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τ-TILTING FINITE SIMPLY CONNECTED ALGEBRAS

QI WANG

On the occasion of Professor Susumu Ariki's 60th birthday.

Abstract. In this paper, we showed that τ-tilting finite simply connected algebras are representation-finite. Consequently, some related algebras are considered, including critical algebras, iterated tilted algebras, tubular algebras, and so on. In particular, we studied the muti-staircase algebras introduced by Magdalena Boos, which are strongly simply connected.

1. Introduction

Throughout, we denote by A a basic finite-dimensional algebra over an algebraically closed field k. In particular, tame always means representation-infinite tame.

τ-tilting theory is introduced by Adachi, Iyama and Reiten [4] as a generalization of the classical tilting theory from the viewpoint of mutations. They introduced support τ-tilting modules for A and constructed the mutations of support τ-tilting modules, which have some nice properties. For more details of τ-tilting theory, we refer to [1], [2], [3], [6], [7], [32], [44], [63] and so on.

We are interested in algebras with a finite number of basic support τ-tilting modules, which are called τ-tilting finite algebras. It is natural that representation-finite algebras are τ-tilting finite, but the converse is not true in general. Therefore, *what kind of algebras satisfies the condition: τ-tilting finite implies representation-finite* becomes an interesting question. There are some answers, for example, cycle-finite algebras [44], gentle algebras [48], tilted and cluster-tilted algebras [64] and commutative ladders [5].

Simply connected (representation-finite) algebras are first introduced by Bongartz and Gabriel [25], they showed that one can use the covering techniques to reduce the representation theory of an arbitrary representation-finite algebra to that of a representation-finite simply connected algebra. More precisely, for any representation-finite algebra A, the indecomposable A-modules can be lifted to indecomposable B-modules over a simply connected algebra B, which is contained inside a certain Galois covering of the standard form $\widetilde{A}$ (in the sense of [28], see also subsection 3.2) of A .

Naturally, Assem and Skowroński [15] expanded the notion *simply connected* to representation-infinite case (see Definition 2.7). This wider class of algebras includes the tubular algebras, the iterated tilted algebras of Dynkin type, the iterated tilted algebras of Euclidean types $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p, (n \geqslant 4, p = 6, 7 \text{ or } 8)$ and so on. So far, the covering techniques on representation-infinite case is little known.

However, a subclass of simply connected algebras has been extensively investigated, which is called strongly simply connected algebras and introduced by Skowroński [54]. First, it is shown in [28] that for the representation-finite case, simply connected and strongly simply connected coincide. Then, the hierarchy (in terms of domestic, polynomial growth and wild) of representation-infinite strongly simply connected algebras has been completely determined, see [22], [29], [45], [46], [49] and [55]. We summarize these results in Proposition 3.12.

Tits form is introduced by Gabriel [35] for the path algebra kQ of a finite connected quiver Q. He showed that such a path algebra is representation-finite if and only if the corresponding Tits form of Q is positive (see subsection 2.2). This provides a possibility that one may determine the representation type of an algebra by its Tits form. For more details, we refer to [50], where the author summarizes many of related results.

See subsection 2.5 for the definition of simply connected algebras.

Theorem 1.1. (Proposition 3.3 and Theorem 3.4) *Let A be a simply connected algebra, then the following are equivalent.*

(1) *A is τ-tilting finite.*
(2) *A is representation-finite.*
(3) *The Tits form q_A is weakly positive.*

Let A be a representation-finite simply connected algebra, any tilted algebra of A or any Auslander algebra of A (see Example 2.10), satisfies the condition: τ-tilting finite implies representation-finite. Moreover, we denote by $T_2(A) := \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$ the algebra of 2×2 upper triangular matrices over A. Then, we have

Theorem 1.2. (Theorem 3.8) *Let A be a representation-finite simply connected algebra, then $T_2(A)$ is τ-tilting finite if and only if $T_2(A)$ is representation-finite.*

Iterated tilted algebras introduced by Assem and Happel [12] (see also subsection 3.4) are natural generalizations of tilted algebras [37]. As we have mentioned, Zito [64] shows that any τ-tilting finite tilted algebra is representation-finite. Therefore, we may generalize Zito's result to the iterated tilted algebras.

Theorem 1.3. (Corollary 3.10) *Let A be an iterated tilted algebra of Dynkin type, or of type $\widetilde{\mathbb{D}}_n$ with $n \geqslant 4$, or of type $\widetilde{\mathbb{E}}_p$ with $p = 6, 7$ or 8, then A is τ-tilting finite if and only if A is representation-finite.*

The following algebras played an important role in the study of tame strongly simply connected algebras.

