COUNTING Domineering POSITIONS

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Abstract. Domineering is a two player game played on a checkerboard in which one player places dominoes vertically, while the other places them horizontally. We will give bivariate generating polynomials enumerating Domineering positions given the number of pieces placed by each player. Specifically, we enumerate all positions, maximal positions, and Right or Left ending positions. We also discuss how this relates to other counting problems, including matchings on grid graphs.

1. Introduction

Combinatorial games are 2-player games with perfect information and no chance devices, such as Chess or Go. Many combinatorial games have, for a fixed starting position, a finite number of options and the game is guaranteed to end in a finite number of moves; in theory we could determine by computer which player would win if both players play perfectly. In practice game theorists and computer scientists have not determined the outcomes of games under perfect-play because of the complexity of the required search.

Enumeration of positions has been studied, directly or indirectly, for several combinatorial games. The first few papers on counting game positions considered the problem of enumerating specific types of positions—Go end positions in [7, 6, 16] and second-player win positions for two games in [8, 9]. In the game Node Kayles, played on a graph, the two players alternate choosing vertices not adjacent to any previously chosen ones, thus forming an independent set. Therefore the independence polynomial of a graph is equal to the generating polynomial for positions of Node Kayles on that graph. Similarly, in the play of Arc Kayles the players form a matching. The enumeration of matchings has been studied for many graphs (see Section 1.1). In *partizan* games, where players may have distinct options at some point during play, the convention is to call the two players **Left** (who uses bLue pieces) and **Right** (who uses Red pieces). In Col, which is played on a graph, a move for Left is to colour a vertex blue, while a move for Right is to colour a

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vertex red, and no two vertices of the same colour may be adjacent. Oh and Lee
[12] call such a position a bipartite independent vertex set and give the generating
polynomial for grid graphs. Brown et al. [4] give the generating polynomial, which
they call the polynomial profile, for several games, including closed forms for Col
and the game Snort (like Col, but two vertices of *different* colours may not be
adjacent) played on paths.

In this paper we consider the game **Domineering**. This game is played on a
checkerboard. The two players alternately place dominoes; Left places vertically
and Right places horizontally. The game ends when the player whose turn it is
cannot place a piece; this player loses — this is the Normal play convention. We
will count all Domineering positions, as well as Domineering positions with
certain properties, following the method used by Oh and Lee in [11, 12]. In newer
work Oh also considers monomer-dimer tilings in [10], thus implicitly counting
Domineering positions, which overlaps with some of our work as discussed in
Section 2.

Domineering, also called Dominoes or Crosscram in older work, was intro-
duced by Göran Andersson and popularized by Martin Gardner in the 1970s. There
has been significant interest in this game since then (see for example [2, 3, 17, 18, 19]
and references therein). Some simplification techniques are known and can be com-
bined with computer search for analysis, but no complete solution is available. It is
known who wins on many rectangular boards up to size 11×11 and some nonrect-
angular boards, as well as some values (essentially who has the advantage and by
how much) and temperatures (essentially the urgency of moving). For the reader
further interested in combinatorial game theory techniques, see [14].

As a game of Domineering progresses, it often naturally breaks into smaller
components, see Figure 1. A player on their turn has to decide which component
they want to play in and make their move there. This is called the **disjunctive
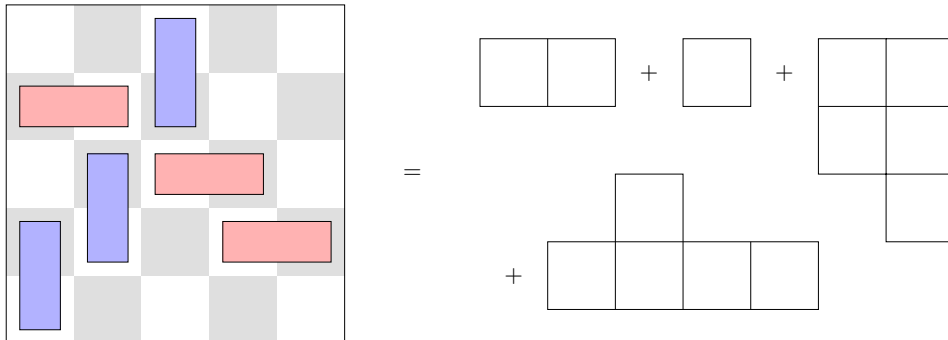sum** of these components.

Figure 1. During play a Domineering position may decompose
into a disjunctive sum of Domineering positions that are not
necessarily rectangular. The board on the left breaks into the
disjunctive sum of the components on the right.

If the two players choose different components in the disjunctive sum it can
happen that the same player makes two consecutive turns in the same component.
For example in Figure 1, if it is Right's turn, he could choose to play in the third

component, Left may choose the fourth, and Right choose the third again. Due to this, we are also interested in Domineering positions in which the two players do not necessarily alternate turns, hence we consider plays where the difference between the number of Left dominoes and Right dominoes may be larger than 1.

In Section 2 we will find the generating function for the number of positions on an $m \times n$ rectangular board which we denote by

$$D_{m,n}(x,y) = \sum d(a,b)x^a y^b$$

where $d(a,b)$ is the number of positions with a Left dominoes and b Right dominoes. We then demonstrate in Section 3 how to generalize the technique employed to find $D_{m,n}(x,y)$ to calculate the generating function for non-rectangular boards.

We are also particularly interested in enumerating positions at the end of the game. A **Left end** is a position in which Left can no longer play a piece, while Right potentially still has moves; a **Right end** is defined similarly. A **maximal position** is a position which is both a Left end and a Right end — a position in which no player can place a domino. In Section 4 we will find the generating function for maximal positions, which we denote by $F_{m,n}(x,y)$. In Section 5 we consider the generating function for Right ends; this is essentially the same as the generating function for Left ends.

Finally, in Section 6 we will discuss some other questions related to the technique used within and the problem of enumerating Domineering positions.

1.1. **Related Problems.** The checkerboard corresponds to a grid graph by using a vertex for each square and connecting them with an edge if the two squares are horizontally or vertically adjacent. Playing Domineering on the checkerboard is then equivalent to forming a matching (also called an edge independent set) on the grid graph, with a distinction made between horizontal and vertical edges.

If we make no distinction between Left and Right dominoes, that is, we enumerate the positions with a fixed number of dominoes, this is equivalent to enumerating the number of matchings in a grid graph or enumerating the monomer-dimer tilings of the chessboard (for a summary of connections to physics and chemistry see [13]). In particular, this means that $D_{m,n}(x,x)$ is the generating function for matchings in an $m \times n$ grid graph, and that $D_{m,n}(1,1)$ gives the total number of such matchings. The latter, also known as the Hosoya index of the grid graph, is known (for example see [1] or the OEIS[1] sequences A030186 for $2 \times n$, A033506 for $3 \times n$, and A028420 for $n \times n$). If a perfect matching exists in the $m \times n$ grid graph, then the number of perfect matchings is the leading coefficient of $D_{m,n}(x,x)$ as this is the number of maximum matchings.

