Fibonacci, Motzkin, Schröder, Fuss-Catalan and other Combinatorial Structures: Universal and Embedded Bijections

R. Brak* and N. Mahony†

School of Mathematics and Statistics,
The University of Melbourne
Parkville, Victoria 3052,
Australia

September 23, 2019

Abstract

A combinatorial structure, $\mathcal{F}$, with counting sequence $\{a_n\}_{n\geq 0}$ and ordinary generating function $G_{\mathcal{F}} = \sum_{n\geq 0} a_n x^n$, is positive algebraic if $G_{\mathcal{F}}$ satisfies a polynomial equation $G_{\mathcal{F}} = \sum_{k=0}^{N} p_k(x)\, G_{\mathcal{F}}^k$ and $p_k(x)$ is a polynomial in x with *non-negative integer coefficients*. We show that every such family is associated with a normed $\boldsymbol{n}$-magma. An $\boldsymbol{n}$-magma with $\boldsymbol{n} = (n_1, \ldots, n_k)$ is a pair $\mathcal{M}$ and $\mathcal{F}$ where $\mathcal{M}$ is a set of combinatorial structures and $\mathcal{F}$ is a tuple of n_i-ary maps $f_i : \mathcal{M}^{n_i} \to \mathcal{M}$. A norm is a super-additive size map $\|\cdot\| : \mathcal{M} \to \mathbb{N}$.

If the normed $\boldsymbol{n}$-magma is free then we show there exists a recursive, norm preserving, universal bijection between all positive algebraic families $\mathcal{F}_i$ with the same counting sequence. A free $\boldsymbol{n}$-magma is defined using a universal mapping principle. We state a theorem which provides a combinatorial method of proving if a particular $\boldsymbol{n}$-magma is free. We illustrate this by defining several $\boldsymbol{n}$-magmas: eleven $(1,1)$-magmas (the Fibonacci families), seventeen $(1,2)$-magmas (nine Motzkin and eight Schröder families) and seven (3)-magmas (the Fuss-Catalan families). We prove they are all free and hence obtain a universal bijection for each $\boldsymbol{n}$. We also show how the $\boldsymbol{n}$-magma structure manifests as an embedded bijection.

Keywords:

*rb1@unimelb.edu.au
†nedm@unimelb.edu.au

Contents

List of combinatorial families

1 Definitions and general results

We generalise the Catalan results of Brak [8] to arbitrary positive algebraic combinatorial families. A combinatorial structure, $\mathcal{F}$, with counting sequence $\{a_n\}_{n\geq 0}$ and ordinary generating function $G_{\mathcal{F}}(x) = \sum_{n\geq 0} a_n x^n$ is positive algebraic if $G_{\mathcal{F}}(x)$ satisfies a polynomial equation

$$\sum_{k=0}^{N} p_k(x)\, G_{\mathcal{F}}^k = G_{\mathcal{F}} \tag{1}$$

and $p_k(x) = \sum_{i=0}^{m_k} b_i^{(k)} x^i$ is a degree m_k polynomial in x with *non-negative integer coefficients* (ie. all $b_i^{(k)} \geq 0$). This case contains a large number of well known combinatorial families such as Fibonacci, Catalan, Motzkin, Schröder and Fuss-Catalan.

We show that positive algebraic families are associated with particular normed $\boldsymbol{n}$-magmas (a norm is a super-additive size map). An $\boldsymbol{n}$-magma with $\boldsymbol{n} = (n_1, \ldots, n_k)$ is a pair $(\mathcal{M}, \mathcal{F})$ where $\mathcal{M}$ is a set and $\mathcal{F}$ is a tuple of n_i-ary maps $f_i : \mathcal{M}^{n_i} \to \mathcal{M}$. To each monomial $b_i^{(k)} x^i G_{\mathcal{F}}^k$ in (1) we associate $b_i^{(k)}$ unique k-ary maps. These are the maps that constitute the n_i-ary maps of the $\boldsymbol{n}$-magma. These maps have to be carefully defined for each combinatorial family to ensure they satisfy certain required properties.

1.1 Magma definitions

In this section we generalise a number of definitions from [8].

A magma is an algebraic structure defined in [7] as a pair $(\mathcal{M}, \star)$ where $\mathcal{M}$ is a non-empty countable set called the **base set** and $\star$ is a **product map**

$$\star : \mathcal{M} \times \mathcal{M} \to \mathcal{M}.$$

If $(\mathcal{N}, \bullet)$ is a magma, then a **magma morphism** θ from $\mathcal{M}$ to $\mathcal{N}$ is a map $\theta : \mathcal{M} \to \mathcal{N}$ such that for all $m, m' \in \mathcal{M}$,

$$\theta(m \star m') = \theta(m) \bullet \theta(m').$$

We generalise these definitions to allow for arbitrary n-ary maps.

Definition 1 ($\boldsymbol{n}$-magma)**.** *Let $\mathcal{M}$ be a non-empty countable set called the **base set**. An **$\boldsymbol{n}$-magma** defined on $\mathcal{M}$, where $\boldsymbol{n} = (n_1, \ldots, n_k)$ with $n_1 \leq \cdots \leq n_k$, is a $(k+1)$-tuple $(\mathcal{M}, f_1, \ldots, f_k)$ where*

$$f_i : \mathcal{M}^{n_i} \to \mathcal{M}$$

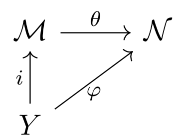
*is an n_i-ary map, for each $i = 1, \ldots, k$. If $(\mathcal{M}, f_1, \ldots, f_k)$ and $(\mathcal{N}, g_1, \ldots, g_k)$ are two $\boldsymbol{n}$-magmas, then an **$\boldsymbol{n}$-magma morphism** from $\mathcal{M}$ to $\mathcal{N}$ is a map $\theta : \mathcal{M} \to \mathcal{N}$ such that for all $i \in \{1, \ldots, k\}$ and all $m_1, \ldots, m_{n_i} \in \mathcal{M}$,*

$$\theta(f_i(m_1, \ldots, m_{n_i})) = g_i(\theta(m_1), \ldots, \theta(m_{n_i})).$$

In this definition, we adopt the convention that we write the maps in the same order for all distinct $\boldsymbol{n}$-magmas for the same $\boldsymbol{n}$. Hence if $(\mathcal{M},\ f_1,\ \ldots,\ f_k)$ and $(\mathcal{N},\ g_1,\ \ldots,\ g_k)$ are two $\boldsymbol{n}$-magmas, then f_i and g_i have the same arity, for each $i = 1, \ldots, k$. We also require that if we have more than one map with the same arity, then the order in which we write the maps is significant and thus permuting the maps defines a different $\boldsymbol{n}$-magma. For example, the two (n_1, n_1)-magmas $(\mathcal{M},\ f_1,\ f_2)$ and $(\mathcal{M}',\ f_1',\ f_2')$ are equal if and only if $\mathcal{M}' = \mathcal{M}$, $f_1' = f_1$ and $f_2' = f_2$. This distinction is important later when we prove that the Cartesian $(1, 1)$-magma is free.

We are interested in when an $\boldsymbol{n}$-magma is free. To this end we start with the universal mapping definition of free, but will later give a theorem – Theorem 1 – enabling us to give a combinatorial, rather than a universal mapping, proof of when an $\boldsymbol{n}$-magma is free.

Definition 2 (Free $\boldsymbol{n}$-magma Universal Mapping Principle)**.** *Let $\boldsymbol{n} = (n_1, \ldots, n_k)$ and let $(\mathcal{M},\ f_1,\ \ldots,\ f_k)$ be an $\boldsymbol{n}$-magma. Then $(\mathcal{M},\ f_1,\ \ldots,\ f_k)$ is **free** if the following is true: There exists a set Y and a map $i : Y \to \mathcal{M}$ such that for all $\boldsymbol{n}$-magmas $(\mathcal{N},\ f_1',\ \ldots,\ f_k')$ and for all maps $\varphi : Y \to \mathcal{N}$, there exists a unique (up to isomorphism) $\boldsymbol{n}$-magma morphism $\theta : \mathcal{M} \to \mathcal{N}$ such that $\varphi = \theta \circ i$, that is, the diagram*



*commutes. The image of the set Y in $\mathcal{M}$, $X = Img(Y)$, will be called the **set of generators** of $\mathcal{M}$.*

We now define a size function on the base set of the $\boldsymbol{n}$-magma. We will call the function a norm as we will require it to be "super-additive" (defined below) with respect to each of the $\boldsymbol{n}$-magma maps. The norm is used in two essential ways. Firstly, it partitions the base set $\mathcal{M}$ into sets of elements which have the same size. The size of the sets in this partition defines the counting sequence. Secondly, the super-additivity of the norm map is used, along with unique factorisation, to give a combinatorial characterisation of free $\boldsymbol{n}$-magmas. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$ denote the set of positive integers.

Definition 3 (Norm)**.** *Let $(\mathcal{M},\ f_1,\ \ldots,\ f_k)$ be an $\boldsymbol{n}$-magma, where $\boldsymbol{n} = (n_1, \ldots, n_k)$. A **norm** is a map $\|\cdot\| : \mathcal{M} \to \mathbb{N}$ that satisfies the **super-additive** conditions:*

1. *All unary maps $f_i : \mathcal{M} \to \mathcal{M}$ must satisfy*
$$\|f_i(m)\| > \|m\| \ .$$
2. *All maps $f_i : \mathcal{M}^{n_i} \to \mathcal{M}$, $n_i > 1$, must satisfy*
$$\|f_i(m_1, \ldots, m_{n_i})\| \geq \sum_{j=1}^{n_i} \|m_j\| \ .$$

*If $(\mathcal{M},\ f_1,\ \ldots,\ f_k)$ has a norm, then it will be called a **normed $\boldsymbol{n}$-magma**.*

Note, it is important that there are *no* elements in $\mathcal{M}$ of size zero, ie. we use the set $\mathbb{N}$ in the norm definition and *not* the set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. Furthermore, any norm defined on an $\boldsymbol{n}$-magma is not necessarily unique. The non-uniqueness is significant as for example there exist Motzkin and Schröder families given by the same $(1,2)$-magma. Both families have the same base set, as well as the same unary and binary maps, however they have different norm maps.

We now consider the problem of factorisation in $\boldsymbol{n}$-magmas and discuss how it is related to the existence of a norm. We show that the existence of a norm guarantees that recursive factorisation of any $\boldsymbol{n}$-magma element will terminate. We begin by defining reducible and irreducible elements.

Definition 4 (Reducible, irreducible elements)**.** *Let $(\mathcal{M}, f_1, \ldots, f_k)$ be an $\boldsymbol{n}$-magma, where $\boldsymbol{n} = (n_1, \ldots, n_k)$. The image of the maps f_i in $\mathcal{M}$, is called the **set of reducible elements**. Letting $\mathcal{M}_i^+ = Img(f_i)$ for each $i = 1, \ldots, k$, we can define the set of reducible elements as $\mathcal{M}^+ = \bigcup_{i=1}^k \mathcal{M}_i^+$. The elements of the set $\mathcal{M}^0 = \mathcal{M} \backslash \mathcal{M}^+$ are called **irreducible elements** and the set $\mathcal{M}^0$ is called the **set of irreducibles**.*

A unique factorisation $\boldsymbol{n}$-magma describes when every element of the base set can be written uniquely in terms of the irreducible elements and the $\boldsymbol{n}$-magma maps.

Definition 5 (Unique factorisation)**.** *Let $(\mathcal{M}, f_1, \ldots, f_k)$ be an $\boldsymbol{n}$-magma, where $\boldsymbol{n} = (n_1, \ldots, n_k)$. If*

(i) every map $f_i : \mathcal{M}^{n_i} \to \mathcal{M}$ is injective, and

(ii) $\mathcal{M}_i^+ \cap \mathcal{M}_j^+ = \emptyset$ for all $i, j \in \{1, \ldots, k\}$ such that $i \neq j$,

*then we will call $(\mathcal{M}, f_1, \ldots, f_k)$ a **unique factorisation $\boldsymbol{n}$-magma**.*

Note that (ii) requires that the images of the maps forms a partition of the set of reducible elements $\mathcal{M}^+$.

We will be interested only in unique factorisation $\boldsymbol{n}$-magmas since this property holds for all combinatorial structures we consider.

Thus in the remainder of this paper we assume all $\boldsymbol{n}$-magma are unique factorisation $\boldsymbol{n}$-magmas.

1.2 General $\boldsymbol{n}$-magma theorems

In this section we present a number of general results about $\boldsymbol{n}$-magmas. These results will prove useful in later sections and allow us to describe certain properties which $\boldsymbol{n}$-magmas may possess. Propositions 1, 2 and 3, along with Theorem 1, are generalisations of results stated and proven in [8].

Proposition 1. *Let $(\mathcal{M}, f_1, \ldots, f_k)$ be a normed $(n_1, \ldots, n_k)$-magma with non-empty base set $\mathcal{M}$, and let the set of elements with minimal norm be $\mathcal{M}_{min} \subset \mathcal{M}$. Then $\mathcal{M}_{min}$ is non-empty and all elements of $\mathcal{M}_{min}$ are irreducible.*

Proof. Clearly we have that $\mathcal{M}_{\min}$ is non-empty since $\mathcal{M}$ is non-empty and $\mathcal{M}_{\min}$ is taken to be the subset of $\mathcal{M}$ whose elements have minimal norm.

To prove that all elements of $\mathcal{M}_{\min}$ are irreducible, proceed by contradiction. Assume that there exists some $m \in \mathcal{M}_{\min}$ such that m is reducible. Therefore $m \in \mathrm{Img}(f_i)$ for some $i \in \{1, \ldots, k\}$, so there exists $m_1, \ldots, m_{n_i}$ such that $m = f_i(m_1, \ldots, m_{n_i})$. If $n_i = 1$ (that is, f_i is a unary map) then $m = f_i(m_1)$, and so $\|m\| > \|m_1\|$. This contradicts the fact that $m \in \mathcal{M}_{\min}$. If $n_i > 1$, then $\|m\| \geq \sum_{j=1}^{n_i} \|m_j\|$, and since $\mathrm{Img}(\|\cdot\|) \subseteq \mathbb{N}$, we must have that $\|m\| > \|m_j\|$ for each $j \in \{1, \ldots, n_i\}$. This again contradicts the fact that $m \in \mathcal{M}_{\min}$. Thus we conclude that every element of $\mathcal{M}_{\min}$ is irreducible. □

Now let $\boldsymbol{n} = (n_1, \ldots, n_k)$ and consider an arbitrary unique factorisation $\boldsymbol{n}$-magma $(\mathcal{M}, f_1, \ldots, f_k)$. We have that for all reducible elements $m \in \mathcal{M}^+$, there exists a unique $i \in \{1, \ldots, k\}$ and unique $m_1, \ldots, m_{n_i} \in \mathcal{M}$ such that

$$m = f_i(m_1, \ldots, m_{n_i}).$$

We can recursively define a function π as

$$\pi(m) = \begin{cases} [\pi(m_1), \ldots, \pi(m_{n_i})], & \text{if } m \in \mathcal{M}^+ \text{ with } m = f_i(m_1, \ldots, m_{n_i}), \\ m, & \text{if } m \in \mathcal{M}^0. \end{cases} \tag{2}$$

The bracketed expression $[\pi(m_1), \ldots, \pi(m_{n_i})]$ is considered an n_i-tuple. The square parentheses are used to avoid possible ambiguity arising from the use of round parentheses later.

We are interested in when the recursion (2) terminates. This motivates the following definition.

Definition 6 (Finite decomposition $\boldsymbol{n}$-magma)**.** *Let $(\mathcal{M}, f_1, \ldots, f_k)$ be an $\boldsymbol{n}$-magma. If, for all elements $m \in \mathcal{M}$, the recursive function* (2) *terminates then $(\mathcal{M}, f_1, \ldots, f_k)$ will be called a **finite decomposition $\boldsymbol{n}$-magma**. If $(\mathcal{M}, f_1, \ldots, f_k)$ is a finite decomposition $\boldsymbol{n}$-magma, then $\pi(m)$ will be called the **decomposition** of m.*

If the unique factorisation $\boldsymbol{n}$-magma has a norm, then the recursive function (2) will always terminate, as given by the following proposition.

Proposition 2. *Let $(\mathcal{M}, f_1, \ldots, f_k)$ be a unique factorisation $\boldsymbol{n}$-magma. Then it is a normed $\boldsymbol{n}$-magma if and only if it is a finite decomposition $\boldsymbol{n}$-magma.*

Proof. Forward: Since $(\mathcal{M}, f_1, \ldots, f_k)$ is normed, there exists a function $\|\cdot\| : \mathcal{M} \to \mathbb{N}$ which satisfies Definition 3. Now taking any $m \in \mathcal{M}$, either $m \in \mathcal{M}^0$ and so the recursion terminates immediately or there exists a unique $i \in \{1, \ldots, k\}$ and unique $m_1, \ldots, m_{n_i} \in \mathcal{M}$ such that $m = f_i(m_1, \ldots, m_{n_i})$. In this case, $\|m\| \geq \sum_{j=1}^{n_i} \|m_j\|$, and so $\|m\| > \|m_j\|$ for each $j \in \{1, \ldots, n_i\}$ because $\|\cdot\|$ takes values in $\mathbb{N}$. This recursive procedure continues until a factor is in $\mathcal{M}^0$ at which point it terminates from (2), or until a factor has minimal norm. In this case Proposition 1 states that this factor must be in $\mathcal{M}^0$ and thus the recursion terminates. This must occur in a finite number of steps since $\mathrm{Img}(\|\cdot\|) \subseteq \mathbb{N}$ and, since $\mathbb{N}$ is a well ordered set, any subset of $\mathbb{N}$ has a least element. Thus $(\mathcal{M}, f_1, \ldots, f_k)$ is a finite decomposition $\boldsymbol{n}$-magma.

Reverse: Since ($\mathcal{M}$, f_1, …, f_k) is a finite decomposition $\boldsymbol{n}$-magma, the recursion (2) must terminate for all $m \in \mathcal{M}$. We can define a norm $\|\cdot\| : \mathcal{M} \to \mathbb{N}$ on ($\mathcal{M}$, f_1, …, f_k) as follows. For each $m \in \mathcal{M}^0$, define $\|m\| = 1$. For each $m \in \mathcal{M}^+$, we know that $m = f_i(m_1, \dots, m_{n_i})$ for unique $i \in \{1, \dots, k\}$ and unique $m_1, \dots, m_{n_i} \in \mathcal{M}$. Defining $\|m\| = \sum_{j=1}^{n_i} \|m_j\|$ for such m, we have that $\|\cdot\|$ is well-defined since $\pi(m)$ terminates and thus contains a finite number of occurrences of elements of $\mathcal{M}^0$ (for which we have already defined the value of $\|\cdot\|$). Further, we have that $\|\cdot\|$ is a norm since it satisfies the conditions of Definition 3. □

Note that the above proposition holds even if an $\boldsymbol{n}$-magma is *not* a unique factorisation $\boldsymbol{n}$-magma. We chose to prove it only for the case of a unique factorisation $\boldsymbol{n}$-magma as this is the only result we require. Making this assumption also simplifies the proof considerably.

We now prove the following result, which gives us a combinatorial way to characterise when an $\boldsymbol{n}$-magma is free.

Theorem 1. *Let $\boldsymbol{n} = (n_1, \dots, n_k)$. If ($\mathcal{M}$, f_1, …, f_k) is a unique factorisation normed $\boldsymbol{n}$-magma with non-empty finite set of irreducibles, then ($\mathcal{M}$, f_1, …, f_k) is a free $\boldsymbol{n}$-magma generated by the irreducible elements.*

Proof. Let ($\mathcal{M}$, f_1, …, f_k) be a unique factorisation normed $\boldsymbol{n}$-magma with non-empty finite set of irreducibles, $\mathcal{M}^0$. Take any set Y such that $|Y| = |\mathcal{M}^0|$. Let ($\mathcal{N}$, f'_1, …, f'_k) be an arbitrary $\boldsymbol{n}$-magma and let $\varphi : Y \to \mathcal{N}$ be any map. We are required to show that there exists a map $i : Y \to \mathcal{M}^0$ such that there exists a unique $\boldsymbol{n}$-magma morphism $\theta : \mathcal{M} \to \mathcal{N}$ with the property that $\varphi = \theta \circ i$. We take $i : Y \to \mathcal{M}^0$ to be any bijection from Y to $\mathcal{M}^0 \subset \mathcal{M}$ (such a bijection exists since φ was chosen so that $|Y| = |\mathcal{M}^0|$). We define $\theta : \mathcal{M} \to \mathcal{N}$ as follows:

(i) For all $m \in \mathcal{M}^0$, we have $m = i(y)$ for some $y \in Y$. For such m, define

$$\theta(m) = \varphi(y). \tag{3}$$

(ii) For all $m \in \mathcal{M}^+$, where $\mathcal{M}^+ = \mathcal{M} \backslash \mathcal{M}^0$, we recursively define the image under θ as follows: if $m = f_i(m_1, \dots, m_{n_i})$, then define

$$\theta(m) = f'_i(\theta(m_1), \dots, \theta(m_{n_i})). \tag{4}$$

From (4), we have that θ is an $\boldsymbol{n}$-magma morphism, while (3) ensures that $\varphi = \theta \circ i$. We have that θ is unique (given the choice of the maps i and φ). This is because the value of θ on the set of irreducibles $\mathcal{M}^0$ along with the recursive definition of θ for all other elements of $\mathcal{M}$ uniquely specifies the value of θ for all elements of $\mathcal{M}$. This also ensures that θ is the only map which satisfies both (4) and $\varphi = \theta \circ i$. □

Theorem 1 shows that the following is sufficient to show that a set $\mathcal{M}$ along with k maps $f_1, \dots, f_k$, where $f_i : \mathcal{M}^{n_i} \to \mathcal{M}$ for each $i \in \{1, \dots, k\}$, is a free $(n_1, \dots, n_k)$-magma:

(i) Show that all maps f_i are injective.

(ii) Show that the images of the maps are disjoint, that is, for all $i, j \in \{1, \dots, k\}$ such that $i \neq j$, we have $\mathrm{Img}(f_i) \cap \mathrm{Img}(f_j) = \emptyset$.

(iii) Show that there exists a map $\|\cdot\| : \mathcal{M} \to \mathbb{N}$ which satisfies Definition 3.

(iv) Determine the set of generators (usually by first determining the range of the maps $\mathcal{M}^+ = \bigcup_{i=1}^{k} \mathcal{M}_i^+$ and then taking its complement, $\mathcal{M}^0 = \mathcal{M} \setminus \mathcal{M}^+$).

Note, *any* norm will suffice to prove a particular $\boldsymbol{n}$-magma is free (using Theorem 1).

The next result is well-known for free structures. It states that there exists an isomorphism between any pair of $\boldsymbol{n}$-magmas for the same $\boldsymbol{n}$. Moreover, it states that this isomorphism is unique up to the choice of bijection between the generators. We will use this result later to define universal bijections between our combinatorial families.

Proposition 3. *Let Y be a set and let $(\mathcal{M}, f_1, \ldots, f_k)$ and $(\mathcal{N}, f'_1, \ldots, f'_k)$ be free $\boldsymbol{n}$-magmas satisfying free $\boldsymbol{n}$-magma universal mapping diagrams as follows:*

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\theta} & \mathcal{M}' \\ {\scriptstyle i}\uparrow & \nearrow{\scriptstyle f} & \\ Y & & \end{array} \qquad \begin{array}{ccc} \mathcal{N} & \xrightarrow{\phi} & \mathcal{N}' \\ {\scriptstyle j}\uparrow & \nearrow{\scriptstyle g} & \\ Y & & \end{array} \tag{5}$$

Then there exists a unique $\boldsymbol{n}$-magma isomorphism $\Gamma : \mathcal{M} \to \mathcal{N}$ such that $\Gamma \circ i = j$ and $\Gamma^{-1} \circ j = i$.

We repeat the standard proof since it is a constructive proof and hence provides the basis for the universal bijection algorithm stated later.

Proof. Since $(\mathcal{M}, f_1, \ldots, f_k)$ is a free $\boldsymbol{n}$-magma, the left diagram of (5) commutes for all $i : Y \to \mathcal{M}$ and all $f : Y \to \mathcal{M}'$. Taking $\mathcal{M}' = \mathcal{N}$ and $f = j$ gives

$$j = \theta \circ i. \tag{6}$$

Similarly, since $(\mathcal{N}, f'_1, \ldots, f'_k)$ is a free $\boldsymbol{n}$-magma, the right diagram of (5) commutes for all $j : Y \to \mathcal{N}$ and all $g : Y \to \mathcal{N}'$. Taking $\mathcal{N}' = \mathcal{M}$ and $j = i$ gives

$$i = \phi \circ j. \tag{7}$$

Together, (6) and (7) give $j = \theta \circ \phi \circ j$ and $i = \phi \circ \theta \circ i$. Therefore $\theta \circ \phi = \mathrm{id}_{\mathcal{M}}$ and $\phi \circ \theta = \mathrm{id}_{\mathcal{N}}$, and thus $\Gamma = \theta$ and $\Gamma^{-1} = \phi$ are isomorphisms. □

Suppose that $(\mathcal{M}, f_1, \ldots, f_k)$ and $(\mathcal{N}, f'_1, \ldots, f'_k)$ are free $\boldsymbol{n}$-magmas with one generator, for the same $\boldsymbol{n}$. Since $|Y| = 1$, we have that i and j are unique. From this it follows that the $\boldsymbol{n}$-magma isomorphism $\Gamma : \mathcal{M} \to \mathcal{N}$ from Proposition 3 is unique.

Proposition 3 proves to be very useful since all of the the combinatorial families considered in the appendix are free $\boldsymbol{n}$-magmas generated by a single element for some $\boldsymbol{n}$. Suppose that we have two combinatorial families which are both free $\boldsymbol{n}$-magmas generated by a single element for the same $\boldsymbol{n}$. Then this unique isomorphism gives us a one-to-one map between the objects of the two families. This map preserves the recursive structure of the objects. We will also show that this unique isomorphism preserves the norm of the objects it maps. Thus we have a size-preserving recursive bijection between the two combinatorial families. This will be used to define the universal bijections.

2 Combinatorial structures

We apply the above results to several well known combinatorial structures counted by Fibonacci, Motzkin, Schröder and Fuss-Catalan numbers.

We begin by defining these combinatorial families which we will generalise to those counted by the "p-analogue" of the sequences. This was done in [8] for the Catalan numbers by defining the p-Catalan numbers, where $p \in \mathbb{N} = \{1, 2, 3, \ldots\}$, by $C_0(p) = p$ and

$$C_n(p) = p^{n+1} \frac{1}{n+1} \binom{2n}{n}, \qquad n \geq 1. \tag{8}$$

A natural interpretation of these p-analogue sequences (in most instances) is as a variation of the combinatorial family where we allow some part of the object to be coloured in any of p colours. In particular, we will see that it is the part which corresponds to the generator of the family. The p-analogue sequences arise naturally when using certain set constructions which give the original sequences. The well known cases correspond to $p = 1$.