- Critical algebras arising from graded trees, see subsection 3.1.
- Tubular algebras, see subsection 3.5.
- *pg*-critical algebras, see subsection 3.6.
- Hypercritical algebras, see subsection 3.7.

We have the following result.

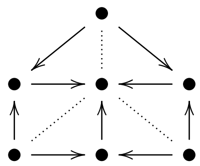
Theorem 1.4. (Corollary 3.2 and Theorem 3.13) *All critical algebras arising from graded trees, tubular algebras, pg-critical algebras and hypercritical algebras are τ-tilting infinite.*

A commutative ladder of degree n is a bounded quiver algebra presented by the following quiver with all possible commutative relations.

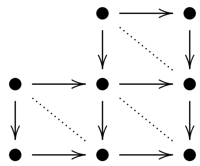
$$\begin{array}{ccccccccc} 1 & \longrightarrow & 2 & \longrightarrow & 3 & \longrightarrow & \cdots & \longrightarrow & n \\ \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\ 1' & \longrightarrow & 2' & \longrightarrow & 3' & \longrightarrow & \cdots & \longrightarrow & n' \end{array} .$$

Since readers may not be familiar with multi-staircase algebras, we give two examples to clarify our intentions (see Section 4 for the formal definition of multi-staircase algebras).

Multi-staircase algebras can be presented by, for example, the following quivers with all possible commutative relations.



Example 4.3



Example 5.1

Therefore, multi-staircase algebras are natural generalizations of commutative ladders. Multi-staircase algebras are introduced by Boos [26] and [27]. These algebras are basic, connected, finite dimensional, so that Boos can give a complete classification in terms of their representation type.

Since multi-staircase algebras are strongly simply connected (Proposition 4.4), they are τ-tilting finite if and only if representation-finite. We may give detailed research for τ-tilting finite multi-staircase algebras. We refer to Section 4 and 5 for precise definitions.

Theorem 1.5. *Let $A(\Lambda)$ be a proper multi-staircase algebra parameterized by a m-dimensional Young diagram Λ with $3 \leqslant m \in \mathbb{N}$, it is τ-tilting finite if and only if $m = 3$, $\Lambda = (\lambda, \mu)$ is flat and $(\lambda, \mu) \preceq (\lambda', \mu')$, where (λ', μ') comes up in the following $(x \in \mathbb{N})$:*

$$(1, \underline{5}, 2), (4, \underline{4}, 1), (1, \underline{5}, 1^2), (2, \underline{2}, 2^2), (1, \underline{2}, 2^3), (3, \underline{3}, 1^3), (1, \underline{3}, 1^4),$$
$$(1, \underline{2}, 2, 1^3), (2^2, \underline{2}, 1^3), (1^2, \underline{2}, 1^3), (1^2, \underline{2}, 1^4), (1, \underline{x}, 1), (1, \underline{2}, 1^x), (2, \underline{2}, 1^x).$$

Moreover, we have

Λ	$(1, \underline{x}, 1)$	$(1, \underline{2}, 1^x)$	$(2, \underline{2}, 1^x)$
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A(\Lambda)$	$\left[\begin{smallmatrix} 2x+3 \\ x+1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 2x+5 \\ x+2 \end{smallmatrix}\right]$	*Theorem 6.12*

where $\binom{x}{y}$ is the binomial coefficient and $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \frac{x+y}{x}\binom{x}{y}$ for any $x, y \in \mathbb{N}$.

Proof. The proof follows Theorem 3.4, Proposition 4.4, Proposition 4.6, Proposition 4.7 and Theorem 4.8. □

Theorem 1.6. *A staircase algebra $A(\lambda)$ parameterized by a Young diagram $\lambda \vdash n$ is τ-tilting finite if and only if one of the following holds:*

(1) $\lambda \in \{(n), (n-k, 1^k), (n-2, 2), (2^2, 1^{n-4})\}$*, where* $k \leqslant n$.
(2) $n \leqslant 8$ *and* $\lambda \notin \{(4, 3, 1), (3^2, 2), (3, 2^2, 1), (4, 2, 1^2)\}$.

