]. For a residue i modulo n, let $\Lambda_i^{(n)}$ denote the ith fundamental weight of $A_n^{(1)}$. For $\Lambda = \sum_{i=0}^n m_i \Lambda_i^{(n)} \in P^+_{\mathrm{cl},\ell}$, let Λ' be the dominant integral weight of $A_{\ell-1}^{(1)}$ defined by

$$\Lambda' = \sum_{i=0}^n \Lambda^{(\ell-1)}_{m_i+m_{i+1}+\cdots+m_n} \in P^+_{\mathrm{cl},n+1}.$$

With this setting, Frenkel found the following duality between the q-specialized characters of $V(\Lambda)$ and $V(\Lambda')$:

$$\dim_q(V(\Lambda)) \prod_{k=0}^{\infty} \frac{1}{1-q^{(n+1)k}} = \dim_q(V(\Lambda')) \prod_{k=0}^{\infty} \frac{1}{1-q^{\ell k}},$$

where $\dim_q(V)$ is the q-specialized character of V (see [7, Theorem 2.3] or [22, Subsection 4.4]).

Now we will show that

$$\left|\max^+_{A_n^{(1)}}(\Lambda)\right| = \left|\max^+_{A_{\ell-1}^{(1)}}(\Lambda')\right|.$$

Recall ev_s in (2.13). Letting $\Lambda \in P^+_{\mathrm{cl},\ell}((\ell-1)\Lambda_0 + \Lambda_{i_0})$ and $\Lambda' \in P^+_{\mathrm{cl},n+1}(n\Lambda_0 + \Lambda_{j_0})$, then $\mathsf{ev}_s(\Lambda) \equiv_{n+1} i_0$ and $\mathsf{ev}_{s'}(\Lambda') \equiv_\ell j_0$ by Theorem 2.14. Here $\boldsymbol{S} = (1,2,\ldots,n)$ and $\boldsymbol{S}' = (1,2,\ldots,\ell-1)$. On the other hand, by the definition of Λ', we have

$$\mathsf{ev}_{s'}(\Lambda') \equiv_\ell \sum_{i=0}^{n}(i+1)m_i,$$

which implies that

$$\mathsf{ev}_{s'}(\Lambda') - \mathsf{ev}_{s}(\Lambda) \equiv_\ell \sum_{i=0}^{n}(i+1)m_i - \sum_{i=0}^{n} i m_i = \sum_{i=0}^{n} m_i = \ell \equiv_\ell 0. \tag{6.7}$$

By (6.7), we have

$$j_0 - i_0 \equiv_{(n+1,\ell)} \mathsf{ev}_{s'}(\Lambda') - \mathsf{ev}_{s}(\Lambda) \equiv_{(n+1,\ell)} 0.$$

Therefore $(n+1, \ell, i_0) = (\ell, n+1, j_0)$ and our assertion follows from (6.6).

Appendix A. Recursive formulae for $|\max^+(\Lambda)|$

$A_{2n}^{(2)}$ **type.** Define $T_0^{A_{\text{even}}^{(2)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_0^{A_{\text{even}}^{(2)}}(n,0) = T_0^{A_{\text{even}}^{(2)}}(n,1) = 1 \ (n \geqslant 0), \quad T_0^{A_{\text{even}}^{(2)}}(0,\ell) = 1 \ (\ell \geqslant 2), \text{ and}$$

$$T_0^{A_{\text{even}}^{(2)}}(n,\ell) = T_0^{A_{\text{even}}^{(2)}}(n,\ell-2) + T_0^{A_{\text{even}}^{(2)}}(n-1,\ell). \tag{A.1}$$

Then for all $n \in \mathbb{Z}_{\geqslant 2}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{A_{\text{even}}^{(2)}}(n,\ell) = |\max^+_{A_{2n}^{(2)}}(\ell\Lambda_0)|$. In particular, as a triangular array, T_0 can be described as follows:

