THE γ-COEFFICIENTS OF BRANDEN'S (p,q)-EULERIAN POLYNOMIALS AND ANDRÉ PERMUTATIONS

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ABSTRACT. P. Brändén (European J. Combin. 29 (2008), no. 2, 514–531) studied a (p,q)-analogue of the classical Eulerian polynomials $A_n(p,q,t)$ and conjectured that its γ-coefficient $a_{n,k}(p,q)$ is divisible by $(p+q)^k$. The aim of this paper is to show that the quotient $d_{n,k}(p,q) := a_{n,k}(p,q)/(p+q)^k$ is the enumerative polynomial of André permutations of the second kind of size n with k descents. In particular, our result leads to a combinatorial model for G.-N. Han's recent q-Euler numbers (arXiv:1906.00103v1).

1. Introduction

André permutations are variants of alternating permutations, and are enumerated by the famous Euler numbers E_n defined by the generating function, see [FS73, FS76, FH15],

$$\tan(x) + \sec(x) = \sum_{n\geq 0} E_n \frac{x^n}{n!}. \tag{1}$$

The first few values of E_n are 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, ... (sequence A000111 in the OEIS).

Let $\mathfrak{S}_n$ be the set of permutations of $[n] := \{1,\dots,n\}$. A permutation $\sigma : i \to \sigma_i$ of $[n]$ will be identified with the *word* $\sigma = \sigma_1 \dots \sigma_n$. The entry $i \in [n]$ is called a *descent* (resp. *ascent*) of σ if $i < n$ and $\sigma_i > \sigma_{i+1}$ (resp. $\sigma_i < \sigma_{i+1}$), and the number of descents (resp. ascents) in σ is denoted by $\operatorname{des}\sigma$ (resp. $\operatorname{asc}\sigma$). A *double descent* (resp. *valley*) of σ is a triple $(\sigma_i, \sigma_{i+1}, \sigma_{i+2})$ with $\sigma_i > \sigma_{i+1} > \sigma_{i+2}$ (resp. $\sigma_i > \sigma_{i+1} < \sigma_{i+2}$) for $1 \leq i \leq n-2$. Given a permutation σ, we denote by $\sigma_{[k]}$ the *subword* of σ consisting of $1,\dots,k$ in the order they appear in σ. A permutation σ of $[n]$ is called *André permutation (of the second kind)* if $\sigma_{[k]}$ has no double descents and ends with an ascent for all $1 \leq k \leq n$. For example, the permutation $\sigma = 43512$ is not André since the subword 4312 of σ contains a double descent $(4,3,1)$, while the permutation $\tau = 31245$ is André. The set of André permutations of $[n]$ is denoted by $\mathcal{A}_n$. For instance, the set $\mathcal{A}_4$ consists of five permutations $1234, 1423, 3124, 3412$, and 4123.

The *Eulerian polynomials* $A_n(t)$ and *André polynomials* $D_n(t)$ are defined by

$$A_n(t) = \sum_{\sigma\in\mathfrak{S}_n} t^{\operatorname{des}\sigma} \quad \text{and} \quad D_n(t) = \sum_{\sigma\in\mathcal{A}_n} t^{\operatorname{des}\sigma}.$$

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The first few values of $A_n(t)$ and $D_n(t)$ are

$$A_1(t)=1,\quad A_2(t)=1+t,\quad A_3(t)=1+4t+t^2,$$
$$A_4(t)=1+11t+11t^2+t^3,\quad A_5(t)=1+26t+66t^2+26t^3+t^4;$$

and

$$D_1(t)=D_2(t)=1,\quad D_3(t)=1+t,\quad D_4(t)=1+4t,\quad D_5(t)=1+11t+4t^2,$$
$$D_6(t)=1+26t+34t^2,\quad D_7(t)=1+57t+180t^2+34t^3.$$

The two kinds of polynomials are connected through the so-called γ-expansion:

$$A_n(t)=\sum_{k=0}^{(n-1)/2} d_{n,k}(2t)^k(t+1)^{n-1-2k}, \tag{2}$$

where $d_{n,k}$ is the number of André permutations of $[n]$ with k descents, namely, the coefficient of t^k in $D_n(t)$, see [FS73, FH01, CS11].

