

TWO CONGRUENCES CONCERNING APÉRY NUMBERS

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ABSTRACT. Let n be a nonnegative integer. The n -th Apéry number is defined by

$$A_n := \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2.$$

Z.-W. Sun ever investigated the congruence properties of Apéry numbers and posed some conjectures. For example, Sun conjectured that for any prime $p \geq 7$

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}$$

and for any prime $p \geq 5$

$$\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 + 4p^4 H_{p-1} + \frac{6}{5}p^8 B_{p-5} \pmod{p^9},$$

where $H_n = \sum_{k=1}^n 1/k$ denotes the n -th harmonic number and B_0, B_1, \dots are the well-known Bernoulli numbers. In this paper we shall confirm these two conjectures.

1. INTRODUCTION

The well-known Apéry numbers given by

$$A_n := \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 = \sum_{k=0}^n \binom{n+k}{2k}^2 \binom{2k}{k}^2 \quad (n \in \mathbb{N} = \{0, 1, \dots\}),$$

were first introduced by Apéry to prove the irrationality of $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ (see [1, 6]).

In 2012, Z.-W. Sun introduced the Apéry polynomials

$$A_n(x) = \sum_{k=0}^n \binom{n+k}{k}^2 \binom{n}{k}^2 x^k \quad (n \in \mathbb{N})$$

and deduced various congruences involving sums of such polynomials. (Clearly, $A_n(1) = A_n$.)

For example, for any odd prime p and integer x , he obtained that

$$\sum_{k=0}^{p-1} (2k+1)A_k(x) \equiv p \left(\frac{x}{p}\right) \pmod{p^2}, \quad (1.1)$$

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where $(-)$ denotes the Legendre symbol. Letting $x = 1$ and for any prime $p \geq 5$, Sun established the following generalization of (1.1):

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}, \quad (1.2)$$

where B_0, B_1, \dots are the well-known Bernoulli numbers defined as follows:

$$B_0 = 0, \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \quad (n = 2, 3, \dots).$$

In 1850 Kummer (cf. [4]) proved that for any odd prime p and any even number b with $b \not\equiv 0 \pmod{p-1}$

$$\frac{B_{k(p-1)+b}}{k(p-1)+b} \equiv \frac{B_b}{b} \pmod{p} \quad \text{for } k \in \mathbb{N}. \quad (1.3)$$

For $m \in \mathbb{Z}^+ = \{1, 2, \dots\}$ the n -th harmonic numbers of order m are defined by

$$H_n^{(m)} := \sum_{k=1}^n \frac{1}{k^m} \quad (n = 1, 2, \dots)$$

and $H_0^{(m)} := 0$. For the sake of convenience we often use H_n instead of $H_n^{(1)}$. From [3] we know that $H_{p-1} \equiv -p^2 B_{p-3}/3 \pmod{p^3}$ for any prime $p \geq 5$. Thus (1.2) has the following equivalent form

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^5}. \quad (1.4)$$

Via some numerical computation, Sun [10, Conjecture 4.2] conjectured that (1.4) also holds modulo p^6 provided that $p \geq 7$. This is our first theorem.

Theorem 1.1. *For any prime $p \geq 7$*

$$\sum_{k=0}^{p-1} (2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6}. \quad (1.5)$$

Motivated by Sun's work on Apéry polynomials, V.J.W. Guo and J. Zeng studied the divisibility of the following sums:

$$\sum_{k=0}^{n-1} (2k+1)^{2r+1} A_k \quad (n \in \mathbb{Z}^+ \text{ and } r \in \mathbb{N}).$$

Particularly, for $r = 1$, they obtained

$$\sum_{k=0}^{n-1} (2k+1)^3 A_k \equiv 0 \pmod{n^3} \quad (1.6)$$

and

$$\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 \pmod{2p^6}, \quad (1.7)$$

where $p \geq 5$ is a prime. As an extension to (1.7), Sun [8, Conjecture A65] proposed the following challenging conjecture.

Conjecture 1.1. *For any prime $p \geq 5$ we have*

$$\sum_{k=0}^{p-1} (2k+1)^3 A_k \equiv p^3 + 4p^4 H_{p-1} + \frac{6}{5} p^8 B_{p-5} \pmod{p^9}.$$

This is our second theorem.

Theorem 1.2. *Conjecture 1.1 is true.*

Proofs of Theorems 1.1–1.2 will be given in Sections 2–3 respectively.