Moreover, we have

λ	(n)	$(n-k, 1^k)$	$(n-2, 2)$	$(2^2, 1^{n-4})$
$\#\mathsf{s}\tau\text{-}\mathsf{tilt}\ A(\lambda)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$		*Thm. 6.17 and Conj. 6.18*	

Proof. See Theorem 3.4, Proposition 4.4, Proposition 5.2 and Theorem 5.3. □

This paper is organized as follows. In Section 2, we review basic concepts of τ-tilting theory and simply connected algebras. In Section 3, we show the main result on simply connected algebras. Then, we apply the main result to some related algebras. In Section 4 and 5, we focus on multi-staircase algebras. In Section 6, we determine the number of basic support τ-tilting modules for two special τ-tilting finite multi-staircase algebras.

2. Preliminaries

Any finite-dimensional algebra A can be considered as a bounded quiver algebra kQ/I, with a finite connected quiver $Q = (Q_0, Q_1)$ and an admissible ideal I. We may refer to [19] for more background materials on representation theory and quiver theory.

We denote by $\mathsf{mod}\ A$ the category of finitely generated right A-modules and by $\mathsf{proj}\ A$ the full subcategory consisting of projective A-modules. Besides, we shall denote by Γ_A the Auslander-Reiten quiver of A, whose vertices are the isomorphism classes of indecomposable right A-modules, and in which there is an arrow (and only one) from the class of M to that of N whenever there exists an irreducible map from M to N.

Bongartz and Gabriel [25] showed another perspective that we may regard $A \simeq kQ/I$ as a k-category. The class of objects of this category is the set Q_0 of vertices in Q, and the class $A(i, j)$ of morphisms from i to j is the k-vector space $kQ(i, j)$ of linear combinations of paths in Q with source i and target j, modulo the subspace $I(i, j) = I \cap kQ(i, j)$.

Now, we recall some well-known definitions without further reference.

- A is called *triangular* if Q is acyclic.
- A full subcategory B of A is called *convex* if any path in Q_A with source and sink in Q_B lies entirely in Q_B.
- A relation $\rho = \sum_{i=1}^{n} \lambda_i \omega_i \in I$ with $\lambda_i \neq 0$ is called *minimal* if $n \geqslant 2$ and for each non-empty proper subset $J \subset \{1, 2, \ldots, n\}$, we have $\sum_{j\in J} \lambda_j \omega_j \notin I$.
- The *support algebra* $\mathsf{supp}\ M$ of a right A-module M is the factor algebra of A by modulo the ideal which is generated by all idempotents satisfying $Me_i = 0$.

2.1. **τ-tilting theory.** Let $M \in \mathsf{mod}\ A$, we denote by $\mathsf{add}(M)$ (respectively, $\mathsf{Fac}(M)$) the full subcategory of $\mathsf{mod}\ A$ whose objects are direct summands (respectively, factor modules) of finite direct sums of copies of M. Let $\tau := \mathrm{DTr}(-)$ be the Auslander-Reiten translation and $|M|$ the number of isomorphism classes of indecomposable direct summands of M.

Definition 2.1. ([4, Definition 0.1]) (1) M is called τ-rigid if $\mathsf{Hom}_A(M, \tau M) = 0$.

(2) M is called τ-tilting if M is τ-rigid and $|M| = |A|$.

(3) M is called support τ-tilting if M is a τ-tilting $(A/\langle e\rangle)$-module for an idempotent e of A. Equivalently, let $P := eA$, then (M, P) is called a support τ-tilting pair.

We denote by $\tau\text{-}\mathsf{rigid}\ A$ ($\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$) the set of isomorphism classes of indecomposable τ-rigid (support τ-tilting) A-modules. Here, by a (support) τ-tilting module is always meant a basic (support) τ-tilting module. The following is the core of τ-tilting theory.

Definition 2.2. ([4, Definition 2.19, Theorem 2.30]) Let $T = M \oplus N$ be a τ-tilting A-module with an indecomposable summand M satisfying $M \notin \mathsf{Fac}(N)$. We take an exact sequence with a minimal left $\mathsf{add}(N)$-approximation f,

$$M \xrightarrow{f} N' \longrightarrow U \longrightarrow 0,$$

then $\mu_M^-(T) := U \oplus N$ is called the left mutation of T with respect to M.

We define the support τ-tilting quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ of A as follows.

- The set of vertices is $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$.
- We draw an arrow from M to N if N is a left mutation of M.

It is shown in [4] that $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ A)$ is the Hasse quiver of poset $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$. In fact, there is a patrial order on $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ defined by $\mathsf{Fac}(N) \subseteq \mathsf{Fac}(M)$, for any $M, N \in \mathsf{s}\tau\text{-}\mathsf{tilt}\ A$.

Remark 2.3. In Definition 2.2, Zhang [63, Theorem 1.2] showed that if $U \neq 0$, then U is indecomposable and $U \notin \mathsf{add}(T)$. Since A is the maximal element in poset $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$, U cannot be a projective module if $U \neq 0$.