Definition 7. *Let $p \in \mathbb{N}$, and define the following sequences:*

(i) Define the ***p-Fibonacci numbers*** *$F_n(p)$ by the recurrence relation*

$$F_n(p) = F_{n-1}(p) + F_{n-2}(p), \qquad n \geq 2, \tag{9}$$

with $F_0(p) = 0$ and $F_1(p) = p$.

(ii) Define the ***p-Motzkin numbers*** *$M_n(p)$ by the recurrence relation*

$$M_n(p) = M_{n-1}(p) + \sum_{k=0}^{n-2} M_k(p)\, M_{n-k-2}(p), \qquad n \geq 2, \tag{10}$$

with $M_0(p) = M_1(p) = p$.

(iii) Define the (little) ***p-Schröder numbers*** *$S_n(p)$ by the recurrence relation*

$$S_n(p) = S_{n-1}(p) + \sum_{k=0}^{n-1} S_k(p)\, S_{n-k-1}(p), \qquad n \geq 1, \tag{11}$$

with $S_0(p) = p$.

(iv) Define the ***order 3 p-Fuss-Catalan numbers*** *(from here onwards, p-Fuss-Catalan numbers) $T_n(p)$ by the recurrence relation*

$$T_n(p) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1-i} T_i(p)\, T_j(p)\, T_{n-1-i-j}(p), \qquad n \geq 1, \tag{12}$$

with $T_0(p) = p$.

We will refer to the order 3 Fuss-Catalan numbers simply as the *Fuss-Catalan numbers*. We do not consider any other order of Fuss-Catalan number, although one could easily extend these ideas to higher orders in an obvious way. See [2] for further discussion of the general Fuss-Catalan numbers.

For each of these four p-sequences we state propositions giving the algebraic equation satisfied by their generating functions and an expression for the counting sequences. The former is derived in the standard way from the above defining recurrence relations and the latter are derived using the Lagrange inversion formula. Since these are standard computations we do not provide any details.

Proposition 4. *The generating function $F(x) = \sum_{n\geq 0} F_n(p)\, x^n$ for the p-Fibonacci numbers satisfies the algebraic equation*

$$F(x) = px + xF(x) + x^2F(x).$$

and

$$F_n(p) = p \sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n-k-1}{k}.$$

Proposition 5. *The generating function $M(x) = \sum_{n\geq 0} M_n(p)\, x^n$ for the p-Motzkin numbers satisfies the algebraic equation*

$$M(x) = p + xM(x) + x^2M(x)^2.$$

and

$$M_n(p) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \binom{n}{2k} C_k(p),$$

where $C_k(p)$ is the k^{th} p-Catalan number (8)*.*

Proposition 6. *The generating function $S(x) = \sum_{n\geq 0} S_n(p)\, x^n$ for the p-Schröder numbers satisfies the algebraic equation*

$$S(x) = p + xS(x) + xS(x)^2.$$

and

$$S_n(p) = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k}(p),$$

where $C_k(p)$ is the k^{th} p-Catalan number (8)*.*

Proposition 7. *The generating function $T(x) = \sum_{n\geq 0} T_n(p)\, x^n$ for the p-Fuss-Catalan numbers satisfies the algebraic equation*

$$T(x) = p + xT(x)^3.$$

and

$$T_n(p) = \frac{1}{3n+1}\binom{3n+1}{n} p^{2n+1}.$$

3 Fibonacci normed (1,1)-magmas

In this section, we discuss (1,1)-magmas and show if an appropriate norm is defined they are related to combinatorial families which are enumerated by the Fibonacci numbers. We first define the Cartesian (1,1)-magma and show its relationship with the p-Fibonacci numbers. We then define a universal bijection between any two Fibonacci normed (1,1)-magmas.

3.1 Cartesian (1,1)-magma

We define arguably the simplest (1,1)-magma, which we call the Cartesian (1,1)-magma.

Definition 8 (Cartesian (1,1)-magma)**.** *Let X be a non-empty finite set. Define the sequence $\mathcal{W}_n(X)$ of sets of words in the alphabet $\{u_1, u_2\} \cup X$ as follows:*

$$\mathcal{W}_1(X) = X, \tag{13a}$$
$$\mathcal{W}_2(X) = \{u_1 w : w \in \mathcal{W}_1(X)\}, \tag{13b}$$
$$\mathcal{W}_n(X) = \{u_1 w : w \in \mathcal{W}_{n-1}(X)\} \cup \{u_2 w : w \in \mathcal{W}_{n-2}(X)\}, \qquad n \geq 3, \tag{13c}$$

where $u_i w$ is the concatenation of the symbol u_i with the word w, for $i = 1, 2$.

Let $\mathcal{W}_X = \bigcup_{i \geq 1} \mathcal{W}_i(X)$ and define two unary maps,

$$\mu_1 : \mathcal{W}_X \to \mathcal{W}_X,$$
$$\mu_2 : \mathcal{W}_X \to \mathcal{W}_X,$$

as follows:

$$\mu_1(w) = u_1 w,$$
$$\mu_2(w) = u_2 w.$$

The triple $(\mathcal{W}_X, \mu_1, \mu_2)$ will be called the ***Cartesian (1,1)-magma generated by X****.*

If $X = \{\epsilon\}$, the sequence of sets $\mathcal{W}_n(X)$ defining the base set $\mathcal{W}_X$ of the Cartesian (1,1)-magma begins as follows:

$$\begin{aligned}
\mathcal{W}_1(X) &= \{\epsilon\},\\
\mathcal{W}_2(X) &= \{u_1\epsilon\},\\
\mathcal{W}_3(X) &= \{u_1u_1\epsilon,\ u_2\epsilon\},\\
\mathcal{W}_4(X) &= \{u_1u_1u_1\epsilon,\ u_1u_2\epsilon,\ u_2u_1\epsilon\},\\
\mathcal{W}_5(X) &= \{u_1u_1u_1u_1\epsilon,\ u_1u_1u_2\epsilon,\ u_1u_2u_1\epsilon,\ u_2u_1u_1\epsilon,\ u_2u_2\epsilon\},\\
&\vdots
\end{aligned}$$

Having defined the Cartesian (1,1)-magma, it is possible to prove that it is free directly from Definition 2 without introducing any norm. However we will take the shorter route by showing there exists a norm, hence along with the other conditions required by Theorem 1, we will prove it is free.

Theorem 2. *The Cartesian (1,1)-magma of Definition 8 is a free (1,1)-magma.*

We will show that $(\mathcal{W}_X,\, \mu_1,\, \mu_2)$ is a unique factorisation normed (1,1)-magma with set of irreducibles equal to X. Then, by Theorem 1, we will have that $(\mathcal{W}_X,\, \mu_1,\, \mu_2)$ is a free (1,1)-magma generated by X.

Proof. Suppose that $w_1, w_2 \in \mathcal{W}_X$ are such that $\mu_1(w_1) = \mu_1(w_2)$. We have $\mu_1(w_1) = u_1 w_1$ and $\mu_1(w_2) = u_1 w_2$ and hence $u_1 w_1 = u_1 w_2$ thus have $w_1 = w_2$. Similarly we have that μ_2 is injective since $\mu_2(w_1) = \mu_2(w_2)$ for $w_1, w_2 \in \mathcal{W}_X$ implies $u_2 w_1 = u_2 w_2$ and hence $w_1 = w_2$. Thus we have that both μ_1 and μ_2 are injective. To show $\mathrm{Img}(\mu_1) \cap \mathrm{Img}(\mu_2) = \emptyset$, proceed by contradiction. Suppose that

$$\mathrm{Img}(\mu_1) \cap \mathrm{Img}(\mu_2) \neq \emptyset$$

and take $w \in \mathrm{Img}(\mu_1) \cap \mathrm{Img}(\mu_2)$. Since $w \in \mathrm{Img}(\mu_1)$, we have $w = \mu_1(w_1) = u_1 w_1$ for some $w_1 \in \mathcal{W}_X$. Similarly, since $w \in \mathrm{Img}(\mu_2)$, we must have $w = \mu_2(w_2) = u_2 w_2$ for some $w_2 \in \mathcal{W}_X$. Therefore $u_1 w_1 = u_2 w_2$ and hence $u_1 = u_2$, thus giving a contradiction.

Thus the maps μ_1 and μ_2 are injective and $\mathrm{Img}(\mu_1) \cap \mathrm{Img}(\mu_2) = \emptyset$, so $(\mathcal{W}_X,\, \mu_1,\, \mu_2)$ is a unique factorisation (1,1)-magma.

The set of irreducibles is X since this is the complement of $\mathrm{Img}(\mu_1) \cup \mathrm{Img}(\mu_2)$.

Take $w \in \mathcal{W}_X$, supposing that $w \in \mathcal{W}_n(X)$ and hence that $\|w\| = n$. We have from (13c) that $\mu_1(w) \in \mathcal{W}_{n+1}(X)$ and $\mu_2(w) \in \mathcal{W}_{n+2}(X)$. Therefore $\|\mu_1(w)\| = n+1$ and $\|\mu_2(w)\| = n+2$. Thus $\|\mu_1(w)\| > \|w\|$ and $\|\mu_2(w)\| > \|w\|$ and so by Definition 3, $\|\cdot\|$ is a norm. Therefore $(\mathcal{W}_X,\, \mu_1,\, \mu_2)$ is a normed (1,1)-magma. □

Proposition 8. *Let $(\mathcal{W}_X,\, \mu_1,\, \mu_2)$ be the Cartesian (1,1)-magma generated by the set X, where $|X| = p$, and define the map $\|\cdot\|_F : \mathcal{W}_X \to \mathbb{N}$ by $\|m\|_F = n$ when $m \in \mathcal{W}_n(X)$. If*

$$N_n = \{m \in \mathcal{W}_X : \|m\|_F = n\}, \quad n \geq 1,$$

then

$$|N_n| = F_n(p), \qquad n \geq 1,$$

where $F_n(p)$ is the nth p-Fibonacci number of Definition 7.

Proof. First, note that $N_n = \mathcal{W}_n(X)$ since $\|m\|_F = n$ if and only if $m \in \mathcal{W}_n(X)$. Now, we have $\mathcal{W}_1(X) = X$ so $|N_1| = |\mathcal{W}_1(X)| = |X| = p$, and $\mathcal{W}_2(X) = \{u_1 w : w \in \mathcal{W}_1(X)\}$ so

$$|N_2| = |\mathcal{W}_2(X)| = |\{u_1 w : w \in \mathcal{W}_1(X)\}| = |\mathcal{W}_1(X)| = p.$$

For $n \geq 3$, (13c) gives

$$\begin{aligned} |N_n| &= |\{u_1 w : w \in \mathcal{W}_{n-1}(X)\}| + |\{u_2 w : w \in \mathcal{W}_{n-2}(X)\}| \\ &= |\mathcal{W}_{n-1}(X)| + |\mathcal{W}_{n-2}(X)| \\ &= |N_{n-1}| + |N_{n-2}| \end{aligned}$$

This is exactly the p-Fibonacci recurrence (9). □

Corollary 1. *Let $(\mathcal{W}_\epsilon,\ \mu_1,\ \mu_2)$ be the Cartesian (1,1)-magma generated by the single element ϵ. If N_n and $\|\cdot\|_F : \mathcal{W}_\epsilon \to \mathbb{N}$ are as defined in Proposition 8, then*

$$|N_n| = F_n, \qquad n \geq 1,$$

where F_n is the nth Fibonacci number.

Take the Cartesian (1,1)-magma generated by $\{\epsilon\}$, $(\mathcal{W}_\epsilon,\ \mu_1,\ \mu_2)$ and note that the norm $\|\cdot\|_F : \mathcal{W}_\epsilon \to \mathbb{N}$ from above could equivalently be defined by requiring that

$$\|\epsilon\|_F = 1, \tag{14a}$$
$$\|\mu_1(w)\|_F = \|w\|_F + 1, \tag{14b}$$
$$\|\mu_2(w)\|_F = \|w\|_F + 2, \tag{14c}$$

for all $w \in \mathcal{W}_\epsilon$.

This motivates the following definition. We seek to characterise when a (1,1)-magma is associated with the Fibonacci numbers. As we have noted, we can define many norms on the same (1,1)-magma. Defining a norm $\|\cdot\|_F$ which satisfies (14) gives us the Fibonacci numbers. Thus we define a Fibonacci normed (1,1)-magma to be a free (1,1)-magma with a single generator *along with* a particular norm function which satisfies (14) for the relevant (1,1)-magma generator and unary maps.

Definition 9 (Fibonacci normed (1,1)-magma)**.** *Let $(\mathcal{M},\ f_1,\ f_2)$ be a unique factorisation normed (1,1)-magma with only one irreducible element, ϵ. Let $\|\cdot\|_F : \mathcal{M} \to \mathbb{N}$ be a norm satisfying*

$$\|\epsilon\|_F = 1, \tag{15a}$$
$$\|f_1(m)\|_F = \|m\|_F + 1, \tag{15b}$$
$$\|f_2(m)\|_F = \|m\|_F + 2, \tag{15c}$$

*for all $m \in \mathcal{M}$. Then $(\mathcal{M},\ f_1,\ f_2)$ with the norm $\|\cdot\|_F$ is called a **Fibonacci normed (1,1)-magma**.*

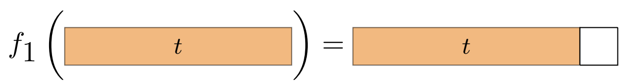
In the remainder of this section, we reference a number of combinatorial structures which are counted by the Fibonacci numbers. Further details of these are provided in Appendix 7.1. We take the convention that the two unary maps are called f and g. We make it clear that we are using the maps specific to a certain family by placing a subscript on each of the maps. This subscript contains the number assigned to that family in the appendix. We choose to call the unique generator in each family ϵ, and make it clear which family it comes from via the subscript. We name the maps in such a way that our (1,1)-magma is $(\mathcal{M},\ f,\ g)$, and hence

$$\|f(m)\| = \|m\| + 1, \qquad \|g(m)\| = \|m\| + 2, \tag{16}$$

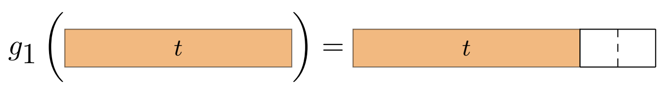
for all $m \in \mathcal{M}$.

We have seen that any Fibonacci normed (1,1)-magma is such that the number of elements of the base set with norm n is given by the nth Fibonacci number. We now present a simple example of a Fibonacci normed (1,1)-magma, taking the family of Fibonacci tilings $\mathbb{F}_1$. This family arises by considering the number of ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes. The number of ways to tile such a board is given by F_n. We can define the Fibonacci tiling (1,1)-magma $(\mathcal{F}_1, f_1, g_1)$ as follows:

- Take the base set $\mathcal{F}_1$ to be the set of all tilings of a $1 \times n$ board using 1×1 squares and 1×2 dominoes, for all $n \in \mathbb{N}_0$. Note that we consider the trivial empty tiling to be the only way to tile a 1×0 board (i.e. an empty board).
- Define one unary map to take a tiling of a $1 \times n$ board and add a single 1×1 square to the right to give a tiling of a $1 \times (n+1)$ board. Call this map f_1. Schematically:



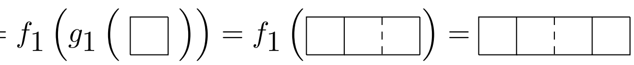
- Define the other unary map to take a tiling of a $1 \times n$ board and add a 1×2 domino to the right to give a tiling of a $1 \times (n+2)$ board. Call this map g_1. Schematically:



- The only generator is the trivial empty tiling of an empty board: $\epsilon_1 = \emptyset$. This is the only element in the base set which is not in the image of one of the two maps.

It is simple to see that $(\mathcal{F}_1, f_1, g_1)$ is a unique factorisation (1,1)-magma. This is because we can form any tiling of a $1 \times n$ board by applying a unique sequence of compositions of the two maps to the generator. Take for example the following tiling:

We see that this can be constructed as follows:

$$f_1(g_1(f_1(\epsilon))) = f_1(g_1(f_1(\emptyset))) =$$ 

We define the norm $\|\cdot\|_F$ of a tiling of a $1 \times n$ board to be $n+1$. Thus we see that the norm satisfies (15) of Definition 9. That is,

$$\|\epsilon_1\|_F = 1,$$
$$\|f_1(t)\|_F = \|t\|_F + 1,$$
$$\|g_1(t)\|_F = \|t\|_F + 2,$$

for all $t \in \mathcal{F}_1$.

Therefore the (1,1)-magma $(\mathcal{F}_1, f_1, g_1)$ along with the norm just defined is indeed a Fibonacci normed (1,1)-magma.

Some Fibonacci families are such that there is no natural interpretation of the generator ϵ. This is usually the case when there are F_{n+2} objects with traditional size parameter n. This is because the generator then corresponds to an object with traditional size parameter equal to -1. In such cases, we define the (1,1)-magma $(\mathcal{M},\ f,\ g)$ by the following procedure. We simply take ϵ to be an arbitrary symbol denoting the generator and define the two maps f and g as follows:

(i) Define $f(\epsilon)$.

(ii) Define $g(\epsilon)$.

(iii) For all $m \in \mathcal{M}\backslash\{\epsilon\}$, define $f(m)$.

(iv) For all $m \in \mathcal{M}\backslash\{\epsilon\}$, define $g(m)$.

This process completely specifies the two maps and the base set, so we have a well-defined free (1,1)-magma. For examples of when this procedure is required, see the following Fibonacci families in Appendix 7.1:

- $\mathbb{F}_9$: Binary sequences with no consecutive 1's,
- $\mathbb{F}_{11}$: Subsets with no consecutive integers.

3.2 Free (1,1)-magma isomorphisms and a universal bijection

We have seen that any Fibonacci normed (1,1)-magma is such that the number of elements of the base set with norm n is given by the nth Fibonacci number. Using the result of Proposition 3, there exists a unique (1,1)-magma isomorphism between any two Fibonacci normed (1,1)-magmas. As we will see, this isomorphism preserves the norms of the objects in the case that both (1,1)-magmas are Fibonacci normed (1,1)-magmas. Thus this gives a size-preserving bijection between the two Fibonacci families. We therefore make explicit just how this isomorphism is defined so that we are able to make use of it for specific families.

Definition 10 (Universal bijection)**.** *Let* $(\mathcal{M}, f, g)$ *and* $(\mathcal{N}, f', g')$ *be free (1,1)-magmas with generating sets* $X_{\mathcal{M}}$ *and* $X_{\mathcal{N}}$ *respectively, with* $|X_{\mathcal{M}}| = |X_{\mathcal{N}}|$*. Let* $\sigma : X_{\mathcal{M}} \to X_{\mathcal{N}}$ *be any bijection, and define the map* $\Upsilon : \mathcal{M} \to \mathcal{N}$ *as follows: for all* $m \in \mathcal{M} \setminus X_{\mathcal{M}}$*,*

(i) Decompose m *into an expression in terms of generators* $\epsilon_i \in X_{\mathcal{M}}$ *and the unary maps* f *and* g*.*

(ii) In the decomposition of m*, replace every occurrence of* ϵ_i *with* $\sigma(\epsilon_i)$*, every occurence of* f *with* f' *and every occurrence of* g *with* g'*. Call this expression* $\upsilon(m)$*.*

(iii) Define $\Upsilon(m)$ *to be* $\upsilon(m)$*, that is, evaluate all maps in* $\upsilon(m)$ *to give an element of* $\mathcal{N}$*.*

This leads to the following proposition. The proposition follows immediately from the fact that Υ is equal to the map Γ from Proposition 3.

Proposition 9. *Let* $\Upsilon : \mathcal{M} \to \mathcal{N}$ *be the map of Definition 10. Then* Υ *is a free (1,1)-magma isomorphism.*

Schematically, we can write Υ as follows:

$$m \xrightarrow{\text{decompose}} \underset{\epsilon_i \to \sigma(\epsilon_i),\ f \to f',\ g \to g'}{\text{substitute}} \xrightarrow{\text{evaluate}} n. \tag{17}$$

Since Υ is an isomorphism between the free (1,1)-magmas $(\mathcal{M},\ f,\ g)$ and $(\mathcal{N},\ f',\ g')$, we have that Υ defines a bijection between the base sets $\mathcal{M}$ and $\mathcal{N}$. Further to this, it gives us that this bijection is recursive: if $m = f(m_0)$, then

$$\Upsilon(m) = f'(\Upsilon(m_0)),$$

and if $m = g(m_0)$, then

$$\Upsilon(m) = g'(\Upsilon(m_0)).$$

It is also important to note that if the free (1,1)-magmas have norms satisfying certain conditions, then the norm is preserved under the map Υ. Suppose that $(\mathcal{M},\ f,\ g)$ and $(\mathcal{N},\ f',\ g')$ have norms $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{N}$ and $\|\cdot\|_{\mathcal{N}} : \mathcal{N} \to \mathbb{N}$ respectively. If:

(i) $\|m\|_{\mathcal{M}} = \|\sigma(m)\|_{\mathcal{N}}$ for all $m \in X_{\mathcal{M}}$,

(ii) $\|f(m)\|_{\mathcal{M}} = \|m\|_{\mathcal{M}} + \kappa_1$ for all $m \in \mathcal{M}$ and $\|f'(n)\|_{\mathcal{N}} = \|n\|_{\mathcal{N}} + \kappa_1$ for all $n \in \mathcal{N}$, where $\kappa_1 \in \mathbb{N}$, and

(iii) $\|g(m)\|_{\mathcal{M}} = \|m\|_{\mathcal{M}} + \kappa_2$ for all $m \in \mathcal{M}$ and $\|g'(n)\|_{\mathcal{N}} = \|n\|_{\mathcal{N}} + \kappa_2$ for all $n \in \mathcal{N}$, where $\kappa_2 \in \mathbb{N}$,

then we have

$$\|m\|_{\mathcal{M}} = \|\Upsilon(m)\|_{\mathcal{N}}, \qquad m \in \mathcal{M}.$$

This result follows immediately by considering the decomposed expressions for m and $\Upsilon(m)$.

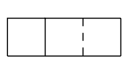
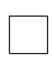
Combinatorially we are primarily interested in bijections between structures of the same "size" and thus we are interested in bijections which preserve the norm. This is the case for Fibonacci normed (1,1)-magmas which are invariant under the map Υ.

We now present a number of examples illustrating the universal bijection of Definition 10. We will consider the following Fibonacci normed (1,1)-magmas:

- Fibonacci tilings $(\mathcal{F}_1, f_1, g_1)$,
- Reflections through two plates of glass $(\mathcal{F}_{10}, f_{10}, g_{10})$,
- Binary sequences with no consecutive 1's $(\mathcal{F}_9, f_9, g_9)$,
- Compositions with no 1's $(\mathcal{F}_4, f_4, g_4)$.

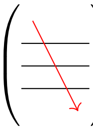
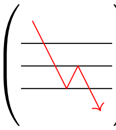
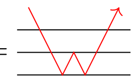
See Appendix 7.1 for the definitions and details of each of these (1,1)-magmas.

We begin by demonstrating the bijection from the family of Fibonacci tilings to the family of reflections through two plates of glass. First, take a Fibonacci tiling, and decompose it down into its factorised form:

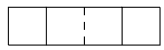
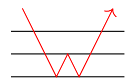
 $= f_1\left($  $\right) = f_1\left(g_1\left($  $\right)\right) = f_1\left(g_1\left(f_1\left(\emptyset\right)\right)\right) = f_1\left(g_1\left(f_1\left(\epsilon_1\right)\right)\right)$

Next, we replace every occurrence of the generator ϵ_1 with the generator of the reflections through two plates of glass, ϵ_{10}. We also replace each map f_1 with f_{10} and each map g_1 with

g_{10}. After making these substitutions, evaluate the resulting expression to obtain an element of the second family:

$f_{10}\left(g_{10}\left(f_{10}\left(\epsilon_{10}\right)\right)\right) = f_{10}\left(g_{10}\left(\right.\right.$  $\left.\left.\right)\right) = f_{10}\left(\right.$  $\left.\right) =$ 

Thus the universal bijection maps

 $\mapsto$ 

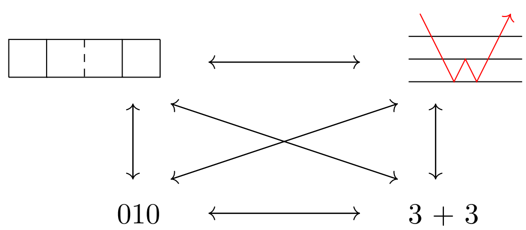
If instead we were seeking a bijection between Fibonacci tilings and binary sequences with no consecutive 1's, then we would simply replace all parts of the factorised expression for the Fibonacci tiling with the parts corresponding to binary sequences with no consecutive 1's, as follows:

$$f_9\left(g_9\left(f_9\left(\epsilon_9\right)\right)\right) = f_9\left(g_9\left(\emptyset\right)\right) = f_9\left(01\right) = 010.$$

Similarly for the Fibonacci family of compositions containing no 1's:

$$f_4\left(g_4\left(f_4\left(\epsilon_4\right)\right)\right) = f_4\left(g_4\left(f_4\left(2\right)\right)\right) = f_4\left(g_4\left(3\right)\right) = f_4\left(3+2\right) = 3+3.$$

So we see that the universal bijection gives each of the following bijections:



This demonstrates how useful this universal bijection is. Rather than simply obtaining bijections between families one by one, we see that each time we determine the (1,1)-magma structure of a Fibonacci family, it immediately gives us a bijection to each other Fibonacci family whose (1,1)-magma structure is known. As a result of this, we are able to very quickly build up a large number of bijections.

4 Motzkin and Schröder normed (1,2)-magmas

In this section we consider (1,2)-magmas and show how these relate to Motzkin numbers and Schröder numbers. We adopt the convention that the binary map is always written as an in-fix operator. We begin by constructing an example of a free (1,2)-magma, which we call the Cartesian (1,2)-magma.

4.1 Cartesian (1,2)-magma

First, we introduce and discuss some notation that will be used throughout this section. We will use square parentheses when writing n-tuples to avoid possible ambiguity arising from the use of round parentheses later. Thus we take the n-ary Cartesian product to be the following:

$$X_1 \times \cdots \times X_n = \{[x_1, \ldots, x_n] : x_i \in X_i,\ i \in \{1, \ldots, n\}\}.$$

and the notation $[X]$ to mean the set

$$[X] = \{[x] : x \in X\}.$$

This gives us a set notation for unary maps which we will use to define an explicit (1,2)-magma.