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&1\\
&&&&&&&1&&1\\
&&&&&&1&&1&&1\\
&&&&&1&&1&&2&&1\\
&&&&1&&1&&3&&2&&1\\
&&&1&&1&&4&&3&&3&&1\\
&&1&&1&&5&&4&&6&&3&&1\\
&1&&1&&6&&5&&10&&6&&4&&1\\
\iddots&&\iddots&&\iddots&&\iddots&&\vdots&&\ddots&&\ddots&&\ddots&&\ddots
\end{array},$$

which is known in [20, A065941]. It is Pascal's triangle with duplicated diagonals, i.e., $T_0^{A_{\text{even}}^{(2)}}(n,2\ell) = T_0^{A_{\text{even}}^{(2)}}(n,2\ell+1) = \binom{n+\ell}{\ell}$ for $n, \ell \geqslant 0$. The recursive condition (A.1) says if we add the circled 1 and the circled 2 then we get double circled 3.

$B_n^{(1)}$ **type.** Define $T_0^{B^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_0^{B^{(1)}}(n,0) = 1 \ (n \geqslant 1), \quad T_0^{B^{(1)}}(n,1) = 2 \ (n \geqslant 1), \quad T_0^{B^{(1)}}(0,\ell) = 2 \ (\ell \geqslant 0), \text{ and}$$

$$T_0^{B^{(1)}}(n,\ell) = T_0^{B^{(1)}}(n,\ell-2) + T_0^{B^{(1)}}(n-1,\ell) \ (n \geqslant 1, \ \ell \geqslant 2).$$

Then for all $n \in \mathbb{Z}_{\geqslant 3}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{B^{(1)}}(n,\ell) = |\max^+_{B_n^{(1)}}(\ell\Lambda_0)|$. In particular, as a triangular array, $T_0^{B^{(1)}}$ can be described as follows:

$$\begin{array}{ccccccccccccccccc}
&&&&&&&&2\\
&&&&&&&1&&2\\
&&&&&&1&&2&&2\\
&&&&&1&&2&&3&&2\\
&&&&1&&2&&4&&4&&2\\
&&&1&&2&&5&&6&&5&&2\\
&&1&&2&&6&&8&&9&&6&&2\\
&1&&2&&7&&10&&14&&12&&7&&2\\
\iddots&&\iddots&&\iddots&&\iddots&&\vdots&&\ddots&&\ddots&&\ddots&&\ddots
\end{array}.$$

This array is the triangular array obtained by removing the left boundary diagonal from the triangular array in [20, A129714] whose row sums are the Fibonacci numbers.

Define $T_n^{B^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_n^{B^{(1)}}(n,0) = 0 \ (n \geqslant 0) \quad \text{and} \quad T_n^{B^{(1)}}(n,\ell) = T_0^{B^{(1)}}(n,\ell-1) \quad \text{for } n \geqslant 0, \ell > 0.$$

Then for all $n \in \mathbb{Z}_{\geqslant 3}$, $\ell \in \mathbb{Z}_{>0}$, we have $T_n^{B^{(1)}}(n,\ell) = |\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_n)|$. In particular, as a triangular array, $T_n^{B^{(1)}}$ can be described as follows:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&\color{red}0\\
&&&&&&&\color{red}0&&2\\
&&&&&&\color{red}0&&1&&2\\
&&&&&\color{red}0&&1&&2&&2\\
&&&&\color{red}0&&1&&2&&3&&2\\
&&&\color{red}0&&1&&2&&4&&4&&2\\
&&\color{red}0&&1&&2&&5&&6&&5&&2\\
&\color{red}0&&1&&2&&6&&8&&9&&6&&2\\
\iddots&&\iddots&&\iddots&&\iddots&&\vdots&&\ddots&&\ddots&&\ddots&&\ddots
\end{array}.
$$