2. PROOF OF THEOREM 1.1

The proofs in this paper strongly depend on the congruence properties of harmonic numbers and the Bernoulli numbers. (The readers may consult [4, 7, 9, 11] for the properties of them.) Below we first list some congruences involving harmonic numbers and the Bernoulli numbers which may be used later.

Lemma 2.1. [2, Remark 3.2] *For any prime $p \geq 5$ we have*

$$2H_{p-1} + pH_{p-1}^{(2)} \equiv \frac{2}{5} p^4 B_{p-5} \pmod{p^5}.$$

From [7, Theorems 5.1&5.2], we have the following congruences.

Lemma 2.2. *For any prime $p \geq 7$ we have*

$$\begin{aligned} H_{(p-1)/2} &\equiv -2q_p(2) \pmod{p}, \\ H_{p-1}^{(2)} &\equiv \left(\frac{4}{3} B_{p-3} - \frac{1}{2} B_{2p-4} \right) p + \left(\frac{4}{9} B_{p-3} - \frac{1}{4} B_{2p-4} \right) p^2 \pmod{p^3}, \\ H_{(p-1)/2}^{(2)} &\equiv \left(\frac{14}{3} B_{p-3} - \frac{7}{4} B_{2p-4} \right) p + \left(\frac{14}{9} B_{p-3} - \frac{7}{8} B_{2p-4} \right) p^2 \pmod{p^3}, \\ H_{p-1}^{(3)} &\equiv -\frac{6}{5} p^2 B_{p-5} \pmod{p^3}, \quad H_{(p-1)/2}^{(3)} \equiv 6 \left(\frac{2B_{p-3}}{p-3} - \frac{B_{2p-4}}{2p-4} \right) \pmod{p^2}, \\ H_{p-1}^{(4)} &\equiv \frac{4}{5} p B_{p-5} \pmod{p^2}, \quad H_{(p-1)/2}^{(4)} \equiv 0 \pmod{p}, \quad H_{p-1}^{(5)} \equiv 0 \pmod{p^2}. \end{aligned}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

Remark 2.1. By Kummer's congruence (1.3), we know $B_{2p-4} \equiv 4B_{p-3}/3 \pmod{p}$. Then the congruences of $H_{p-1}^{(2)}$ and $H_{(p-1)/2}^{(2)}$ can be reduced to

$$H_{p-1}^{(2)} \equiv \left(\frac{4}{3}B_{p-3} - \frac{1}{2}B_{2p-4} \right) p + \frac{1}{9}p^2 B_{p-3} \pmod{p^3}$$

and

$$H_{(p-1)/2}^{(2)} \equiv \left(\frac{14}{3}B_{p-3} - \frac{7}{4}B_{2p-4} \right) p + \frac{7}{18}p^2 B_{p-3} \pmod{p^3}$$

respectively. By Lemma 2.1, we immediately obtain that $H_{p-1} \equiv -pH_{(p-1)/2}^{(2)}/2 \pmod{p^4}$. Thus

$$H_{p-1} \equiv \left(\frac{1}{4}B_{2p-4} - \frac{2}{3}B_{p-3} \right) p^2 - \frac{1}{18}p^3 B_{p-3} \pmod{p^4}. \quad (2.1)$$

Recall that the Bernoulli polynomials $B_n(x)$ are defined as

$$B_n(x) := \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \quad (n \in \mathbb{N}). \quad (2.2)$$

Clearly, $B_n = B_n(0)$. Also, we have

$$\sum_{k=1}^{n-1} k^{m-1} = \frac{B_m(n) - B_m}{m} \quad (2.3)$$

for any positive integer n and m .

Let $d > 0$ and $\mathbf{s} := (s_1, \dots, s_d) \in (\mathbb{Z} \setminus \{0\})^d$. The alternating multiple harmonic sum [12] is defined as follows

$$H(\mathbf{s}; n) := \sum_{1 \leq k_1 < k_2 < \dots < k_d \leq n} \prod_{i=1}^d \frac{\text{sgn}(s_i)^{k_i}}{k_i^{|s_i|}}.$$

Clearly, $H_n^{(m)} = H(m; n)$.