For $\sigma=\sigma_1\dots\sigma_n\in\mathfrak{S}_n$, the statistic $(31\text{-}2)\,\sigma$ (resp. $(13\text{-}2)\,\sigma$) is the number of pairs (i,j) such that $2\le i<j\le n$ and $\sigma_{i-1}>\sigma_j>\sigma_i$ (resp. $\sigma_{i-1}<\sigma_j<\sigma_i$). Similarly, the statistic $(2\text{-}13)\,\sigma$ (resp. $(2\text{-}31)\,\sigma$) is the number of pairs (i,j) such that $1\le i<j\le n-1$ and $\sigma_{j+1}>\sigma_i>\sigma_j$ (resp. $\sigma_{j+1}<\sigma_i<\sigma_j$). In [Bra08] Brändén considered the following (p,q)-analog of Eulerian polynomials

$$A_n(p,q,t)=\sum_{\sigma\in\mathfrak{S}_n} p^{(2\text{-}31)\,\sigma}q^{(13\text{-}2)\,\sigma}t^{\operatorname{des}\sigma}, \tag{3}$$

which can be recasted as (cf. [SZ12, (9)])

$$A_n(p,q,t)=\sum_{\sigma\in\mathfrak{S}_n} p^{(2\text{-}13)\,\sigma}q^{(31\text{-}2)\,\sigma}t^{\operatorname{des}\sigma}. \tag{4}$$

Brändén showed that there are polynomials $a_{n,k}(p,q)\in\mathbb{N}[p,q]$ satisfy

$$A_n(p,q,t)=\sum_{k=0}^{\lfloor (n-1)/2\rfloor} a_{n,k}(p,q)t^k(1+t)^{n-1-2k}, \tag{5}$$

and further conjectured the divisibility of $a_{n,k}(p,q)$ by $(p+q)^k$ for $0\le k\le \lfloor (n-1)/2\rfloor$, see [Bra08, Conjecture 10.3]. This conjecture was proved by Shin and Zeng [SZ12] using the combinatorial theory (cf. [CSZ97, Fl80, Vi83]) of continued fractions of Jacobi type, i.e., a formal power series defined by

$$\sum_{n=0}^{\infty} a_n t^n=\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\cdots}}}.$$

When $\gamma_n = 0$ the corresponding continued fractions are called Stieltjes type:

$$\sum_{n=0}^{\infty} a_n t^n = \cfrac{1}{1 - \cfrac{\alpha_1 t}{1 - \cfrac{\alpha_2 t}{1 - \cdots}}}.$$

Actually, Shin and Zeng [SZ12] proved the stronger result that there are polynomials $d_{n,k}(p,q) \in \mathbb{N}[p,q]$ such that

$$d_{n,k}(p,q) = \frac{a_{n,k}(p,q)}{(p+q)^k} \quad \text{for} \quad 0 \le k \le \lfloor (n-1)/2 \rfloor. \tag{6}$$

In view of (2) and (6) it is natural to seek a combinatorial interpretation for $d_{n,k}(p,q)$. The aim of this paper is to provide such an interpretation within the model of André permutations. Let us define the (p,q)-*André polynomials* $D_n(p,q,t)$ by

$$D_n(p,q,t) := \sum d_{n,k}(p,q) t^k \qquad (0 \le k \le \lfloor (n-1)/2 \rfloor). \tag{7}$$

The first few values of $D_n(p,q,t)$ are $D_1(p,q) = D_2(p,q) = 1$, and

$$D_3(p,q,t) = 1 + t, \quad D_4(p,q,t) = 1 + (p+q+2)t,$$
$$D_5(p,q,t) = 1 + \left((p+q)^2 + 2(p+q) + 3\right) t + \left(p^2 + pq + q^2 + 1\right) t^2.$$

For $n \in \mathbb{N}$ we define the (p,q)-analogue of the integer n by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q} = \sum_{i+j=n-1} p^i q^j,$$

and the (p,q)-analogue of the binomial coefficient $\binom{n}{k}$ by

$$\binom{n}{k}_{p,q} = \frac{[n]_{p,q} \ldots [n-k+1]_{p,q}}{[1]_{p,q} \ldots [k]_{p,q}} \qquad (0 \le k \le n).$$

The following three theorems are our main results.

Theorem 1. *The following formula holds*

$$\sum_{n=0}^{\infty} D_{n+1}(p,q,t) x^n = \cfrac{1}{1 - x - \cfrac{\binom{2}{2}_{p,q} t x^2}{1 - [2]_{p,q} x - \cfrac{\binom{3}{2}_{p,q} t x^2}{1 - [3]_{p,q} x - \cfrac{\binom{4}{2}_{p,q} t x^2}{1 - [4]_{p,q} x - \cdots}}}}, \tag{8}$$

where $\gamma_n = [n+1]_{p,q}$ *and* $\beta_n = \binom{n+1}{2}_{p,q}$.