Note that Table 6.3 says

$$
\left|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \binom{n+\lfloor\frac{\ell-1}{2}\rfloor}{n} + \binom{n+\lfloor\frac{\ell-2}{2}\rfloor}{n} = \left|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0)\right| \quad \text{for } \ell > 0,
$$

which explains the reason why we define $T_n^{B^{(1)}}(n,\ell)$ as $T_0^{B^{(1)}}(n,\ell-1)$ for $n \geqslant 0, \ell > 0$. To emphasize this, we denote by $\color{red}0$ the zeros in the left boundary diagonal. Note also that $T_0^{B^{(1)}}(n,\ell)$ and hence $T_n^{B^{(1)}}(n,\ell)$ can be obtained from $T_0^{A^{(2)}_{\rm even}}$ as follows:

$$
T_0^{B^{(1)}}(n,\ell) = T_0^{A^{(2)}_{\rm even}}(n,\ell) + T_0^{A^{(2)}_{\rm even}}(n,\ell-1) \quad \text{for } n,\ell \geqslant 1.
$$

$D_n^{(1)}$ **type.** Define $T_0^{D^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$
\begin{aligned}
&T_0^{D^{(1)}}(n,0) = T_0^{D^{(1)}}(n,1) = 1 \ (n \geqslant 0), T_0^{D^{(1)}}(0,2) = 3,\ T_0^{D^{(1)}}(0,2\ell-1) = 2 \ (\ell \geqslant 2),\ T_0^{D^{(1)}}(0,2\ell) = 4 \ (\ell \geqslant 2),\\
&T_0^{D^{(1)}}(n,\ell) = T_0^{D^{(1)}}(n,\ell-2) + T_0^{D^{(1)}}(n-1,\ell)\\
&\qquad\qquad + (-1)^n\delta(\ell \equiv_2 0)(1+\delta(n \equiv_2 1))T_0^{A^{(2)}_{\rm even}}\left(\left\lfloor\frac{n-1}{2}\right\rfloor, \frac{\ell}{2}-1\right) \quad (n \geqslant 1, \ell \geqslant 2).
\end{aligned}
$$

Then for all $n \in \mathbb{Z}_{\geqslant 4}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{D^{(1)}}(n,\ell) = |\max^+_{D_n^{(1)}}(\ell\Lambda_0)|$. In particular, as a triangular array, T_0 can be described as follows:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&1\\
&&&&&&&1&&1\\
&&&&&&1&&1&&3\\
&&&&&1&&1&&2&&2\\
&&&&1&&1&&4&&3&&4\\
&&&1&&1&&3&&4&&4&&2\\
&&1&&1&&5&&5&&9&&5&&4\\
&1&&1&&4&&6&&10&&9&&6&&2\\
\iddots&&\iddots&&\iddots&&\iddots&&\vdots&&\ddots&&\ddots&&\ddots&&\ddots
\end{array}.
$$

Define $T_1^{D^{(1)}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$
\begin{aligned}
&T_1^{D^{(1)}}(n,0) = 0 \ (n \geqslant 0), \quad T_1^{D^{(1)}}(n,1) = 1 \ (n \geqslant 0), T_1^{D^{(1)}}(0,2\ell-1) = 2 \ (\ell \geqslant 2),\ T_1^{D^{(1)}}(0,2\ell) = 0 \ (\ell \geqslant 1),\\
&T_1^{D^{(1)}}(n,\ell) = T_1^{D^{(1)}}(n,\ell-2) + T_1^{D^{(1)}}(n-1,\ell)\\
&\qquad\qquad + (-1)^{n+1}\delta(\ell \equiv_2 0)(1+\delta(n \equiv_2 1))T_0^{A^{(2)}_{\rm even}}\left(\left\lfloor\frac{n-1}{2}\right\rfloor, \frac{\ell}{2}-1\right) \quad (n \geqslant 1, \ell \geqslant 2).
\end{aligned}
$$