Definition 11 (Cartesian (1,2)-magma)**.** *Let X be a non-empty finite set. Define the sequence $\mathcal{W}_n(X)$ of sets of nested 1- and 2-tuples by*

$$\mathcal{W}_1(X) = X, \tag{18a}$$
$$\mathcal{W}_2(X) = [\mathcal{W}_1(X)], \tag{18b}$$
$$\mathcal{W}_n(X) = [\mathcal{W}_{n-1}(X)] \ \cup \ \bigcup_{k=1}^{n-2} (\mathcal{W}_k(X) \times \mathcal{W}_{n-k-1}(X)), \qquad n \geq 3. \tag{18c}$$

Let $\mathcal{W}_X = \bigcup_{n\geq 1} \mathcal{W}_n(X)$ and $\mathcal{W}_X^+ = \mathcal{W}_X \backslash X$. Define the unary map $\mu : \mathcal{W}_X \to \mathcal{W}_X$ by

$$\mu(w) = [w], \qquad w \in \mathcal{W}_X, \tag{19}$$

and the binary map $\diamond : \mathcal{W}_X \times \mathcal{W}_X \to \mathcal{W}_X$ by

$$w_1 \diamond w_2 = [w_1, w_2], \qquad w_1, w_2 \in \mathcal{W}_X. \tag{20}$$

*The triple $(\mathcal{W}_X,\ \mu,\ \diamond)$ is called the **Cartesian (1,2)-magma generated by X**.*

If $X = \{\epsilon\}$, the sequence of sets $\mathcal{W}_n(X)$ defining the base set $\mathcal{W}_X$ of the Cartesian (1,2)-magma begins as follows:

$$\begin{aligned}
\mathcal{W}_1(X) &= \{\epsilon\}, \\
\mathcal{W}_2(X) &= \{[\epsilon]\}, \\
\mathcal{W}_3(X) &= \{[[\epsilon]], \quad [\epsilon, \epsilon]\}, \\
\mathcal{W}_4(X) &= \{[[[\epsilon]]], \quad [[\epsilon, \epsilon]], \quad [\epsilon, [\epsilon]], \quad [[\epsilon], \epsilon]\}, \\
&\ \ \vdots
\end{aligned}$$

We now prove that the Cartesian (1,2)-magma is free.

Theorem 3. *The Cartesian (1,2)-magma $(\mathcal{W}_X,\ \mu,\ \diamond)$ is a free (1,2)-magma.*

We will show that $(\mathcal{W}_X,\ \mu,\ \diamond)$ is a unique factorisation normed (1,2)-magma with set of irreducibles equal to X. Then, by Theorem 1, we will have that $(\mathcal{W}_X,\ \mu,\ \diamond)$ is a free (1,2)-magma generated by X.

Proof. Suppose that $w, w' \in \mathcal{W}_X$ are such that $\mu(w) = \mu(w')$. Then $[w] = [w']$ and hence $w = w'$. Thus μ is injective. Now suppose $w_1, w_2, w_1', w_2' \in \mathcal{W}_X$ are such that $w_1 \diamond w_2 = w_1' \diamond w_2'$. Then $[w_1, w_2] = [w_1', w_2']$ and hence $w_1 = w_1', w_2 = w_2'$ and so $\diamond$ is injective. Clearly we have $\mathrm{Img}(\mu) \cap \mathrm{Img}(\diamond) = \emptyset$ since $\mathrm{Img}(\mu)$ contains only 1-tuples and $\mathrm{Img}(\diamond)$ contains only 2-tuples. Thus $(\mathcal{W}_X,\ \mu,\ \diamond)$ is a unique factorisation (1,2)-magma.

The set of irreducibles is X since this is the complement of $\mathrm{Img}(\mu) \cup \mathrm{Img}(\diamond)$.

For each $w \in \mathcal{W}_X$, define $\|w\| = n$ if $w \in \mathcal{W}_n(X)$. Now take $w \in \mathcal{W}_n(X)$. Since $\mu(w) = [w] \in [\mathcal{W}_n(X)]$, we have $\mu(w) \in \mathcal{W}_{n+1}(X)$ from (18). Therefore

$$\|\mu(w)\| = n + 1 > \|w\| .$$

Now consider $w_1 \in \mathcal{W}_{n_1}(X)$ and $w_2 \in \mathcal{W}_{n_2}(X)$. We have $w_1 \diamond w_2 = [w_1, w_2]$ and hence $w_1 \diamond w_2 \in \mathcal{W}_{n_1+n_2+1}(X)$ from (18). Therefore

$$\|w_1 \diamond w_2\| = n_1 + n_2 + 1 > \|w_1\| + \|w_2\| .$$

Therefore $\|\cdot\| : \mathcal{W}_X \to \mathbb{N}$ is a norm. □

4.2 Motzkin normed (1,2)-magmas

For the purpose of proving Theorem 3 we were required to demonstrate that there exists a norm on the (1,2)-magma $(\mathcal{W}_X,\ \mu,\ \diamond)$. While we could have chosen any function which satisfies Definition 3, this particular norm was chosen since it gives rise to the Motzkin numbers. In Section 4.3 we will define a different norm on the *same* base set which will give rise to the Schröder numbers.

Proposition 10. *Let $(\mathcal{W}_X,\ \mu,\ \diamond)$ be the Cartesian (1,2)-magma generated by the set X, where $|X| = p$. Define the map $\|\cdot\|_M : \mathcal{W}_X \to \mathbb{N}$ by $\|m\|_M = n$ when $m \in \mathcal{W}_n(X)$, where $\mathcal{W}_n(X)$ is as defined in Definition 11 for $n \in \mathbb{N}$. If*

$$N_n = \{m \in \mathcal{W}_X : \|m\|_M = n\}, \qquad n \geq 1,$$

then

$$|N_n| = M_{n-1}(p), \qquad n \geq 1,$$

where $M_n(p)$ are the p-Motzkin numbers from Definition 7.

Proof. Since $\|m\|_M = n$ if and only if $m \in \mathcal{W}_n(X)$, we have $N_n = \mathcal{W}_n(X)$. We have $N_1 = \mathcal{W}_1(X) = X$ so $|N_1| = |X| = p$, and $N_2 = \mathcal{W}_2(X) = [X]$ so $|N_2| = |[X]| = p$. Now, for $n \geq 3$, (18c) gives

$$\begin{aligned} |N_n| &= \left| [\mathcal{W}_{n-1}(X)] \ \cup \ \bigcup_{k=1}^{n-2} (\mathcal{W}_k(X) \times \mathcal{W}_{n-k-1}(X)) \right| \\ &= |N_{n-1}| + \sum_{k=1}^{n-2} |N_k| \cdot |N_{n-k-1}| \end{aligned}$$

This is equivalent to the p-Motzkin recurrence (10). □

Corollary 2. *Let $(\mathcal{W}_\epsilon,\ \mu,\ \diamond)$ be the Cartesian (1,2)-magma generated by the single element, ϵ. If N_n and $\|\cdot\|_M : \mathcal{W}_\epsilon \to \mathbb{N}$ are as defined in Proposition 10, then*

$$|N_n| = M_{n-1}, \qquad n \geq 1,$$

where M_n is the Motzkin number from Definition 7.

The norm function $\|\cdot\|_M : \mathcal{W}_\epsilon \to \mathbb{N}$ from the above corollary could equivalently be defined recursively as follows: Let $\|\cdot\|_M : \mathcal{W}_\epsilon \to \mathbb{N}$ be such that

$$\|\epsilon\|_M = 1, \tag{21a}$$
$$\|\mu(w)\|_M = \|w\|_M + 1, \tag{21b}$$
$$\|w \diamond w'\|_M = \|w\|_M + \|w'\|_M + 1, \tag{21c}$$

for all $w, w' \in \mathcal{W}_\epsilon$.

We see that a free (1,2)-magma generated by a single element is partitioned by the norm function into sets given by the Motzkin numbers only when the norm satisfies (21). Thus we define a Motzkin normed (1,2)-magma to be a free (1,2)-magma with a single generator *along with* a norm function which satisfies (21) for the relevant (1,2)-magma.

Definition 12 (Motzkin normed (1,2)-magma)**.** *Let $(\mathcal{M},\ f,\ \star)$ be a unique factorisation normed (1,2)-magma with one irreducible element, ϵ. Let $\|\cdot\|_M : \mathcal{M} \to \mathbb{N}$ be a norm satisfying*

$$\|\epsilon\|_M = 1, \tag{22a}$$
$$\|f(m)\|_M = \|m\|_M + 1, \tag{22b}$$
$$\|m \star m'\|_M = \|m\|_M + \|m'\|_M + 1 \tag{22c}$$

for all $m, m' \in \mathcal{M}$. Then $(\mathcal{M},\ f,\ \star)$ with the norm $\|\cdot\|_M : \mathcal{M} \to \mathbb{N}$ is called a ***Motzkin normed (1,2)-magma****.*

If $(\mathcal{M},\ f,\ \star)$ is a Motzkin normed (1,2)-magma with unique generator ϵ, then the base set $\mathcal{M}$ begins (sorting by norm) by evaluating the following expressions:

Norm 1: ϵ.
Norm 2: $f(\epsilon)$.
Norm 3: $f(f(\epsilon))$, $\quad \epsilon \star \epsilon$.
Norm 4: $f(f(f(\epsilon)))$, $\quad f(\epsilon \star \epsilon)$, $\quad \epsilon \star f(\epsilon)$, $\quad f(\epsilon) \star \epsilon$.
Norm 5: $f(f(f(f(\epsilon))))$, $\quad f(f(\epsilon \star \epsilon))$, $\quad f(\epsilon \star f(\epsilon))$, $\quad f(f(\epsilon) \star \epsilon)$, $\epsilon \star f(f(\epsilon))$, $\quad \epsilon \star (\epsilon \star \epsilon)$, $\quad f(\epsilon) \star f(\epsilon)$, $\quad f(f(\epsilon)) \star \epsilon$, $(\epsilon \star \epsilon) \star \epsilon$.

We see that the Motzkin norm $\|\cdot\|_M : \mathcal{M} \to \mathbb{N}$ can be informally stated as follows, where $m \in \mathcal{M}$:

$$\|m\|_M = (\text{number of } \epsilon\text{'s}) + (\text{number of } f\text{'s}) + (\text{number of } \star\text{'s}).$$

For each Motzkin family the norm is usually a simple function of the conventional size parameter for that family. This can be seen in Appendix 7.2, where the details of a number of Motzkin normed (1,2)-magmas can be found.

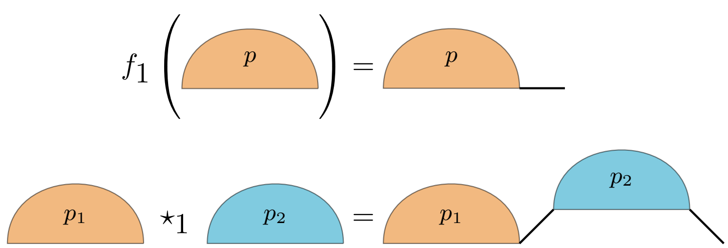
We will consider a number of combinatorial families which are listed in Appendix 7.2. We adopt the following convention for any Motzkin normed (1,2)-magma: (i) the unary map is denoted f and (ii) the binary map is denoted $\star$ and written using in-fix notation. We make it clear that we are using the maps specific to a certain family by referencing that family in the subscript of the map which is done via the number assigned to that family in the appendix.

We now discuss another example of a Motzkin normed (1,2)-magma, the Motzkin paths $\mathbb{M}_1$. A Motzkin path of length n is a path from $(0,0)$ to $(n,0)$ using steps $U=(1,1)$, $D=(1,-1)$ and $H=(1,0)$ which remains above the line $y=0$. The number of such paths of length n is given by the nth Motzkin number M_n. For the corresponding Motzkin normed (1,2)-magma $(\mathcal{M}_1, f_1, \star_1)$, the base set $\mathcal{M}_1$ is given by all Motzkin paths from $(0,0)$ to $(n,0)$, for all $n \in \mathbb{N}_0$. The empty path, $n=0$, is taken to be a single vertex.

The generator of this family is the empty path,

$$\epsilon_1 = \bullet,$$

and the two maps are defined schematically as follows:



Thus the unary map adds a single horizontal step after the path and the binary map concatenates the two paths while adding a pair of up and down steps as shown. These are the usual right factorisations of a Motzkin path corresponding to the recursive structure of all Motzkin paths, but now interpreted as maps. All Motzkin paths can be constructed using sequences of compositions of these two maps applied to the generator. Therefore this is a unique factorisation (1,2)-magma.

The norm $\|m\|_M$ of any Motzkin path m is defined as the length of the path + 1. We can immediately see that this norm satisfies (22):

$$\begin{aligned}
\|\epsilon\|_M &= 1, \\
\|f_1(p_1)\|_M &= \|p_1\|_M + 1, \\
\|p_1 \star_1 p_2\|_M &= \|p_1\|_M + \|p_2\|_M + 1,
\end{aligned}$$

for all Motzkin paths $p_1, p_2 \in \mathcal{M}_1$.

Thus we see that $(\mathcal{M}_1, f_1, \star_1)$ is a Motzkin normed (1,2)-magma.

4.3 Schröder normed (1,2)-magmas

In this section, we show how the Schröder numbers are related to free (1,2)-magmas which we do by defining a different norm on the Cartesian (1,2)-magma. In order to clearly define the norm in terms of the Cartesian base set, we provide an alternative method for constructing the Cartesian (1,2)-magma. Note, the resulting set is the same set as that defined by (18) but constructed differently.

Let X be a non-empty finite set, and define the sequence $\mathcal{Y}_n(X)$ of sets of nested 1- and 2-tuples by

$$\mathcal{Y}_1(X) = X, \tag{23a}$$

$$\mathcal{Y}_n(X) = [\mathcal{Y}_{n-1}(X)] \ \cup \ \bigcup_{k=1}^{n-1} \left(\mathcal{Y}_k(X) \times \mathcal{Y}_{n-k}(X)\right), \qquad n \geq 2. \tag{23b}$$

If $X = \{\epsilon\}$, then the sequence of sets $\mathcal{Y}_n(X)$ begins as follows:

$$\begin{aligned}
&\mathcal{Y}_1(X) = \{\epsilon\}, \\
&\mathcal{Y}_2(X) = \{[\epsilon], \quad [\epsilon,\epsilon]\}, \\
&\mathcal{Y}_3(X) = \{[[\epsilon]], \quad [[\epsilon,\epsilon]], \quad [\epsilon,[\epsilon]], \quad [\epsilon,[\epsilon,\epsilon]], \quad [[\epsilon],\epsilon], \quad [[\epsilon,\epsilon],\epsilon]\}, \\
&\quad\vdots
\end{aligned}$$

Letting $\mathcal{Y}_X = \bigcup_{n\geq 1} \mathcal{Y}_n(x)$, we have that $(\mathcal{Y}_X,\, \mu,\, \diamond)$ is equal to the Cartesian (1,2)-magma $(\mathcal{W}_X,\, \mu,\, \diamond)$ of Definition 11. This follows from the fact that $\mathcal{Y}_X$ and $\mathcal{W}_X$ both contain all nested 1- and 2-tuples containing elements of the set X. Thus we have the following proposition.

Proposition 11. *Let $\mathcal{Y}_X = \bigcup_{n\geq 1} \mathcal{Y}_n(x)$. Define the unary map $\mu : \mathcal{W}_X \to \mathcal{W}_X$ by*

$$\mu(w) = [w], \qquad w \in \mathcal{W}_X,$$

and the binary map $\diamond : \mathcal{W}_X \times \mathcal{W}_X \to \mathcal{W}_X$ by

$$w_1 \diamond w_2 = [w_1, w_2], \qquad w_1, w_2 \in \mathcal{W}_X.$$

Then $(\mathcal{Y}_X,\, \mu,\, \diamond)$ is the Cartesian (1,2)-magma of Definition 11.

The link between the Cartesian (1,2)-magma and the Schröder numbers is given by the following proposition.

Proposition 12. *Let $(\mathcal{W}_X,\, \mu,\, \diamond)$ be the Cartesian (1,2)-magma generated by the set X, where $|X| = p$. Define the map $\|\cdot\|_S : \mathcal{W}_X \to \mathbb{N}$ by $\|m\|_S = n$ when $m \in \mathcal{Y}_n(X)$, where $\mathcal{Y}_n(X)$ is as defined in* (23)*. If*

$$N_n = \{m \in \mathcal{W}_X :\ \|m\|_S = n\}, \qquad n \geq 1,$$

then

$$|N_n| = S_{n-1}(p), \qquad n \geq 1,$$

where $S_n(p)$ are the p-Schröder numbers from Definition 7.

Proof. Since $\|m\|_S = n$ if and only if $m \in \mathcal{Y}_n(X)$, we have $N_n = \mathcal{Y}_n(X)$. Thus we have $|N_1| = |\mathcal{Y}_1(X)| = |X| = p$. Now, for $n \geq 2$, (23b) gives

$$\begin{aligned} |N_n| &= \left| \mathcal{Y}_{n-1}(X) \ \cup \ \bigcup_{k=1}^{n-1} \mathcal{Y}_k(X) \times \mathcal{Y}_{n-k}(X) \right| \\ &= |N_{n-1}| + \sum_{k=1}^{n-1} |N_k| \cdot |N_{n-k}| \end{aligned}$$

This is equivalent to the p-Schröder recurrence (11). □

Corollary 3. *Let $(\mathcal{W}_\epsilon,\ \mu,\ \diamond)$ be the Cartesian (1,2)-magma generated by the single element, ϵ. If N_n and $\|\cdot\|_S : \mathcal{W}_\epsilon \to \mathbb{N}$ are as defined in Proposition 12, then*

$$|N_n| = S_{n-1}, \qquad n \geq 1,$$

where S_n is the Schröder number from Definition 7.

Note the norm function $\|\cdot\|_S : \mathcal{W}_\epsilon \to \mathbb{N}$ from the above corollary could equivalently be defined recursively as follows: Let $\|\cdot\|_S : \mathcal{W}_\epsilon \to \mathbb{N}$ be such that

$$\|\epsilon\|_S = 1, \tag{24a}$$
$$\|\mu(w)\|_S = \|w\|_S + 1, \tag{24b}$$
$$\|w \diamond w'\|_S = \|w\|_S + \|w'\|_S\,, \tag{24c}$$

for all $w, w' \in \mathcal{W}_\epsilon$.

This motivates the following definition.

Definition 13 (Schröder normed (1,2)-magma)**.** *Let $(\mathcal{M},\ f,\ \star)$ be a unique factorisation normed (1,2)-magma with only one irreducible element, ϵ. Let $\|\cdot\|_S : \mathcal{M} \to \mathbb{N}$ be a norm satisfying*

$$\|\epsilon\|_S = 1, \tag{25a}$$
$$\|f(m)\|_S = \|m\|_S + 1, \tag{25b}$$
$$\|m \star m'\|_S = \|m\|_S + \|m'\|_S \tag{25c}$$

for all $m, m' \in \mathcal{M}$. Then $(\mathcal{M},\ f,\ \star)$ with the norm $\|\cdot\|_S : \mathcal{M} \to \mathbb{N}$ is called a ***Schröder normed (1,2)-magma****.*

If $(\mathcal{M},\ f,\ \star)$ is a Schröder normed (1,2)-magma with unique generator ϵ, then the base set $\mathcal{M}$ begins (sorting by norm) by evaluating the following expressions:

Norm 1: ϵ.
Norm 2: $f(\epsilon)$, $\quad \epsilon \star \epsilon$.
Norm 3: $f(f(\epsilon))$, $\quad f(\epsilon \star \epsilon)$, $\quad \epsilon \star f(\epsilon)$, $\quad \epsilon \star (\epsilon \star \epsilon)$,
$f(\epsilon) \star \epsilon$, $\quad (\epsilon \star \epsilon) \star \epsilon$.

Informally, the Schröder norm $\|\cdot\|_S : \mathcal{M} \to \mathbb{N}$ can be defined as follows, where $m \in \mathcal{M}$:

$$\|m\|_S = (\text{number of } f\text{'s}) + (\text{number of } \star\text{'s}) + 1.$$

For each Schröder family the norm is usually a simple function of the conventional size parameter for that family. The details of a number of Schröder normed (1,2)-magmas can be found in Appendix 7.3.

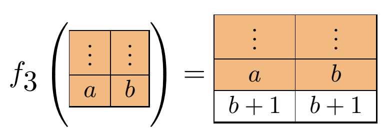
In the remainder of this section, we will reference a number of Schröder families. Further details of these families are provided in Appendix 7.3. We call the unary map f and the binary map $\star$ which will be written in in-fix form. We make it clear that we are using the maps specific to a certain family by referencing that family in the subscript of the maps which we do via the number assigned to that family in the appendix.

As an example of a Schröder normed (1,2)-magma, consider the Schröder family of semi-standard Young Tableaux (SSYT) of shape $n \times 2$, $\mathbb{S}_3$ as defined in Appendix 7.3. The nth Schröder number S_n is equal to the number of such tableaux.

We construct the Schröder normed (1,2)-magma of SSYT of shape $n \times 2$, $(\mathcal{S}_3, f_3, \star_3)$, as follows:

- Take the base set $\mathcal{S}_3$ to be the set of all semi-standard Young Tableaux of shape $n \times 2$, for every $n \in \mathbb{N}_0$. We take the trivial empty tableau $\emptyset$ to be the only SSYT of shape 0×2.

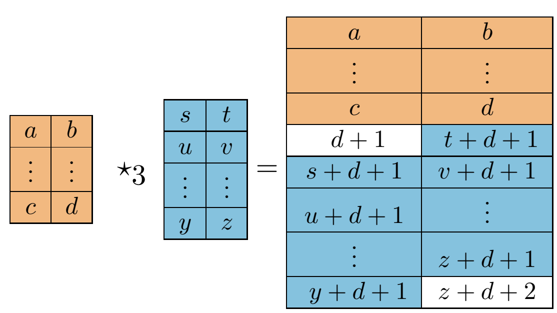
- Define the unary map f_3 as follows:



  with the convention that the empty tableau is considered to have all entries equal to 0. Thus we have

$$f_3(\emptyset) = \boxed{1\,|\,1}$$

- Define the binary map $\star_3$ as follows:



  Again, note that when applying this to the empty tableau, we consider any entries of the empty tableau to be equal to 0.

Note that the empty SSYT ϵ is the only element in the base set which is not in the image of one of the two maps. Thus this is the only generator, so we can define $\epsilon_3 = \emptyset$. Then we can see that the base set $\mathcal{S}_3$ is generated by ϵ_3 via the two maps f_3 and $\star_3$.

Any SSYT of shape $n \times 2$ factorises uniquely in terms of the generator ϵ_3 and the two maps f_3 and $\star_3$. For example, consider the following SSYT, which we can decompose as follows:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & 5 \\ \hline 6 & 6 \\ \hline \end{array} = f_3\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}\right) = f_3\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline \end{array} \star_3 \emptyset\right) = f_3\left(\left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \star_3 \emptyset\right) \star_3 \emptyset\right) = f_3\left(\left(f_3\left(\emptyset\right) \star_3 \emptyset\right) \star_3 \emptyset\right)$$

We define the norm $\|\cdot\|_S$ of a tableau of shape $n \times 2$ to be $n+1$. With this definition we obtain:

$$\begin{aligned} \|\epsilon\|_S &= 1, \\ \|f_3(t)\|_S &= \|t\|_S + 1, \quad t \in \mathcal{S}_3, \\ \|t_1 \star_3 t_2\|_S &= \|t_1\|_S + \|t_2\|_S\,, \quad t_1, t_2 \in \mathcal{S}_3. \end{aligned}$$

Thus $(\mathcal{S}_3, f_3, \star_3)$ is a unique factorisation normed (1,2)-magma with a finite non-empty set of irreducibles (and hence a free (1,2)-magma). The norm $\|\cdot\|_S : \mathcal{S}_3 \to \mathbb{N}$ satisfies (25). Therefore this is a Schröder normed (1,2)-magma.

4.4 Free (1,2)-magma isomorphisms and a universal bijection

We now apply Proposition 3 which states that there exists a unique (1,2)-magma isomorphism between any free (1,2)-magmas generated by sets of the same size. We demonstrate how this defines a universal bijection between any pair of Motzkin families or any pair of Schröder families.

Definition 14 (Universal bijection)**.** *Suppose that $(\mathcal{M},\ f,\ \star)$ and $(\mathcal{N},\ g,\ \ltimes)$ are free (1,2)-magmas with generating sets $X_{\mathcal{M}}$ and $X_{\mathcal{N}}$ respectively, with $|X_{\mathcal{M}}| = |X_{\mathcal{N}}|$. Let $\sigma : X_{\mathcal{M}} \to X_{\mathcal{N}}$ be any bijection, and define the map $\Upsilon : \mathcal{M} \to \mathcal{N}$ as follows:*
For all $m \in \mathcal{M} \setminus X_{\mathcal{M}}$,

(i) Decompose m into an expression in terms of generators $\epsilon_i \in X_{\mathcal{M}}$ and the maps f and $\star$.

(ii) In the decomposition of m, replace every occurrence of ϵ_i with $\sigma(\epsilon_i)$, every occurence of f with g and every occurrence of $\star$ with $\ltimes$. Call this expression $\upsilon(m)$.

(iii) Define $\Upsilon(m)$ to be $\upsilon(m)$, that is, evaluate all maps in $\upsilon(m)$ to give an element of $\mathcal{N}$.

This leads to the following proposition, which follows from the fact that Υ is exactly the map Γ from Proposition 3.

Proposition 13. *Let $\Upsilon : \mathcal{M} \to \mathcal{N}$ be the map of Definition 14. Then Υ is a free (1,2)-magma isomorphism.*

Schematically, we can write Υ as follows:

$$m \xrightarrow{\text{decompose}} \underset{\epsilon_i \to \sigma(\epsilon_i),\ f \to g,\ \star \to \ltimes}{\text{substitute}} \xrightarrow{\text{evaluate}} n. \tag{26}$$

Since Υ is an isomorphism between the free (1,2)-magmas $(\mathcal{M},\ f,\ \star)$ and $(\mathcal{N},\ g,\ \ltimes)$, we have that Υ defines a bijection between the base sets $\mathcal{M}$ and $\mathcal{N}$. Furthermore, it gives us that this bijection is recursive: if $m = f(m_0)$, then

$$\Upsilon(m) = g(\Upsilon(m_0)),$$

and if $m = m_1 \star m_2$, then

$$\Upsilon(m) = \Upsilon(m_1) \ltimes \Upsilon(m_2).$$

Note, Υ preserves the norm when the two (1,2)-magmas are equipped with suitable norms. Suppose that $(\mathcal{M},\ f,\ \star)$ and $(\mathcal{N},\ g,\ \ltimes)$ are free (1,2)-magmas with the same number of generators and that they have respective norms $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{N}$ and $\|\cdot\|_{\mathcal{N}} : \mathcal{N} \to \mathbb{N}$. If the following conditions are satisfied:

(i) $\|m\|_{\mathcal{M}} = \|\sigma(m)\|_{\mathcal{N}}$ for all $m \in X_{\mathcal{M}}$,

(ii) for $\kappa_1 \in \mathbb{N}$, $\|f(m)\|_{\mathcal{M}} = \|m\|_{\mathcal{M}} + \kappa_1$ for all $m \in \mathcal{M}$, and $\|g(n)\|_{\mathcal{N}} = \|n\|_{\mathcal{N}} + \kappa_1$ for all $n \in \mathcal{N}$, and

(iii) for $\kappa_2 \in \mathbb{N}_0$, $\|m_1 \star m_2\|_{\mathcal{M}} = \|m_1\|_{\mathcal{M}} + \|m_2\|_{\mathcal{M}} + \kappa_2$ for all $m_1, m_2 \in \mathcal{M}$, and $\|n_1 \ltimes n_2\|_{\mathcal{N}} = \|n_1\|_{\mathcal{N}} + \|n_2\|_{\mathcal{N}} + \kappa_2$ for all $n_1, n_2 \in \mathcal{N}$,

then we have

$$\|m\|_{\mathcal{M}} = \|\Upsilon(m)\|_{\mathcal{N}}, \qquad m \in \mathcal{M}.$$

This follows by considering the decomposed expressions for m and $\Upsilon(m)$, assuming that (i)-(iii) hold.