Theorem 2. *For* $0 \le k \le \lfloor (n-1)/2 \rfloor$ *we have*

$$D_n(p,q,t) = \sum_{\sigma \in \mathcal{A}_n} p^{(2-13)\sigma} q^{(31-2)\sigma - \operatorname{des}\sigma} t^{\operatorname{des}\sigma}.$$

Example 1. We list the André permutations in $\mathcal{A}_3$ and $\mathcal{A}_4$ with their number of patterns $2-13$ and $31-2$. The valleys are in boldface.

$n=3$

$\sigma \in \mathcal{A}_3$	$(2-13)\sigma$	$(31-2)\sigma$
123	0	0
3**1**2	0	1

$n=4$

$\sigma \in \mathcal{A}_4$	$(2-13)\sigma$	$(31-2)\sigma$
1234	0	0
14**2**3	0	1
3**1**24	1	1
34**1**2	0	1
4**1**23	0	2

As the cardinality of $\mathcal{A}_n$ is the Euler number E_n, we have

$$D_n(1,1,1)=E_n \quad \text{for} \quad n\geq 1,$$

and derive from (8) the following classical formula, see [St90, (14)] and [So18, Note 9]),

$$\sum_{n=0}^{\infty} E_{n+1}x^n = \cfrac{1}{1-x-\cfrac{x^2}{1-2x-\cfrac{3x^2}{1-3x-\cfrac{6x^2}{1-\cdots}}}} \tag{9}$$

with $\gamma_n = n+1$ and $\beta_n = n(n+1)/2$. Recently Han [Han19] defined a q-analogue of Euler numbers $E_n(q)$ using a q-version of (9):

$$\sum_{n=0}^{\infty} E_{n+1}(q)x^n = \cfrac{1}{1-x-\cfrac{x^2}{1-[2]_q x-\cfrac{[3]_q x^2}{1-[3]_q x-\cfrac{[6]_q x^2}{1-\cdots}}}} \tag{10}$$

with $\gamma_n = [n+1]_q$ and $\beta_n = [n]_q[n+1]_q/[2]_q$.

At the end of his paper Han raised the question of finding a combinatorial model for the new q-Euler numbers. Comparing (8) and (10) we see that

$$E_n(q) = D_n(1,q,1) = D_n(q,1,1) \qquad (n\geq 1),$$

and derive from Theorem 2 the following interpretations for Han's q-Euler numbers.

Corollary 3. *We have*

$$E_n(q) = \sum_{\sigma\in\mathcal{A}_n} q^{(2-13)\sigma} = \sum_{\sigma\in\mathcal{A}_n} q^{(31-2)\sigma-\operatorname{des}\sigma}. \tag{11}$$

We shall give a triple sum formula for $D_n(1,q,t)$ in Theorem 13 and also prove the following explicit formula when $q=-1$.

Theorem 4. *For $n \geq 1$ we have*

$$D_n(1,-1,t) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} k! t^k. \tag{12}$$

Remark 1. If $t=1$, in view of (10), the above result is equivalent to

$$E_n(-1) = \sum_{k=0}^{n-1} \binom{n-1-k}{k} k!, \tag{13}$$

which was posted by P. Barry, see A122852 in OEIS [OEIS]. Han [Han19, Theorem 7.1] gave a non-trivial (sic) proof of (13) by showing that both sides of (13) satisfy the same recurrence relation, which had been conjectured by R. J. Mathar. Our proof of (12) is insightful for the summation formula in (12) and combinatorial in nature.

2. Proof of Theorem 1

For $\sigma \in \mathfrak{S}_n$, let $\sigma_0 = \sigma_{n+1} = 0$. Then any index $i \in [n]$ can be classified according to one of the four cases:

- a *peak* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i > \sigma_{i+1}$;
- a *valley* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double ascent* if $\sigma_{i-1} < \sigma_i$ and $\sigma_i < \sigma_{i+1}$;
- a *double descent* if $\sigma_{i-1} > \sigma_i$ and $\sigma_i > \sigma_{i+1}$.