Then for all $n \in \mathbb{Z}_{\geqslant 4}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_1^{D^{(1)}}(n,\ell) = |\max^+_{D_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_1)|$. In particular, as a triangular array, $T_1^{D^{(1)}}$ can be described as follows:

$$
\begin{array}{ccccccccccccccccc}
&&&&&&&&0\\
&&&&&&&0&&1\\
&&&&&&0&&1&&0\\
&&&&&0&&1&&2&&2\\
&&&&0&&1&&1&&3&&0\\
&&&0&&1&&3&&4&&4&&2\\
&&0&&1&&2&&5&&4&&5&&0\\
&0&&1&&4&&6&&9&&9&&6&&2\\
\iddots&&\iddots&&\iddots&&\iddots&&\vdots&&\ddots&&\ddots&&\ddots&&\ddots
\end{array}.
$$

Set $T_n^{D_n^{(1)}} = T_n^{B_n^{(1)}}$. Then, for all $n \in \mathbb{Z}_{\geqslant 4}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$, we have

$$|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_n)| = T_n^{B^{(1)}}(n,\ell) = T_n^{D^{(1)}}(n,\ell) = |\max^+_{D_n^{(1)}}((\ell-1)\Lambda_0 + \Lambda_{n-\epsilon})| \quad (\epsilon \in \{0,1\}).$$

$A^{(2)}_{2n-1}$ **type.** Define $T_0 : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_0^{A^{(2)}_{\mathrm{odd}}}(n,0) = T_0^{A^{(2)}_{\mathrm{odd}}}(n,1) = 1 \ (n \geqslant 0), \quad T_0^{A^{(2)}_{\mathrm{odd}}}(0,\ell) = 1 \ (\ell \geqslant 2), \text{ and}$$

$$T_0^{A^{(2)}_{\mathrm{odd}}}(n,\ell) = T_0^{A^{(2)}_{\mathrm{odd}}}(n,\ell-2) + T_0^{A^{(2)}_{\mathrm{odd}}}(n-1,\ell) - \delta(n \equiv_2 1, \ell \equiv_2 0, n > 1) T_0^{A^{(2)}_{\mathrm{even}}}\left(\frac{n-1}{2}, \frac{\ell}{2} - 1\right) \ (n \geqslant 1, \ \ell \geqslant 2).$$

Then for all $n \in \mathbb{Z}_{\geqslant 3}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{A^{(2)}_{\mathrm{odd}}}(n,\ell) = |\max^+_{A^{(2)}_{2n-1}}(\ell\Lambda_0)|$. In particular, as a triangular array, T_0 can be described as follows:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 2 \quad 1 \\
1 \quad 1 \quad 3 \quad 2 \quad 1 \\
1 \quad 1 \quad 3 \quad 3 \quad 3 \quad 1 \\
1 \quad 1 \quad 4 \quad 4 \quad 6 \quad 3 \quad 1 \\
1 \quad 1 \quad 4 \quad 5 \quad 8 \quad 6 \quad 4 \quad 1 \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \vdots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \ .
\end{array}$$

Define $T_1^{A^{(2)}_{\mathrm{odd}}} : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_1^{A^{(2)}_{\mathrm{odd}}}(n,0) = 0 \ (n \geqslant 0), \quad T_1^{A^{(2)}_{\mathrm{odd}}}(n,1) = 1 \ (n \geqslant 0), \quad T_1^{A^{(2)}_{\mathrm{odd}}}(0,\ell) = 1 \ (\ell \geqslant 1), \text{ and}$$

$$T_1^{A^{(2)}_{\mathrm{odd}}}(n,\ell) = T_1^{A^{(2)}_{\mathrm{odd}}}(n,\ell-2) + T_1^{A^{(2)}_{\mathrm{odd}}}(n-1,\ell) + \delta(n \equiv_2 1, \ell \equiv_2 0, n > 1) T_0^{A^{(2)}_{\mathrm{even}}}\left(\frac{n-1}{2}, \frac{\ell}{2} - 1\right) \ (n \geqslant 1, \ \ell \geqslant 2).$$