Let A, B, D, E, F be defined as in [12, Section 6], i.e.,

$$A := \sum_{k=2}^{p-3} B_k B_{p-3-k}, \quad B := \sum_{k=2}^{p-3} 2^k B_k B_{p-3-k}, \quad D := \sum_{k=2}^{p-3} \frac{B_k B_{p-3-k}}{k},$$

$$E := \sum_{k=2}^{p-3} \frac{2^k B_k B_{p-3-k}}{k}, \quad F := \sum_{k=2}^{p-3} \frac{2^{p-3-k} B_k B_{p-3-k}}{k}.$$

Lemma 2.3. *For any prime $p \geq 7$ we have*

$$D - 4F \equiv 2B - 2A - q_p(2)B_{p-3} \pmod{p}.$$

Proof. In [12, Section 6], Tauraso and Zhao proved that

$$H(1, -3; p-1) \equiv B - A \equiv 2E - 2D + 2q_p(2)B_{p-3} \pmod{p}$$

and

$$\frac{5}{2}D - 2E - 2F - \frac{3}{2}q_p(2)B_{p-3} \equiv 0 \pmod{p}.$$

Combining the above two congruences we immediately obtain the desired result. \square

Lemma 2.4. *Let $p \geq 7$ be a prime. Then we have*

$$H(3, 1; (p-1)/2) \equiv H_{(p-1)/2}^{(3)} H_{(p-1)/2} - 4B + 4A \pmod{p}. \quad (2.4)$$

Proof. By Lemma 2.2, it is easy to check that

$$\begin{aligned} H_{(p-1)/2}^{(3)} H_{(p-1)/2} &= H(1, 3; (p-1)/2) + H(3, 1; (p-1)/2) + H_{(p-1)/2}^{(4)} \\ &\equiv H(1, 3; (p-1)/2) + H(3, 1; (p-1)/2) \pmod{p}. \end{aligned} \quad (2.5)$$

Thus it suffices to evaluate $H(1, 3; (p-1)/2)$ modulo p . By Fermat's little theorem, (2.2) and (2.3) we arrive at

$$\begin{aligned} H(1, 3; (p-1)/2) &= \sum_{1 \leq j < k \leq (p-1)/2} \frac{1}{jk^3} \equiv \sum_{1 \leq j < k \leq (p-1)/2} \frac{j^{p-2}}{k^3} = \sum_{1 \leq k \leq (p-1)/2} \frac{B_{p-1}(k) - B_{p-1}}{k^3(p-1)} \\ &= \sum_{1 \leq k \leq (p-1)/2} \frac{\sum_{i=1}^{p-1} \binom{p-1}{i} k^{i-3} B_{p-1-i}}{p-1} = \sum_{i=1}^{p-1} \frac{\binom{p-1}{i} B_{p-1-i}}{p-1} \sum_{k=1}^{(p-1)/2} k^{i-3} \\ &\equiv \frac{\binom{p-1}{2} B_{p-3}}{p-1} H_{(p-1)/2} + \sum_{i=4}^{p-1} \frac{\binom{p-1}{i} B_{p-1-i}}{p-1} \cdot \frac{B_{i-2}(\frac{1}{2}) - B_{i-2}}{i-2} \pmod{p}, \end{aligned}$$

where the last step follows from the fact $B_n = 0$ for any odd $n \geq 3$. By [4] we know that $B_n(1/2) = (2^{1-n} - 1)B_n$. Thus

$$\begin{aligned} H(1, 3; (p-1)/2) &\equiv -B_{p-3} H_{(p-1)/2} - \sum_{i=4}^{p-1} \frac{(2^{3-i} - 2)B_{p-1-i}B_{i-2}}{i-2} \\ &= -B_{p-3} H_{(p-1)/2} - \sum_{i=2}^{p-3} \frac{(2^{1-i} - 2)B_{p-3-i}B_i}{i} \\ &\equiv -B_{p-3} H_{(p-1)/2} - 8F + 2D \pmod{p}. \end{aligned}$$

With helps of Lemmas 2.2 and 2.3, we have

$$H(1, 3; (p-1)/2) \equiv 4B - 4A \pmod{p}.$$

Combining this with (2.5), we have completed the proof of Lemma 2.4. \square

Lemma 2.5. *Let $p \geq 7$ be a prime. Then we have*

$$\sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \equiv \frac{3}{2p^2} H_{p-1} + \frac{1}{2} H_{(p-1)/2}^{(3)} + \frac{1}{2} H_{p-1}^{(2)} H_{(p-1)/2} - p H_{(p-1)/2}^{(3)} H_{(p-1)/2} + 4p(B - A) \pmod{p}.$$

Proof. By [5, Eq. (3.13)] we know that for any odd prime p

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \equiv \frac{3}{p^2} H_{p-1} \pmod{p^2}. \quad (2.6)$$

On the other hand,

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} = \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} + \sum_{k=1}^{(p-1)/2} \frac{H_{p-k}^{(2)}}{p-k}.$$