Let $\operatorname{peak}\sigma$ (resp. $\operatorname{valley}\sigma$, $\operatorname{da}\sigma$, $\operatorname{dd}\sigma$) denote the number of peaks (resp. valleys, double ascents, double descents) in σ. Note that $\operatorname{peak}\sigma = \operatorname{valley}\sigma + 1$. Generalizing the (p,q)-Eulerian polynomials $A_n(p,q,t)$ Shin and Zeng [SZ12] introduced more general Eulerian polynomials $A_n(p,q,t,u,v,w)$ defined by

$$A_n(p,q,t,u,v,w) := \sum_{\sigma \in \mathfrak{S}_n} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma} t^{\operatorname{des}\sigma} u^{\operatorname{da}\sigma} v^{\operatorname{dd}\sigma} w^{\operatorname{valley}\sigma}. \tag{14}$$

We recall the following result in [SZ12, Theorem 2].

Lemma 5 (Shin-Zeng). *The polynomials $a_{n,k}(p,q)$ in (5) have the following interpretation*

$$a_{n,k}(p,q) = \sum_{\sigma \in DD_{n,k}} p^{(2\text{-}13)\,\sigma} q^{(31\text{-}2)\,\sigma} \tag{15}$$

where $DD_{n,k} := \{\sigma \in \mathfrak{S}_n : \operatorname{valley}\sigma = k,\ \ \operatorname{dd}\sigma = 0\}$, and

$$A_n(p,q,t,u,v,w) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k}(p,q)(tw)^k (u+vt)^{n-1-2k}. \tag{16}$$

Moreover, for all $0 \leq k \leq \lfloor (n-1)/2 \rfloor$, the quotient $a_{n,k}(p,q)/(p+q)^k$ is a polynomial in p and q with integral coefficients.

Proof of Theorem 1. Specializing (t,u,v,w) with $(p+q,0,1,w)$ in (16), by (6) and (7) we see that

$$\sum_{k=0}^{\lfloor (n-1)/2\rfloor} d_{n,k}(p,q)w^k=\frac{A_n(p,q,p+q,0,1,w)}{(p+q)^{n-1}}.$$

On the other hand, it is known [SZ12, Eq. (28)] that

$$\sum_{n\geq 1}A_n(p,q,t,u,v,w)x^{n-1}= \frac{1}{1-(u+tv)[1]_{p,q}x-\cfrac{[1]_{p,q}[2]_{p,q}twx^2}{1-(u+tv)[2]_{p,q}x-\cfrac{[2]_{p,q}[3]_{p,q}twx^2}{\cdots}}} \tag{17}$$

with $\gamma_n=(u+tv)[n+1]_{p,q}$ and $\beta_n=[n]_{p,q}[n+1]_{p,q}tw$.

Formula (8) follows from (17) by the substitution $(t,u,v,w)\to(p+q,0,1,w)$ and $x\to x/p+q$. □

3. Proof of Theorem 2

We need some definitions from [FS73] and [FS74]. Recall that a permutation w of a finite subset $\{a_1<a_2<\cdots<a_n\}$ of $\mathbb{N}$ is a word $w=x_1\dots x_n$. The word u obtained by juxtaposing two words v and w in this order is written $u=vw$. The word v (resp. w) is the left (resp. right) factor of u. More generally, a factorization of length q ($q>0$) of a word w is any sequence $(w_1,w_2,...,w_q)$ of words (some of them possibly empty) such that the juxtaposition product $w_1w_2...w_q$ is equal to w.

Lemma 6 ([FS74])**.** *Let $w=x_1x_2...x_n$ $(n>0)$ be a permutation and x be one of the letters x_i $(1<i<n)$. Then w has a unique factorization (w_1,w_2,x,w_4,w_5) of length 5, called its x-factorization, which is characterized by the three properties*

(i) *w_1 is empty or its last letter is less than x;*
(ii) *w_2 (resp. w_4) is empty or all its letters are greater than x;*
(iii) *w_5 is empty or its first letter is less than x.*

It is also known that André permutations can be defined using x-factorization.

Definition 7 ([FS73])**.** A permutation $\sigma\in\mathfrak{S}_n$ is an André permutation of the second kind if it is empty or satisfies the following:
(i) σ has no double descents,
(ii) $n-1$ is not a descent, i.e. $\sigma_{n-1}<\sigma_n$,
(iii) If $i\in\{2,\cdots,n\}$ is a valley of σ with σ_i-factorization $\sigma=w_1w_2\sigma_iw_4w_5$, then $\min(w_2)>\min(w_4)$, i.e.,the minimum letter of w_2 is larger than the minimum letter of w_4.