Then for all $n \in \mathbb{Z}_{\geqslant 3}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_1^{A^{(2)}_{\mathrm{odd}}}(n,\ell) = |\max^+_{A^{(2)}_{2n-1}}((\ell-1)\Lambda_0 + \Lambda_1)|$. In particular, as a triangular array, $T_1^{A^{(2)}_{\mathrm{odd}}}$ can be described as follows:

$$\begin{array}{c}
0 \\
0 \quad 1 \\
0 \quad 1 \quad 1 \\
0 \quad 1 \quad 1 \quad 1 \\
0 \quad 1 \quad 1 \quad 2 \quad 1 \\
0 \quad 1 \quad 2 \quad 3 \quad 2 \quad 1 \\
0 \quad 1 \quad 2 \quad 4 \quad 3 \quad 3 \quad 1 \\
0 \quad 1 \quad 3 \quad 5 \quad 6 \quad 6 \quad 3 \quad 1 \\
\iddots \quad \iddots \quad \iddots \quad \iddots \quad \vdots \quad \ddots \quad \ddots \quad \ddots \quad \ddots \ .
\end{array}$$

$D^{(2)}_{n+1}$ **type.** Set $T_0^{D^{(2)}} := T_0^{A^{(2)}_{\mathrm{even}}}$. Then for all $n \in \mathbb{Z}_{\geqslant 2}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_0^{D^{(2)}}(n,\ell) = |\max^+_{D^{(2)}_{n+1}}(\ell\Lambda_0)|$. Note that Table 6.3 says

$$\left|\max^+_{D^{(2)}_{n+1}}(\ell\Lambda_0)\right| = \binom{n + \lfloor \frac{\ell}{2} \rfloor}{n} = \left|\max^+_{A^{(2)}_{2n}}(\ell\Lambda_0)\right|,$$

which explains the reason why we define $T_0^{D^{(2)}}$ as $T_0^{A^{(2)}_{\mathrm{even}}}$.

Define $T_n : \mathbb{Z}_{\geqslant 0} \times \mathbb{Z}_{\geqslant 0} \to \mathbb{Z}_{\geqslant 0}$ by

$$T_n^{D^{(2)}}(n,0) = \color{red}{0} \ (n \geqslant 0) \quad \text{and} \quad T_n^{D^{(2)}}(n,\ell) = T_0^{A^{(2)}_{\mathrm{even}}}(n,\ell-1) \ (n \geqslant 0, \ell > 0).$$

Then for all $n \in \mathbb{Z}_{\geqslant 2}$, $\ell \in \mathbb{Z}_{\geqslant 0}$, we have $T_n^{D^{(2)}}(n,\ell) = |\max^+_{D^{(2)}_{n+1}}((\ell-1)\Lambda_0 + \Lambda_n)|$. Note that Table 6.3 says

$$\left|\max^+_{D^{(2)}_{n+1}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \binom{n + \lfloor \frac{\ell-1}{2} \rfloor}{n} = \left|\max^+_{A^{(2)}_{2n}}((\ell-1)\Lambda_0)\right| \quad \text{for } \ell > 0,$$

which explains the reason why we define $T_n^{D^{(2)}}(n,\ell)$ as $T_0^{A^{(2)}_{\mathrm{even}}}(n,\ell-1)$ for $n \geqslant 0, \ell > 0$. To emphasize this, we denote by $\color{red}{0}$ the zeros in the left boundary diagonal.

Appendix B. Level-rank duality

$B_n^{(1)}$ **type.** From Table 6.3 it follows that

$$\left|\max^+_{B_n^{(1)}}(\ell\Lambda_0)\right| = \binom{n+\lfloor\frac{\ell}{2}\rfloor}{n} + \binom{n+\lfloor\frac{\ell-1}{2}\rfloor}{n}.$$

Hence, for $n \geqslant 3$, $\ell \geqslant 7$ and $\ell \equiv_2 1$, we have

$$\left|\max^+_{B_n^{(1)}}(\ell\Lambda_0)\right| = \left|\max^+_{B_{(\ell-1)/2}^{(1)}}((2n+1)\Lambda_0)\right|,$$

i.e., when we exchange n with $(\ell-1)/2$, the number of dominant maximal weights remains same.