Both the Motzkin norm, (22) and the Schröder norm (25) satisfy (i), (ii) and (iii) above and thus both norms are invariant under Υ.

4.4.1 Universal bijections for Motzkin families

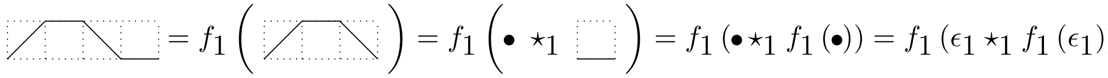
In this section we consider a number of examples illustrating this universal bijection between free (1,2)-magmas.

Using a simple example, we demonstrate how the universal bijections work for Motzkin normed (1,2)-magmas. We consider the following Motzkin normed (1,2)-magmas:

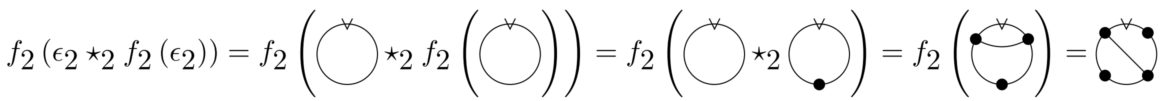
- Motzkin paths $(\mathcal{M}_1, f_1, \star_1)$,
- Non-intersecting chords $(\mathcal{M}_2, f_2, \star_2)$,
- Unary-binary trees $(\mathcal{M}_3, f_3, \star_3)$.

See Appendix 7.3 for the definitions of these families, as well as details of the relevant (1,2)-magmas and norms.

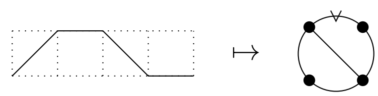
Take a Motzkin path and decompose it into its factorised form:



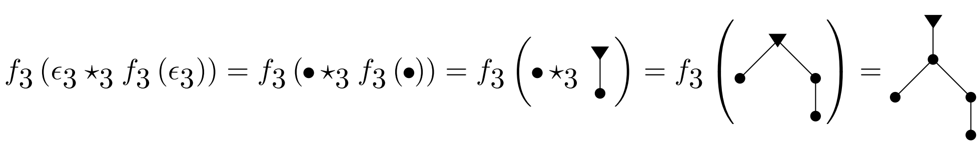
Now substitute generators and maps then evaluate to obtain an object from the Motzkin family of non-intersecting chords:



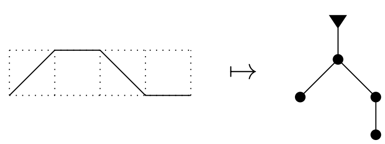
So the universal bijection maps



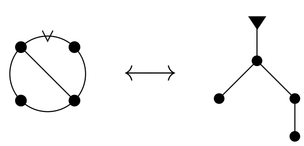
If we were instead seeking a bijection from Motzkin paths to Motzkin unary-binary trees, then we would simply replace the maps and generators in the factorised expression for the Motzkin path with the maps and generators from the (1,2)-magma corresponding to Motzkin unary-binary trees as follows:



Thus we see that the universal bijection gives



and also



4.4.2 Universal bijections for Schröder families

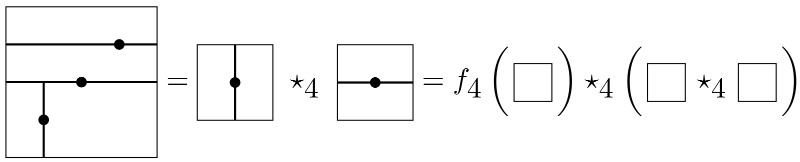
Consider the following Schröder normed (1,2)-magmas:

- Rectangulations $(\mathcal{S}_4, f_4, \star_4)$,

- Semi-standard Young tableaux of shape $n \times 2$ $(\mathcal{S}_3, f_3, \star_3)$,
- Unary-binary trees $(\mathcal{S}_5, f_5, \star_5)$.

See Appendix 7.3 for the definitions of these families and details of the relevant (1,2)-magmas and norms.

Take a rectangulation and factorise it:



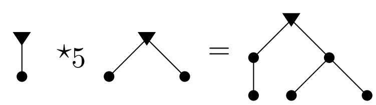
$= f_4\,(\epsilon_4) \star_4 (\epsilon_4 \star_4 \epsilon_4)$

Now to obtain the image in the family of semi-standard Young tableaux of shape $2 \times n$ via the universal bijection we replace the generators and the maps with the respective generators and maps from the Schröder normed (1,2)-magma of SSYT of shape $2 \times n$:

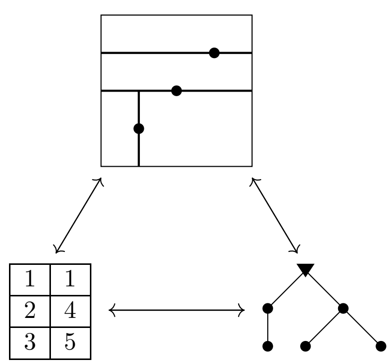
$$f_3\,(\epsilon_3) \star_3 (\epsilon_3 \star_3 \epsilon_3) = f_3\,(\emptyset) \star_3 (\emptyset \star_3 \emptyset) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array} \star_3 \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}$$

Finally, to obtain a bijection to Schröder unary-binary trees, replace the maps and generators in the factorised expression for the rectangulation as follows:

$$f_5\,(\epsilon_5) \star_5 (\epsilon_5 \star_5 \epsilon_5) = f_5\,(\bullet) \star_5 (\bullet \star_5 \bullet) =$$



Thus we see that we have the following bijections:



5 Fuss-Catalan normed (3)-magmas

In this section we consider (3)-magmas and discuss how these relate to the order 3 Fuss-Catalan sequence. Recall that a (3)-magma $(\mathcal{M}, t)$ is a set $\mathcal{M}$ along with a ternary map $t : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$. We begin this section by constructing an example of a free (3)-magma, which we call the Cartesian (3)-magma.

5.1 Cartesian (3)-magma

The Cartesian (3)-magma is one of the simplest (3)-magmas and is constructed using Cartesian products. As in section 4, we use square parentheses to represent a 3-tuple, so we denote the ternary Cartesian product as follows:

$$X_1 \times X_2 \times X_3 = \{[x_1, x_2, x_3] : x_1 \in X_1,\, x_2 \in X_2,\, x_3 \in X_3\}.$$

Definition 15 (Cartesian (3)-magma)**.** *Let X be a non-empty finite set. Define the sequence $\mathcal{W}_n(X)$ of sets of nested 3-tuples by*

$$\mathcal{W}_1(X) = X, \tag{27a}$$

$$\mathcal{W}_{2n+1}(X) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-i-1} \left(\mathcal{W}_{2i+1}(X) \times \mathcal{W}_{2j+1}(X) \times \mathcal{W}_{2(n-i-j-1)+1}(X)\right), \qquad n \geq 1. \tag{27b}$$

Let $\mathcal{W}_X = \bigcup_{n\geq 0} \mathcal{W}_{2n+1}(X)$ and $\mathcal{W}_X^+ = \mathcal{W}_X \backslash X$. Define $\tau : \mathcal{W}_X \times \mathcal{W}_X \times \mathcal{W}_X \to \mathcal{W}_X$ by

$$\tau(w_1, w_2, w_3) = [w_1, w_2, w_3], \qquad w_1, w_2, w_3 \in \mathcal{W}_X.$$

The pair $(\mathcal{W}_X,\, \tau)$ is called the ***Cartesian (3)-magma generated by X****.*

If $X = \{\epsilon\}$, the sets $\mathcal{W}_n(X)$ in the above definition begin:

$$\begin{aligned}
\mathcal{W}_1(X) &= \{\epsilon\},\\
\mathcal{W}_3(X) &= \{[\epsilon,\epsilon,\epsilon]\},\\
\mathcal{W}_5(X) &= \{[\epsilon,\epsilon,[\epsilon,\epsilon,\epsilon]], \quad [\epsilon,[\epsilon,\epsilon,\epsilon],\epsilon], \quad [[\epsilon,\epsilon,\epsilon],\epsilon,\epsilon]\},\\
\mathcal{W}_7(X) &= \{[\epsilon,\epsilon,[\epsilon,\epsilon,[\epsilon,\epsilon,\epsilon]]], \quad [\epsilon,\epsilon,[\epsilon,[\epsilon,\epsilon,\epsilon],\epsilon]], \quad [\epsilon,\epsilon,[[\epsilon,\epsilon,\epsilon],\epsilon,\epsilon]]\\
&\qquad [\epsilon,[\epsilon,\epsilon,\epsilon],[\epsilon,\epsilon,\epsilon], \quad [\epsilon,[\epsilon,\epsilon,[\epsilon,\epsilon,\epsilon]],\epsilon], \quad [\epsilon,[\epsilon,[\epsilon,\epsilon,\epsilon],\epsilon],\epsilon],\\
&\qquad [\epsilon,[[\epsilon,\epsilon,\epsilon],\epsilon,\epsilon],\epsilon], \quad [[\epsilon,\epsilon,\epsilon],\epsilon,[\epsilon,\epsilon,\epsilon]], \quad [[\epsilon,\epsilon,\epsilon],[\epsilon,\epsilon,\epsilon],\epsilon]\\
&\qquad [[\epsilon,\epsilon,[\epsilon,\epsilon,\epsilon]],\epsilon,\epsilon], \quad [[\epsilon,[\epsilon,\epsilon,\epsilon],\epsilon],\epsilon,\epsilon], \quad [[[\epsilon,\epsilon,\epsilon],\epsilon,\epsilon],\epsilon,\epsilon]\\
&\ \vdots
\end{aligned}$$

We now prove that the Cartesian (3)-magma is free.

Theorem 4. *The Cartesian (3)-magma $(\mathcal{W}_X,\, \tau)$ is a free (3)-magma.*

We will show that $(\mathcal{W}_X,\, \tau)$ is a unique factorisation normed (3)-magma with set of irreducibles X. Then, by Theorem 1, we will have that $(\mathcal{W}_X,\, \tau)$ is a free (3)-magma generated by X.

Proof. Suppose that $\tau(w_1, w_2, w_3) = \tau(w_1', w_2', w_3')$ for $w_i, w_i' \in \mathcal{W}_X$, $i = 1, 2, 3$. Therefore $[w_1, w_2, w_3] = [w_1', w_2', w_3']$ and hence $(w_1, w_2, w_3) = (w_1', w_2', w_3')$. Thus we have that τ is injective and so $(\mathcal{W}_X,\, \tau)$ is a unique factorisation (3)-magma. The set of irreducibles is given by X since this is the complement of $\text{Img}(\tau)$. We have $\|\tau(w_1, w_2, w_3)\| = \|w_1\| + \|w_2\| + \|w_3\|$ as $[w_1, w_2, w_3] \in \mathcal{W}_{n_1+n_2+n_3}(X)$ if $w_i \in \mathcal{W}_{n_i}(X)$, for $i \in \{1, 2, 3\}$. Thus $(\mathcal{W}_X,\, \tau)$ is a normed (3)-magma. □

Note, the norm we have defined in the above proof differs slightly to the traditional size parameter for combinatorial families counted by this sequence as it is more natural for us to have only objects with odd norm. This ensures that the norm is strictly additive with respect to the ternary map. If we instead wanted our base set to be partitioned into 'norms' given by the natural numbers, we would require our norm to be sub-additive with respect to the ternary map ie. the map would then not be a norm.

We now show how the norm used in the proof of Theorem 4 partitions the Cartesian (3)-magma into sets whose size is equal to the p-Fuss-Catalan numbers.

Proposition 14. *Let $(\mathcal{W}_X, \tau)$ be the Cartesian (3)-magma generated by the set X, where $|X| = p$, and define the map $\|\cdot\|_T : \mathcal{W}_X \to \mathbb{N}$ by $\|m\|_T = n$ when $m \in \mathcal{W}_n(X)$. If*

$$N_n = \{m \in \mathcal{W}_X : \|m\|_T = n\}, \qquad n \geq 1,$$

then

$$|N_{2n+1}| = T_n(p), \qquad n \geq 0.$$

Proof. Begin by noting that $\|m\|_T = n$ if and only if $m \in \mathcal{W}_n(X)$, and hence we have $N_n = \mathcal{W}_n(X)$. Therefore $|N_1| = |\mathcal{W}_1(X)| = |X| = p$. For $n \geq 1$, (27b) gives

$$\begin{aligned} |N_{2n+1}| &= \left| \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-i-1} \big(\mathcal{W}_{2i+1}(X) \times \mathcal{W}_{2j+1}(X) \times \mathcal{W}_{2(n-i-j-1)+1}(X)\big) \right| \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} |N_{2i+1}| \cdot |N_{2j+1}| \cdot \left|N_{2(n-i-j-1)+1}\right| \end{aligned}$$

This is equivalent to the p-Fuss-Catalan recurrence (7). □

Corollary 4. *Let $(\mathcal{W}_\epsilon, \tau)$ be the Cartesian (3)-magma generated by the single element ϵ. If N_n and $\|\cdot\|_M : \mathcal{W}_\epsilon \to \mathbb{N}$ are as defined in Proposition 14, then*

$$|N_{2n+1}| = T_n, \qquad n \geq 0,$$

with T_n as defined in Definition 12.

Note, the norm function $\|\cdot\|_T : \mathcal{W}_\epsilon \to \mathbb{N}$ from the above corollary could equivalently be defined as follows: Let $\|\cdot\|_T : \mathcal{W}_\epsilon \to \mathbb{N}$ be such that

$$\begin{aligned} \|\epsilon\|_T &= 1, \\ \|\tau(w_1, w_2, w_3)\|_T &= \|w_1\|_T + \|w_2\|_T + \|w_3\|_T, \quad w_1, w_2, w_3 \in \mathcal{W}_\epsilon. \end{aligned}$$

We have seen that a free (3)-magma generated by a single element with a norm function satisfying (28) is partitioned into sets given by the Fuss-Catalan numbers. Thus we are now ready to characterise when a (3)-magma is associated with the Fuss-Catalan numbers.

Definition 16 (Fuss-Catalan normed (3)-magma)**.** *Let* $(\mathcal{M},\, t)$ *be a unique factorisation normed (3)-magma with one irreducible element,* ϵ*. Let* $\|\cdot\|_T : \mathcal{M} \to \mathbb{N}$ *be a norm which satisfies* $\|\epsilon\|_T = 1$ *and*

$$\|t(m_1, m_2, m_3)\|_T = \|m_1\|_T + \|m_2\|_T + \|m_3\|_T\,, \tag{29}$$

for all $m_1, m_2, m_3 \in \mathcal{M}$*. Then* $(\mathcal{M},\, t)$ *with the norm* $\|\cdot\|_T : \mathcal{M} \to \mathbb{N}$ *is called a* ***Fuss-Catalan normed (3)-magma****.*

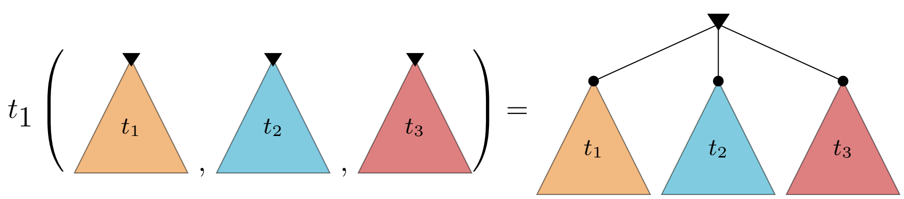
Let $(\mathcal{M},\, t)$ be a Fuss-Catalan normed (3)-magma. Then the base set $\mathcal{M}$ begins by evaluating the following expressions:

Norm 1: ϵ.
Norm 3: $t(\epsilon,\, \epsilon,\, \epsilon)$.
Norm 5: $t(\epsilon,\, \epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon)),\;\; t(\epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon),\;\; t(t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon,\, \epsilon)$.
Norm 7: $t(\epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon),\, t(\epsilon,\, \epsilon,\, \epsilon)),\;\; t(t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon)),\;\; t(t(\epsilon,\, \epsilon,\, \epsilon),\, t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon),$
$t(\epsilon,\, \epsilon,\, t(\epsilon,\, \epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon))),\;\; t(\epsilon,\, \epsilon,\, t(\epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon)),\;\; t(\epsilon,\, \epsilon,\, t(t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon,\, \epsilon)),$
$t(\epsilon,\, t(\epsilon,\, \epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon)),\, \epsilon),\;\; t(\epsilon,\, t(\epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon),\, \epsilon),\;\; t(\epsilon,\, t(t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon,\, \epsilon),\, \epsilon),$
$t(t(\epsilon,\, \epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon)),\, \epsilon,\, \epsilon),\;\; t(t(\epsilon,\, t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon),\, \epsilon,\, \epsilon),\;\; t(t(t(\epsilon,\, \epsilon,\, \epsilon),\, \epsilon,\, \epsilon),\, \epsilon,\, \epsilon)$.

Another example of a Fuss-Catalan normed (3)-magma is the Fuss-Catalan family of ternary trees $\mathbb{T}_1$. The generator is the tree containing no edges which we represent by a single node:

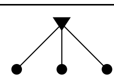
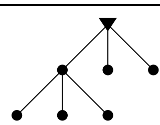
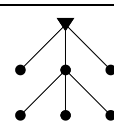
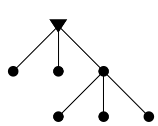
$$\epsilon_1 = \bullet$$

The base set is the set of all ternary trees and the ternary map is defined as follows:



The norm of any ternary tree is defined to be the number of leaves in the tree. Notice that this norm is additive as the number of leaves in a ternary tree is equal to the sum of the number of leaves in each of its three factors.

Sorting by norm, the base set of the ternary tree Fuss-Catalan normed (3)-magma begins as follows:

Norm 1:	$\bullet = \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

Clearly, this is a unique factorisation (3)-magma with only one irreducible element. This is immediate from the definition of the ternary map, as we see that any ternary tree has unique left, middle and right subtrees. Noting that there exists a norm, we have from Theorem 1 that this is a free (3)-magma. Since the norm satisfies (29), we have that this is indeed a Fuss-Catalan normed (3)-magma.

5.2 Free (3)-magma isomorphisms and a universal bijection

Proposition 3 tells us that there is a unique (3)-magma ismorphism between any two free (3)-magmas with the same number of generators. This allows us to define a universal bijection between any two Fuss-Catalan normed (3)-magmas.

Definition 17 (Universal bijection)**.** *Let ($\mathcal{M}$, t) and ($\mathcal{N}$, t') be free (3)-magmas with generating sets $X_{\mathcal{M}}$ and $X_{\mathcal{N}}$ respectively, with $|X_{\mathcal{M}}| = |X_{\mathcal{N}}|$. Let $\sigma : X_{\mathcal{M}} \to X_{\mathcal{N}}$ be any bijection, and define the map $\Upsilon : \mathcal{M} \to \mathcal{N}$ as follows. For all $m \in \mathcal{M} \setminus X_{\mathcal{M}}$,*

(i) Decompose m into an expression in terms of generators $\epsilon_i \in X_{\mathcal{M}}$ and the map t.

(ii) In the decomposition of m replace every occurrence of ϵ_i with $\sigma(\epsilon_i)$ and every occurrence of t with t'. Call this expression $\upsilon(m)$.

(iii) Define $\Upsilon(m)$ to be $\upsilon(m)$, that is, evaluate all maps in $\upsilon(m)$ to give an element of $\mathcal{N}$.

From the above definition, we immediately get the following proposition. As noted in previous sections, this comes from the fact that the map Υ of Definition 17 is equal to the map Γ of Proposition 3.

Proposition 15. *Let $\Upsilon : \mathcal{M} \to \mathcal{N}$ be the map of Definition 17. Then Υ is a free (3)-magma isomorphism.*

Schematically, we can write Υ as follows:

$$m \quad \xrightarrow{\text{decompose}} \quad \underset{\epsilon_i \to \sigma(\epsilon_i),\ t \to t'}{\text{substitute}} \quad \xrightarrow{\text{evaluate}} \quad n. \tag{30}$$

Υ is an isomorphism between the free (3)-magmas ($\mathcal{M}$, t) and ($\mathcal{N}$, t'), and therefore defines a bijection between the base sets $\mathcal{M}$ and $\mathcal{N}$. It also gives us that this bijection is recursive:

$$\Upsilon\left(t(m_1, m_2, m_3)\right) = t'\left(\Upsilon(m_1), \Upsilon(m_2), \Upsilon(m_3)\right).$$

It is also important to note that Υ is a norm-preserving map, subject to some conditions on the norms defined on the two (3)-magmas. Suppose that ($\mathcal{M}$, t) and ($\mathcal{N}$, t') are free (3)-magmas with generating sets of the same cardinality which have norms $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{N}$ and $\|\cdot\|_{\mathcal{N}} : \mathcal{N} \to \mathbb{N}$. If the norms $\|\cdot\|_{\mathcal{M}}$ and $\|\cdot\|_{\mathcal{N}}$ are such that:

(i) $\|m\|_{\mathcal{M}} = \|\sigma(m)\|_{\mathcal{N}}$ for all $m \in X_{\mathcal{M}}$, and

(ii) for all $m_1, m_2, m_3 \in \mathcal{M}$, $\|t(m_1, m_2, m_3)\|_{\mathcal{M}} = \sum_{i=1}^{3} \|m_i\|_{\mathcal{M}} + \kappa$, and for all $n_1, n_2, n_3 \in \mathcal{N}$, $\|t'(n_1, n_2, n_3)\|_{\mathcal{M}} = \sum_{i=1}^{3} \|n_i\|_{\mathcal{M}} + \kappa$, for some $\kappa \in \mathbb{N}_0$, then we have

$$\|m\|_{\mathcal{M}} = \|\Upsilon(m)\|_{\mathcal{N}}, \qquad m \in \mathcal{M}.$$

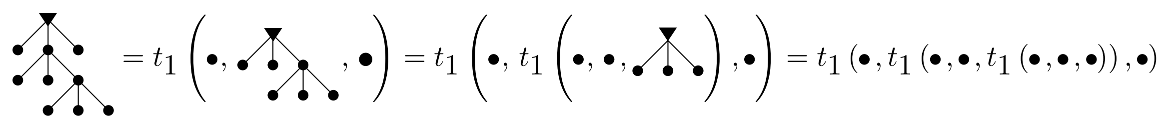
This can be seen by considering the decomposed expressions for m and $\Upsilon(m)$.

This tells us that if $(\mathcal{M},\, t)$ and $(\mathcal{N},\, t')$ are both Fuss-Catalan normed (3)-magmas, then the norm is preserved under the universal bijection Υ.

We now give some examples illustrating this universal bijection between free (3)-magmas generated by a single element. As in previous sections, each map and appearance of a generator references a family from Appendix 7.4 via the subscript. We consider the following Fuss-Catalan normed (3)-magmas:

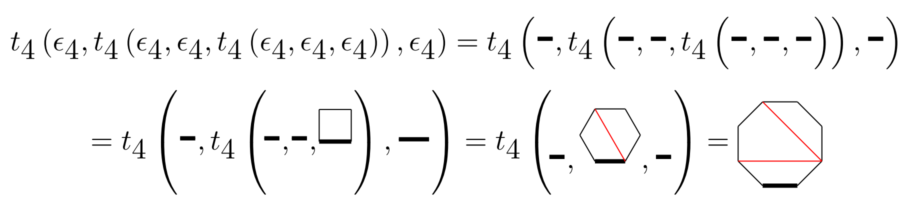
- Ternary trees $(\mathcal{T}_1, t_1)$,
- Quadrillages $(\mathcal{T}_4, t_4)$,
- Non-crossing partitions $(\mathcal{T}_3, t_3)$.

Consider the following ternary tree, which we can factorise as shown:

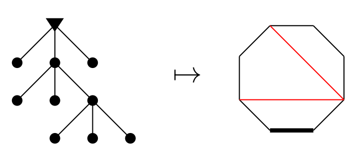


$$= t_1(\epsilon_1, t_1(\epsilon_1, \epsilon_1, t_1(\epsilon_1, \epsilon_1, \epsilon_1)), \epsilon_1)$$

We wish to obtain the image of this ternary tree under the universal bijection to the Fuss-Catalan normed (3)-magma of quadrillages. Thus we replace the generators and the ternary map in this factorised expression with the generators and the ternary map of the quadrillage (3)-magma:

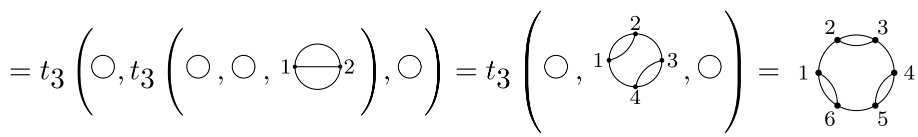


So the universal bijection maps

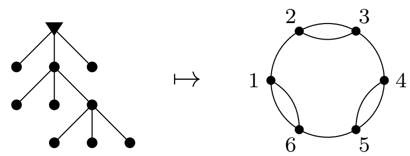


Bijecting the above ternary tree to a non-crossing partition with blocks of even size, we simply replace all occurrences of the generator and the ternary map in the factorised expression for the ternary tree with the generator and the ternary map from the Fuss-Catalan normed (3)-magma of non-crossing partitions with blocks of even size:

$$t_3(\epsilon_3, t_3(\epsilon_3, \epsilon_3, t_3(\epsilon_3, \epsilon_3, \epsilon_3)), \epsilon_3) = t_3\left(\bigcirc, t_3\left(\bigcirc, \bigcirc, t_3\left(\bigcirc, \bigcirc, \bigcirc\right)\right), \bigcirc\right)$$



Thus we see that



under the universal bijection.