Let $\mathcal{A}_{n,k}$ be the set of André permutations of $[n]$ with k descents. For example, the elements of $\mathcal{A}_{5,2}$ are

$$31524,\ 41523,\ 51423,\ 53412.$$

Lemma 8. *For $0 \le k \le (n-1)/2$, we have*

$$d_{n,k}(p,q) = \sum_{\sigma \in \mathcal{A}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma - k}.$$

Proof. We shall prove the equivalent identity (cf. (6))

$$(p+q)^k \, d_{n,k}(p,q) = a_{n,k}(p,q).$$

Let $\mathcal{P}[k]$ be the power set of $[k]$, i.e., the set of all subsets of $[k]$. Since

$$\begin{aligned}(p+q)^k d_{n,k}(p,q) &= \left(\sum_{s \in \mathcal{P}([k])} p^{|s|} q^{k-|s|} \right) \sum_{\sigma \in \mathcal{A}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma - k} \\ &= \sum_{(s,\sigma) \in \mathcal{P}[k] \times \mathcal{A}_{n,k}} p^{|s| + (2-13)\sigma} q^{-|s| + (31-2)\sigma},\end{aligned}$$

the identity to be proved is equivalent to

$$\sum_{(s,\sigma) \in \mathcal{P}[k] \times \mathcal{A}_{n,k}} p^{|s| + (2-13)\sigma} q^{-|s| + (31-2)\sigma} = \sum_{\sigma \in \mathrm{DD}_{n,k}} p^{(2-13)\sigma} q^{(31-2)\sigma}.$$

Hence, it is sufficient to establish a bijection $\phi : P[k] \times \mathcal{A}_{n,k} \to \mathrm{DD}_{n,k}$ such that if $(s,\sigma) \in P[k] \times \mathcal{A}_{n,k}$ and $\phi(s,\sigma) = \tau$ then

$$\begin{aligned} |s| + (2-13)\sigma &= (2-13)\tau \\ -|s| + (31-2)\sigma &= (31-2)\tau. \end{aligned} \tag{18}$$

By definition, if σ_i is a valley of $\sigma \in \mathcal{A}_{n,k}$ with σ_i-factorization $(w_1, w_2, \sigma_i, w_4, w_5)$, then $\min(w_2) > \min(w_4)$, so $(\sigma_{i-1}, \sigma_i, \min(w_4))$ forms a $(31-2)$ pattern, this implies that $(31-2)\sigma \ge k$. Moreover, there are only three types for $y := \min(w_4)$: peak, double ascent or valley. Recall that $\sigma_0 = \sigma_{n+1} = 0$.

We first define the action ϕ on the valley σ_i according to the type of y as follows:

- (i) If y is a peak, then $y = \sigma_{i+1}$ because, otherwise there should be a letter $z < y$ in w_4 between σ_i and y. Then we exchange w_2 and the first letter of w_4, i.e., the new permutation is $\phi(\sigma) = w_1 y \sigma_i w_2 \tilde{w}_4 w_5$ where $w_4 = y\tilde{w}_4$. This transformation eliminates the $(31-2)$ pattern $(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$, and creates a new $(2-13)$ pattern $(y, \sigma_i, w_{2,1})$, where $w_{2,1}$ is the first letter of w_2.
- (ii) If y is a double ascent, then $y = \sigma_{i+1}$ for the same reason as for (i). Let $j < i$ be the largest index such that $\sigma_j < y < \sigma_{j+1}$. Then we move y to the slot between σ_j and σ_{j+1}. This transformation eliminates the $(31-2)$ pattern $(\sigma_{i-1}, \sigma_i, \sigma_{i+1})$ and creates a new $(2-13)$ pattern $(\sigma_{i+1}, \sigma_i, \sigma_{i+2})$.
- (iii) If y is a valley, then we just exchange σ_i and y, and this transformation eliminates the $(31-2)$ pattern $(\sigma_{i-1}, \sigma_i, y)$ and creates a new $(2-13)$ pattern (y, σ_i, y'), where y' is the letter next to the right of y in σ.