Similarly, $F_{m,n}(x,x)$ gives the generating function for the maximal matchings in an $m \times n$ grid graph. The total number of maximal positions $F_{m,n}(1,1)$ are known for some values of m and n (OEIS sequences A000931 for $1 \times n$, A286945 for $2 \times n$, A288028 for $3 \times n$, and A287595 for $n \times n$).

Domineering belongs to the class of strong placement games, and thus one can assign to each game a simplicial complex representing the legal positions (see [5] for details). The coefficients of $D_{m,n}(x,x)$ are then the entries of the f-vector — the vector counting the number of faces of a given dimension — of this simplicial complex, while the coefficients of $F_{m,n}(x,x)$ give the number of maximal faces,

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[1]Online Encyclopedia of Integer Sequences [15]

called facets, of a fixed dimension. Our work in this paper was originally motivated by wanting to determine how many facets there are with a given number of vertices representing Left or Right moves. We solve this problem by finding $F_{m,n}(x, y)$ in Section 4.

2. Counting All Domineering Positions

Given m and n, our goal is to count the number of DOMINEERING positions that can occur as the result of play in an $m \times n$ rectangle. To distinguish this problem from counting specific types of domineering positions later we refer to these as *general* DOMINEERING positions.

We will find a generating function $D_{m,n}(x, y)$ such that

$$D_{m,n}(x, y) = \sum d(a, b)x^a y^b$$

where $d(a, b)$ is the number of DOMINEERING positions with a Left (vertical) dominoes and b Right (horizontal) dominoes.

From the generating function, we will extract information on the play positions, those that can be reached through alternating play, in Section 2.1.

In their paper, Oh [10] considers an equivalent problem to the particular problem of counting general positions using a trivariate generating function, where the third variable counts the number of empty squares. The number of empty squares is not particularly relevant to DOMINEERING. However, this information can be deduced from the number of dominoes of either player and the board size. We use an equivalent method, but as our generating functions are slightly different, our recursions are different as well. We include the recursion here for completeness, particularly as we continue using the matrices in Section 3.

To count rectangular DOMINEERING positions we consider the tilings of rectangles using tiles with edge labels, then count the tilings that correspond to positions.
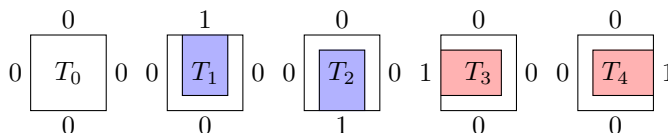
We use the tiles in Figure 2.

FIGURE 2. The tiles used to count general DOMINEERING positions

For a rectangular tiling to form a DOMINEERING position it has to satisfy the following conditions:

(1) *Adjacency condition*: All shared edges of adjacent tiles have the same label.
(2) *Boundary condition*: All boundary edges of the tiling have label 0.

Together these conditions ensure that the tiling has no half-dominoes.

In turn, any DOMINEERING position can be uniquely represented using such a tiling. See Figure 3 for an example.

In the tiling of a DOMINEERING position, such as in Figure 3, the number of Left (vertical) dominoes is equal to the number of T_1 tiles, and the number of Right (horizontal) dominoes is equal to the number of T_3 tiles.
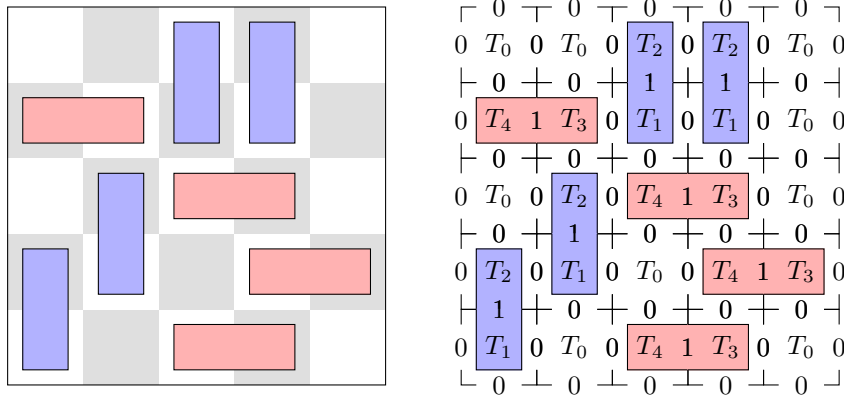
FIGURE 3. A Domineering position on a 5×5 board and its equivalent tiling.

We use the term **mosaic** for tilings that satisfy the adjacency condition, but not necessarily the boundary condition. A $1 \times q$ mosaic is called a **bar mosaic**, that is, a row of q tiles.

We will find generating polynomials by first counting bar mosaics. We then stack these bars to count rectangular mosaics. It is in this step that we will consider the adjacency condition for vertically adjacent tiles. Finally, we will restrict to those tilings that also satisfy the boundary condition.

We will count mosaics instead of just Domineering tilings as a row or column of a Domineering rectangle is not necessarily a Domineering position itself. For example in Figure 3 the second row from the bottom forms the bar mosaic shown in Figure 4—it does not correspond to a Domineering position on its own as the leftmost two tiles are unpaired half-dominoes but shared edges of horizontally adjacent tiles do have the same label.
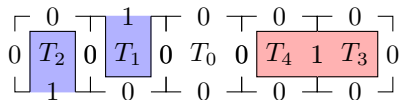
FIGURE 4. A row of a Domineering tiling is a bar mosaic, which satisfies the (horizontal) adjacency condition but not necessarily boundary condition.

The enumeration of bar mosaics will be done using matrices, called **bar-state matrices**. We take care to construct these matrices so that we can count rectangular mosaics by matrix multiplication. As we only want in the end to count mosaics that satisfy the boundary condition we can make due with counting only bar mosaics with label 0 on the right end. The enumeration of the bar mosaics will be done recursively — we use the enumeration of bar mosaics to enumerate longer bar mosaics — so our matrices are recursively defined (using blocks). We use two bar-state matrices; the matrix to count bar mosaics of length q with right label 0 and left label 0 is denoted by $G_{0,q}$ and the matrix to count bar mosaics of length

q with right label 0 and left label 1 is denoted by $G_{1,q}$. We use G here for naming the matrices, standing for 'general', to reserve M for matrices in the subsequent sections of the paper, which include a discussion of maximal positions.