For $k = 1, 2, \dots, (p-1)/2$ we have

$$H_{p-k}^{(2)} = \sum_{j=k}^{p-1} \frac{1}{(p-j)^2} \equiv \sum_{j=k}^{p-1} \left(\frac{1}{j^2} + \frac{2p}{j^2} \right) \equiv H_{p-1}^{(2)} - H_{k-1}^{(2)} - 2pH_{k-1}^{(3)} \pmod{p^2}$$

by Lemma 2.2. Thus

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} &\equiv 2 \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} - H_{(p-1)/2}^{(3)} + p \sum_{k=1}^{(p-1)/2} H(2, 2; (p-1)/2) \\ &\quad - H_{p-1}^{(2)} H_{(p-1)/2} + 2pH(3, 1; (p-1)/2) \pmod{p^2}. \end{aligned}$$

In view of Lemma 2.2, we have

$$H(2, 2; (p-1)/2) = \frac{\left(H_{(p-1)/2}^{(2)} \right)^2}{2} - \frac{H_{(p-1)/2}^{(4)}}{2} \equiv 0 \pmod{p}.$$

This together with Lemma 2.4 proves Lemma 2.5. \square

Lemma 2.6. [10, Lemma 2.1] *Let $k \in \mathbb{N}$. Then for $n \in \mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1} (2m+1) \binom{m+k}{2k}^2 = \frac{(n-k)^2}{2k+1} \binom{n+k}{2k}^2.$$

Proof of Theorem 1.1. By Lemma 2.6 it is routine to check that

$$\begin{aligned} \sum_{m=0}^{p-1} (2m+1) A_m &= \sum_{m=0}^{p-1} (2m+1) \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \sum_{m=0}^{p-1} (2m+1) \binom{m+k}{2k}^2 \\ &= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{(p-k)^2}{2k+1} \binom{p+k}{2k}^2 = p^2 \sum_{k=0}^{p-1} \frac{1}{2k+1} \binom{p-1}{k}^2 \binom{p+k}{k}^2. \end{aligned}$$

Note that

$$\begin{aligned} \binom{p-1}{k}^2 \binom{p+k}{k}^2 &= \prod_{j=1}^k \left(1 - \frac{p^2}{j^2} \right)^2 \equiv \prod_{j=1}^k \left(1 - \frac{2p^2}{j^2} + \frac{p^4}{j^4} \right) \\ &\equiv 1 - 2p^2 H_k^{(2)} + p^4 H_k^{(4)} + 4p^4 H(2, 2; k) \pmod{p^5}. \end{aligned}$$

Since $H_{(p-1)/2}^{(4)} \equiv 0 \pmod{p}$ and $H(2, 2; (p-1)/2) \equiv 0 \pmod{p}$, we have

$$\sum_{m=0}^{p-1} (2m+1)A_m \equiv p^2\Sigma_1 - 2p^4\Sigma_2 \pmod{p^6}, \quad (2.7)$$

where

$$\Sigma_1 := \sum_{k=0}^{p-1} \frac{1}{2k+1} \quad \text{and} \quad \Sigma_2 := \sum_{k=0}^{p-1} \frac{H_k^{(2)}}{2k+1}.$$

We first consider Σ_1 modulo p^4 . Clearly,

$$\begin{aligned} \sum_{k=(p+1)/2}^{p-1} \frac{1}{2k+1} &= \sum_{k=0}^{(p-3)/2} \frac{1}{2(p-1-k)+1} \\ &\equiv -8p^3 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^4} - 2p \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^2} \\ &\quad - 4p^2 \sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^3} - \sum_{k=0}^{(p-3)/2} \frac{1}{2k+1} \pmod{p^4}. \end{aligned}$$

For $r \in \{2, 3, 4\}$,

$$\sum_{k=0}^{(p-3)/2} \frac{1}{(2k+1)^r} = H_{p-1}^{(r)} - \frac{1}{2^r} H_{(p-1)/2}^{(r)}.$$

By the above and in view of Lemma 2.2,

$$\begin{aligned} \Sigma_1 &= \frac{1}{p} + \sum_{k=0}^{(p-3)/2} \frac{1}{2k+1} + \sum_{k=(p+1)/2}^{p-1} \frac{1}{2k+1} \\ &\equiv \frac{1}{p} - 2p \left(H_{p-1}^{(2)} - \frac{1}{4} H_{(p-1)/2}^{(2)} \right) + \frac{1}{2} p^2 H_{(p-1)/2}^{(3)} \pmod{p^4}. \end{aligned} \quad (2.8)$$