From Table 6.3 it follows that

$$\left|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_n)\right| = \binom{n+\lfloor\frac{\ell-1}{2}\rfloor}{n} + \binom{n+\lfloor\frac{\ell}{2}\rfloor-1}{n}.$$

Hence, for $n \geqslant 3$, $\ell \geqslant 8$ and $\ell \equiv_2 0$, we have

$$\left|\max^+_{B_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_n)\right| = \left|\max^+_{B_{\ell/2-1}^{(1)}}((2n+1)\Lambda_0+\Lambda_{\ell/2-1})\right|,$$

i.e., when we exchange n with $\ell/2-1$, the number of dominant maximal weights remains same.

$C_n^{(1)}$ **type.** From Table 6.3 it follows that for any $\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$,

$$\left|\max^+_{C_n^{(1)}}(\Lambda)\right| = \frac{1}{2}\left(\binom{\ell+n}{n} + (-1)^i\delta(n\ell \equiv_2 0)\binom{\lfloor\frac{\ell+n}{2}\rfloor}{\lfloor\frac{n}{2}\rfloor}\right).$$

Hence, for $n \geqslant 2$ and $\ell \geqslant 2$, we have

$$\left|\max^+_{C_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_i)\right| = \left|\max^+_{C_\ell^{(1)}}((n-1)\Lambda_0+\Lambda_i)\right| \quad \text{for } i = 0, 1,$$

i.e., when we exchange n with ℓ, the number of dominant maximal weights remains same.

$D_n^{(1)}$ **type.** In case where $\ell \equiv_2 0$, from Table 6.2, it follows that for $i = n-1, n$

$$\left|\max^+_{D_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_i)\right| = \frac{1}{4}\left(\mathsf{m}(1^4, 2^{n-3}) - \mathsf{m}(2^{n-1})\right).$$

Using Lemma 6.2, one can see that

$$\mathsf{m}(1^4, 2^{n-3}) = \binom{n+\frac{\ell}{2}-2}{\frac{\ell}{2}} + 8\binom{n+\frac{\ell}{2}-1}{\frac{\ell}{2}-1} \quad \text{and} \quad \mathsf{m}(2^{n-1}) = \binom{n+\frac{\ell}{2}-2}{\frac{\ell}{2}}$$

and thus

$$\left|\max^+_{D_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_i)\right| = 2\binom{n+\frac{\ell}{2}-1}{\frac{\ell}{2}-1}.$$

Hence, for $n \geqslant 4$, $\ell \geqslant 9$ and $\ell \equiv_2 0$, we have

$$\left|\max^+_{D_n^{(1)}}((\ell-1)\Lambda_0+\Lambda_n)\right| = \left|\max^+_{D_{\ell/2-1}^{(2)}}((2n+1)\Lambda_0+\Lambda_{\ell/2-1})\right|,$$

i.e., for an even integer ℓ, when we exchange n with $\ell/2-1$, the number of dominant maximal weights remains same.

$A_{2n-1}^{(2)}$ **type.** In case where $\ell \equiv_2 1$, from Table 6.2, it follows that for any $\Lambda = (\ell-1)\Lambda_0 + \Lambda_i \in \mathtt{DR}(P^+_{\mathrm{cl},\ell})$,

$$\left|\max^+_{A_{2n-1}^{(2)}}(\Lambda)\right| = \frac{\mathsf{m}(1^2, 2^{n-1})}{2} = \frac{1}{2}\left(\binom{n+\lfloor\frac{\ell-1}{2}\rfloor}{\lfloor\frac{\ell-1}{2}\rfloor} + \binom{n+\lfloor\frac{\ell}{2}\rfloor}{\lfloor\frac{\ell}{2}\rfloor}\right).$$