Theorem 2.1. *The polynomial profile of* DOMINEERING *on an $m \times n$ board is the (1,1) entry of $G_{0,n}^m$ where $G_{0,0} = \begin{bmatrix} 1 \end{bmatrix}$, $G_{1,0} = \begin{bmatrix} 0 \end{bmatrix}$, and*

$$G_{0,q+1} = \begin{bmatrix} \begin{matrix} G_{0,q} \\ +G_{1,q} \end{matrix} & xG_{0,q} \\ G_{0,q} & \mathbf{0} \end{bmatrix} \qquad G_{1,q+1} = \begin{bmatrix} yG_{0,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

Proof. We will first prove that the bar-state matrices give the generating polynomials for bar mosaics.

In a bar-state matrix the rows and columns are indexed by binary strings of length q. Thus each matrix has size $2^q \times 2^q$. We order these strings lexicographically. Thus, the strings of length 3 are ordered as

$$000, 001, 010, 011, 100, 101, 110, 111.$$

An entry in a bar-state matrix is the generating polynomial of the bar whose top edge labels are the column index, and whose bottom edge labels are the row index; the right edge label is 0 and the left edge label corresponds to the matrix (the first subscript of G). The generating polynomial of the bar is the monomial $x^a y^b$ where a is the number of T_1 in the bar and b is the number of T_3 in the bar.

For $q = 0$, so the length is 0, it is only possible to satisfy the adjacency condition if both the starting and ending labels are the same. We thus have $G_{0,0} = \begin{bmatrix} 1 \end{bmatrix}$ and $G_{1,0} = \begin{bmatrix} 0 \end{bmatrix}$.

Our construction for larger q uses blocks and we name them as in Figure 5 to avoid proliferating subscripts. We need to show that in each matrix the 4 blocks are as we claim. Our argument is recursive and follows by first considering the leftmost tile in each bar. The block considered determines the top label and bottom label of the leftmost tile in bar mosaics enumerated as in Figure 5.
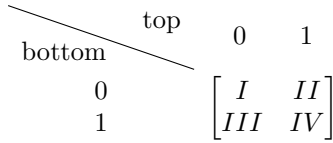

FIGURE 5. The block construction of the state-matrices for general DOMINEERING positions.

For $q = 1$ the bars we are counting consist of a single tile which must have right label 0 (or cannot have right label 1), that is, one of the four tiles 0–3; you may wish to reference Figure 2. Thus between the bar-state matrices $G_{0,1}$ and $G_{1,1}$ we have four nonzero blocks — determined by the top and bottom labels of tiles 0–3. Recall that each block has a corresponding top and bottom label of the leftmost tile. Thus

$$G_{0,1} = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix} \qquad G_{1,1} = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix},$$

and this shows that the recursion is correct for the first step.

The bar-state matrix $G_{0,q+1}$ enumerates mosaics where the leftmost tile has left label 0 (and right label 0). We consider possible leftmost tiles which are tiles 0–2 and 4, as these are the only tiles with left label 0. Of these four tiles, those tiles that meet the conditions for block I are tile 0 and tile 4. As the right label of tile 0 is 0, the possible completions of our bar when the leftmost tile is tile 0 are given by $G_{0,q}$. As the right label of tile 4 is 1, the possible completions of our bar are given by $G_{1,q}$. So block I is $G_{0,q} + G_{1,q}$.

The conditions for block II (that the leftmost tile have top label 1 and bottom label 0) require T_1 as the leftmost tile in the bar mosaic and the entry is $xG_{0,q}$ because the right label of tile 1 is 0 and $G_{0,q}$ counts the bar mosaics with left label 0 and right label 0 (recall that for each T_1 we have an x).

The conditions for block III force T_2 as the leftmost tile, giving the entry $G_{0,q}$. For block IV, we would need a tile with bottom and top label 1 but no tiles have such labels and so our entry is **0**.

In $G_{1,q+1}$, we have a left label of 1. As our recursion starts by consider possible left tiles, we need only consider tile 3, as this is the only tile with 1 as a left label.

Thus the only block for which tile 3 has appropriate top and bottom labels is block I. Since T_3 is forced and its left label is 0, the possible completions of the bar are given by $G_{0,q}$ and the entry in block one is $yG_{0,q}$. All other blocks, having no possible leftmost tiles, count no mosaics and thus their blocks are **0**.

We turn our attention to rectangular $m \times n$ tilings and in particular those that satisfy the left and right boundary conditions, so we will only consider the bar mosaics of length n with left label 0 in addition to having the right label 0; these are counted by $G_{0,n}$.

First consider $2 \times n$ mosaics that satisfy the adjacency condition. Such a mosaic consists of two bar mosaics where the string of bottom labels of one mosaic matches the string of top labels of the other. Suppose A is the bar-state matrix for the top bar and B is the bar-state matrix for the bottom bar. To count all the $2 \times n$ mosaics that satisfy the adjacency condition we need to consider all 2^n strings of matched labels along the stacked bar mosaics. That is, we take columns in A and rows in B. Since the other labels of the bar mosaics have to match, we take the dot product of the two vectors to get the polynomial for this mosaic. Thus the matrix for the $2 \times n$ mosaic is BA.

To get mosaics that also satisfy the boundary conditions and thus correspond to DOMINEERING positions we can start with bar mosaics that satisfy the left and right boundary conditions; in our case $A = B = G_{0,n}$, so the state matrix for $2 \times n$ mosaics where the left and right boundary conditions are also satisfied is $G_{0,n}^2$.

For larger mosaics this works the same way; we stack more bars and find that the matrix for a $m \times n$ mosaic is $G_{0,n}^m$.

Fixing the string of top labels and of bottom labels for the $2 \times n$ mosaic corresponds to specifying an entry in the matrix. Finally, to satisfy the boundary condition on top and bottom we need to restrict to all top and bottom labels being 0. This means we take the entry in the top left corner, entry $(1, 1)$. □

Example 2.2. Now we compute the number of general positions on a 4×3 DOMINEERING board. We need the bar-state matrix for bar mosaics of length 3 satisfying the left and right boundary conditions, $G_{0,3}$, which we find recursively:

$$G_{0,1} = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix} \qquad G_{1,1} = \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix}$$

$$G_{0,2} = \left[\begin{array}{cc:cc} y+1 & x & x & x^2 \\ 1 & 0 & x & 0 \\ \hdashline 1 & x & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \qquad G_{1,2} = \left[\begin{array}{cc:cc} y & xy & 0 & 0 \\ y & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$G_{0,3} = \left[\begin{array}{cccc:cccc} 2y+1 & xy+x & x & x^2 & xy+x & x^2 & x^2 & x^3 \\ y+1 & 0 & x & 0 & x & 0 & x^2 & 0 \\ 1 & x & 0 & 0 & x & x^2 & 0 & 0 \\ 1 & 0 & 0 & 0 & x & 0 & 0 & 0 \\ \hdashline y+1 & x & x & x^2 & 0 & 0 & 0 & 0 \\ 1 & 0 & x & 0 & 0 & 0 & 0 & 0 \\ 1 & x & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Calculating $G_{0,3}^4$ gives the state matrix for 4×3 boards and and taking the $(1,1)$ entry, gives the generating function for 4×3 DOMINEERING positions:

$$\begin{aligned} D_{4,3}(x,y) = \; & x^6 + 9x^5 + 6x^4y^2 + 20x^4y + 30x^4 + 46x^3y^2 + 84x^3y + 45x^3 \\ & + 4x^2y^4 + 24x^2y^3 + 100x^2y^2 + 100x^2y + 30x^2 + 24xy^4 \\ & + 72xy^3 + 90xy^2 + 48xy + 9x + 16y^4 + 32y^3 + 24y^2 + 8y + 1. \end{aligned}$$