Definition 2.4. ([32, Corollary 2.9]) An algebra A is said to be τ-tilting finite if there are only finitely many isomorphism classes of indecomposable τ-rigid A-modules, or equivalently, if $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ is a finite set.

Proposition 2.5. ([4, Theorem 2.14]) *There exists a poset isomorphism between* $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A$ *and* $\mathsf{s}\tau\text{-}\mathsf{tilt}\ A^{op}$.

We have the following obvious observation.

Lemma 2.6. *Let A be a τ-tilting finite algebra, then eAe is also τ-tilting finite for any idempotent e of A.*

2.2. **Tits form.** Let $A = kQ/I$ be a triangular algebra, the Tits form $q_A : \mathbb{Z}^{Q_0} \to \mathbb{Z}$ is the integral quadratic form defined by

$$q_A(v) = \sum_{i\in Q_0} v_i^2 - \sum_{i\to j\in Q_1} v_iv_j + \sum_{i,j\in Q_0} r(i,j)v_iv_j,$$

where $v := (v_i) \in \mathbb{Z}^{Q_0}$ and $r(i,j) = |R \cap I(i,j)|$ for a minimal set $R \subseteq \bigcup_{i,j\in Q_0} I(i,j)$ of generators of the admissible ideal I. Then, the Tits form q_A is called

- (weakly) positive if $q_A(v) > 0$ for any $v \neq 0$ in $\mathbb{Z}^{Q_0}$ ($v \neq 0$ in $\mathbb{N}^{Q_0}$, respectively),
- (weakly) non-negative if $q_A(v) \geqslant 0$ for any $v \in \mathbb{Z}^{Q_0}$ ($v \in \mathbb{N}^{Q_0}$, respectively).

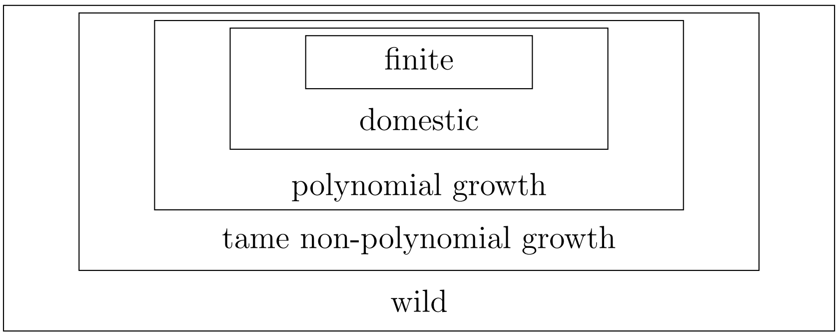
2.3. **Separation property.** Let $A = kQ/I$ be a triangular algebra, we denote by P_i the indecomposable projective module at vertex i and $\mathsf{rad}\ P_i$ its radical. Then P_i is said to be separated if $\mathsf{rad}\ P_i$ is a direct sum of pairwise non-isomorphic indecomposable modules whose supports are contained in pairwise different connected components of $Q(i)$, where $Q(i)$ is the subquiver of Q obtained by deleting all vertices of Q being a source of a path in Q with target i (including the trivial path from i to i). We say that A satisfies the separation property if every indecomposable projective module is separated.

2.4. **Representation type.** It is well-known that A is representation-finite if there are only finitely many isomorphism classes of indecomposable A-modules.

From Drozd [33], all finite-dimensional algebras may be divided into two disjoint cases:

1. A is *tame* if for any dimension d, there exists a finite number of $k[x]$-A-bimodules M_i, $(1 \leqslant i \leqslant n_d)$, which are finitely generated and free as left $k[x]$-modules, such that all but finitely many isomorphism classes of indecomposable right A-modules of dimension d, are of the form $k[x]/\{x-\omega\} \otimes_{k[x]} M_i$ with some $\omega \in k$ and some $i \in \{1, 2, \dots, n_d\}$. Let $\mu_A(d)$ be the least number of $k[x]$-A-bimodules satisfying the above condition for d. Then,
   - A is representation-finite ([20], [24]) if and only if $\mu_A(d) = 0$ for all $d \geqslant 1$.
   - A is of *domestic type* ([31]) if there is a constant C such that $\mu_A(d) \leqslant C$ for all $d \geqslant 1$.
   - A is of *polynomial growth type* ([56]) if there are positive integer m and constant C such that $\mu_A(d) \leqslant Cd^m$ for all $d \geqslant 1$.
2. A is *wild* if there is a finitely generated $k\langle X, Y\rangle$-A-bimodule M which is free over $k\langle X, Y\rangle$ and sends non-isomorphic indecomposable $k\langle X, Y\rangle$-modules via the functor $- \otimes_{k\langle X,Y\rangle} M$ to non-isomorphic indecomposable A-modules.