We now extend the above action ϕ to each subset s of the set of the valleys of σ, that we identify with $[k]$ as follows: if $(i_1, \ldots, i_k)$ is the sequence of valleys of σ ordered in increasing order, then any subset s of $[k]$ is identified with the set $\{i_j\}_{j \in s}$. We first deal with the elements of cases (i) and (ii), and then the case (iii); and when we deal with the elements of case (iii) we will apply ϕ to the elements in decreasing order, i.e., from the largest element to the smallest element. As the transformation can be reversed, we have constructed a bijection ϕ satisfying (18). □

For the reader's convenience, we run the bijection ϕ from $\mathcal{P}[2]\times\mathcal{A}_{5,2}$ to $\mathrm{DD}_{5,2}$ in Figure 1.

$\sigma \in \mathcal{A}_{5,2}$	$(2-13)$	$(31-2)$	$s \in \mathcal{P}[2]$	$\tau \in \mathrm{DD}_{5,2}$	$(2-13)$	$(31-2)$
31524	2	2	$\emptyset$	3**1**5**2**4	2	2
3**1**524	2	2	$\{1\}$	3**2**5**1**4	3	1
315**2**4	2	2	$\{2\}$	3**1**4**2**5	3	1
3**1**5**2**4	2	2	$\{1,2\}$	3**2**4**1**5	4	0
41523	1	3	$\emptyset$	4**1**5**2**3	1	3
4**1**523	1	3	$\{1\}$	4**2**5**1**3	2	2
415**2**3	1	3	$\{2\}$	4**1**3**2**5	2	2
4**1**5**2**3	1	3	$\{1,2\}$	4**2**3**1**5	3	1
51423	0	4	$\emptyset$	5**1**4**2**3	0	4
5**1**423	0	4	$\{1\}$	5**2**4**1**3	1	3
514**2**3	0	4	$\{2\}$	5**1**3**2**4	1	3
5**1**4**2**3	0	4	$\{1,2\}$	5**2**3**1**4	2	2
53412	0	2	$\emptyset$	5**3**4**1**2	0	2
534**1**2	0	2	$\{1\}$	**21**5**3**4	1	1
5**3**412	0	2	$\{3\}$	4**3**5**1**2	1	1
5**3**4**1**2	0	2	$\{1,3\}$	**21**4**3**5	2	0

FIGURE 1. The bijection $\phi : (s,\sigma) \mapsto \tau$ from $\mathcal{P}[2] \times \mathcal{A}_{5,2}$ to $\mathrm{DD}_{5,2}$

For example, if $\sigma = 31524 \in \mathcal{A}_{5,2}$ and $s = \{1,2\}$, then

- for the valley 1, the corresponding $y_1 := min(w_4)$ is 2, which is a valley,
- for the valley 2, the corresponding $y_2 := min(w_4)$ is 4, which is a peak.

So, we should first deal with the valley 2, the 2-factorization is $(w_1,\, w_2,\, x,\, w_4,\, w_5) = (31, 5,\, 2,\, 4,\, \emptyset)$ according to case (i) of ϕ, we just exchange 4 and 5, and get 31425; then we apply ϕ to the valley 1 in 31425, the 1-factorization is $(w_1,\, w_2,\, x,\, w_4,\, w_5) = (\emptyset, 3,\, 1,\, 425,\, \emptyset)$, which is case (iii), we just exchange 1 and 2, and get $\phi(s,\sigma) = 32415$.

4. A FORMULA FOR $D_n(1,-1,t)$ AND $D_n(1,q,t)$

A *Motzkin path* of length n is a sequence of points $\gamma := (\gamma_0, \ldots, \gamma_n)$ in the integer plane $\mathbb{Z} \times \mathbb{Z}$ such that

- $\gamma_0 = (0,0)$ and $\gamma_n = (n,0)$,
- $\gamma_i - \gamma_{i-1} \in \{(1,0),(1,1),(1,-1)\}$,
- $\gamma_i := (x_i, y_i) \in \mathbb{N} \times \mathbb{N}$ for $i = 0, \ldots, n$.

In other words, a Motzkin path of length n is a lattice path starting at $(0,0)$, ending at $(n,0)$, and never going below the x-axis, consisting of up steps $U=(1,1)$, horizontal steps $H=(1,0)$, and down steps $D=(1,-1)$. Let $\mathcal{P}_n$ be the set of Motzkin paths of length n. Clearly we can identify Motzkin paths of length n with words w on $\{U,H,D\}$ of length n such that all prefixes of w contain no more $D's$ than $U's$ and the number of $U's$ equals the number of $D's$. The height of a step (γ_i,γ_{i+1}) is the coordinate of the starting point γ_i. Given a Motzkin path $p\in\mathcal{P}_n$ and two sequences (γ_i) and (β_i) of some commutative ring, we weight each up step by 1, and each horizontal step (resp. down step) at height i by γ_i (resp. β_i) and define the weight $w(p)$ of p by the product of the weights of all its steps. The following result of Flajolet [Fl80] is our starting point.