For example, that on a 4×3 DOMINEERING board there are 90 positions with 1 vertical domino and 2 horizontal dominoes is shown by the term $90xy^2$ and that there is 1 position with 6 vertical dominoes is shown by the term x^6.

2.1. **Play positions.** The polynomials we have found count all legal DOMINEER-ING positions. That is, as mentioned previously, we do not assume alternating play. To enumerate positions which may occur during alternating play, which we call **play positions**, we have to restrict to those terms in $D_{m,n}(x,y)$ where the difference in the powers of x and y is at most 1.

Example 2.3. Using Example 2.2 we can see that the polynomial for the play positions on a 4×3 board is

$$46x^3y^2 + 24x^2y^3 + 100x^2y^2 + 100x^2y + 90xy^2 + 48xy + 9x + 8y + 1.$$

Setting $x = y = 1$ in this polynomial we get the total number of play positions, in this case 426.

The number of play positions on an $n \times n$ board are given in Table 1. We also include the ratio of all positions that are play positions, truncated to 5 decimals. Recall that the number of all positions, being equal to the number of matchings in a grid graph, are given in the OEIS sequence A028420.

The ratio computed in Table 1 gives a sense of the cost of considering all plays from a position, as is standard in combinatorial game theory, compared to considering only plays reachable in alternating play which is common in practice.

We expect that the ratio of play positions decreases as the board size increases because on larger boards there are an increasing number of ways to have a board with the number of dominoes for the two players differing by more than one. The ratios apparently do not follow an arithmetic or geometric sequence.

n	Number of play positions	Ratio of play positions
1	1	1
2	5	0.71428
3	75	0.57251
4	4,632	0.46264
5	1,076,492	0.38299
6	963,182,263	0.32222
7	3,317,770,165,381	0.27774
8	43,809,083,383,524,391	0.24367
9	2,209,112,327,971,366,587,064	0.21689
10	424,273,291,301,040,427,702,718,109	0.19532

TABLE 1. The number and ratio of play positions on small square boards.

We expect the ratios of play positions to general positions to approach 0. While it is interesting to consider how this ratio changes as the board size grows, it is also potentially interesting to observe how this ratio changes during play.

3. NON-RECTANGULAR BOARDS

The techniques we use to find the generating polynomials of rectangular boards can be used to find the generating polynomials of non-rectangular boards as well.

We think of a non-rectangular board as being a rectangular board with missing squares and treat the missing squares as squares forced to be empty. For example, the board on the left in Figure 6 we think of as being contained in a 2×3 board with forced empty squares on the top left and top middle.
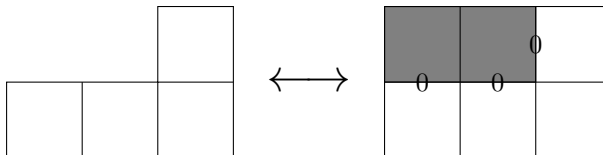

FIGURE 6. The equivalence of a non-rectangular board and a rectangular board with forced empty squares.

We will create a (bar-state) matrix R_i for each row, then multiply these matrices to get the state matrix for the entire mosaic. In the board in Figure 6 the matrix for row 1 is $R_1 = G_{0,1}$. For row 2 we will use $G_{0,3}$, but restrict those columns which have 0 in the first and second position for the label. We denote this as $R_2 = G_{0,3}|^{00\text{-}}$. For restrictions on bottom labels we will use a subscript. Example 3.2 shows this procedure but before that we consider rows with missing squares in the middle as in Example 3.1. If a row contains an missing square in the middle, the Kronecker product of the bar-state matrices of the shorter strips is the bar-state matrix for the row. The Kronecker product of two matrices A and B is

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

If missing squares in a row break it into three or more parts we use the Kronecker product multiple times.

The tilings/play in the left part of the broken row are independent of those in the right part. The state-matrix for the whole row is a block matrix where the blocks are determined by possible labels of the first part of the row, while the order of the second part in each block is the same and determined by the possible top and bottom labels in that part. Thus, taking the Kronecker product of the matrix for the first strip with the matrix for the second strip is exactly what we need here.

Example 3.1. Consider the strip

The matrix for this row is

$$G_{0,1} \otimes G_{0,2}|_{\_0} = \begin{bmatrix} 1 & x \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} y+1 & x & x & x^2 \\ 1 & x & 0 & 0 \end{bmatrix} = \left[\begin{array}{cccc|cccc} y+1 & x & x & x^2 & xy+x & x^2 & x^2 & x^3 \\ 1 & x & 0 & 0 & x & x^2 & 0 & 0 \\ \hline y+1 & x & x & x^2 & 0 & 0 & 0 & 0 \\ 1 & x & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Finally, we multiply the matrices for each row such that the bottom row matrix is the left-most one, while the top row matrix is the right-most one, allowing us to match the top/column label of each row with the bottom/row label of the row above.

Example 3.2. Consider the board
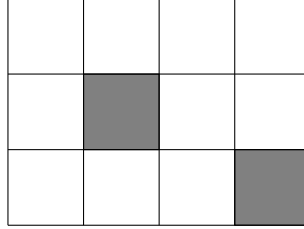
For row 1 we have

$$R_1 = G_{0,4}|_{\_0\_};$$

for row 2 as in the board above

$$R_2 = G_{0,1} \otimes G_{0,2}|_{\_0};$$

and for row 3

$$R_3 = G_{0,3}|^{\text{-0-}}.$$

The matrix for the entire mosaic is then $G = R_3 \times R_2 \times R_1$. To apply the restriction of having only 0s for top and bottom labels we take only the $(1,1)$ entry of the matrix. Thus we get the generating polynomial

$$D = 2x^3y^2 + 8x^3y + 4x^3 + 12x^2y^2 + 22x^2y + 8x^2 + xy^4 + 10xy^3 + 26xy^2$$
$$+ 21xy + 5x + 2y^4 + 9y^3 + 12y^2 + 6y + 1.$$

Using this technique for non-rectangular boards, we are thus able to find the generating polynomial counting positions that can be reached from a partially played board. We will do so in Example 3.4, where we will also use the following simplification for disjunctive sums.