Hence, for $n \geqslant 3$, $\ell \geqslant 7$ and $\ell \equiv_2 1$, we have

$$\left|\max^+_{A_{2n-1}^{(2)}}((\ell-1)\Lambda_0+\Lambda_i)\right| = \left|\max^+_{A_{2((\ell-1)/2)}^{(2)}}(2n\Lambda_0+\Lambda_i)\right| \quad \text{for } i = 0, 1,$$

i.e., for an odd integer ℓ, when we exchange n with $(\ell-1)/2$, the number of dominant maximal weights remains same.

$A^{(2)}_{2n}$ **type.** From Table 6.3 it follows that

$$\left|\max^+_{A^{(2)}_{2n}}(\ell\Lambda_0)\right| = \binom{\lfloor \frac{\ell}{2} \rfloor + n}{n}.$$

Hence, for $n \geqslant 2$, $\ell \geqslant 4$ and $\ell \equiv_2 0$ (resp. $\ell \equiv_2 1$), we have

$$\left|\max^+_{A^{(2)}_{2n}}(\ell\Lambda_0)\right| = \left|\max^+_{A^{(2)}_{2(\ell/2)}}(2n\Lambda_0)\right| \quad \left(\text{resp. } \left|\max^+_{A^{(2)}_{2n}}(\ell\Lambda_0)\right| = \left|\max^+_{A^{(2)}_{2((\ell-1)/2)}}((2n+1)\Lambda_0)\right|\right),$$

i.e., for an even (resp. odd) integer ℓ, when we exchange n with $\ell/2$ (resp. $(\ell-1)/2$), the number of dominant maximal weights remains same.

$D^{(2)}_{n+1}$ **type.** From Table 6.3 it follows that

$$\left|\max^+_{D^{(2)}_{n+1}}(\ell\Lambda_0)\right| = \binom{\lfloor \frac{\ell}{2} \rfloor + n}{n}.$$

Hence, for $n \geqslant 2$, $\ell \geqslant 4$ and $\ell \equiv_2 0$ (resp. $\ell \equiv_2 1$), we have

$$\left|\max^+_{D^{(2)}_{n+1}}(\ell\Lambda_0)\right| = \left|\max^+_{D^{(2)}_{\ell/2}}(2n\Lambda_0)\right| \quad \left(\text{resp. } \left|\max^+_{D^{(2)}_{n+1}}(\ell\Lambda_0)\right| = \left|\max^+_{D^{(2)}_{(\ell-1)/2}}((2n+1)\Lambda_0)\right|\right),$$

i.e., for an even (resp. odd) integer ℓ, when we exchange n with $\ell/2$ (resp. $(\ell-1)/2$), the number of dominant maximal weights remains same.

From Table 6.3 it follows that

$$\left|\max^+_{D^{(2)}_{n+1}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \binom{\lfloor \frac{\ell-1}{2} \rfloor + n}{n}.$$

Hence, for $n \geqslant 2$, $\ell \geqslant 5$ and $\ell \equiv_2 0$ (resp. $\ell \equiv_2 1$),

$$\left|\max^+_{D^{(2)}_{n+1}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \left|\max^+_{D^{(2)}_{\ell/2-1}}((2n+1)\Lambda_0 + \Lambda_{\ell/2-1})\right|$$
$$\left(\text{resp. } \left|\max^+_{D^{(2)}_{n+1}}((\ell-1)\Lambda_0 + \Lambda_n)\right| = \left|\max^+_{D^{(2)}_{(\ell-1)/2}}(2n\Lambda_0 + \Lambda_{(\ell-1)/2})\right|\right),$$

i.e., for an even (resp. odd) integer ℓ, when we exchange n with $\ell/2-1$ (resp. $(\ell-1)/2$), the number of dominant maximal weights remains same.

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