By a tame algebra is meant a representation-infinite tame algebra. For representation types of algebras, we have the following hierarchy and each of the inclusions is proper.



2.5. **Simply connected algebras.** We recall from [15] the construction of simply connected algebras. Let $A = kQ/I$ be a triangular algebra with a quiver $Q = (Q_0, Q_1, s, t)$ and an admissible ideal I. For each arrow $\alpha \in Q_1$, let α^- be its formal inverse with $s(\alpha^-) = t(\alpha)$ and $t(\alpha^-) = s(\alpha)$. Then, we set

$$Q_1^- = \{\alpha^- \mid \alpha \in Q_1\}.$$

A walk is a formal composition $w = w_1 w_2 \dots w_n$ with $w_i \in Q_1 \cup Q_1^-$ for all $1 \leqslant i \leqslant n$. We set $s(w) = s(w_1)$, $t(w) = t(w_n)$ and denote by 1_x the trivial path at vertex x.

For walks w and u with $s(u) = t(w)$, the composition wu is defined in the obvious way. Then, let $\sim$ be the smallest equivalence relation on the set of all walks in Q satisfying the following conditions:

- For each $\alpha : x \to y$ in Q_1, we have $\alpha\alpha^- \sim 1_x$ and $\alpha^-\alpha \sim 1_y$.
- For each minimal relation $\sum_{i=1}^n \lambda_i \omega_i$ in I, we have $\omega_i \sim \omega_j$ for all $1 \leqslant i, j \leqslant n$.
- If u, v, w and w' are walks and $u \sim v$, then $wuw' \sim wvw'$ whenever these compositions are defined.

We denote by $[w]$ the equivalence class of a walk w.

For a given $x \in Q_0$, the set $\Pi_1(Q, I, x)$ of equivalence classes of all walks w with $s(w) = t(w) = x$ becomes a group via $[u] \cdot [v] = [uv]$, and it is independent of the choice of x. Then, the fundamental group of (Q, I) is defined as follows.

$$\Pi_1(Q, I) := \Pi_1(Q, I, x).$$

Definition 2.7. ([15, Definition 1.2]) A triangular algebra A is said to be simply connected if, for any presentation $A = kQ/I$ as a bounded quiver algebra, the fundamental group $\Pi_1(Q, I)$ is trivial.

It follows [28] and [43] that if A is representation-finite, then the above definition coincides with the definition introduced by Bongartz and Gabriel [25]. It is not easy to recognise whether a given algebra is simply connected or not, because little is known about the characterization of simply connected.

Proposition 2.8. *Let A be a triangular algebra.*

(1) ([53, (4.2)]) *A is simply connected if and only if it does not admit a proper Galois covering.*
(2) ([54, (2.3)]) *If A has the separation property, then it is simply connected.*

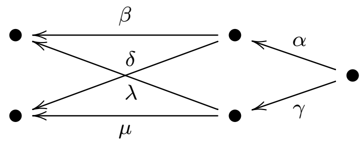
2.6. **Strongly simply connected algebras.** A triangular algebra A is called strongly simply connected [54, (2.2)] if every convex subcategory of A is simply connected. The characterization of strongly simply connected algebras has been extensively investigated.

Proposition 2.9. *Let A be a triangular algebra. The following are equivalent.*

(1) *A is strongly simply connected.*
(2) ([54, (4.1)]) *Every convex subcategory of A (or A^{op}) has the separation property.*
(3) ([54, (4.1)]) *The first Hochschild cohomology space $H^1(B)$ of any convex subcategory B of A vanishes.*
(4) ([14, Theorem 1.3]) *There is a presentation (Q, I) of A such that $\Pi_1(Q', I')$ is trivial for any connected full convex bounded subquiver (Q', I') of (Q, I).*

Example 2.10. We have the following examples.