Proposition 3.3. *Suppose G is the game of* DOMINEERING *on the board B_1 and H is* DOMINEERING *on B_2. Then the generating polynomial of $G+H$ is the product of the generating polynomials of G and H.*

Proof. Let the coefficient of $x^i y^j$ in the generating polynomial of G be $a_{i,j}$, of H be $b_{i,j}$, and of $G + H$ be $c_{i,j}$.

Consider a position in $G + H$ with u Left pieces and v Right pieces. Say i Left pieces and j Right pieces of these have been played in G and the rest in H. There are $a_{i,j}b_{u-i,v-j}$ such positions, and thus

$$c_{u,v} = \sum_{\substack{0 \leq i \leq u \\ 0 \leq j \leq v}} a_{i,j}b_{u-i,v-j}.$$

Therefore, the generating polynomial of $G + H$ is the product of those of G and H. □

This proposition again emphasizes that it makes sense to enumerate all positions, not just play positions, due to disjunctive sums allowing a difference in the number of pieces larger than 1.

Example 3.4. The position in Figure 7 occurred partway through Game 1 of the finals of the DOMINEERING tournament at the 1994 workshop "Games of No Chance" at MSRI (see [18, Fig. 3]). We will determine the polynomial for all play positions that would have to be analyzed from this point on.
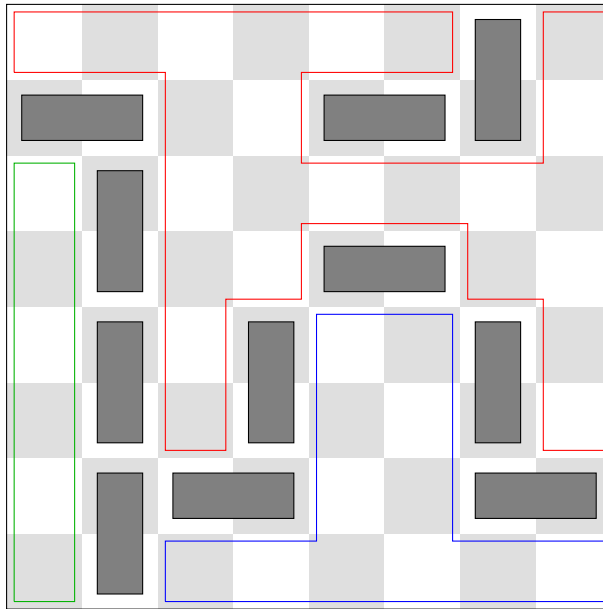

FIGURE 7. A board position from Game 1 of the 1994 tournament finals position with disjunctive components highlighted.

We start by finding the generating polynomials for general positions of each of the three disjunctive sum component as we have done in Example 3.2.

The "bottom" component, bordered in blue, as a mosaic has the matrix

$$G_{0,6}|^{00\text{-}\text{-}00} \times G_{0,2}^3$$

(recall that we multiply the matrixes for each row from the bottom up). Thus the generating polynomial, the $(1,1)$ entry of this matrix, is

$$x^4y^2 + 2x^4y + x^4 + 10x^3y^2 + 16x^3y + 6x^3 + 3x^2y^4 + 24x^2y^3 + 58x^2y^2$$
$$+ 46x^2y + 11x^2 + 8xy^4 + 42xy^3 + 62xy^2 + 34xy + 6x + y^6 + 9y^5$$
$$+ 26y^4 + 35y^3 + 24y^2 + 8y + 1.$$

The "top" component, bordered in red, has the matrix

$$(G_{0,1} \otimes G_{0,1}) \times (G_{0,1} \otimes G_{0,1}) \times (G_{0,2}|_{\text{-}0} \otimes G_{0,2}|_{0\text{-}}) \times (G_{0,6}|^{\text{-}\text{-}000\text{-}}_{\text{-}\text{-}00\text{-}\text{-}})$$
$$\times (G_{0,2} \otimes G_{0,1}) \times (G_{0,6}|00\text{-}\text{-}00 \otimes G_{0,1}).$$

The generating polynomial is

$$x^9y^3 + 3x^9y^2 + 3x^9y + x^9 + 23x^8y^3 + 62x^8y^2 + 55x^8y + 16x^8 + 4x^7y^5$$
$$+ 43x^7y^4 + 291x^7y^3 + 555x^7y^2 + 401x^7y + 98x^7 + 60x^6y^5 + 539x^6y^4$$
$$+ 1874x^6y^3 + 2564x^6y^2 + 1469x^6y + 295x^6 + 4x^5y^7 + 71x^5y^6 + 650x^5y^5$$
$$+ 2908x^5y^4 + 6250x^5y^3 + 6303x^5y^2 + 2885x^5y + 481x^5 + 34x^4y^7$$
$$+ 467x^4y^6 + 2590x^4y^5 + 7406x^4y^4 + 11148x^4y^3 + 8637x^4y^2 + 3218x^4y$$
$$+ 452x^4 + x^3y^9 + 20x^3y^8 + 222x^3y^7 + 1421x^3y^6 + 4991x^3y^5$$
$$+ 9909x^3y^4 + 11109x^3y^3 + 6812x^3y^2 + 2098x^3y + 251x^3 + 3x^2y^9 + 55x^2y^8$$
$$+ 437x^2y^7 + 1909x^2y^6 + 4790x^2y^5 + 7083x^2y^4 + 6172x^2y^3 + 3063x^2y^2$$
$$+ 789x^2y + 81x^2 + 3xy^9 + 50xy^8 + 330xy^7 + 1123xy^6 + 2181xy^5$$
$$+ 2533xy^4 + 1774xy^3 + 726xy^2 + 158xy + 14x + y^9 + 15y^8 + 85y^7 + 237y^6$$
$$+ 373y^5 + 353y^4 + 204y^3 + 70y^2 + 13y + 1.$$

Finally, the component on the left, bordered in green, is simply a 6×1 board, thus having matrix $G_{0,1}^6$ and generating function $D_{1,6}(x,y) = x^3 + 6x^2 + 5x + 1$.