6 Embedded bijections

In order to use the universal bijection, we are first required to factorise an object. This procedure is simple in some families, but for others this procedure can be difficult or slow to do “by hand”. For families where the factorisation process is not straightforward, it may be the case that there exists a simple bijection to a different family which is easy to factorise.

For families which can be represented geometrically (in contrast to say ‘pure’ sequence families) in certain cases we can give a geometric representation of the product structure. This representation will then additionally provide the factorisation required to apply a universal bijection. To this end we use an idea from category theory where the existence of a (binary) product object is defined via a product diagram (and a universal mapping principle). Thus if some categorical object A is a product of two other objects B and C, the diagram

$$B \longleftarrow A \longrightarrow C$$

is used as part of the categorical definition[1]. This idea was used in [8] for Catalan objects which resulted in the embedding of complete binary trees into other Catalan objects. Here we consider Motzkin, Schröder and Fuss-Catalan structures which give rise to unary-binary tree and ternary tree embeddings. The generalisation to other positive algebraic structures is clear.

6.1 Embedded bijections for Motzkin and Schröder geometric structures

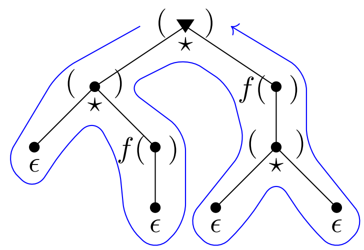
We can use the (1,2)-magma structure of any Motzkin (respectively Schröder) family to define an embedding of some other Motzkin (Schröder) object inside any object in that family. This occurs in such a way that the recursive structure of these families is respected.

In Section 4.4, we showed how unary-binary trees correspond to both Motzkin and Schröder normed (1,2)-magmas. Assuming that unary-binary trees correspond to the free (1,2)-magma

[1]We don’t provide the full categorical definition as this is not required here - only the diagram.

$(\mathcal{M},\ f,\ \star)$ with the single generator ϵ, we label each leaf and each internal node of a unary-binary tree as follows:

The labels given to an internal node are determined by whether that node has out-degree 1 or 2. After labelling in this way, counter-clockwise traversal of the tree gives its factorisation. For example, consider the following unary-binary tree which has been labelled:



By traversing this tree counter-clockwise we find that its factorisation is

$$((\epsilon \star f\,(\epsilon)) \star f\,((\epsilon \star \epsilon)))\,.$$

Since we are able to determine the factorisation of a unary-binary tree by simple tree traversal, we focus on embedding unary-binary trees into other geometric families (both Motzkin and Schröder).

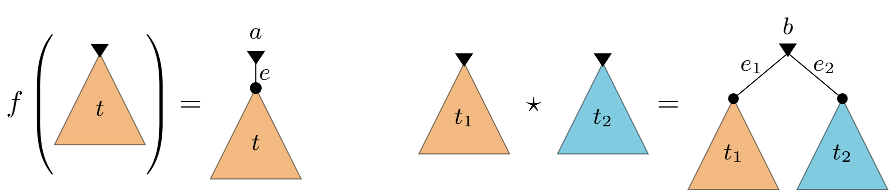
In this section we assume that all free (1,2)-magmas have a single generator and are geometric (meaning they are defined pictorially in some way rather than as sequences). Given that a family is geometric, the details of its (1,2)-magma structure give us the following:

(i) We know which part of the geometry is associated with the generator. We will call this the **generator geometry**.

(ii) From the definition of the unary map, we know which part of the geometry is added each time the unary map is applied. We will call this the **unary map geometry**.

(iii) From the defintion of the binary map, we know which part of the geometry is added each time the binary map is applied. We will call this the **binary map geometry**.

For example, we know that for unary-binary trees, the generators correspond to leaves, and thus the generator geometry is a leaf. From the definitions of the two maps,



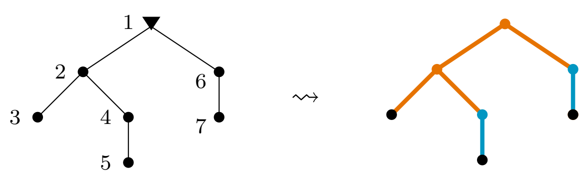
we can see that the unary map geometry is the new root node a and the edge e. The binary map geometry is the new root node b and the two edges e_1 and e_2.

For any geometric family, it is clear that there may be many different ways of drawing the same objects, with each representation differing only slightly and these being trivially in bijection. In order to make it clear exactly which parts of an object's geometry correspond to the generator and which arise from the two maps, we will choose a canonical way of drawing an object in a given family. In particular, we choose a way of drawing our objects so that the points of any object can be partitioned into disjoint subsets. Thus for a (1,2)-magma we require the objects to be drawn such that there are three disjoint subsets: one subset corresponding to the generator geometry, one to the unary map geometry and one to the binary map geometry. We call this canonical form the **(1,2)-magma form**.

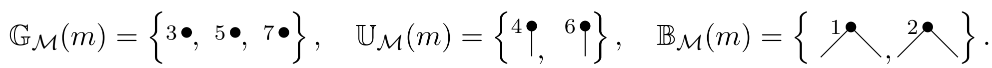
Let $(\mathcal{M}, f, \star)$ be a free (1,2)-magma with a single generator. For $m \in \mathcal{M}$, define the following sets:

- Let $\mathbb{G}_{\mathcal{M}}(m)$ be the set of all elements of generator geometry.
- Let $\mathbb{U}_{\mathcal{M}}(m)$ be the set of all elements of unary map geometry.
- Let $\mathbb{B}_{\mathcal{M}}(m)$ be the set of all elements of binary map geometry.

For example, if we choose to draw unary-binary trees as follows, with generator $\bullet$, unary map geometry drawn in blue and binary map geometry drawn in orange, then it is clear that any unary-binary tree can be partitioned into these sets, as shown in the following example:



This gives the partition

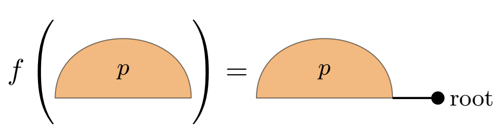


For each piece of generator geometry we mark a point, called the **leaf**. For some families, the generator is the empty object. In these cases, we enforce that the generator is associated with some geometry by trivially modifying how the object is drawn. This ensures that we are able to mark this point. For each piece of unary map geometry and each piece of binary map geometry, we also mark a point. In both cases, this marked point is called the **root**.

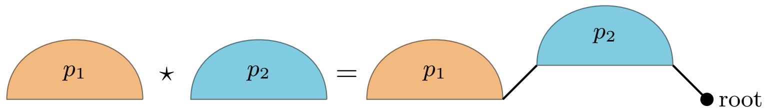
For example, for the family of Motzkin paths $\mathbb{M}_1$ the generator is the empty path so we choose to represent it by a single node, which is also our marked leaf:

$$\epsilon = \bullet$$

We mark the root in the definition of the unary map, noting that this root is part of the unary map geometry:



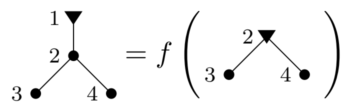
Similarly, we mark the root in the definition of the binary map, again noting that this root is part of the binary map geometry:



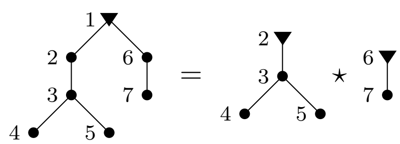
Now, for any object m from a (1,2)-magma $(\mathcal{M}, f, \star)$ such that $m = f(m_0)$, let the **subroot** of m be the root of m_0 if it exists (that is, if $m_0 \neq \epsilon$) and the leaf of m_0 if $m_0 = \epsilon$. If $m = m_1 \star m_2$, then define the following:

- Let the **left subroot** of m be the root of m_1 if $m_1 \neq \epsilon$ and the leaf of m_1 if $m_1 = \epsilon$.
- Let the **right subroot** of m be the root of m_2 if $m_2 \neq \epsilon$ and the leaf of m_2 if $m_2 = \epsilon$.

For example, if we are considering the (1,2)-magma of unary-binary trees (considered as either a Motzkin family or a Schröder family), then the tree



has subroot node 2, while for the tree

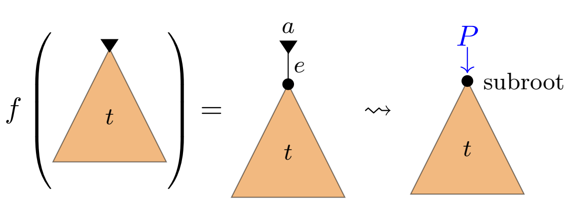


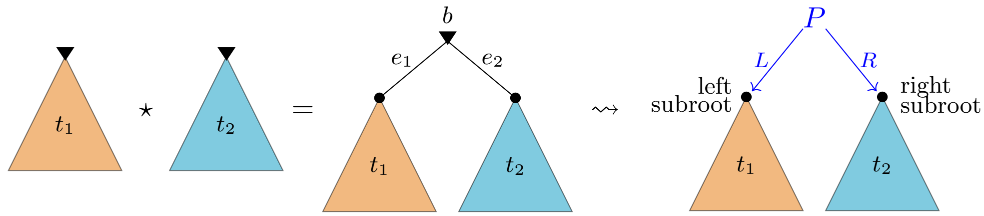
the left subroot is node 2 and the right subroot is node 6.

We now define our embedded bijections via the following recursive procedure of embedding pairs $(P, \rightarrow)$ and triples $(\overset{L}{\leftarrow}, P, \overset{R}{\rightarrow})$ into some Motzkin or Schröder object m. This is done as follows:

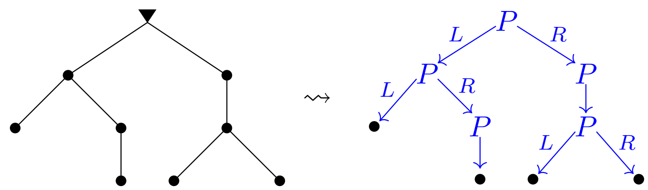
- If $m = f(m_0)$, then attach P to the root of the unary map f. The arrow $\rightarrow$ in the pair $(P, \rightarrow)$ points from P to the subroot of m.
- If $m = m_1 \star m_2$, then attach P to the root of the binary map $\star$. The left arrow $\overset{L}{\leftarrow}$ of the triple $(\overset{L}{\leftarrow}, P, \overset{R}{\rightarrow})$ points from P to the left subroot and the right arrow $\overset{R}{\rightarrow}$ points from P to the right subroot.

We can represent this embedding schematically by drawing the root (of either the unary or binary map) and the subroots in the definition of the maps. For example, for unary-binary trees we have the following:





Repeating the embedding recursively gives, for example,

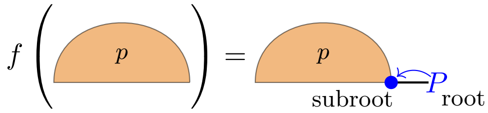


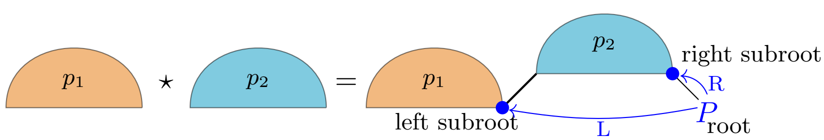
It is clear that this embedded representation is trivially an alternative way of drawing a unary-binary tree. Thus by recursively embedding the pair $(P, \rightarrow)$ and triple $(\stackrel{L}{\leftarrow}, P, \stackrel{R}{\rightarrow})$ we have effectively embedded a unary-binary tree inside our object. Note that the unary-binary tree which is embedded is precisely the unary-binary tree which the object maps to under the universal bijection. Thus the object and the embedded unary-binary tree have the same factorisation. This gives us a simple way of decomposing the object, since we can simply traverse the embedded unary-binary tree as illustrated previously.

We now illustrate how these embedded bijections work with a number of examples, working with both Motzkin normed (1,2)-magmas and Schröder normed (1,2)-magmas.

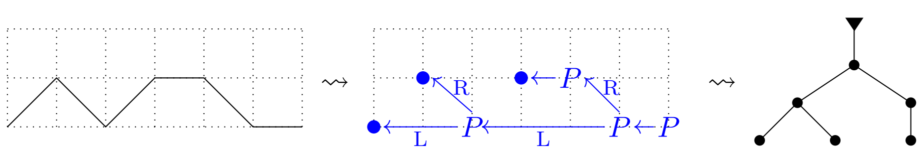
6.1.1 Motzkin embedded bijections

$\mathbb{M}_1$**: Motzkin paths** Consider the family of Motzkin paths. The generator is the empty path which we represent by a single node: $\epsilon = \bullet$. The unary and binary maps are as follows, and we have labelled these diagrams with the roots and details of the embedding:



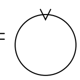


From this, we see that we can embed a unary-binary tree inside a Motzkin path as illustrated in the following example:

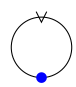


This is an important example as it demonstrates why we must keep track of which is the left subroot and which is the right. If we had have omitted the labels on the arrows, then it would seem that we have embedded a different tree (namely the tree which is obtained by reflecting the embedded tree across a vertical line travelling through the root).

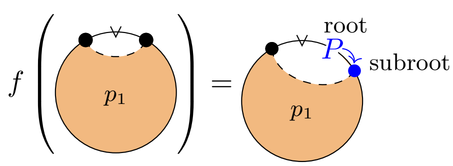
$\mathbb{M}_2$**: Non-intersecting chords** Consider the Motzkin family of non-intersecting chords joining n points on a circle. Note that on each diagram, we place a mark at the top of the circle to fix its orientation. The generator is represented by:

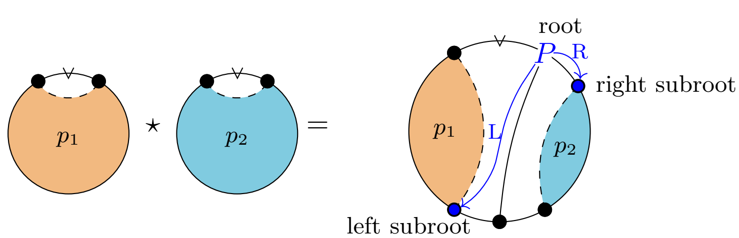
$\epsilon =$ 

We need to mark a point on this generator to be the leaf, so we take it to be the marked point shown in blue below:

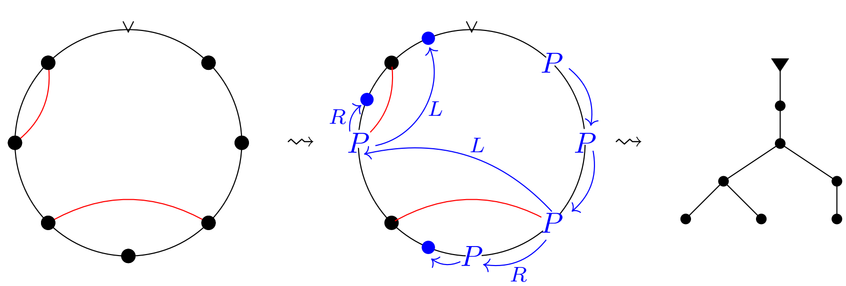


Note, the marked (in blue) point is *not* a node of the circle used for connecting chords but, as seen below, is a point between such nodes. We take the convention that the n points on the circle which we are connecting are drawn in black. The two maps are as follows, with the roots, subroots and embedded arrows all labelled:



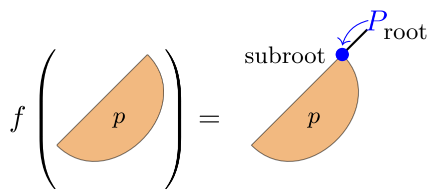


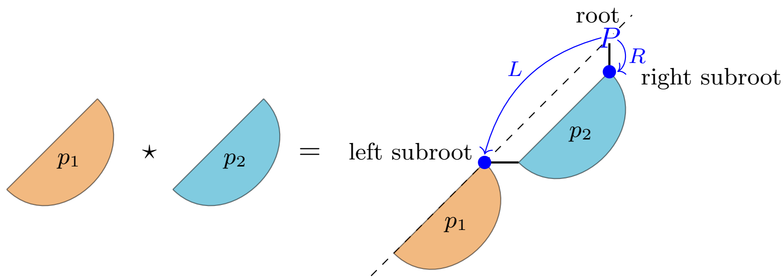
From these, we can see that we can embed a unary-binary tree inside a non-intersecting chord diagram as follows:



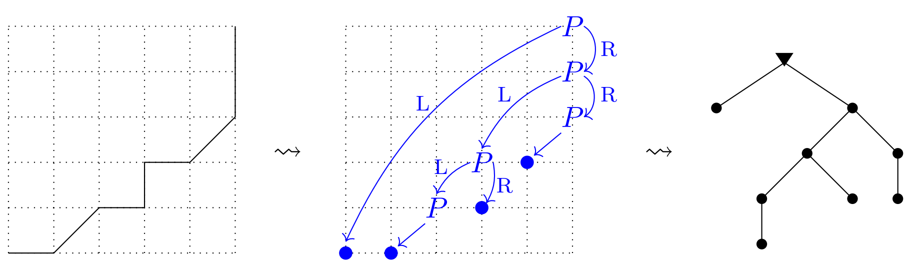
6.1.2 Schröder embedded bijections

$\mathbb{S}_1$: Schröder paths To embed a Schröder unary-binary tree inside a Schröder path. The generator for Schröder paths is taken to be a single vertex: $\epsilon = \bullet$. The embedding of the pairs $(P, \rightarrow)$ and triples $(\overset{L}{\leftarrow}, P, \overset{R}{\rightarrow})$ is shown in the following diagram:

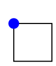




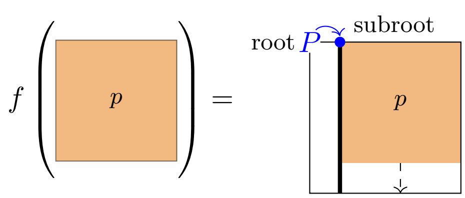
Thus we see that we can embed a unary-binary tree inside a Schröder path as shown in the following example:

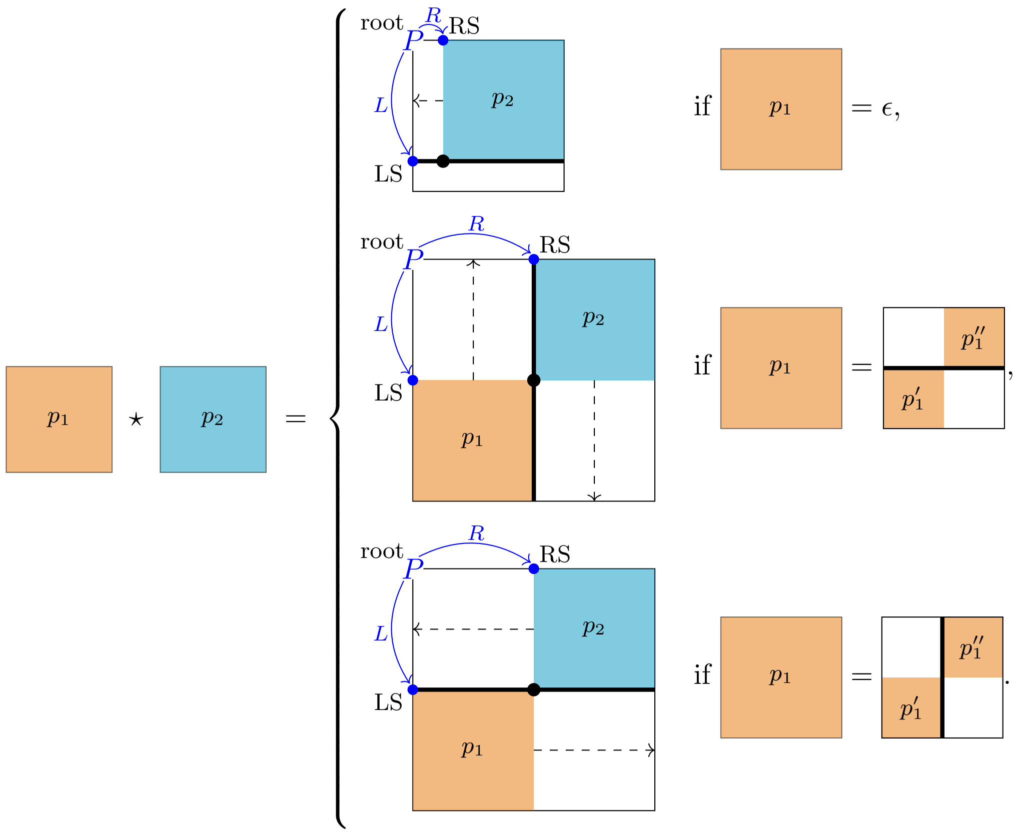


$\mathbb{S}_4$: Rectangulations Finally, we will show how to embed a unary-binary tree inside objects from the Schröder family of rectangulations. The generator is the following, with the marked point shown being the leaf:

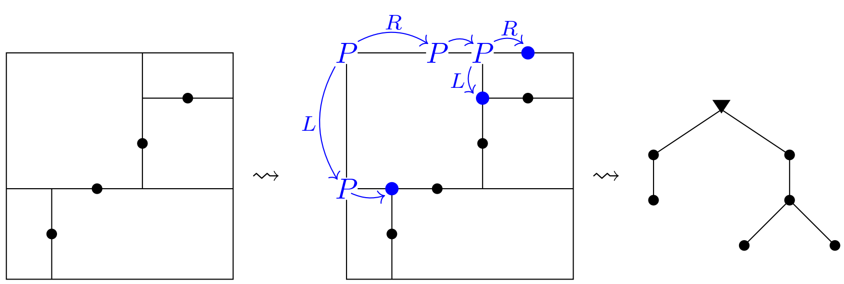
$\epsilon =$ 

The unary and binary maps have the following geometry, where LS and RS are used to denote the left subroot and the right subroot respectively:





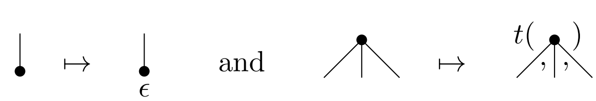
Thus we can embed a unary-binary tree inside a rectangulation as demonstrated in the following example:



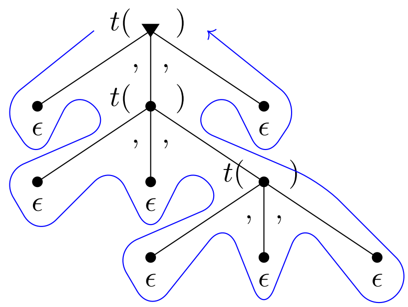
6.2 Embedded bijections for Fuss-Catalan (3)-magmas

In this section we consider how to embed ternary trees inside objects from other Fuss-Catalan families.

Assume the ternary map in the ternary tree (3)-magma is t and the unique generator is ϵ. If we label each leaf and each internal node in a ternary tree as follows:



then a counter-clockwise traversal of the tree gives its factorisation, as shown in the following example:



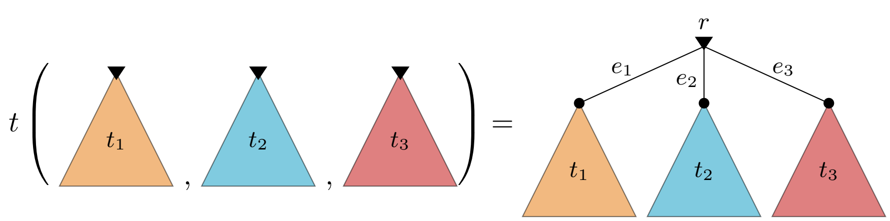
Traversing this tree, we can see that its factorisation is $t\left(\epsilon, t\left(\epsilon, \epsilon, t\left(\epsilon, \epsilon, \epsilon\right)\right), \epsilon\right)$.

From the definition of any geometric (3)-magma, we have the following information:

(i) We know which part of the geometry is associated with the generator. We will call this the **generator geometry**.

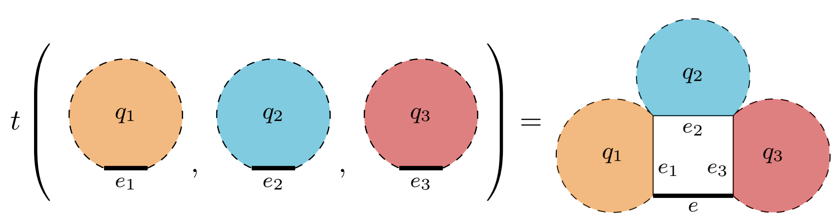
(ii) From the definition of the ternary map, we know which part of the geometry is added each time the ternary map is applied. We will call this the **ternary map geometry**.

Take the family of ternary trees $\mathbb{T}_1$ for example. We can see that the generators correspond to leaves of the tree, while from the ternary map definition,



we observe that the ternary map geometry is the new root r and the three added edges e_1, e_2 and e_3.

For the Fuss-Catalan family of quadrillages $\mathbb{T}_4$, the generators correspond to sides of the polygon (with the exception of the single marked side for all polygons with more than one edge). From the ternary map ,



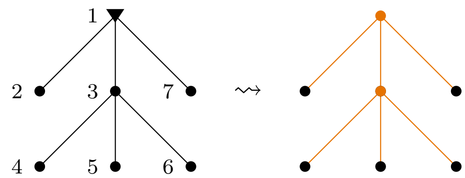
we can see that the ternary map geometry is just the new marked side e.

As in Section 6.1, we wish to draw our objects in such a way that it is clear exactly which parts of an object's geometry correspond to the generator and which parts arise from the ternary map. We choose to draw our objects so that the points of the object can be partitioned into two disjoint subsets, with one containing all pieces of generator geometry and the other all pieces of ternary map geometry. We will say that families defined in such a way that their objects can be partitioned into these subsets are in **(3)-magma form**.

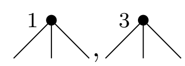
For any free (3)-magma $(\mathcal{M}, t)$ with a single generator and every object $m \in \mathcal{M}$, define the following sets:

- Let $\mathbb{G}_{\mathcal{M}}(m)$ to be the set of all pieces of generator geometry.
- Let $\mathbb{T}_{\mathcal{M}}(m)$ to be the set of all pieces of ternary map geometry.