Now we turn to Σ_2 modulo p^2 . By Lemma 2.2,

$$\begin{aligned} \sum_{k=(p+1)/2}^{p-1} \frac{H_k^{(2)}}{2k+1} &= \sum_{k=0}^{(p-3)/2} \frac{H_{p-1-k}^{(2)}}{2(p-1-k)+1} \\ &\equiv \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} + 2p \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{(2k+1)^2} + \frac{1}{2} H_{p-1}^{(2)} H_{(p-1)/2} + 2p \sum_{k=0}^{(p-3)/2} \frac{H_k^{(3)}}{2k+1} \pmod{p^2}. \end{aligned}$$

Thus

$$\Sigma_2 \equiv \frac{H_{(p-1)/2}^{(2)}}{p} + 2\sigma_1 + \frac{1}{2} H_{p-1}^{(2)} H_{(p-1)/2} + 2p\sigma_2 \pmod{p^2},$$

where

$$\sigma_1 := \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{2k+1} + p \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{(2k+1)^2}$$

and

$$\sigma_2 := \sum_{k=0}^{(p-3)/2} \frac{H_k^{(3)}}{2k+1}.$$

It is easy to see that

$$\begin{aligned} \sigma_1 &\equiv - \sum_{k=0}^{(p-3)/2} \frac{H_k^{(2)}}{p-1-2k} = - \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(2)}}{2k} \\ &\equiv - \frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(2)} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \left(\frac{4}{(2j+1)^2} + \frac{8p}{(2j+1)^3} \right) \\ &= - \frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(2)} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(4H_{2k}^{(2)} - H_k^{(2)} + 8pH_{2k}^{(3)} - pH_k^{(3)} \right) \pmod{p^2}. \end{aligned}$$

Also,

$$\begin{aligned} \sigma_2 &\equiv - \sum_{k=0}^{(p-3)/2} \frac{H_k^{(3)}}{p-1-2k} = - \sum_{k=1}^{(p-1)/2} \frac{H_{(p-1)/2-k}^{(3)}}{2k} \\ &\equiv - \frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(3)} + \frac{1}{2} \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{j=0}^{k-1} \frac{-8}{(2j+1)^3} \\ &= - \frac{1}{2} H_{(p-1)/2} H_{(p-1)/2}^{(3)} - 4 \sum_{k=1}^{(p-1)/2} \frac{1}{k} \left(H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) \pmod{p}. \end{aligned}$$

Combining the above we deduce that

$$\begin{aligned} \Sigma_2 &\equiv \frac{H_{(p-1)/2}^{(2)}}{p} - H_{(p-1)/2} H_{(p-1)/2}^{(2)} + 4 \sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{k} - \sum_{k=1}^{(p-1)/2} \frac{H_k^{(2)}}{k} \\ &\quad + \frac{1}{2} H_{p-1}^{(2)} H_{(p-1)/2} - pH_{(p-1)/2} H_{(p-1)/2}^{(3)} \pmod{p^2}. \end{aligned} \tag{2.9}$$

Note that

$$\sum_{k=1}^{(p-1)/2} \frac{H_{2k}^{(2)}}{k} = \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} + H(2, -1; p-1) + \frac{1}{4} H_{(p-1)/2}^{(3)} - H_{p-1}^{(3)}. \tag{2.10}$$

By [12, Proposition 7.3] we know that

$$H(2, -1; p-1) \equiv -\frac{3}{2}X - \frac{7}{6}pq_p(2)B_{p-3} + p(B-A) \pmod{p^2}, \tag{2.11}$$

where $X := B_{p-3}/(p-3) - B_{2p-4}/(4p-8)$. Now combining (2.9)–(2.11), Lemmas 2.2 and 2.5 we obtain that

$$\Sigma_2 \equiv \frac{H_{(p-1)/2}^{(2)}}{p} + \frac{21H_{p-1}}{2p^2} \pmod{p^2}. \quad (2.12)$$

Substituting (2.8) and (2.12) into (2.7) and in light of (2.6) and Lemma 2.2 we have

$$\begin{aligned} \sum_{m=0}^{p-1} (2m+1)A_m &\equiv p - 2p^3H_{p-1}^{(2)} - \frac{3}{2}p^3H_{(p-1)/2}^{(2)} + \frac{1}{2}p^4H_{(p-1)/2}^{(3)} - 21p^2H_{p-1} \\ &\equiv p - \frac{7}{2}p^2H_{p-1} \pmod{p^6}. \end{aligned}$$

The proof of Theorem 1.1 is complete now. \square

3. PROOF OF THEOREM 1.2

In order to show Theorem 1.2, we need the following results.