(1) All tree algebras are strongly simply connected, see [9].
(2) A hereditary algebra is simply connected if and only if its quiver is a tree, see [9].
(3) Let A be a representation-finite simply connected algebra and T_A a tilting A-module, the tilted algebra $B = \mathrm{End}_A(T_A)$ is simply connected, see [18].
(4) Let A be a representation-finite algebra and $\{M_1, M_2, \ldots, M_s\}$ a complete set of representatives of the isomorphism classes of indecomposable A-modules, then A is simply connected if and only if its Auslander algebra $\mathrm{End}_A(\oplus_{i=1}^{s} M_i)$ is strongly simply connected, see [11].
(5) Let $A := kQ/I$ with $I := <\alpha\beta - \gamma\delta, \alpha\lambda - \gamma\mu>$ and the following quiver Q:



then A is simply connected but not strongly simply connected, see [9].

We may distinguish the class of representation-finite simply connected algebras.

Proposition 2.11. *Let A be a representation-finite triangular algebra, then the following conditions are equivalent.*

(1) *A is simply connected.*
(2) ([28]) *A is strongly simply connected.*
(3) ([30]) *The first Hochschild cohomology space $H^1(A)$ of A vanishes.*
(4) ([43]) *The fundamental group $\Pi_1(|\Gamma_A|)$ of the geometric realization $|\Gamma_A|$ of the Auslander-Reiten quiver Γ_A of A is trivial.*

2.7. **Minimal representation-infinite algebras.** An algebra A is said to be minimal representation-infinite if A is representation-infinite, but A/AeA is representation-finite for any non-zero idempotent e of A. Happel and Vossieck [40] have classified the minimal representation-infinite algebras with preprojective component by using the theory of tilted algebras. We recall their constructions as follows.

Let Γ_A be the Auslander-Reiten quiver of A. A connected component C of Γ_A is called preprojective if there is no oriented cycle in C, and any module in C is of form $\tau^{-n}(P)$ for a $n \in \mathbb{N}$ and an indecomposable projective module P.

A tilted algebra of type Q is the endomorphism algebra of a tilting module T (see subsection 3.4) over a hereditary algebra kQ. If moreover, T is contained in a preprojective component C of Γ_A, then we call $\mathrm{End}_{kQ}T$ a concealed algebra of type Q.

Proposition 2.12. ([40, Theorem 2]) *A minimal representation-infinite algebra with preprojective component is either a n-Kronecker algebra* $(n \geqslant 2)$ *or a tame concealed algebra, which is a concealed algebra of type* $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n (n \geqslant 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ *or* $\widetilde{\mathbb{E}}_8$.

It follows [15, Remark 1.2] that minimal representation-infinite algebras with preprojective component are simply connected.

Proposition 2.13. ([64, Theorem 1.1]) *A tilted algebra is* τ*-tilting finite if and only if it is representation-finite.*

Lemma 2.14. ([5, Corollary 4.2]) *All minimal representation infinite algebras with preprojective component are* τ*-tilting infinite.*

At the end of this section, we recall the following well-known result.

Lemma 2.15. ([47, Theorem 1]) *Let* Q *be a quiver of Dynkin type and* $\mathsf{s\text{-}tilt}\ kQ$ *the set of isomorphism classes of basic support tilting* kQ*-modules. Then,* $\#\mathsf{s\text{-}tilt}\ kQ$ *is independent of the orientation of* Q *and* $\#\mathsf{s\text{-}tilt}\ kQ = \#\mathsf{s\tau\text{-}tilt}\ kQ$. *Moreover, we have*

Q	$\mathbb{A}_n$	$\mathbb{D}_n (n \geqslant 4)$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
$\#\mathsf{s\tau\text{-}tilt}\ kQ$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\left[\begin{smallmatrix}2n-1\\ n-1\end{smallmatrix}\right]$	833	4160	25080

where $\binom{x}{y}$ *is the binomial coefficient and* $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right] = \frac{x+y}{x}\binom{x}{y}$.

3. Simply Connected Algebras

3.1. **Critical algebras.** We recall Bongartz's construction from [23]. The grading of a tree $T := (T_0, T_1)$ is defined to be a function $g : T_0 \to \mathbb{N}$ satisfying

- $g^{-1}(0) \neq \varnothing$.
- $g(x) - g(y) \in 1 + 2\mathbb{Z}$, whenever x and y are neighbours in T.

A graded tree (T, g) is given by a tree T and a grading g. It is shown in [25, Corollary 6.5] that there is a bijection between the isomorphism classes of representation-finite graded trees and the isomorphism classes of representation-finite simply connected algebras.

Definition 3.1. ([23]) An algebra A is called critical if it is representation-infinite, but any proper convex subcategory is representation-finite.