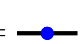
For example, drawing ternary trees with generator $\bullet$ and ternary tree geometry in orange, we can see how any ternary tree can be partitioned into these subsets. For example, for the ternary tree



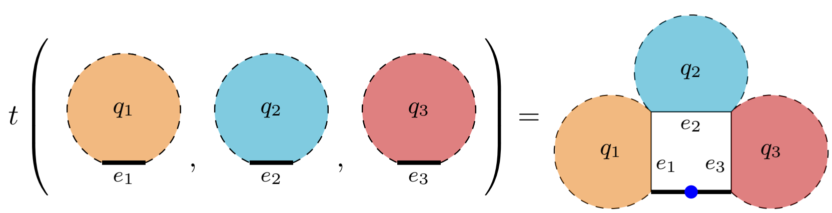
this partition is

$$\mathbb{G}_{\mathcal{M}}(m) = \left\{ {}^{2}\bullet,\ {}^{4}\bullet,\ {}^{5}\bullet,\ {}^{6}\bullet,\ {}^{7}\bullet \right\}, \quad \mathbb{T}_{\mathcal{M}}(m) = \Big\{$$  $$\Big\}.$$

Now, for each piece of generator geometry we shall mark a point and call this point the **leaf**. In some cases, the generator is the empty object. In these instances, we enforce that the generator is associated with some geometry so that we are able to mark this point. We also mark a particular point in each piece of ternary map geometry, calling this point the **root**. For example, for the Fuss-Catalan family of quadrillages, the generator is a single edge, so we represent this by the following:

$\epsilon =$ 

Notice that we have marked a point on this edge in blue. This point is the leaf. Similarly, we can mark the root in our definition of the ternary map, noting that this root is part of the ternary map geometry. This is shown in blue in the following figure:

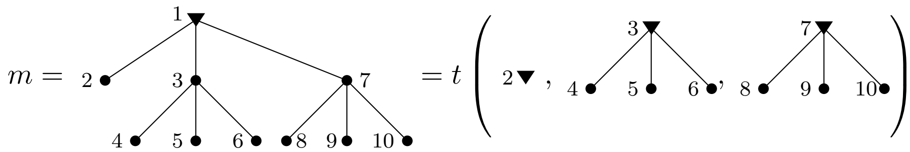


Now, for any object m from a Fuss-Catalan family with (3)-magma $(\mathcal{M}, t)$ such that $m = t(m_1, m_2, m_3)$, define the following:

- Let the **left subroot** be the root of m_1 if it exists (that is, if $m_1 \neq \epsilon$) and the leaf of m_1 if $m_1 = \epsilon$.

- Let the **middle subroot** be the root of m_2 if $m_2 \neq \epsilon$ and the leaf of m_2 if $m_2 = \epsilon$.
- Let the **right subroot** be the root of m_3 if $m_3 \neq \epsilon$ and the leaf of m_3 if $m_3 = \epsilon$.

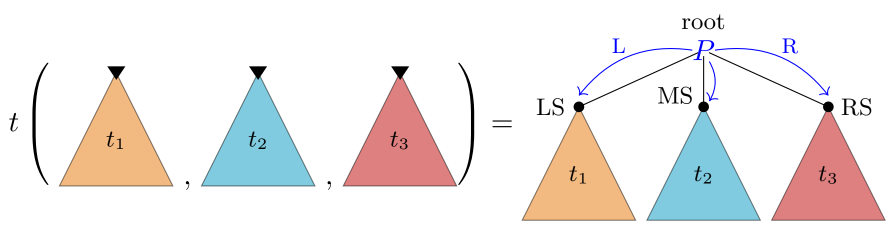
For example, if we are considering the family of ternary trees and we have



then the left subroot is node 2, the middle subroot is node 3 and the right subroot is node 7.

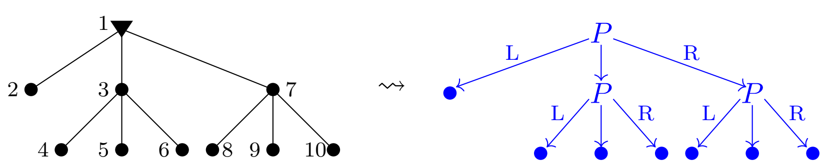
We can now define our embedded bijections via the following recursive procedure of embedding 4-tuples $(P, \overset{L}{\leftarrow}, \downarrow, \overset{R}{\rightarrow})$ inside the Fuss-Catalan object m. If $m = t(m_1, m_2, m_3)$, then attach P to the root of the ternary map. The arrow $\overset{L}{\leftarrow}$ points from P to the left subroot, the arrow $\downarrow$ points from P to the middle subroot and the arrow $\overset{R}{\rightarrow}$ points from P to the right subroot.

We can represent this embedding process schematically by drawing the roots, subroots and embedded arrows in the definition of the ternary map. For example, for ternary trees we have the following:



Here, LS, MS and RS denote the left subroot, middle subroot and right subroot respectively.

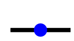
Repeating this embedding recursively gives, for example,



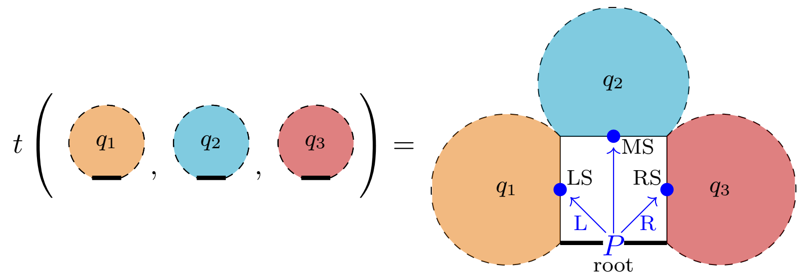
This clearly defines another tree which trivially differs from a ternary tree. We see that by recursively embedding 4-tuples $(P, \overset{L}{\leftarrow}, \downarrow, \overset{R}{\rightarrow})$ inside an object we are effectively embedding ternary trees. Moreover, we are embedding precisely the ternary tree which is in bijection with that object via the universal bijection. As a result we are then able to easily factorise the original object simply by factorising the embedded ternary tree.

We now demonstrate some explicit embedded bijections for different Fuss-Catalan families.

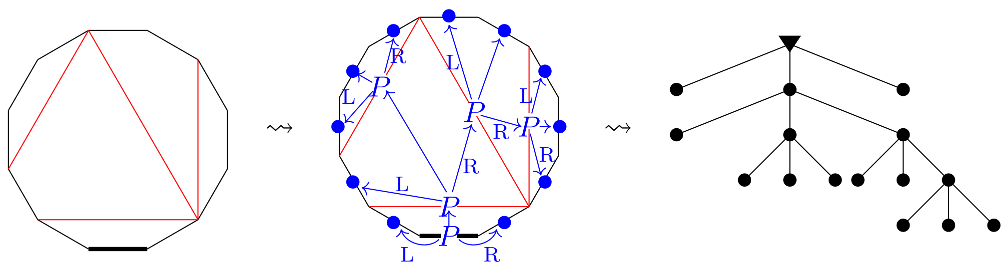
$\mathbb{T}_4$**: Quadrillages** The generator of this family is a single edge, drawn here with a marked point for the leaf:

$\epsilon =$ 

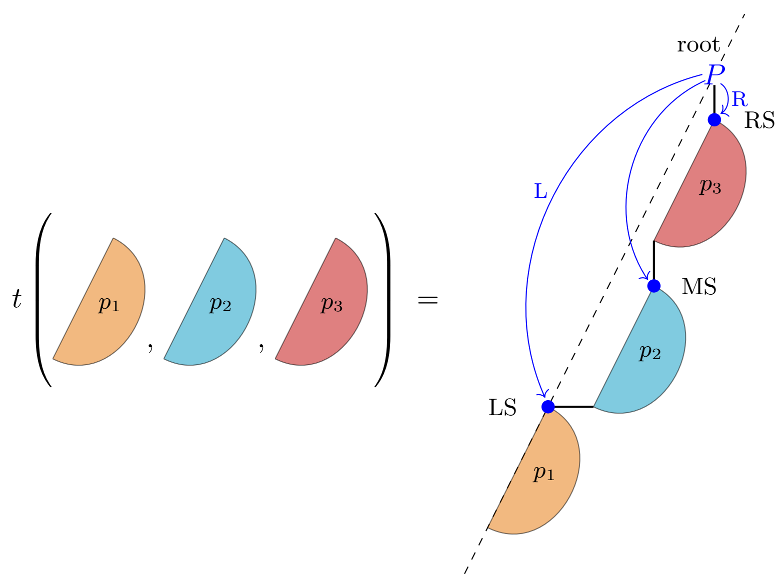
The ternary map is as follows, with details of the embedding drawn in:



We can see from this how we can embed a ternary tree inside a quadrillage. The following is an example of this embedding:

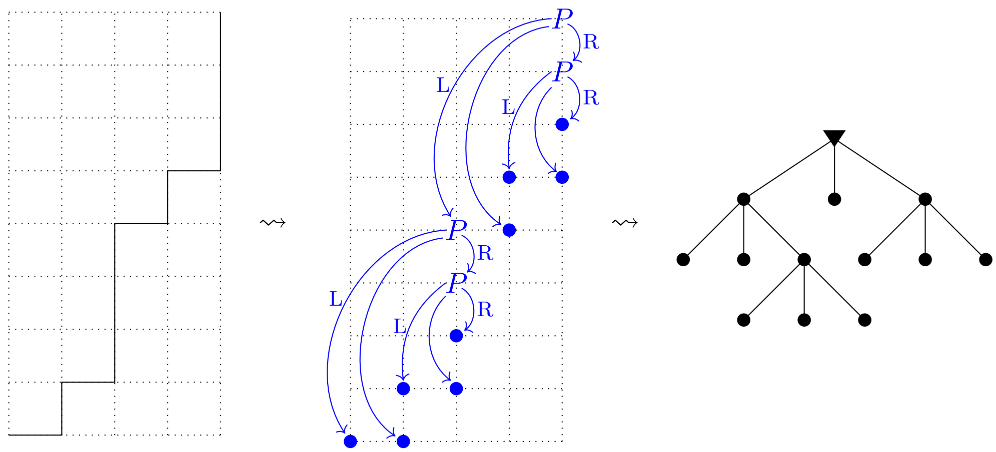


$\mathbb{T}_6$**: Lattice paths** We will show how to embed a ternary tree inside a lattice path of the type enumerated by the Fuss-Catalan numbers. These are paths from $(0,0)$ to $(n,2n)$ consisting of n East steps $(1,0)$ and $2n$ North steps $(0,1)$ that lie weakly below the line $y=2x$. The generator for these paths is taken to be a single vertex: $\epsilon = \bullet$. The ternary map is as follows, with the root, subroots and embedded arrows all labelled:



The dotted line in the above schematic diagram is the line $y=2x$.

We see that we can embed a ternary tree inside a lattice path of this type as shown in the following example:



7 Appendix

In the following appendices, we list a number of families which are enumerated by each of the Fibonacci numbers (Section 7.1), Motzkin numbers (Section 7.2), Schröder numbers (Section 7.3) and order 3 Fuss-Catalan numbers (Section 7.4). For each family, we provide a reference and give a brief definition before detailing the following:

- The generator of the relevant $\boldsymbol{n}$-magma.
- A definition of all of the relevant $\boldsymbol{n}$-magma maps. These are in most cases schematic diagrams.
- A definition of the norm, in terms of some natural parameter of the objects of that family.
- A list of some elements of the corresponding $\boldsymbol{n}$-magma. In each family, we list the elements with the smallest norms alongside their decomposition in terms of the generator and the $\boldsymbol{n}$-magma maps. This information provides concrete examples of how to apply the maps.

One can then use the information provided to obtain a bijection between any two families which are enumerated by the same integer sequence. This can be done using the relevant universal bijection (Definition 10 for Fibonacci families, Definition 14 for Motzkin or Schröder families and Definition 17 for Fuss-Catalan families).

7.1 Fibonacci Families

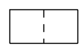
We present a number of Fibonacci normed (1,1)-magmas for well-known families of objects enumerated by the Fibonacci numbers. We take the convention that each generator is called ϵ and the two unary maps are called f and g. These are chosen in such a way that

$$\|f(m)\| = \|m\| + 1, \qquad \|g(m)\| = \|m\| + 2,$$

for all $m \in \mathcal{M}$, where $\mathcal{M}$ is the base set of the (1,1)-magma. Despite the same names being used for each family presented, it is clear that these are all different maps and that the generators are all different. The choice to use the same names was made for the sake of clarity and simplicity. At any point in this paper where these maps or generators are referenced, they appear with a subscript indicating which family they correspond to.

$\mathbb{F}_1$: **Fibonacci tilings [4, 5]**

The Fibonacci number F_{n+1} is the number of ways to tile a $1 \times n$ board using 1×1 squares and 1×2 dominoes. We will represent a 1×1 square and a 1×2 domino by

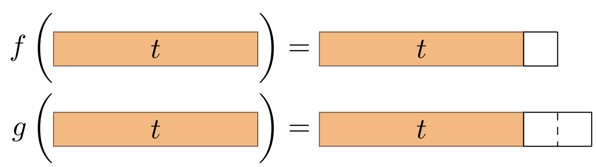
 and 

respectively.

Generator: The empty tiling:

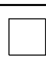
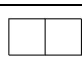
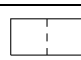
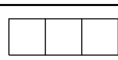
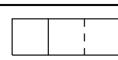
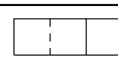
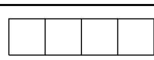
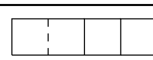
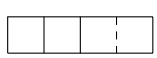
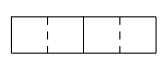
$$\epsilon = \emptyset.$$

Unary maps:



Norm: If t is a tiling of a $1 \times n$ board, then $\|t\| = n + 1$.

(1,1)-magma: The (1,1)-magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= g(\epsilon)$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= g(f(\epsilon))$,  $= f(g(\epsilon))$
Norm 5:	 $= f(f(f(f(\epsilon))))$,  $= f(f(g(\epsilon)))$,  $= f(g(f(\epsilon)))$,  $= g(f(f(\epsilon)))$,  $= g(g(\epsilon))$

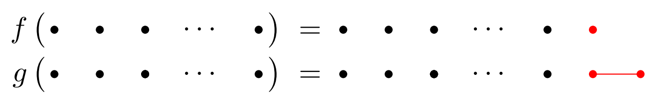
$\mathbb{F}_2$: **Path graph matchings [12]**

F_{n+1} is the number of matchings in a path graph on n vertices, P_n. This is a tree with two nodes of degree 1 and the other $n-2$ nodes of degree 2.

Generator: The empty matching on zero vertices:

$$\epsilon = \emptyset.$$

Unary maps: Let $g = \bullet \quad \bullet \quad \bullet \quad \cdots \quad \bullet$ be a matching on a path graph on n vertices. Define two unary maps as follows:



Norm: If g is a matching in a path graph on n vertices, then $\|g\| = n + 1$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	• $= f(\epsilon)$
Norm 3:	• • $= f(f(\epsilon))$, •—• $= g(\epsilon)$
Norm 4:	• • • $= f(f(f(\epsilon)))$, •—• • $= f(g(\epsilon))$, • •—• $= g(f(\epsilon))$
Norm 5:	• • • • $= f(f(f(f(\epsilon))))$, •—• • • $= f(f(g(\epsilon)))$, • •—• • $= f(g(f(\epsilon)))$, • • •—• $= g(f(f(\epsilon)))$, •—• •—• $= g(g(\epsilon))$

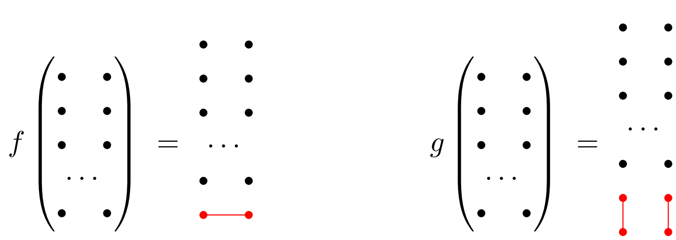
$\mathbb{F}_3$: **Perfect matchings in a ladder graph [22]**

F_{n+1} is the number of perfect matchings in the ladder graph $L_n = P_2 \times P_n$.

Generator: The empty matching in the ladder graph on zero vertices:

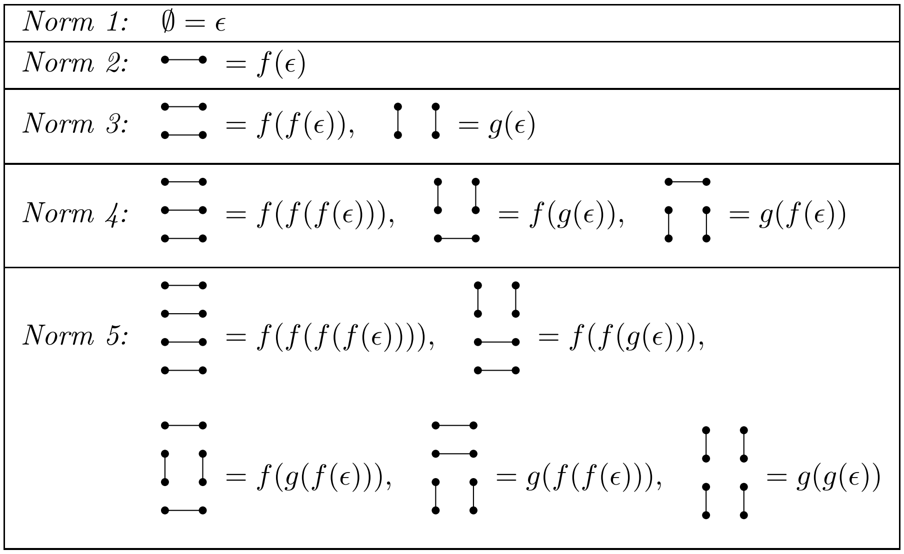
$$\epsilon = \emptyset.$$

Unary maps:



Norm: If M is a perfect matching in L_n, then $\|M\| = n + 1$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:



$\mathbb{F}_4$: **Compositions with no 1's [26]**

F_n is the number of compositions of $n+1$ with no part equal to 1. A composition of an integer n is a way of writing n as the sum of a sequence of strictly positive integers. Two sequences that differ in the order of their terms define different compositions of their sum.

Generator: The composition of 2 into one part:

$$\epsilon = 2.$$

Unary maps: Let $\alpha_1+\cdots+\alpha_k$ be a composition of n, where $\alpha_i \in \mathbb{N}\backslash\{1\}$ for each $i \in \{1,\ldots,k\}$. Then define two unary maps as follows:

$$f(\alpha_1 + \cdots + \alpha_k) = \alpha_1 + \cdots + (\alpha_k + 1),$$
$$g(\alpha_1 + \cdots + \alpha_k) = \alpha_1 + \cdots + \alpha_k + 2.$$

Norm: If c is a composition of the integer n, then $\|c\| = n - 1$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$2 = \epsilon$
Norm 2:	$3 = f(\epsilon)$
Norm 3:	$4 = f(f(\epsilon))$, $2+2 = g(\epsilon)$
Norm 4:	$5 = f(f(f(\epsilon)))$, $2+3 = f(g(\epsilon))$, $3+2 = g(f(\epsilon))$
Norm 5:	$6 = f(f(f(f(\epsilon))))$, $2+4 = f(f(g(\epsilon)))$, $3+3 = f(g(f(\epsilon)))$, $4+2 = g(f(f(\epsilon)))$, $2+2+2 = g(g(\epsilon))$

$\mathbb{F}_5$: **Compositions with no part greater than 2 [16]**

F_n is the number of compositions of $n-1$ with no part greater than 2.

Generator: The empty composition:

$$\epsilon = \emptyset.$$

Unary maps: Let c be a composition, and define two unary maps as follows:

$$f(c) = c + 1,$$
$$g(c) = c + 2.$$

Norm: If c is a composition of the integer n, then $\|c\| = n + 1$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	$1 = f(\epsilon)$
Norm 3:	$1+1 = f(f(\epsilon)),\quad 2 = g(\epsilon)$
Norm 4:	$1+1+1 = f(f(f(\epsilon))),\quad 2+1 = f(g(\epsilon)),$ $1+2 = g(f(\epsilon))$
Norm 5:	$1+1+1+1 = f(f(f(f(\epsilon)))),\quad 2+1+1 = f(f(g(\epsilon))),$ $1+2+1 = f(g(f(\epsilon))),\quad 1+1+2 = g(f(f(\epsilon))),$ $2+2 = g(g(\epsilon))$

$\mathbb{F}_6$: **Compositions using odd parts [26]**

F_n is the number of compositions of n into odd parts.

Generator: The composition of 1 into one part:

$$\epsilon = 1.$$

Unary maps: Let $\alpha_1 + \ldots + \alpha_k$ be a composition, and define two unary maps as follows:

$$f(\alpha_1 + \ldots + \alpha_k) = \alpha_1 + \ldots + \alpha_k + 1,$$
$$g(\alpha_1 + \ldots + \alpha_k) = \alpha_1 + \ldots + (\alpha_k + 2).$$

Norm: If c is a composition of the integer n, then $\|c\| = n$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$1 = \epsilon$
Norm 2:	$1+1 = f(\epsilon)$
Norm 3:	$1+1+1 = f(f(\epsilon)), \quad 3 = g(\epsilon)$
Norm 4:	$1+1+1+1 = f(f(f(\epsilon))), \quad 3+1 = f(g(\epsilon)), \quad 1+3 = g(f(\epsilon))$
Norm 5:	$1+1+1+1+1 = f(f(f(f(\epsilon)))), \quad 3+1+1 = f(f(g(\epsilon))),$ $1+3+1 = f(g(f(\epsilon))), \quad 1+1+3 = g(f(f(\epsilon))), \quad 5 = g(g(\epsilon))$

$\mathbb{F}_7$: **Binary words with odd run lengths [22]**

F_n is the number of binary words (words in the alphabet $\{0,1\}$) of length n beginning with 0 and having all run lengths odd. A run is a subword containing only 0's or only 1's which is maximal (meaning that we cannot extend the subword and still have the property that it contains only 0's or only 1's).

Generator:

$$\epsilon = 0.$$

Unary maps: Let w be a binary word. Define two unary maps as follows:

$$f(w) = \begin{cases} w0, & \text{if length of } w \text{ is even,} \\ w1, & \text{if length of } w \text{ is odd,} \end{cases}$$

$$g(w) = \begin{cases} w11, & \text{if length of } w \text{ is even,} \\ w00, & \text{if length of } w \text{ is odd.} \end{cases}$$

Norm: If w is a binary word of length n, then $\|w\| = n$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$0 = \epsilon$
Norm 2:	$01 = f(\epsilon)$
Norm 3:	$010 = f(f(\epsilon)), \quad 000 = g(\epsilon)$
Norm 4:	$0101 = f(f(f(\epsilon))), \quad 0001 = f(g(\epsilon)), \quad 0111 = g(f(\epsilon))$
Norm 5:	$01010 = f(f(f(f(\epsilon)))), \quad 00010 = f(f(g(\epsilon))), \quad 01110 = f(g(f(\epsilon))),$ $01000 = g(f(f(\epsilon))), \quad 00000 = g(g(\epsilon))$

$\mathbb{F}_8$: **Permutations with $|p_k - k| \le 1$ [21]**

F_{n+1} is the number of permutations $p_1 p_2 \cdots p_n$ of $\{1, \dots, n\}$ such that

$$|p_k - k| \le 1, \qquad k = 1, \dots, n.$$

Generator: The empty permutation ($n = 0$):

$$\epsilon = \emptyset.$$

Unary maps: Let $p_1p_2\cdots p_n$ be a permutation of $\{1,\ldots,n\}$. Define two unary maps as follows:

$$f(p_1p_2\cdots p_n) = p_1p_2\cdots p_n(n+1),$$
$$g(p_1p_2\cdots p_n) = p_1p_2\cdots p_n(n+2)(n+1).$$

Norm: If p is a permutation of $\{1,\ldots,n\}$, then $\|p\| = n+1$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	$1 = f(\epsilon)$
Norm 3:	$12 = f(f(\epsilon))$, $21 = g(\epsilon)$
Norm 4:	$123 = f(f(f(\epsilon)))$, $213 = f(g(\epsilon))$, $132 = g(f(\epsilon))$
Norm 5:	$1234 = f(f(f(f(\epsilon))))$, $2134 = f(f(g(\epsilon)))$, $1324 = f(g(f(\epsilon)))$, $1243 = g(f(f(\epsilon)))$, $2143 = g(g(\epsilon))$

$\mathbb{F}_9$: **Binary sequences with no consecutive 1's [19]**

F_{n+2} is equal to the number of binary sequences (words in the alphabet $\{0,1\}$) of length n that have no consecutive 1's.

Generator: For this family there does not exist a natural representation for the generator ϵ. This is due to the fact that norm n objects correspond to words of length $n-2$ and thus there is no natural way to describe a norm 1 object. For this reason, we describe the (1,1)-magma corresponding to this Fibonacci family in the manner described at the end of Section 3. We define the generator simply to be ϵ (with no further meaning associated with it) and define the following objects:

$$f(\epsilon) = \emptyset \quad \text{(the empty word)},$$
$$g(\epsilon) = 1.$$

Having defined these objects along with the unary and binary maps which follow, we have thus completely specified the relevant (1,1)-magma.

Unary maps: Let w be a binary sequence of length n containing no consecutive 1's. Then define two unary maps as follows:

$$f(w) = w0,$$
$$g(w) = w01.$$

Norm: If w is a binary sequence of length n, then $\|w\| = n+2$.

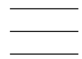
(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	ϵ
Norm 2:	$\emptyset = f(\epsilon)$
Norm 3:	$0 = f(f(\epsilon))$, $1 = g(\epsilon)$
Norm 4:	$00 = f(f(f(\epsilon)))$, $10 = f(g(\epsilon))$, $01 = g(f(\epsilon))$
Norm 5:	$000 = f(f(f(f(\epsilon))))$, $010 = f(g(f(\epsilon)))$, $100 = f(f(g(\epsilon)))$, $001 = g(f(f(\epsilon)))$, $101 = g(g(\epsilon))$

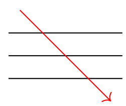
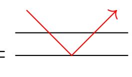
$\mathbb{F}_{10}$: **Reflections across two glass plates [13, 17]**

F_{n+2} is equal to the number of paths through two plates of glass with n reflections (where reflections can occur at plate/plate or plate/air interfaces). These are represented schematically as in [13].