Lemma 3.1. *Let $p \geq 7$ be a prime. Then we have*

$$\sum_{k=1}^{p-1} \frac{H(2, 2; k)}{k} \equiv -\frac{1}{2}B_{p-5} \pmod{p}.$$

Proof. By Remark 2.1 we have

$$\sum_{k=1}^{p-1} \frac{H(2, 2; k)}{k} = \sum_{k=1}^{p-1} \frac{1}{k} \sum_{1 \leq i < j \leq k} \frac{1}{i^2 j^2} \equiv - \sum_{1 \leq i < j \leq p-1} \frac{H_{j-1}}{i^2 j^2} = - \sum_{j=1}^{p-1} \frac{H_{j-1}H_{j-1}^{(2)}}{j^2} \pmod{p}.$$

On one hand,

$$\sum_{j=1}^{p-1} \frac{H_{j-1}H_{j-1}^{(2)}}{j^2} = \sum_{j=1}^{p-1} \frac{H_jH_j^{(2)} - H_j^{(2)}/j - H_j/j^2 + 1/j^3}{j^2}. \quad (3.1)$$

On the other hand, we have

$$\sum_{j=1}^{p-1} \frac{H_{j-1}H_{j-1}^{(2)}}{j^2} = \sum_{j=1}^{p-1} \frac{H_{p-j-1}H_{p-j-1}^{(2)}}{(p-j)^2} \equiv - \sum_{j=1}^{p-1} \frac{H_jH_j^{(2)}}{j^2} \pmod{p}$$

in view of that $H_{p-1-k} \equiv H_k \pmod{p}$ and $H_{p-1-k}^{(2)} \equiv -H_k^{(2)} \pmod{p}$. Combining this with (3.1) we arrive at

$$\sum_{j=1}^{p-1} \frac{H_{j-1}H_{j-1}^{(2)}}{j^2} \equiv \frac{1}{2} \left(-H_{p-1}^{(5)} - H(2, 3; p-1) - H(1, 4; p-1) \right) \pmod{p}.$$

By [12, Theorem 3.1] we have $H(2, 3; p-1) \equiv -2B_{p-5} \pmod{p}$ and $H(1, 4; p-1) \equiv B_{p-5} \pmod{p}$ provided that $p \geq 7$. These together with Lemma 2.2 imply that

$$\sum_{j=1}^{p-1} \frac{H_{j-1} H_{j-1}^{(2)}}{j^2} \equiv \frac{1}{2} B_{p-5} \pmod{p}.$$

This proves the desired Lemma 3.1. \square

Lemma 3.2. *For any prime $p \geq 7$ we have*

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} \equiv \frac{3H_{p-1}}{p^2} - \frac{1}{2} p^2 B_{p-5} \pmod{p^3}. \quad (3.2)$$

Proof. Letting $n = (p-1)/2$ in [5, Lemma 3.1] we obtain that

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} \binom{p-1}{k} \binom{p+k}{k} = -2H_{p-1}.$$

Note that

$$\begin{aligned} \binom{p-1}{k} \binom{p+k}{k} &= \prod_{j=1}^k \frac{p^2 - j^2}{j^2} = (-1)^k \prod_{j=1}^k \left(1 - \frac{p^2}{j^2}\right) \\ &\equiv (-1)^k \left(1 - p^2 H_k^{(2)} + p^4 H(2, 2; k)\right) \pmod{p^5}. \end{aligned}$$

It follows that

$$H_{p-1} - p^2 \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} + p^4 \sum_{k=1}^{p-1} \frac{H(2, 2; k)}{k} \equiv -2H_{p-1} \pmod{p^5}.$$

With the help of Lemma 3.1, we obtain (3.2). \square

Lemma 3.3. *For any prime $p \geq 7$ we have*

$$\sum_{k=0}^{p-1} H(2, 2; k) \equiv -\frac{p}{2} H_{p-1}^{(4)} - \frac{3H_{p-1}}{p^2} + H_{p-1}^{(3)} + \frac{1}{2} p^2 B_{p-5} \pmod{p^3}, \quad (3.3)$$

$$\sum_{k=0}^{p-1} (H(2, 4; k) + H(4, 2; k)) \equiv 3B_{p-5} \pmod{p}, \quad (3.4)$$

$$\sum_{k=0}^{p-1} H(2, 2, 2; k) \equiv -\frac{3}{2} B_{p-3} \pmod{p}. \quad (3.5)$$