Lemma 9 (Flajolet)**.** *We have*

$$\sum_{n=0}^{\infty}\left(\sum_{p\in\mathcal{P}_n} w(p)\right)t^n=\cfrac{1}{1-\gamma_0 t-\cfrac{\beta_1 t^2}{1-\gamma_1 t-\cfrac{\beta_2 t^2}{1-\gamma_2 t-\cdots}}}.$$

A Motzkin path without horizontal steps is called a *Dyck path*, and a Motzkin path without horizontal steps at odd height is called an *André path*.

Lemma 10. *Let $\gamma_i=0$ $(i\geq 0)$ and $\beta_i=\lfloor\frac{i+1}{2}\rfloor$ $(i\geq 1)$. Then*

$$n!=\sum_{p\in\mathcal{P}_n} w(p).$$

In other words, the polynomial $n!$ is the generating polynomial of Dyck paths of length $2n$.

Proof. Recall the following formula of Euler:

$$\sum_{n\geq 0} n!x^n=\cfrac{1}{1-\cfrac{1x}{1-\cfrac{1x}{1-\cfrac{2x}{1-\cfrac{2x}{\cdots}}}}} \tag{19}$$

with $\alpha_n=\lfloor(n+1)/2\rfloor$. The result then follows from Lemma 9. □

Remark 2. We can give a bijective proof of Euler's formula (19) using the method in [CSZ97, Fl80, Vi83].

Lemma 11. *Let $\gamma_{2i}=1$, $\gamma_{2i+1}=0$ $(i\geq 0)$ and $\beta_k=\lfloor\frac{k+1}{2}\rfloor t$ $(i\geq 1)$. Then*

$$D_{n+1}(1,-1,t)=\sum_{p\in\mathcal{P}_n} w(p).$$

In other words, the polynomial $D_{n+1}(1,-1,t)$ is the generating polynomial of André paths of length n.

Proof. When $(p,q)=(1,-1)$ formula(8) reduces to

$$\sum_{n=0}^{\infty} D_{n+1}(1,-1,t)x^n = \cfrac{1}{1-x-\cfrac{tx^2}{1-\cfrac{tx^2}{1-x-\cfrac{2tx^2}{1-\cfrac{2tx^2}{1-x-\cdots}}}}} \tag{20}$$

with coefficients $\gamma_{2n}=1$, $\gamma_{2n+1}=0$ and $\beta_n=\lfloor\frac{n+1}{2}\rfloor t$. The result follows from Lemma 9. □

We denote by $\mathcal{AP}_{n,k}$ the set of André paths of length n with k horizontal steps, and $\mathcal{D}_n$ the set of Dyck paths of half length n. Let

$$\mathcal{Y}_{n,k} := \{(y_1,\dots,y_{k+1})\in\mathbb{N}^{k+1} : y_1+\dots+y_{k+1}=n-2k\}.$$

Lemma 12. *For $0\le k\le\lfloor n/2\rfloor$, there is an explicit bijection $\psi:\mathcal{AP}_{n,n-2k}\to\mathcal{Y}_{n,k}\times\mathcal{D}_k$ such that if $\psi(u)=(y,p)$ with for $u\in\mathcal{AP}_{n,n-2k}$ and $(y,p)\in\mathcal{Y}_{n,k}\times\mathcal{D}_k$ then $w(u)=w(p)$, where the weight is associated to the sequences (α_i) and (β_i) with $\alpha_{2i}=1$, $\alpha_{2i+1}=0$ $(i\ge 0)$, and $\beta_k=\lfloor\frac{k+1}{2}\rfloor t$ $(i\ge 1)$.*

Proof. Since an André path (word) on $\{U,D,H\}$ has only level-steps at even height and starts from height 0, so the subword between two consecutive horizontal steps must be of even length and is a word on $\{UU,DD,UD,DU\}$. Thus, any André word $u\in\mathcal{AP}_{n,n-2k}$ can be written in a unique way as follows:

$$u=H^{y_1}w_1H^{y_2}w_2\dots w_kH^{y_{k+1}}\quad\text{with}\quad w_i\in\{UU,DD,UD,DU\},$$

where $(y_1,\dots,y_{k+1})\in\mathcal{Y}_{n,k}$ and $p=w_1\dots w_k\in\mathcal{D}_k$, i.e., the Dyck path p is obtained by removing all the level steps H's from the André path u. Let $\psi(u)=(y,p)$. It is clear that this is the desired bijection because each down step in p has the same height in u. □