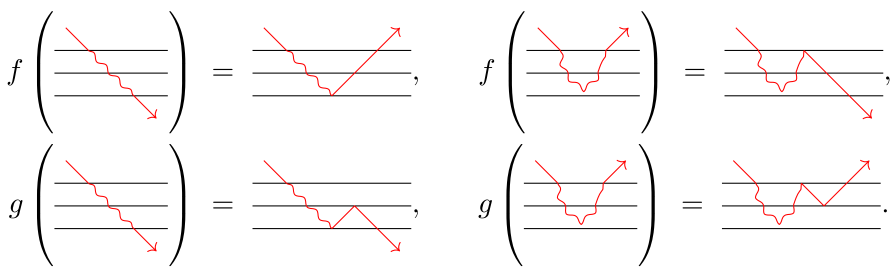
Generator: We represent the generator as follows:

$\epsilon =$ 

Unary maps: For the generator ϵ,

$f(\epsilon) =$  $g(\epsilon) =$ 

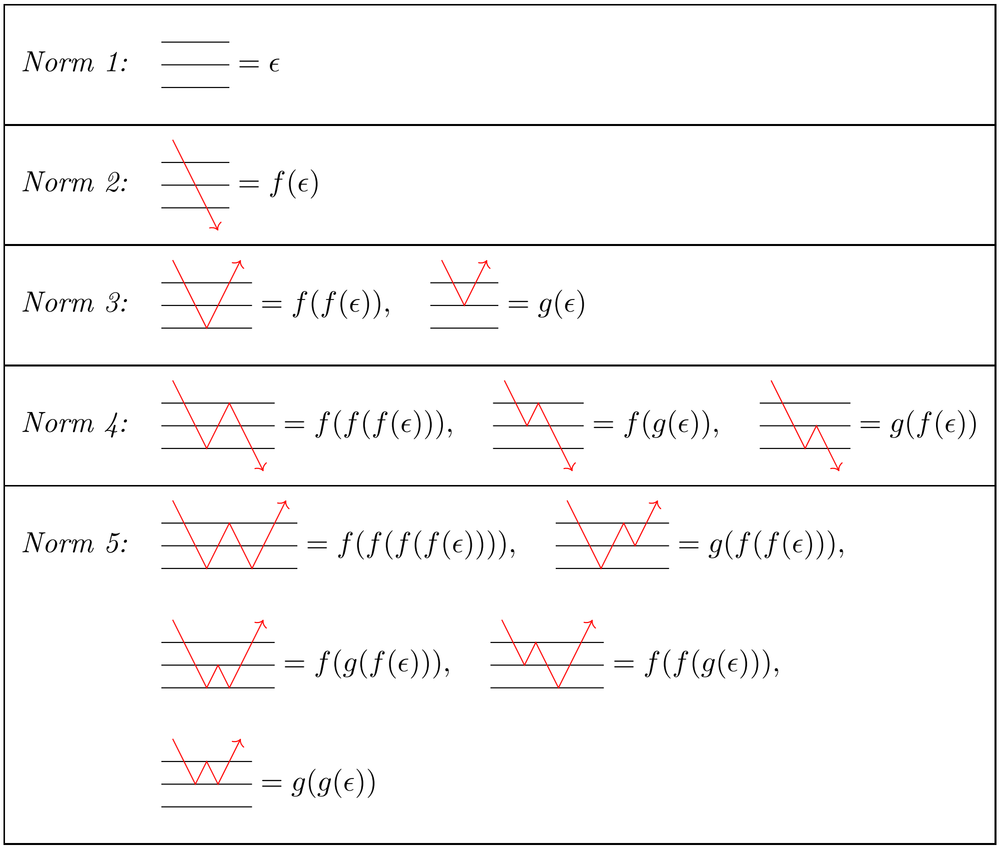
For all elements of the base set other than ϵ, the unary maps can be illustrated schematically as follows:



The map f simply adds one reflection by taking the exiting ray and reflecting it as it leaves the bottom plate. The map g adds two reflections by taking the exiting ray and reflecting it twice, with the second reflection occurring at the centre plate/plate interface. Thus f changes the direction the ray exits the plates whilst g does not change the direction.

Norm: If p is a path with n reflections, then $\|p\| = n + 2$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:



$\mathbb{F}_{11}$: **Subsets with no consecutive integers [26]**

F_{n+2} is equal to the number of subsets of $\{1, 2, \ldots, n\}$ that contain no consecutive integers.

To distinguish between two equal subsets that arise as subsets of two different sized sets we consider $\mathbb{F}_{11}$ to be pairs (n, S) where $S \subseteq \{1, 2, \ldots, n\}$. Thus (n, S) and (m, S) are only equal if $n = m$ (even though both subsets are the same).

Generator: For this family there does not exist a natural representation for the generator ϵ. This is due simply to the nature of the family, since norm n objects correspond to subsets of $\{1, 2, \ldots, n-2\}$ and thus there is no natural way to describe a norm 1 object. For this reason, we describe the (1,1)-magma corresponding to this Fibonacci family in the manner described at the end of Section 3. We define the generator simply to be ϵ (with no further meaning associated with it) and define the following objects:

$$f(\epsilon) = (0, \emptyset),$$
$$g(\epsilon) = (1, \{1\}).$$

Having defined these objects along with the unary and binary maps which follow, we have thus completely specified the relevant (1,1)-magma.

Unary maps: Let S be a subset of $\{1, 2, \ldots, n\}$ containing no consecutive integers. Define two unary maps as follows:

$$f(n, S) = (n + 1, S)$$

$$g(n, S) = (n+2, S \cup \{n+2\}).$$

Norm: $\|(n, S)\| = n + 2$.

(1,1)-magma: The (1,1) magma begins (sorting by norm) as follows:

Norm 1:	ϵ
Norm 2:	$f(\epsilon) = (0, \emptyset)$
Norm 3:	$f(f(\epsilon)) = (1, \emptyset)$, $g(\epsilon) = (1, \{1\})$
Norm 4:	$f(f(f(\epsilon))) = (2, \emptyset)$, $f(g(\epsilon)) = (2, \{1\})$, $g(f(\epsilon)) = (2, \{2\})$
Norm 5:	$f(f(f(f(\epsilon)))) = (3, \emptyset)$, $f(f(g(\epsilon))) = (3, \{1\})$, $f(g(f(\epsilon))) = (3, \{2\})$, $g(f(f(\epsilon))) = (3, \{3\})$, $g(g(\epsilon)) = (3, \{1, 3\})$

7.2 Motzkin Families

We present a number of Motzkin normed (1,2)-magmas. For each (1,2)-magma, we take the convention that the generator is ϵ, the unary map is f and the binary map is $\star$ (and this is always written as an in-fix operator). Despite the same names being used for each (1,2)-magma presented, it is clear that these are all different maps and that the generators are all different.

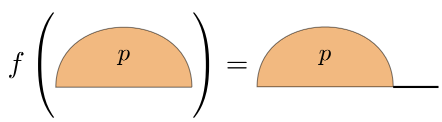
$\mathbb{M}_1$: **Motzkin paths [11]**

M_n is the number of paths from $(0,0)$ to $(n,0)$ using steps $U = (1,1)$, $D = (1,-1)$ and $H = (1,0)$ which remain above the line $y = 0$.

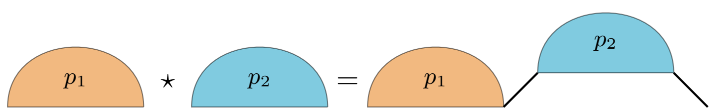
Generator: The empty path which we represent by a single vertex:

$$\epsilon = \bullet$$

Unary map:

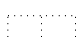
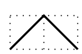
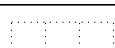
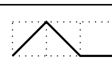
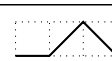
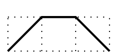


Binary map:



Norm: If p is a Motzkin path from $(0,0)$ to $(n,0)$, then $\|p\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= \epsilon \star f(\epsilon)$

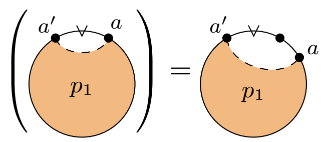
$\mathbb{M}_2$: **Non-intersecting chords [18]**

M_n counts the number of ways of drawing any number of non-intersecting chords joining up to n distinct points on a circle. These n points are called chord points. Note that not all n chord points must be incident with a chord. We mark, with a $\vee$, a unique point at the top of each circle. This fixes the orientation of the circle and distinguishes non-intersecting chord diagrams which only differ by a rotation.

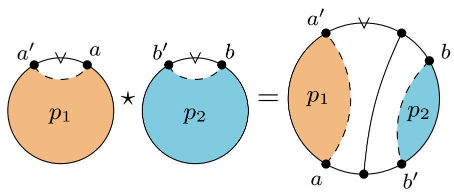
Generator: The trivial way of joining zero chord points on a circle:

$\epsilon =$ 

Unary map:

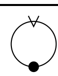
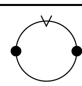
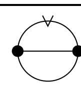
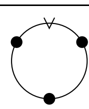
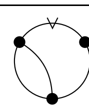
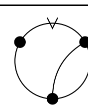
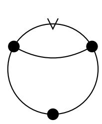


Binary map:



Norm: If p contains n chord points, then $\|p\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= \epsilon \star f(\epsilon)$

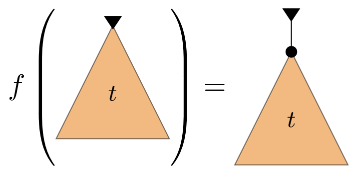
$\mathbb{M}_3$: **Motzkin unary-binary trees [11]**

M_n is equal to the number of rooted trees with n edges in which every vertex has degree at most 3, and in which the root has degree at most 2.

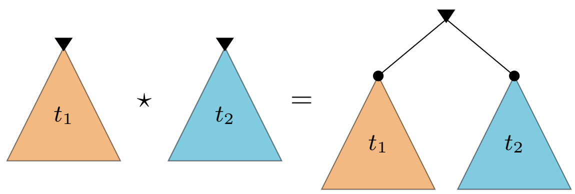
Generator: The generator is a single vertex, that is, the only tree containing no edges:

$$\epsilon = \bullet$$

Unary map:

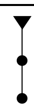
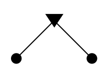
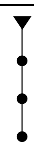
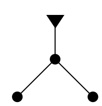
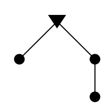
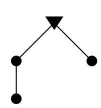


Binary map:



Norm: If t is a tree with n edges, then $\|t\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= f(\epsilon) \star \epsilon$

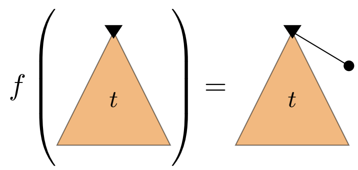
$\mathbb{M}_4$: **Bushes [11]**

M_{n-1} is the number of rooted planar trees with n edges and no degree two nodes, except possibly for the root. Such a tree is called a bush.

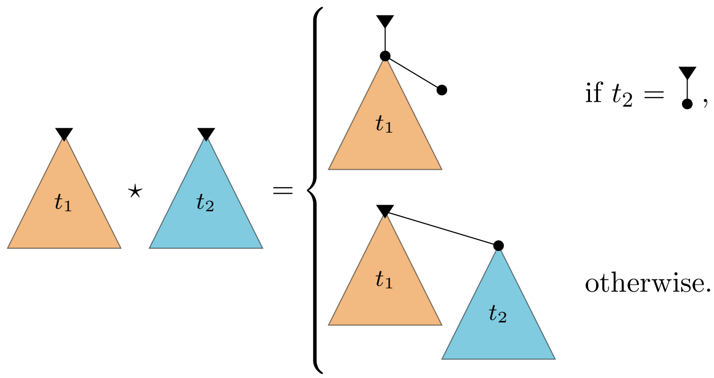
Generator:

$\epsilon =$ 

Unary map:

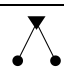
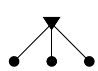
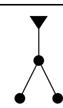
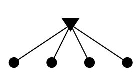
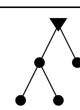
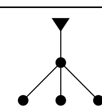
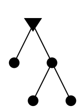


Binary map:



Norm: If b is a bush with n edges, then $\|b\| = n$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= \epsilon \star f(\epsilon)$

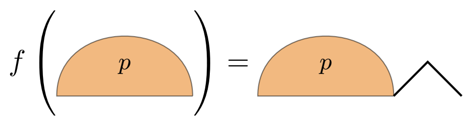
$\mathbb{M}_5$: **UUU-avoiding Dyck paths [23]**

M_n is the number of Dyck paths of length $2n$ with no sequence of three or more consecutive up steps. A Dyck path is a path from $(0,0)$ to $(2n,0)$ using steps $U=(1,1)$ and $D=(1,-1)$ which remain above the line $y=0$.

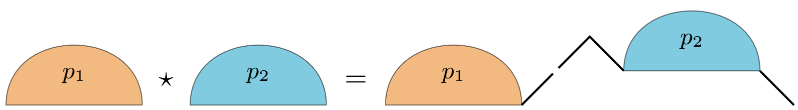
Generator: The empty path which we represent by a single vertex:

$$\epsilon = \bullet$$

Unary map:

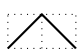
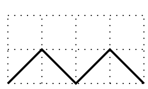
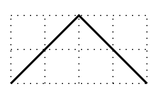
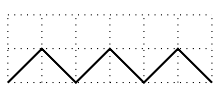
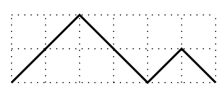
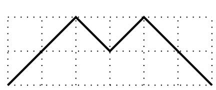
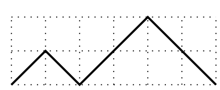


Binary map:



Norm: If p is a Dyck path from $(0,0)$ to $(2n,0)$, then $\|p\| = n+1$.

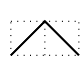
(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= f(\epsilon) \star \epsilon$

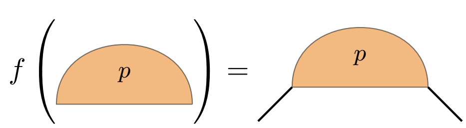
$\mathbb{M}_6$: **UDU-avoiding Dyck paths [23]**

M_{n-1} is the number of Dyck paths of length $2n$ with no sequence of UDU steps.

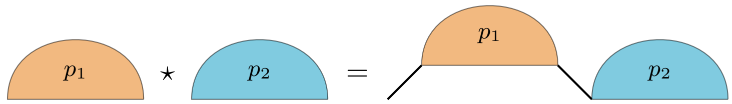
Generator:

$\epsilon =$ 

Unary map:

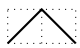
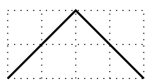
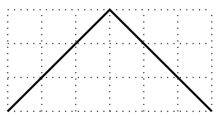
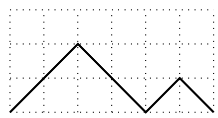
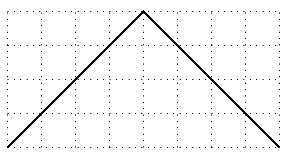
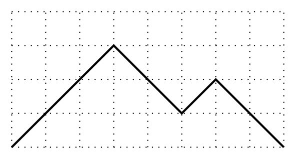
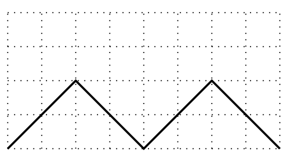
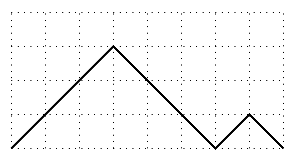


Binary map:



Norm: Let p be a Dyck path from $(0, 0)$ to $(2n, 0)$. Then $\|p\| = n$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= f(\epsilon) \star \epsilon$

$\mathbb{M}_7$: **Recursive set of bracketings [23]**

M_{n-1} is the number of strings of length $2n$ from the following recursively defined set: L contains the string $[\,]$ and, for any strings a and b in L, we also find $[\,a\,]$ and $[\,ab\,]$ in L.

Generator:

$$\epsilon = [\,].$$

Unary map: Let a be a bracketing. Then define the unary map as

$$f(a) = [\,a\,].$$

Binary map: Let a and b be two bracketings. Then define the binary map as

$$a \star b = [\,ab\,].$$

Norm: If a is a bracketing of length $2n$, then $\|a\| = n$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$[\,] = \epsilon$
Norm 2:	$[[\,]] = f(\epsilon)$
Norm 3:	$[[[\,]]] = f(f(\epsilon))$, $\quad [[\,][\,]] = \epsilon \star \epsilon$
Norm 4:	$[[[[\,]]]] = f(f(f(\epsilon)))$, $\quad [[[\,][\,]]] = f(\epsilon \star \epsilon)$, $[[\,][[\,]]] = \epsilon \star f(\epsilon)$, $\quad [[[\,]][\,]] = f(\epsilon) \star \epsilon$

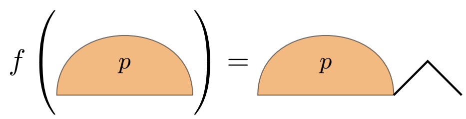
$\mathbb{M}_8$: **Dyck paths with even valleys [23]**

M_n is the number of length $2n$ Dyck paths whose valleys all have even x-coordinates. A valley is a path vertex preceded by a down step and followed by an up step.

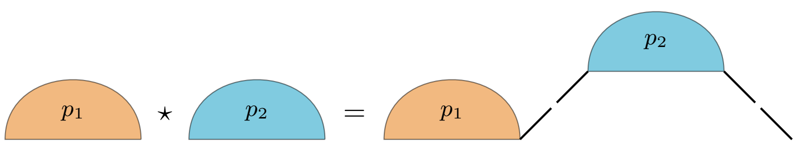
Generator: The empty path which we represent by a single vertex:

$$\epsilon = \bullet$$

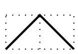
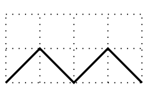
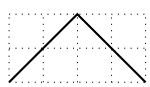
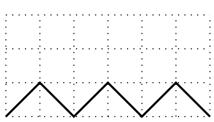
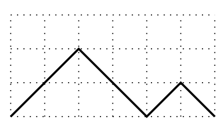
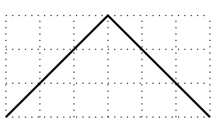
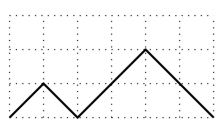
Unary map:



Binary map:



Norm: If p is a Dyck path from $(0,0)$ to $(2n,0)$, then $\|p\| = n+1$.
(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$
Norm 3:	 $= f(f(\epsilon))$,  $= \epsilon \star \epsilon$
Norm 4:	 $= f(f(f(\epsilon)))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= f(\epsilon) \star \epsilon$

$\mathbb{M}_9$: **RNA shapes [15]**

M_n is the number of RNA shapes of size $2n+2$. RNA shapes are Dyck words without "directly nested" motifs of the form $A[[B]]C$ for A, B and C Dyck words. A Dyck word is a word in the alphabet $\{[,]\}$ with an equal number of left and right parentheses with the property that, reading from left to right, the number of right parentheses never exceeds the number of left parentheses.

Generator:

$$\epsilon = [\,].$$

Unary map: Let w be an RNA shape. Then define the unary map as

$$f(w) = w\,[\,].$$

Binary map: Let w_1 and w_2 be RNA shapes. Then define the binary map as

$$w_1 \star w_2 = w_1'\,[\,]\,w_2\,],$$

where $w_1 = w_1'\,]$.

Norm: If w is an RNA shape of length $2n$, then $\|w\| = n$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$[\,] = \epsilon$
Norm 2:	$[\,][\,] = f(\epsilon)$
Norm 3:	$[\,][\,][\,] = f(f(\epsilon))$, $\quad [[\,][\,]] = \epsilon \star \epsilon$
Norm 4:	$[\,][\,][\,][\,] = f(f(f(\epsilon)))$, $\quad [[\,][\,]][\,] = f(\epsilon \star \epsilon)$, $[[\,][\,][\,]] = \epsilon \star f(\epsilon)$, $\quad [\,][[\,][\,]] = f(\epsilon) \star \epsilon$

7.3 Schröder Families

We present a number of Schröder normed (1,2)-magmas. For each (1,2)-magma, we take the convention that the generator is ϵ, the unary map is f and the binary map is $\star$ (and that this is always written as an in-fix operator). Despite the same names being used for each Schröder family presented, it is clear that these are all different maps and the generators are all different.

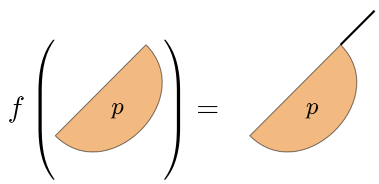
$\mathbb{S}_1$: **Schröder paths [6]**

S_n is the number of paths from $(0,0)$ to (n,n) using only steps $(1,0)$, $(0,1)$ and $(1,1)$ which lie weakly below the line $y=x$. The diagonal $y=x$ is shown as a dashed line in the schematic diagram defining the binary map.

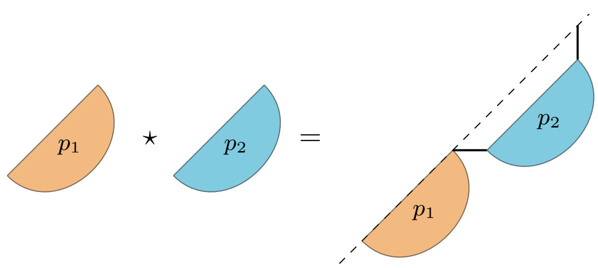
Generator: The empty path which we represent by a single vertex:

$$\epsilon = \bullet$$

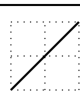
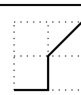
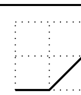
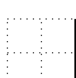
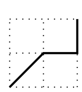
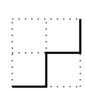
Unary map:



Binary map:



Norm: If p is a Schröder path from $(0,0)$ to (n,n), then $\|p\| = n+1$.
(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

$\mathbb{S}_2$: **Dyck paths with coloured peaks [10]**

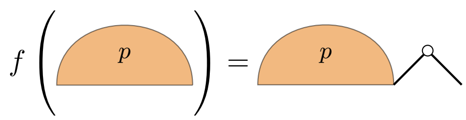
S_n is the number of Dyck paths from $(0, 0)$ to $(2n, 0)$ with each peak coloured black or white. A peak is a point preceded by an up step and followed by a down step.

Generator: The empty path which we represent by a single vertex which is coloured black:

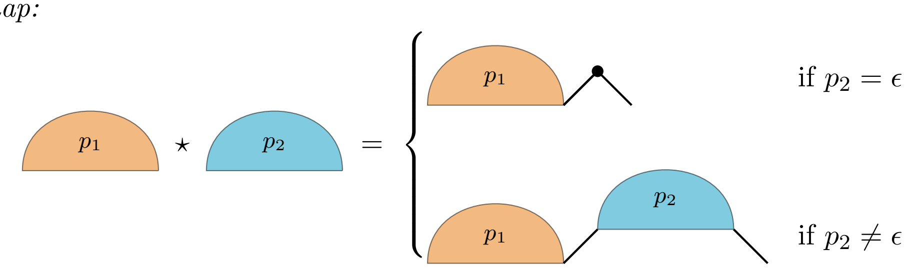
$$\epsilon = \bullet$$

Note that when applying the unary map to ϵ or applying the binary map with ϵ as the left factor that this is simply the empty path since this will not correspond to a peak and thus the node need not be coloured.

Unary map:

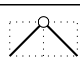
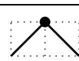
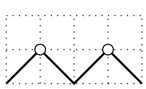
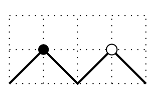
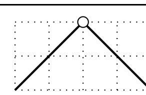
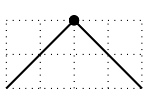
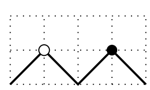
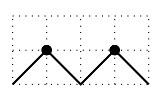


Binary map:



Norm: If p is a Dyck path with coloured peaks from $(0, 0)$ to $(2n, 0)$, then $\|p\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

$\mathbb{S}_3$: **Semi-standard Young tableaux of shape $n \times 2$ [25]**

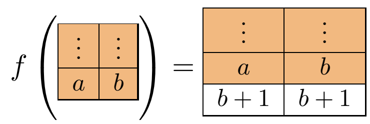
S_n is the number of semi-standard Young tableaux (SSYT) of shape $n \times 2$. Each tableau is filled with entries from $\{1, \ldots, r\}$ for some $r \in \mathbb{N}$, and we require that each number in this set appears in at least one cell. The rows must be weakly increasing from left to right and the columns must be strictly increasing from top to bottom.

Generator: The empty SSYT:

$$\epsilon = \emptyset.$$

When applying either of the two maps to the generator, every entry of the empty tableau is taken to be 0. In the map definitions which follow, we explicitly state what happens when applying the maps to the generator in order to make this clear.

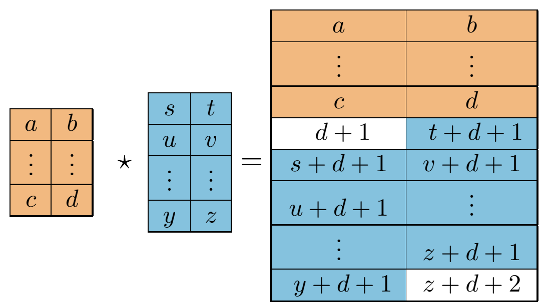
Unary map:



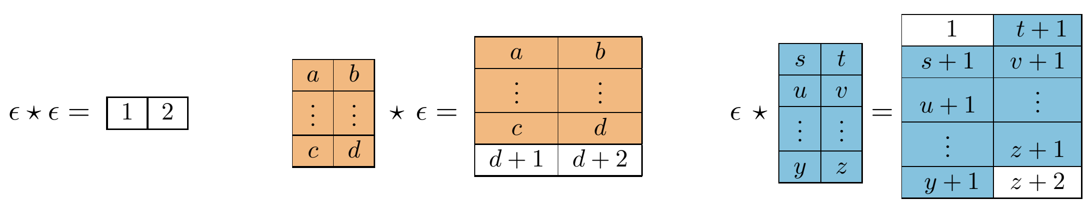
Note that we have

$$f(\epsilon) = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline\end{array}$$

Binary map:



If we apply the binary map with the generator, we have the following:



Norm: If t is a semi-standard Young tableau of size $n \times 2$, then $\|t\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

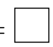
Norm 1:	$\emptyset = \epsilon$
Norm 2:	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array} = f(\epsilon)$, $\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline\end{array} = \epsilon \star \epsilon$
Norm 3:	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 2 & 2 \\ \hline\end{array} = f(f(\epsilon))$, $\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline\end{array} = f(\epsilon \star \epsilon)$, $\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline\end{array} = \epsilon \star f(\epsilon)$, $\begin{array}{\|c\|c\|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} = \epsilon \star (\epsilon \star \epsilon)$, $\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline 2 & 3 \\ \hline\end{array} = f(\epsilon) \star \epsilon$, $\begin{array}{\|c\|c\|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} = (\epsilon \star \epsilon) \star \epsilon$

$\mathbb{S}_4$: **Rectangulations [1]**

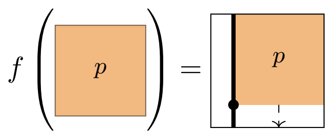
S_n is the number of ways to divide a rectangle into $n+1$ smaller rectangles using n cuts through n points placed equidistant from each other inside the rectangle along the diagonal joining the bottom left corner and the top right corner. Each cut intersects one of the points and divides only a single rectangle in two.

Curiously, this combinatorial family appears on the Schröder numbers Wikipedia page with no citation. After some investigation, it was discovered that these are equivalent to "point-constrained rectangular guillotine partitions" as defined in [1].