Proof. By Lemma 2.2 we arrive at

$$\sum_{k=0}^{p-1} H(2, 2; k) = \sum_{k=1}^{p-1} \sum_{1 \leq i < j \leq k} \frac{1}{i^2 j^2} = \sum_{1 \leq i < j \leq p-1} \frac{p-j}{i^2 j^2}$$

$$\begin{aligned}
&= \frac{p}{2} \left(\left(H_{p-1}^{(2)} \right)^2 - H_{p-1}^{(4)} \right) - \sum_{1 \leq i < j \leq p-1} \frac{1}{i^2 j} \\
&\equiv -\frac{p}{2} H_{p-1}^{(4)} + \sum_{k=1}^{p-1} \frac{H_k}{k^2} \pmod{p^3}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{H_k}{k^2} &= \sum_{k=1}^{p-1} \frac{1}{k^2} \sum_{j=1}^k \frac{1}{j} = \sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=j}^{p-1} \frac{1}{k^2} \\
&\equiv -\sum_{j=1}^{p-1} \frac{H_j^{(2)}}{j} + H_{p-1}^{(3)} \pmod{p^3}.
\end{aligned}$$

From the above and with the help of Lemma 3.2, we obtain (3.3).

Now we turn to prove (3.4). It is easy to see that

$$\begin{aligned}
\sum_{k=0}^{p-1} (H(2, 4; k) + H(4, 2; k)) &= \sum_{1 \leq i < j \leq p-1} \frac{p-j}{i^2 j^4} + \sum_{1 \leq i < j \leq p-1} \frac{p-j}{i^4 j^2} \\
&\equiv -H(2, 3; p-1) - H(4, 1; p-1) \pmod{p}.
\end{aligned}$$

By [12, Theorem 3.1] we have $H(2, 3; p-1) \equiv -2B_{p-5} \pmod{p}$ and $H(4, 1; p-1) \equiv -B_{p-5} \pmod{p}$ for $p \geq 7$. Then (3.4) follows at once.

Finally, we consider (3.5). Clearly,

$$\sum_{k=0}^{p-1} H(2, 2, 2; k) = \sum_{1 \leq i_1 < i_2 < i_3 \leq p-1} \frac{p-i_3}{i_1^2 i_2^2 i_3^2} \equiv -H(2, 2, 1; p-1) \pmod{p}.$$

By [13, Theorem 3.5], we have

$$H(2, 2, 1; p-1) \equiv \frac{3}{2} B_{p-3} \pmod{p}.$$

The proof of Lemma 3.3 is now complete. \square

Lemma 3.4. *Let $k \in \mathbb{N}$. Then for $n \in \mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1} (2m+1)^3 \binom{m+k}{2k}^2 = \frac{(n-k)^2 (2n^2 - k - 1)}{k+1} \binom{n+k}{2k}^2.$$

Proof. It can be verified directly by induction on n . \square

Proof of Theorem 1.2. The case $p = 5$ can be verified directly. Below we assume that $p \geq 7$. By Lemma 3.4 we have

$$\sum_{m=0}^{p-1} (2m+1)^3 A_m = \sum_{m=0}^{p-1} (2m+1)^3 \sum_{k=0}^m \binom{m+k}{2k}^2 \binom{2k}{k}^2 = \sum_{k=0}^{p-1} \binom{2k}{k}^2 \sum_{m=0}^{p-1} (2m+1)^3 \binom{m+k}{2k}^2$$

$$\begin{aligned}
&= \sum_{k=0}^{p-1} \binom{2k}{k}^2 \frac{(p-k)^2(2p^2-k-1)}{k+1} \binom{p+k}{2k}^2 \\
&= p^2 \sum_{k=0}^{p-1} \frac{2p^2-k-1}{k+1} \binom{p-1}{k}^2 \binom{p+k}{k}^2.
\end{aligned}$$

Noting that

$$\begin{aligned}
\binom{p-1}{k}^2 \binom{p+k}{k}^2 &= \prod_{j=1}^k \left(1 - \frac{p^2}{j^2}\right)^2 \equiv \prod_{j=1}^k \left(1 - \frac{2p^2}{j^2} + \frac{p^4}{j^4}\right) \\
&\equiv 1 - 2p^2 H_k^{(2)} + p^4 H_k^{(4)} + 4p^4 H(2, 2; k) - 2p^6 (H(2, 4; k) + H(4, 2; k)) \\
&\quad - 8p^6 H(2, 2, 2; k) \pmod{p^7},
\end{aligned}$$

we arrive at

$$\begin{aligned}
\sum_{m=0}^{p-1} (2m+1)^3 A_m &\equiv 2p^4 \sum_{k=1}^{p-1} \frac{1}{k+1} \left(1 - 2p^2 H_k^{(2)} + p^4 H_k^{(4)} + 4p^4 H(2, 2; k)\right) \\
&\quad - p^2 \sum_{k=0}^{p-1} \left(1 - 2p^2 H_k^{(2)} + p^4 H_k^{(4)} + 4p^4 H(2, 2; k) \right. \\
&\quad \left. - 2p^6 (H(2, 4; k) + H(4, 2; k)) - 8p^6 H(2, 2, 2; k)\right) \pmod{p^9}.
\end{aligned}$$