Let A be an algebra arising from a graded tree (T, g), then [23, Theorem 2] implies that A is critical if and only if T is one of Euclidean diagrams $\widetilde{\mathbb{D}}_n (n \geqslant 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$. Note that the condition of minimal representation-infinite algebras is strictly stronger than the condition of critical algebras, however, the class of minimal representation-infinite algebras with preprojective component and the class of critical algebras arising from graded trees coincide. (This is the reason why we say BHV-list.) Then, such a critical algebra is simply connected and we have the following corollary immediately.

Corollary 3.2. *Critical algebras arising from graded trees are* τ*-tilting infinite.*

Here is a nice criterion for simply connected algebras to be representation-finite.

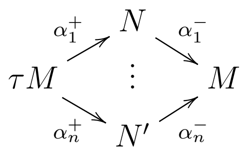
Proposition 3.3. ([21, Theorem 3.3], [22, Theorem 1]) *Let* A *be a simply connected algebra, then* A *is representation-finite if and only if the Tits form* q_A *is weakly positive, if and only if,* A *does not contain a critical convex subcategory which is arising from graded trees.*

Theorem 3.4. *Let A be a simply connected algebra, then it is τ-tilting finite if and only if A is representation-finite.*

Proof. Following Proposition 3.3, if A is representation-infinite, then it contains an idempotent truncation which is critical and arising from a graded tree. Thus, A is τ-tilting infinite from Lemma 2.6 and Corollary 3.2. □

There is another evidence for τ-tilting finiteness of tame strongly simply connected algebras of polynomial growth. In fact, such an algebra is cycle-finite, and, Malicki and Skowroński [44] showed that a cycle-finite algebra is τ-tilting finite if and only if it is representation-finite. Thus, they are τ-tilting infinite.

3.2. **The standard form of a representation-finite algebra.** Assume that A is representation-finite, we consider its Auslander-Reiten quiver Γ_A as a path category $k\Gamma_A$. We have the following sequence in Γ_A for any indecomposable non-projective module M,



then we define $\sigma_M := \sum_{i=1}^{n} \alpha_i^+ \alpha_i^-$ and the mesh-category $k(\Gamma_A) := k\Gamma_A / I_{\Gamma_A}$, which is bounded by the mesh-ideal $I_{\Gamma_A} := \langle \sigma_M \mid \tau M \neq 0 \rangle$.

Definition 3.5. ([25, Definition 5.1]) Let A be a representation-finite algebra, the standard form $\widetilde{A}$ of A is defined to be the full subcategory consisting of the projective points of the mesh-category $k(\Gamma_A)$.

Then, [25, Corollary 5.2] implies that $\widetilde{A}$ is also representation-finite, $\Gamma_{\widetilde{A}} = \Gamma_A$ and $\widetilde{A}$ is the best possible degeneration of A in the sense of algebraic geometry. It worth mentioning that any representation-finite simply connected algebra is standard [25, (6.1)].

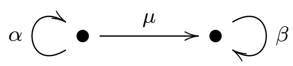
For a representation-finite algebra A, Bretscher and Gabriel [28] further demonstrated the importance of $\widetilde{A}$:

- $\widetilde{A}$ is Morita equivalent to A.
- $\widetilde{A}$ admits a Galois covering $F : B \to B/G := \widetilde{A}$, where B is simply connected and G is the fundamental group $\Pi_1(Q_A, I_A)$, which is a finitely generated free group.

Clearly, $B = \widetilde{A}$ if A is simply connected.

We have to point out that even an algebra admits a (strongly) simply connected Galois covering, it is not necessary to be (strongly) simply connected. For example,

Example 3.6. ([57, Example 3.3]) Let $A = kQ/I$ with $I =< \alpha^4, \beta^4, \alpha\mu - \mu\beta >$ and Q :



then A is of (tame) polynomial growth and admits a strongly simply connected Galois covering. But A is τ-tilting finite [60, Theorem 1.1].

3.3. **Triangular matrix algebras.** Let $T_2(A) := \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ be the algebra of 2×2 upper triangular matrices over an algebra A. Then, the category $\mathsf{mod}\, T_2(A)$ is equivalent to the category whose objects are A-homomorphisms $f : M \to N$ between finite-dimensional A-modules M and N, and morphisms are pairs of homomorphisms making the obvious squares commutative.

Skowroński showed in [52] that $\mathcal{T}_2(A)$ is wild if A is representation-infinite. Therefore, we may consider representation-finite algebras.