Multiplying all three generating polynomials we get the generating polynomial for the entire position. Restricting to only play positions this is

$$17x^{10}y^{11} + 410x^{10}y^{10} + 7690x^{10}y^9 + 6769x^9y^{10} + 76829x^9y^9 + 532379x^9y^8$$
$$+ 436560x^8y^9 + 2217847x^8y^8 + 7453953x^8y^7 + 6030494x^7y^8$$
$$+ 16049771x^7y^7 + 29420257x^7y^6 + 23698832x^6y^7 + 36239078x^6y^6$$
$$+ 38789964x^6y^5 + 31354701x^5y^6 + 29013063x^5y^5 + 18784523x^5y^4$$
$$+ 15287335x^4y^5 + 8768628x^4y^4 + 3451191x^4y^3 + 2830886x^3y^4$$
$$+ 1004132x^3y^3 + 232953x^3y^2 + 192647x^2y^3 + 40779x^2y^2 + 5086x^2y$$
$$+ 4240xy^2 + 487xy + 25x + 21y + 1.$$

In this polynomial we see from the small degree terms the number of possible paths of the game from this position. The high degree terms give some sense of state at the end of the game provided play continues in such a way that the game lasts as long as possible. In this game Left played first; we see that Left (vertical)

has played 6 times whereas Right has played 5 times and thus it is Right's turn from this position. Right has 21 possible moves. Right would likely have been happy to see terms in the polynomial where the degree of y is greater than the degree of x as these indicate the possibility of Right winning; of note then is the term $17x^{10}y^{11}$ with a relatively small coefficient. The actual winner of this game was Left.

Further, by setting $x = y$ in all matrices, we are also able to count the number of matchings of a subgraph of a grid using this technique.

4. Counting Maximal Domineering Positions

Recall that a maximal DOMINEERING positions is one in which neither player has any available moves. In this section we count maximal DOMINEERING positions. We find the generating function

$$F_{m,n}(x,y) = \sum f(a,b)x^a y^b$$

where the board has size $m \times n$ and $f(a,b)$ is the number of maximal DOMINEERING positions with a Left (vertical) dominoes and b Right (horizontal) dominoes. We use F for the generating polynomial of the maximal positions as these correspond to the *facets* (maximal faces) of the simplicial complex representing the position.

We will use the same technique as for the general DOMINEERING positions of counting bar mosaics, multiplying the matrices, and then restricting to tilings, with only a few small differences. The tiles will be different so that with only using adjacency and boundary conditions we can force the tiling to be equivalent to a maximal position, i.e. no two empty spaces may be adjacent, and in turn a position corresponds to a unique tiling. The possible tiles in this case are given in Figure 8.
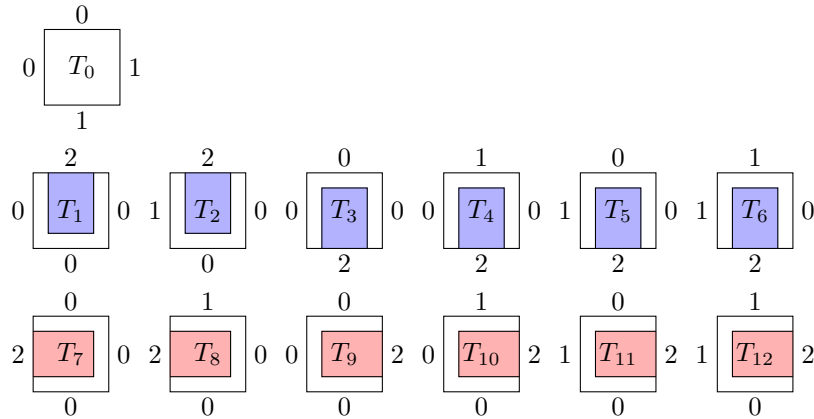

FIGURE 8. The tiles used to count maximal DOMINEERING positions.

For these tiles to form a maximal DOMINEERING position a tiling has to satisfy the following conditions:

(1) *Adjacency condition*: All shared edges of adjacent tiles have the same label.
(2) *Boundary condition*: The left and top boundary edges of the tiling have label 0; the right and bottom boundary edges have label 0 or 1.

In turn, any maximal DOMINEERING position can be uniquely represented using a such a tiling. To decide between the different tiles representing domino halves one simply has to check for empty squares above and to the left. As an example, see the position and equivalent tiling in Figure 9.
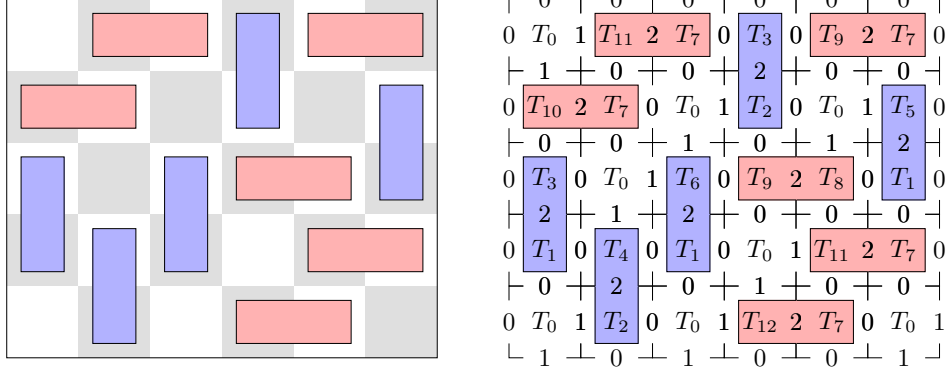

FIGURE 9. A maximal DOMINEERING position on a 5×6 board and its equivalent tiling.

Note that the number of Left (vertical) dominoes is equal to the number of T_1 and T_2 tiles, and the number of Right (horizontal) dominoes is equal to the number of T_7 and T_8 tiles.

In this case we will use M as the label of the bar-state matrices to represent that we are counting *maximal* positions only. Also note that, since the bottom and right edge in a tiling are allowed to have both 0 and 1 as their labels, the generating polynomial will be a sum of entries. A detailed explanation is given in the proof of the following theorem.

Theorem 4.1. *The generating function for the maximal position of an $m \times n$* DOMINEERING *board is*

$$F_{m,n}(x,y) = \sum_{u \in \{0,1\}^n} \left(M_{0,n} + M'_{0,n} \right)^m \left(1 + \sum_{i=1}^n u_i 3^i, 1 \right)$$

where $M_{0,0} = [1]$, $M_{1,0} = [0]$, $M_{2,0} = [0]$, $M'_{0,0} = [0]$, $M'_{1,0} = [1]$, $M'_{2,0} = [0]$, and

$$M_{0,(q+1)} = \begin{bmatrix} M_{2,q} & M_{2,q} & xM_{0,q} \\ M_{1,q} & 0 & 0 \\ M_{0,q} & M_{0,q} & 0 \end{bmatrix} \qquad M'_{0,(q+1)} = \begin{bmatrix} M'_{2,q} & M'_{2,q} & xM'_{0,q} \\ M'_{1,q} & 0 & 0 \\ M'_{0,q} & M'_{0,q} & 0 \end{bmatrix}$$

$$M_{1,(q+1)} = \begin{bmatrix} M_{2,q} & M_{2,q} & xM_{0,q} \\ 0 & 0 & 0 \\ M_{0,q} & M_{0,q} & 0 \end{bmatrix} \qquad M'_{1,(q+1)} = \begin{bmatrix} M'_{2,q} & M'_{2,q} & xM'_{0,q} \\ 0 & 0 & 0 \\ M'_{0,q} & M'_{0,q} & 0 \end{bmatrix}$$

$$M_{2,(q+1)} = \begin{bmatrix} yM_{0,q} & yM_{0,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad M'_{2,(q+1)} = \begin{bmatrix} yM'_{0,q} & yM'_{0,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Proof. The proof will be along the same lines as the proof of Theorem 2.1, thus some of the details will be omitted.