It is clear that

$$\sum_{k=0}^{p-1} \frac{1}{k+1} = H_{p-1} + \frac{1}{p}.$$

With the help of Lemma 3.2 we obtain that

$$\begin{aligned}
\sum_{k=0}^{p-1} \frac{H_k^{(2)}}{k+1} &= \sum_{k=1}^p \frac{H_{k-1}^{(2)}}{k} = \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k} - H_{p-1}^{(3)} + \frac{H_{p-1}^{(2)}}{p} \\
&\equiv \frac{3}{p^2} H_{p-1} + \frac{H_{p-1}^{(2)}}{p} - H_{p-1}^{(3)} - \frac{1}{2} p^2 B_{p-5} \pmod{p^3}.
\end{aligned}$$

Clearly,

$$\sum_{k=0}^{p-1} \frac{H_k^{(4)}}{k+1} = H(4, 1; p-1) + \frac{H_{p-1}^{(4)}}{p} \equiv -B_{p-5} + \frac{H_{p-1}^{(4)}}{p} \pmod{p}.$$

Furthermore,

$$\sum_{k=0}^{p-1} \frac{H(2, 2; k)}{k+1} = \frac{1}{2} \sum_{k=1}^p \frac{1}{k} \left(\left(H_{k-1}^{(2)}\right)^2 - H_{k-1}^{(4)} \right) = \frac{1}{2} \sum_{k=1}^p \frac{1}{k} \left(\left(H_k^{(2)}\right)^2 - H_k^{(4)} - \frac{2H_k^{(2)}}{k^2} + \frac{2}{k^4} \right)$$

$$= \sum_{k=1}^{p-1} \frac{H(2, 2; k)}{k} - H(2, 3; p-1) + \frac{1}{2p} \left(\left(H_{p-1}^{(2)} \right)^2 - H_{p-1}^{(4)} \right).$$

Then by Lemmas 2.2 and 3.1 we arrive at

$$\sum_{k=0}^{p-1} \frac{H(2, 2; k)}{k+1} \equiv \frac{3}{2} B_{p-5} - \frac{1}{2p} H_{p-1}^{(4)} \pmod{p}.$$

For $r = 2, 4$ we have

$$\sum_{k=0}^{p-1} H_k^{(r)} = \sum_{k=1}^{p-1} \sum_{l=1}^k \frac{1}{l^r} = \sum_{l=1}^{p-1} \frac{p-l}{l^r} = p H_{p-1}^{(r)} - H_{p-1}^{(r-1)}.$$

Combining the above and in view of Lemma 3.3 we obtain

$$\begin{aligned} \sum_{m=0}^{p-1} (2m+1)^3 A_m &\equiv 2p^4 H_{p-1} + 2p^3 - 12p^4 H_{p-1} + 4p^6 H_{p-1}^{(3)} - 4p^5 H_{p-1}^{(2)} + 2p^8 B_{p-5} - 2p^8 B_{p-5} \\ &\quad + 2p^7 H_{p-1}^{(4)} + 12p^8 B_{p-5} - 4p^7 H_{p-1}^{(4)} - p^3 + 2p^5 H_{p-1}^{(2)} - 2p^4 H_{p-1} - p^7 H_{p-1}^{(4)} \\ &\quad + p^6 H_{p-1}^{(3)} + 2p^7 H_{p-1}^{(4)} + 12p^4 H_{p-1} - 2p^8 B_{p-5} - 4p^6 H_{p-1}^{(3)} - 6p^8 B_{p-5} \\ &= p^3 - 2p^4 H_{p-1} - 2p^5 H_{p-1}^{(2)} + p^6 H_{p-1}^{(3)} - p^7 H_{p-1}^{(4)} + 4p^8 B_{p-5} \pmod{p^9}. \end{aligned}$$

Then Theorem 1.2 follows from Lemmas 2.1 and 2.2. \square

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