Proof of Theorem 4. By Lemmas 11 and 12 we have

$$D_{n+1}(1,-1,t)=\sum_{k\ge 0}\sum_{(y,p)\in\mathcal{Y}_{n,k}\times\mathcal{D}_k} w(p).$$

Since the cardinality of $\mathcal{Y}_{n,k}$ is $\binom{n-k}{k}$, and the generating polynomial of $\mathcal{D}_k$ is equal to $k!t^k$ by Lemma 10, summing over all $0\le k\le\lfloor n/2\rfloor$ we obtain Theorem 12. □

Example 2. An illustration of the bijection ψ is given in Figure 2.

We can derive a formula for $D_n(1,q,t)$ from a formula of Josuat-Vergès [JV11].

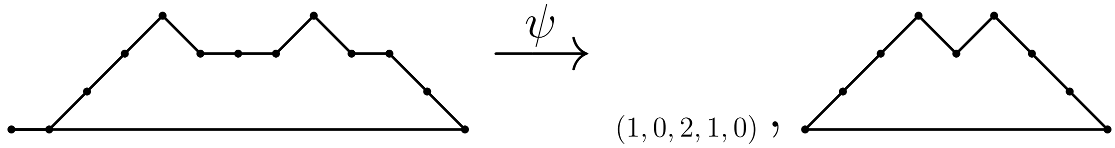

FIGURE 2. The bijection $\psi : \mathcal{AP}_{12,4} \to \mathcal{Y}_{12,4} \times \mathcal{D}_4$

Theorem 13. *For $n \geq 1$ we have*

$$\begin{aligned}
D_n(1,q,t) &= \left(\frac{1+u}{(1+uv)(1+q)}\right)^{n-1} \sum_{\sigma\in\mathfrak{S}_n} v^{\operatorname{des}\sigma} q^{(31-2)\sigma} \\
&= \frac{1}{v(1-q)} \left(\frac{1+u}{(1+uv)(1-q^2)}\right)^{n-1} \\
&\quad \times \sum_{k=0}^{n} (-1)^k \left(\sum_{j=0}^{n-k} v^j \binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1} \right) \cdot \left(\sum_{i=0}^{k} v^i q^{i(k+1-i)} \right),
\end{aligned}$$

where

$$u = \frac{1+q^2-2(1+q)t-(1+q)\sqrt{(1+q)^2-4(1+q)t}}{2(q-t(1+q))}, \tag{21}$$

$$v = \frac{(1+q)-2t-\sqrt{(1+q)^2-4t(1+q)}}{2t}. \tag{22}$$

Proof. Specializing (p,q,t,u,v,w) with $(1,q,q,1,1,t(1+q^{-1}))$ in (16) we obtain

$$D_n(1,q,t) = \frac{A_n(1,q,q,1,1,t(1+q^{-1}))}{(1+q)^{n-1}}.$$

From Corollary 3.2 in [HMZ] we derive

$$A_n(1,q,q,1,1,t(1+q^{-1})) = \left(\frac{1+u}{1+uv}\right)^{n-1} \sum_{\sigma\in\mathfrak{S}_n} v^{\operatorname{des}\sigma} q^{(31-2)\sigma},$$

where u and v are given by (21) and (22). By Theorem 6.3 in [JV11], we have

$$\begin{aligned}
\sum_{\sigma\in\mathfrak{S}_n} y^{\operatorname{asc}\sigma} q^{(13-2)\sigma} &= \sum_{\sigma\in\mathfrak{S}_n} y^{\operatorname{des}\sigma} q^{(31-2)\sigma} \\
&= \frac{1}{y(1-q)^n} \sum_{n=0}^{n} (-1)^k \left(\sum_{j=0}^{n-k} y^j \binom{n}{j}\binom{n}{j+k} - \binom{n}{j-1}\binom{n}{j+k+1} \right) \\
&\quad \times \left(\sum_{i=0}^{k} y^i q^{i(k+1-i)} \right).
\end{aligned}$$

Putting the above three formulae together completes the proof. $\square$

Remark 3. It is a challenge to show directly that Theorem 4 is the limit case of Theorem 13 when $q \to -1$.

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