Generator: The generator is taken to be the trivial way of dividing a rectangle into one rectangle using zero cuts:

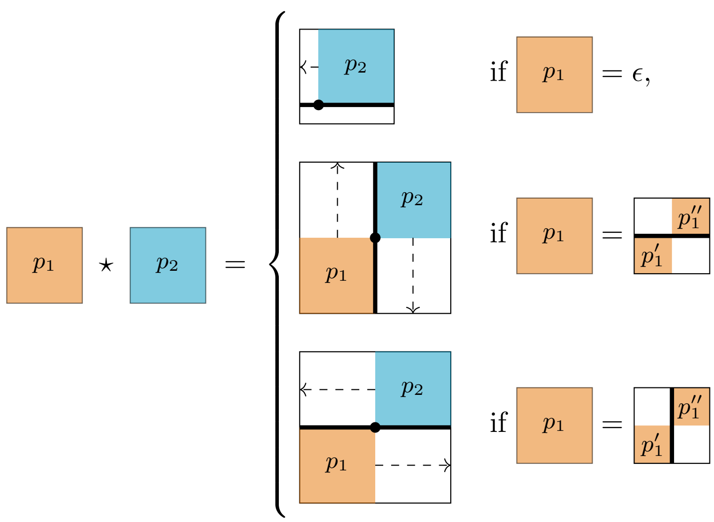
$\epsilon =$ 

Unary map:



After applying the unary map, any vertical cuts in p are extended downwards until they reach the lower boundary. This is represented by the dashed arrow.

Binary map:

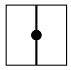
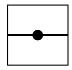
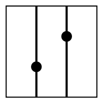
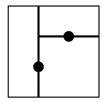
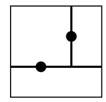
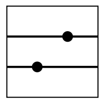
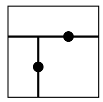
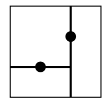


After joining the two rectangulations as shown in the binary map, we then extend all cuts until they meet the boundary. This is represented by the dashed arrows.

Note that p_1' and p_1'' above are not necessarily unique, however the product rule is well defined. In any rectangulation, there will be either a cut which joins the left and right boundaries or there will be a cut which joins the top and bottom boundaries. This follows from the constraint that each cut may divide only a single rectangle in two. It is the unique orientation of any longest cut which determines how we apply the product rule.

Norm: If p is a rectangle dissection using n internal lines (so the rectangle consists of $n+1$ smaller rectangles), then $\|p\| = n+1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

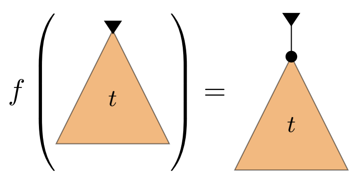
$\mathbb{S}_5$: **Schröder unary-binary trees [25]**

S_n is equal to the number of rooted trees with n non-leaf nodes in which every vertex has degree at most 3, and in which the root has degree at most 2.

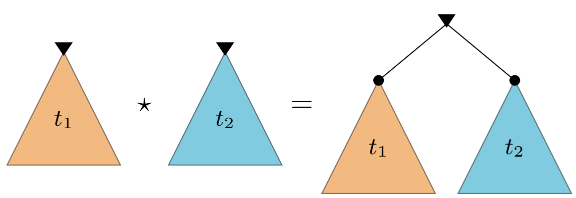
Generator: The generator is a single vertex:

$$\epsilon = \bullet$$

Unary map:

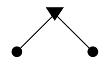
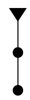
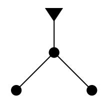
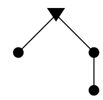
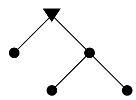
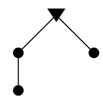
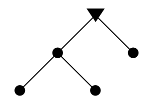


Binary map:



Norm: If t is a tree with n non-leaf nodes, then $\|t\| = n + 1$. Note that we consider the only node in ϵ to be a leaf node, and hence $\|\epsilon\| = 1$.

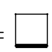
(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

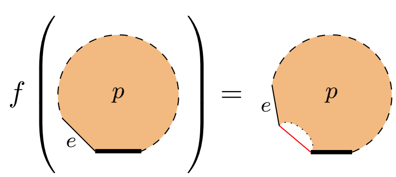
$\mathbb{S}_6$: **Polygon dissections [25]**

S_n is the number of dissections of a regular $(n+4)$-gon by diagonals that do not touch the base. A diagonal is a straight line joining two non-consecutive vertices and dissection means that diagonals are non-crossing though they may share an endpoint. We draw all polygons with a marked side which is denoted by a thick black line. This fixes the orientation of the polygon and distinguishes polygons which only differ by a rotation.

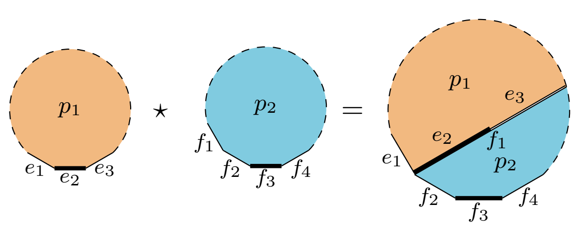
Generator: The trivial dissection of a square obtained by placing no diagonals:

$\epsilon =$ 

Unary map:



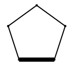
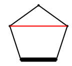
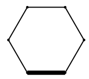
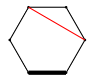
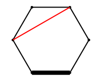
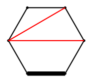
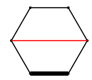
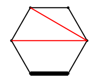
Binary map:



where the edge f_1 coincides with the edges $e_2 e_3$ to form a single diagonal.

Norm: If p is a dissection of an n-gon, then $\|p\| = n - 3$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

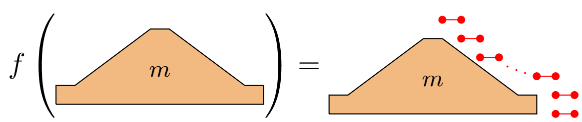
$\mathbb{S}_7$: **Perfect matchings in an Aztec triangle [9]**

S_n is the number of perfect matchings in an Aztec triangle, which is a triangular grid of n^2 squares. This grid is formed by starting with one square, then placing a centred row of 3 squares beneath it, followed by a centred row of 5 squares beneath this and so on. We then place a node at the corner of each square. We allow only perfect matchings in which no two edges cross when all edges are drawn as straight lines connecting two nodes. This enforces that any node is matched with a node placed on an adjacent corner of the same square ie. all matching edges are either a horizontal or vertical and unit length.

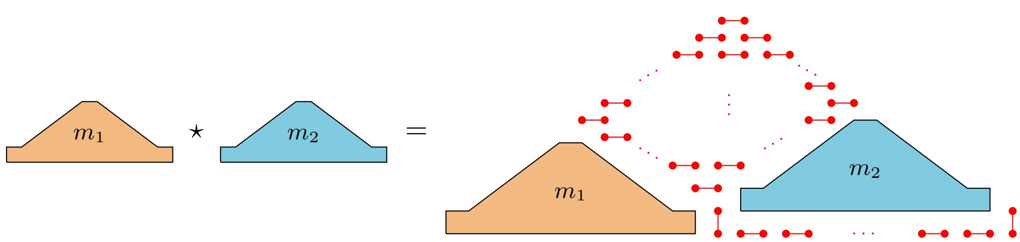
Generator: The trivial empty matching in a triangular grid of zero squares:

$$\epsilon = \emptyset.$$

Unary map:

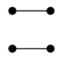
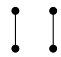
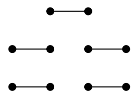
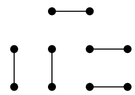
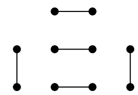
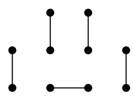
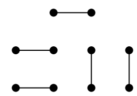
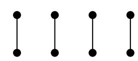


Binary map:



Norm: If m is a perfect matching in an Aztec triangle of n^2 squares, then $\|m\| = n + 1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

$\mathbb{S}_8$: **Coloured parallelogram polyominoes (zebras) [20]**

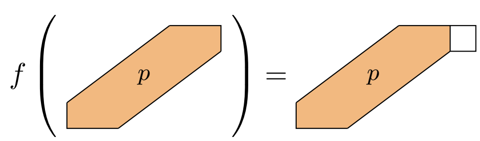
S_n is the number of parallelogram polyominoes of perimeter $2n+2$ with each column coloured black or white. A parallelogram polyomino is a translation invariant array of unit squares bounded by two lattice paths that use the steps $(0,1)$ and $(1,0)$ and that intersect only at their first and last vertices.

Generator: The empty parallelogram polyomino, which we choose to represent as follows:

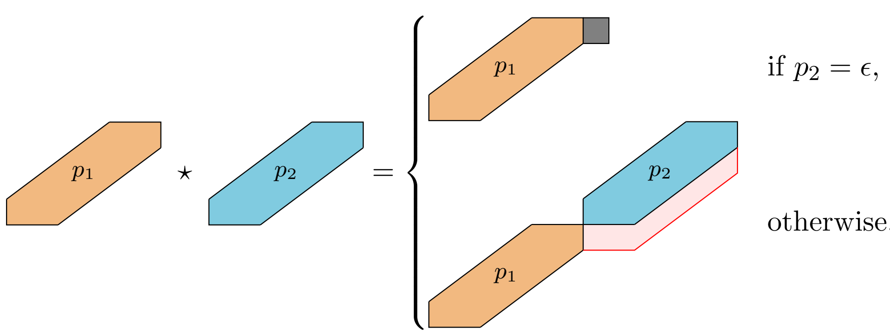
$$\epsilon = |$$

We take this to have perimeter 2.

Unary map: Add a single white square to the top right of the coloured parallelogram polyomino:



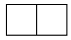
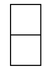
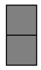
Binary map:



The red section means that we join p_1 and p_2 as shown and then add one cell at the bottom of each column of p_2, with this cell being given the colour of the column it is being added to.

Norm: If p is a coloured parallelogram polyomino of perimeter $2n+2$, then $\|p\| = n+1$.

(1,2)-magma: The (1,2)-magma begins (sorting by norm) as follows:

Norm 1:	$\emptyset = \epsilon$
Norm 2:	 $= f(\epsilon)$,  $= \epsilon \star \epsilon$
Norm 3:	 $= f(f(\epsilon))$,  $= f(\epsilon \star \epsilon)$,  $= \epsilon \star f(\epsilon)$,  $= \epsilon \star (\epsilon \star \epsilon)$,  $= f(\epsilon) \star \epsilon$,  $= (\epsilon \star \epsilon) \star \epsilon$

7.4 Fuss-Catalan Families

We present a number of Fuss-Catalan normed (3)-magmas for combinatorial families enumerated by the order 3 Fuss-Catalan numbers. We take the convention that each generator is called ϵ and that each ternary map is called t. Despite the same names being used for each (3)-magma presented, it is clear that these are all different maps and the generators are all different.

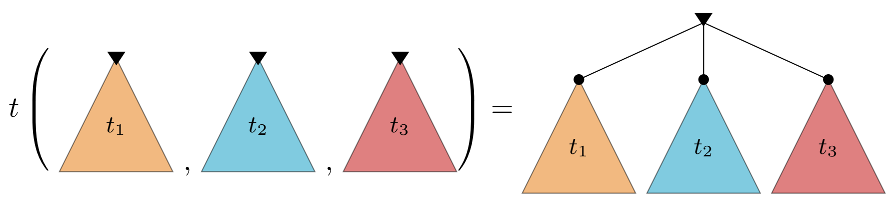
$\mathbb{T}_1$: **Ternary trees [2]**

T_n is the number of complete ternary trees with $2n+1$ leaves. A complete ternary tree is a rooted tree such that every non-leaf node has three children.

Generator: The generator is a single vertex:

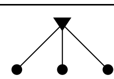
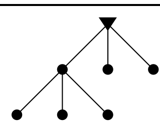
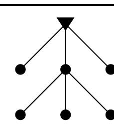
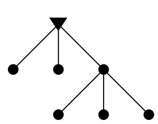
$$\epsilon = \bullet$$

Ternary map:



Norm: If t is a ternary tree with n leaves, then $\|t\| = n$. Note that we consider the only node in ϵ to be a leaf node, and hence $\|\epsilon\| = 1$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

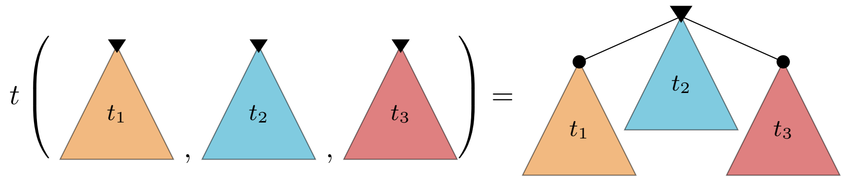
$\mathbb{T}_2$: **Even trees [24]**

T_n is the number of rooted plane trees with $2n$ edges, where every vertex has even out-degree.

Generator: The generator is a single vertex:

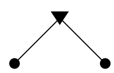
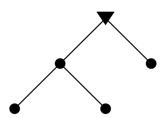
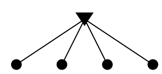
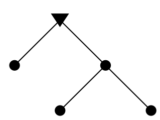
$$\epsilon = \bullet$$

Ternary map:



Norm: If t is a tree with n edges, then $\|t\| = n + 1$.

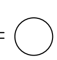
(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

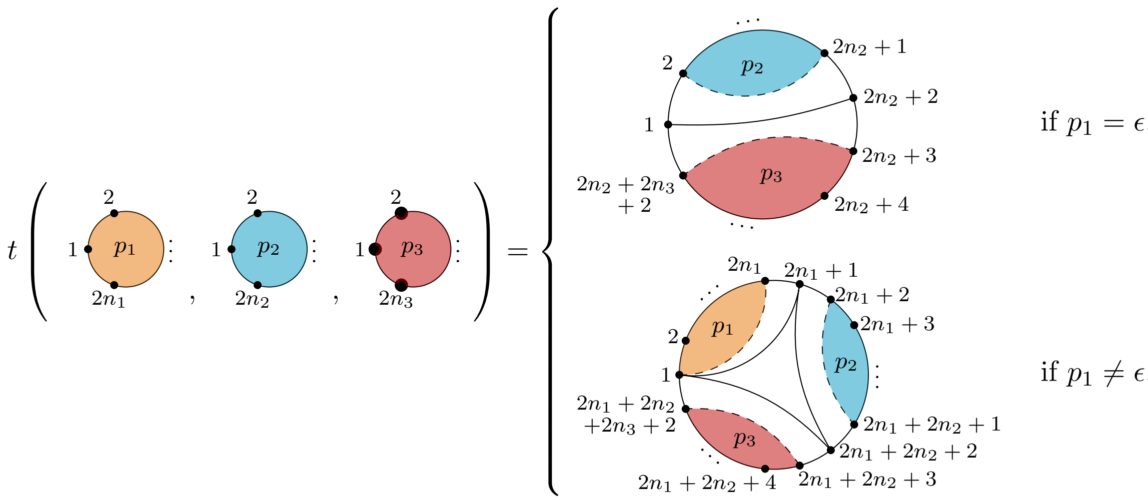
$\mathbb{T}_3$: **Non-crossing partitions with blocks of even size [24]**

T_n is the number of non-crossing partitions of $\{1, 2, \ldots, 2n\}$ with all blocks of even size. We represent such a partition schematically by n marked points on a circle, with a chord connecting points which are in the same block.

Generator: The generator is the trivially empty partition of the empty set:

$\epsilon =$ 

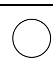
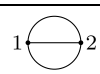
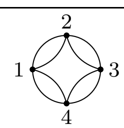
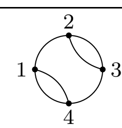
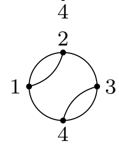
Ternary map:



In the second case, the chord shown connecting node 1 with node $2n_1+1$ means that we add node $2n_1+1$ to the block containing node 1. Note that this will not always result in the chord shown.

Norm: If p is a partition of $\{1, 2, \ldots, 2n\}$, then $\|p\| = 2n + 1$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	 $= \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

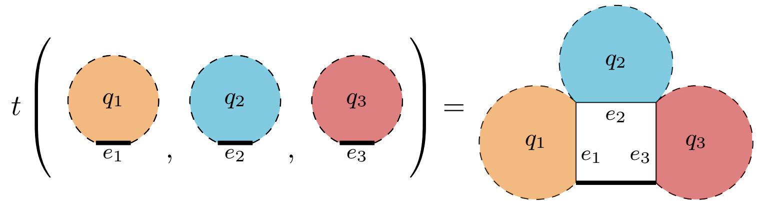
$\mathbb{T}_4$: **Quadrillages [3]**

T_n is the number of quadrillages of a $(2n+2)$-gon. A quadrillage is a dissection of a polygon such that all sub-objects have four sides. We draw all polygons with a marked side which is denoted by a thick black line. This fixes the orientation of the polygon and distinguishes polygons which only differ by a rotation.

Generator: The generator is taken to be a single edge, i.e. a 2-gon:

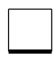
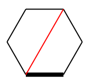
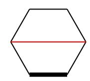
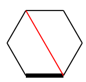
$\epsilon =$ —

Ternary map:



Norm: If q is a quadrillage of a $(2n+2)$-gon, then $\|q\| = 2n+1$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	— $= \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

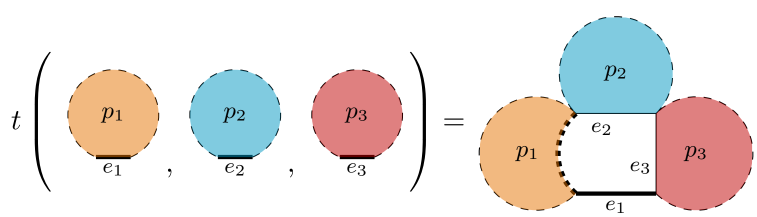
$\mathbb{T}_5$: **Polygon dissection [24]**

T_n is the number of dissections of some convex polygon by non-intersecting chords into polygons with an odd number of sides and having a total number of $2n+1$ edges (sides and diagonals). We draw all polygons with a marked side which is denoted by a thick black line. This fixes the orientation of the polygon and distinguishes polygons which only differ by a rotation.

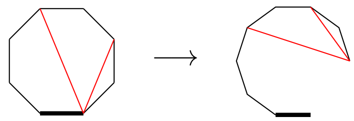
Generator: The generator is taken to be a single edge:

$$\epsilon = \text{—}$$

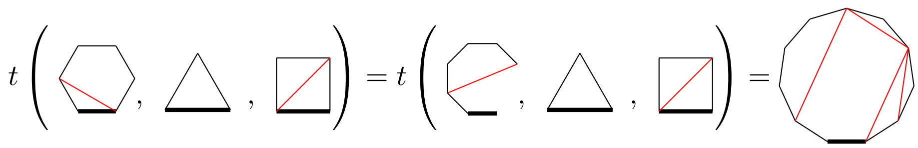
Ternary map:



In the above figure, the dotted internal line for p_1 denotes the fact that we "open up" the polygon p_1 by disconnecting it at the vertex on the right hand end of the marked thick edge. We do this in such a way that any diagonals adjacent to this vertex remain connected to the endpoint of the non-thick edge. For example, we "open up" the following polygon dissection as shown:

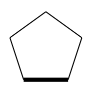
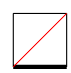
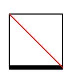


An example of this map is the following:



Norm: If p is a polygon dissection with a total of n of edges (sides and diagonals), then $\|p\| = n$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	**—** $= \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

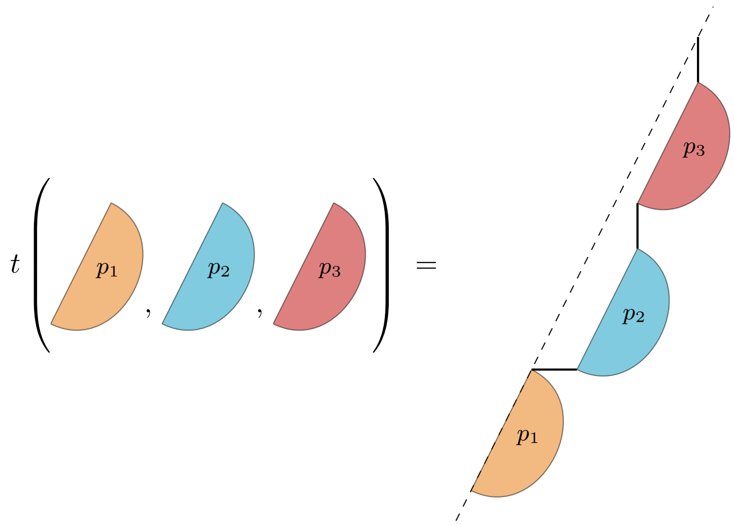
$\mathbb{T}_6$: **Lattice paths [14]**

T_n is the number of lattice paths from $(0, 0)$ to $(n, 2n)$ consisting of n East steps $(1, 0)$ and $2n$ North steps $(0, 1)$ that lie weakly below the line $y = 2x$. The line $y = 2x$ is shown as a dashed line in the schematic diagram defining the map.

Generator: The empty path which we represent by a single vertex:

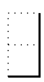
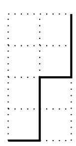
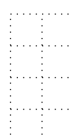
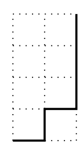
$$\epsilon = \bullet$$

Ternary map:



Norm: If p is a path from $(0,0)$ to $(n,2n)$, then $\|p\| = 2n+1$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 3:	 $= t(\epsilon,\epsilon,\epsilon)$
Norm 5:	 $= t(t(\epsilon,\epsilon,\epsilon),\epsilon,\epsilon)$,  $= t(\epsilon,t(\epsilon,\epsilon,\epsilon),\epsilon)$,  $= t(\epsilon,\epsilon,t(\epsilon,\epsilon,\epsilon))$

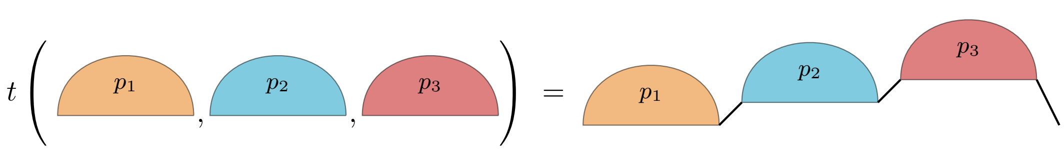
$\mathbb{T}_7$: **2-Dyck paths [2]**

T_n is the number of paths from $(0,0)$ to $(3n,0)$ with steps $(1,1)$ and $(1,-2)$ which remain above the line $y=0$.

Generator: The empty path which we represent by a single vertex:

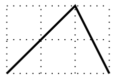
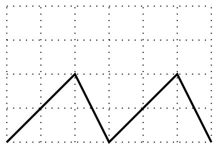
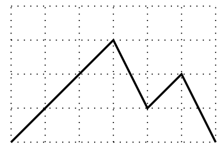
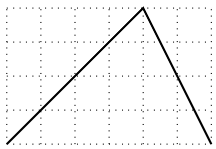
$$\epsilon = \bullet$$

Ternary map:



Norm: If p is a 2-Dyck path from $(0,0)$ to $(3n,0)$, then $\|p\| = 2n+1$.

(3)-magma: The (3)-magma begins (sorting by norm) as follows:

Norm 1:	$\bullet = \epsilon$
Norm 3:	 $= t(\epsilon, \epsilon, \epsilon)$
Norm 5:	 $= t(t(\epsilon, \epsilon, \epsilon), \epsilon, \epsilon)$,  $= t(\epsilon, t(\epsilon, \epsilon, \epsilon), \epsilon)$,  $= t(\epsilon, \epsilon, t(\epsilon, \epsilon, \epsilon))$

References

[1] Ackerman, E., Barequet, G., and Pinter, R. Y. (2004). On the number of rectangular partitions. In *Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 736–745.

[2] Aval, J.-C. (2008). Multivariate Fuss-Catalan numbers. *Discrete Mathematics*, 308(20):4660 – 4669.

[3] Baryshnikov, Y. (2001). *On Stokes sets*, pages 65–86. Springer Netherlands, Dordrecht.

[4] Benjamin, A. and Quinn, J. (2003). *Proofs that Really Count: The Art of Combinatorial Proof*. Mathematical Association of America.

[5] Benjamin, A., Quinn, J., and Su, F. (1998). Phased tilings and generalized Fibonacci identities. *The Fibonacci Quarterly*, 38(3):282 – 288.

[6] Bonin, J., Shapiro, L., and Simion, R. (1993). Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths. *Journal of Statistical Planning and Inference*, 34(1):35 – 55.

[7] Bourbaki, N. (1981). *Algebra II*. Springer- Verlag Berlin Heidelberg 2003, english translation by cohn and howie edition.

[8] Brak, R. (2018). A Universal Bijection for Catalan Structures. *arXiv e-prints*, page arXiv:1808.09078.

[9] Ciucu, M. (1996). Perfect matchings of cellular graphs. *Journal of Algebraic Combinatorics*, 5(2):87–103.

[10] Deutsch, E. (2001). A bijective proof of the equation linking the Schröder numbers, large and small. *Discrete Mathematics*, 241:235–240.

[11] Donaghey, R. and Shapiro, L. (1977). Motzkin numbers. *Journal of Combinatorial Theory, Series A*, 23(3):291 – 301.

[12] Farrell, E. (1986). On the occurrences of fibonacci sequences in the counting of matchings in linear polygonal chains. *The Fibonacci Quarterly*, 24:238–246.

[13] Hoggatt Jr, V. (1979). Reflections across two and three glass plates. *The Fibonacci Quarterly*, 17:118–142.

[14] Lin, C.-H. (2011). Some combinatorial interpretations and applications of Fuss-Catalan numbers. *ISRN Discrete Mathematics*, 2011.

[15] Lorenz, W. A., Ponty, Y., and Clote, P. (2008). Asymptotics of RNA shapes. *Journal of Computational Biology*, 15(1):31–63.

[16] Moser, L. (1963a). Problem B-5. *The Fibonacci Quarterly*, 1(1):74.

[17] Moser, L. (1963b). Some reflections. *The Fibonacci Quarterly*, 1(4):75 – 76.

[18] Motzkin, T. (1948). Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products. *Bull. Amer. Math. Soc.*, 54(4):352–360.

[19] Nyblom, M. (2012). Enumerating binary strings without r-runs of ones. *International Mathematical Forum*, 7(37–40):1865–1876.

[20] Pergola, E. and Sulanke, R. A. (1998). Schröder triangles, paths, and parallelogram polyominoes. *Journal of Integer Sequences*, 1(98.1):7.

[21] Simion, R. and Schmidt, F. W. (1985). Restricted permutations. *European Journal of Combinatorics*, 6(4):383 – 406.

[22] Sloane, N. J. A. (2019a). The On-Line Encyclopedia of Integer Sequences: A000045. https://oeis.org/A000045/.

[23] Sloane, N. J. A. (2019b). The On-Line Encyclopedia of Integer Sequences: A001006. https://oeis.org/A001006/.

[24] Sloane, N. J. A. (2019c). The On-Line Encyclopedia of Integer Sequences: A001764. https://oeis.org/A001764/.

[25] Sloane, N. J. A. (2019d). The On-Line Encyclopedia of Integer Sequences: A006318. https://oeis.org/A006318/.

[26] Stanley, R. P. (2011). *Enumerative Combinatorics*, volume 1. Cambridge University Press, New York, NY, USA, 2nd edition.