The bar-state matrices count bar mosaics of length q. The matrix $M_{k,q}$ counts those bars with starting label k and ending label 0, while the matrix $M'_{k,q}$ counts those with starting label k and ending label 1.

The labels will be ternary strings, ordered lexicographically. Thus, the strings of length 2 are ordered as

$$00, 01, 02, 10, 11, 12, 20, 21, 22.$$

The column labels of the matrix again correspond to the top label of the bar mosaic, while the row label is the bottom label. Each bar is represented by a monomial $x^a y^b$ where a is the number of tiles 1 and 2 in the bar and b the number of tiles 7 and 8.

Each matrix has 9 blocks, which we label using roman numerals as shown in Figure 10, where we also show the corresponding top and bottom labels for the leftmost tile.

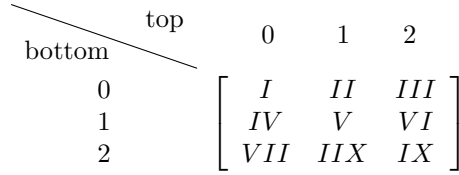| bottom \ top	0	1	2
0	I	II	III
1	IV	V	VI
2	VII	IIX	IX

FIGURE 10. The block construction of the state-matrices for maximal DOMINEERING positions.

In $M_{0,q+1}$, we have a left label of 0. The only tile that meets the conditions for block I (and $M_{0,q+1}$) is T_9. As the right edge label of T_9 is 2, the possible completions of our bar are given by $M_{2,q}$ and so our block I entry is $M_{2,q}$.

The conditions for block II force T_{10} as the starting tile and the entry is again $M_{2,q}$.

The conditions for block III force T_1 — giving the entry $xM_{0,q}$ as T_1 has right label of 0. For block IV, tile T_0 is the only possible starting tile, giving entry $M_{1,q}$. For blocks V and VI we would need a tile with bottom label 1 and top label 1 or 2, respectively, and no such tiles exist so our entry is $\mathbf{0}$.

The conditions for block VII are top label of 0 and bottom label of 2; the only tile that works is T_3 and hence the entry is $M_{0,q}$. The conditions for block VIII are similar but with top label 1 so T_4 is the only tile that works.

The conditions for block IX are top label of 2 and bottom label of 2 which never occurs and thus our entry is $\mathbf{0}$.

In $M_{1,q+1}$, we have a starting (left) label of 1. This forces: T_{11} in block I; T_{12} in block II; T_2 in block III; T_5 in block VII; and T_6 in block VIII. The remaining blocks have no possible tiles. Thus our entries for the matrix are as above.

In $M_{2,q+1}$, we have a starting (left) label of 2. Thus the only possible tiles are T_7 and T_8. In block I the only possible tile is T_7, giving $yM_{0,q}$. In block II the tile is T_8, giving $yM_{0,q}$. All other blocks have no possible tiles, so are $\mathbf{0}$.

Note that the arguments for $M'_{0,q+1}$, $M'_{1,q+1}$, and $M'_{2,q+1}$ are similar as the possible left-most tile is identical and only the ending label changes. Thus we skip these.

Now, restricting to tilings, we will only consider the bar mosaics of length n with left label 0 and right labels 0 or 1, so $M_{0,n} + M'_{0,n}$. As in the general positions case, stacking the bar mosaics corresponds to matrix multiplication. Thus the mosaics with all left labels 0 and right labels 0 or 1 are enumerated in $(M_{0,n} + M'_{0,n})^m$. Finally, we need to restrict to top labels being all 0 and bottom labels 0 or 1. This means we need to sum the entries in the first column which are in rows numbered $1 + \sum_{i=1}^{n} u_i 3^i$ where the u_i are all 0 or 1 (the ternary expansion of the row number -1 has no 2s), giving our result. $\qquad\square$

Example 4.2. Using the recursion we find that

$$M_{0,2} = \left[\begin{array}{ccc:ccc:ccc}
y & y & 0 & y & y & 0 & 0 & 0 & x^2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 \\
\hdashline
0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & x & 0 & 0 & x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0
\end{array}\right].$$

Taking the third power and summing the relevant entries, we get that

$$F_{3,2}(x,y) = 2x^2 y + 2x^2 + y^3.$$

Remark 4.3. As for the general positions case, we can also use these matrices to find the generating polynomial for non-rectangular boards. The technique is the same, thus we do not demonstrate it here.

5. Counting Left and Right Ending Positions

Similar to maximal positions, we are also able to count Left or Right ending positions by choosing the tiles appropriately so that the adjacency condition alone forces such an end. We will just be counting the Right ending positions in this section as it results in bar-state matrices of size $2^q \times 2^q$, while the Left ending positions, using a similar argument, would need bar-state matrices of size $3^q \times 3^q$ (although fewer matrices). Alternatively, the generating polynomial for Left ending positions on an $m \times n$ board can be found using the generating polynomial of Right ending positions on an $n \times m$ board by switching x and y. Further, if we would like to find the number of positions which are Left or Right ending, we add the number of positions of each type and then subtract the number of maximal positions as this is the intersection of Left and Right ending.

For Right ending positions, we may have two empty squares vertically adjacent as Left may potentially still play a domino. But we may not have two empty squares horizontally adjacent as Right cannot have any moves. To achieve this using the adjacency condition, the tiles we use are given in Figure 11.
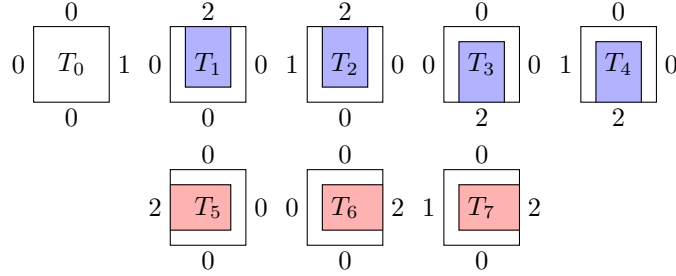
$$
\begin{array}{ccccccccc}
& 0 & & 2 & & 2 & & 0 & & 0 \\
0 & T_0 & 1 \quad 0 & T_1 & 0 \quad 1 & T_2 & 0 \quad 0 & T_3 & 0 \quad 1 & T_4 & 0 \\
& 0 & & 0 & & 0 & & 2 & & 2
\end{array}
$$

$$
\begin{array}{cccccc}
& 0 & & 0 & & 0 \\
2 & T_5 & 0 \quad 0 & T_6 & 2 \quad 1 & T_7 & 2 \\
& 0 & & 0 & & 0
\end{array}
$$

FIGURE 11. The tiles used to count Right ending Domineering positions.

The two conditions for a tiling to be a Domineering Right end are:

(1) *Adjacency condition*: All shared edges of adjacent tiles have the same label.
(2) *Boundary condition*: The left, top, and bottom edges of the tiling have labels 0; the right edge has labels 0 or 1.

Figure 12 contains an example of a position and the corresponding tiling.

We use RE for naming the bar-state matrices. The matrix $RE_{k,q}$ counts those bar mosaics of length q with starting label k and ending label 0, while the matrix $RE'_{k,q}$ counts those with starting label k and ending label 1.

Theorem 5.1. *The generating polynomial of* Domineering *Right ending positions on an $m \times n$ board is the $(1,1)$ entry of $(RE_{0,n} + RE'_{0,n})^m$ where $RE_{0,0} = \begin{bmatrix} 1 \end{bmatrix}$, $RE_{1,0} = \begin{bmatrix} 0 \end{bmatrix}$, $RE_{2,0} = \begin{bmatrix} 0 \end{bmatrix}$, $RE'_{0,0} = \begin{bmatrix} 0 \end{bmatrix}$, $RE'_{1,0} = \begin{bmatrix} 1 \end{bmatrix}$, $RE'_{2,0} = \begin{bmatrix} 0 \end{bmatrix}$, and*

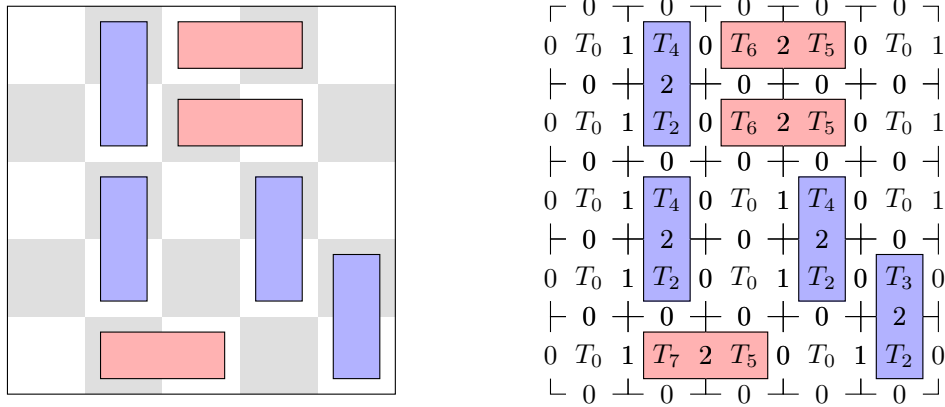FIGURE 12. A Right ending Domineering position on a 5×5 board and its equivalent tiling.

$$RE_{0,q+1} = \begin{bmatrix} \begin{matrix} RE_{1,q} \\ +RE_{2,q} \end{matrix} & xRE_{0,q} \\ RE_{0,q} & \mathbf{0} \end{bmatrix} \qquad RE'_{0,q+1} = \begin{bmatrix} \begin{matrix} RE'_{1,q} \\ +RE'_{2,q} \end{matrix} & xRE'_{0,q} \\ RE'_{0,q} & \mathbf{0} \end{bmatrix}$$

$$RE_{1,q+1} = \begin{bmatrix} RE_{2,q} & xRE_{0,q} \\ RE_{0,q} & \mathbf{0} \end{bmatrix} \qquad RE'_{1,q+1} = \begin{bmatrix} RE'_{2,q} & xRE'_{0,q} \\ RE'_{0,q} & \mathbf{0} \end{bmatrix}$$

$$RE_{2,q+1} = \begin{bmatrix} yRE_{0,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad RE'_{2,q+1} = \begin{bmatrix} yRE'_{0,q} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

We give no proof here as the argument is simply a combination of the arguments for the general and maximal positions.

In Table 2 we give the total number of Right ending positions on an $m \times n$ board (the table for Left ends is transpose of this).

n / m	1	2	3	4	5
1	1	1	2	2	3
2	2	4	11	25	61
3	3	9	48	172	731
4	5	25	227	1,427	10,388
5	8	64	1,054	11,134	140,555
6	13	169	4,921	88,733	1,932,067
7	21	441	22,944	701,926	26,425,981
8	34	1,156	107,017	5,567,467	362,036,629

n / m	6	7	8
1	4	5	7
2	146	351	844
3	2,976	12,039	49,401
4	72,751	510,779	3,604,887
5	1,693,116	20,414,525	248,119,648
6	40,008,789	831,347,033	17,385,222,733
7	941,088,936	33,656,587,715	1,211,649,519,869
8	22,168,654,178	1,365,206,879,940	84,588,476,099,284

TABLE 2. The number of Right ending DOMINEERING positions on an $m \times n$ board.

The number of Right ending positions on a $1 \times n$ board is recursively given by $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_n = a_{n-2} + a_{n-3}$ (the Padovan sequence shifted, OEIS sequence A000931). Consider the leftmost square: If it contains a domino, then the strip to the right of the domino can contain any Right ending position of length $n-2$. If it is an empty square, then a domino has to be immediately to the right of it, and the remaining strip of length $n-3$ can contain any Right ending position.

In the positions of shape $m \times 1$ Right cannot play, thus all positions are Right ending. The sequence of these numbers is the Fibonacci sequence (OEIS sequence A000045) as every position is a combination of empty squares and Left dominoes.

It appears that the number of Right ending positions on an $m \times 2$ board is given by the squared Fibonacci numbers (OEIS sequence A007598) and the number of positions on an $m \times 3$ is given by the recursion $a_m = 4a_{m-1} + 4a_{m-2} - 4a_{m-3} - a_{m-4}$ (OEIS sequence A054894). Other sequences for the number of Right ending positions do not appear in the OEIS at this point.

6. Further Work

We suggest three broad avenues for continuing the work in this paper. One is to explore our original motivation and connect the enumeration of Domineering to questions of the algebraic structure of placement game positions. Secondly, we expect the methods demonstrated in this paper to be useful for other placement games played on grid-like boards (such as Cartesian products of cycles and paths). Lastly, there is the possibility of counting positions to analyze particular game situations either in theory or practice to develop better play strategies.

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