Proposition 3.7. ([42, Theorem 4.1]) *Let A be a standard representation-finite algebra with a simply connected Galois covering $F : B \to A$, then $\mathcal{T}_2(A)$ is representation-finite if and only if $\mathcal{T}_2(B)$ does not contain a convex subcategory which is tame concealed.*

Theorem 3.8. *Let A be a representation-finite simply connected algebra, then $\mathcal{T}_2(A)$ is τ-tilting finite if and only if $\mathcal{T}_2(A)$ is representation-finite.*

Proof. It follows Lemma 2.6, Lemma 2.14, Proposition 3.7 and Subsection 3.2. □

3.4. **Iterated tilted algebras.** Let A be an algebra, we recall from [37] that an A-module T is called tilting (respectively, cotilting) if it satisfies $|T| = |\Lambda|$, $\mathsf{proj.dim}\ T \leqslant 1$ (respectively, $\mathsf{inj.dim}\ T \leqslant 1$) and $\mathrm{Ext}_A^1(T, T) = 0$.

Then, two algebras A and B are said to be tilting-cotilting equivalent if there exists a sequence of algebras $A = A_0, A_1, \ldots, A_m = B$ and a sequence of modules $T_{A_i}^i, (0 \leqslant i \leqslant m)$ such that $A_{i+1} = \mathrm{End}_{A_i} T_{A_i}^i$ and $T_{A_i}^i$ is either a tilting or cotilting module.

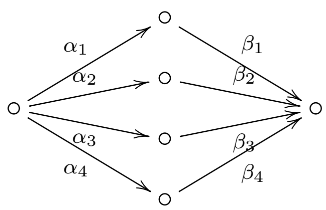
Definition 3.9. ([12, (1.4)] and [39, Theorem 3]) Let kQ be a hereditary algebra, then we call A iterated tilted of type Q if A is tilting-cotilting equivalent to kQ.

It is shown by [8, Proposition 3.5] that iterated tilted algebras of Dynkin type are simply connected and by [15, Corollary 1.4] that an iterated tilted algebra of Euclidean type is simply connected if and only if Q is of types $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p, (n \geqslant 4, p = 6, 7 \text{ or } 8)$. Therefore,

Corollary 3.10. *Let A be an iterated tilted algebra of Dynkin type and types $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_p, (n \geqslant 4, p = 6, 7$ or $8)$, then A is τ-tilting finite if and only if it is representation-finite.*

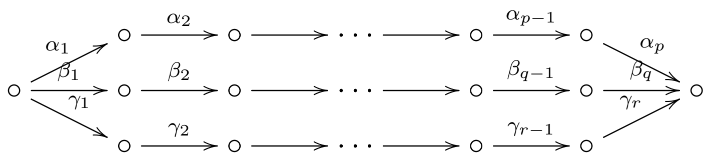
3.5. **Tubular algebras.** Following Ringel [51],

(1) The canonical algebra $\mathcal{C}(2, 2, 2, 2)$ is defined by the following quiver and relations.



$$\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0,$$
$$\alpha_1\beta_1 + \lambda\alpha_2\beta_2 + \alpha_4\beta_4 = 0,$$
$$\lambda \in k/\{0, 1\}.$$

(2) The canonical algebra $\mathcal{C}(p, q, r)$ with $p \leqslant q \leqslant r$ is given by the quiver



bounded by $\alpha_1\alpha_2 \ldots \alpha_p + \beta_1\beta_2 \ldots \beta_q + \gamma_1\gamma_2 \ldots \gamma_r = 0$.

In particular, we call four special cases $\mathcal{C}(2, 2, 2, 2)$, $\mathcal{C}(3, 3, 3)$, $\mathcal{C}(2, 4, 4)$ and $\mathcal{C}(2, 3, 6)$ the tubular canonical algebras.

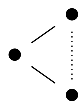
Definition 3.11. ([36, Corollary 1.7], [38] and [51]) An algebra A is said to be a tubular algebra if it is tilting-cotilting equivalent to one of tubular canonical algebras.

It is well-known that tubular algebras are of global dimension 2 and have only 6, 8, 9 or 10 simple modules. Then, Assem and Skowroński [15, Corollary 1.4] showed that tubular algebras are simply connected. Furthermore, [55, Proposition 2.4] implies that every tubular algebra is tame, non-domestic, of polynomial growth and of linear growth.

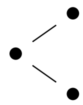
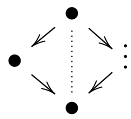
3.6. *pg*-**critical algebras.** In order to find the criteria for simply connected algebras to be of (tame) polynomial growth, Nörenberg and Skowroński [46] introduced the polynomial growth critical algebras (briefly *pg*-critical algebras), that is, tame simply connected algebras which are not of polynomial growth but every proper convex subcategory is.

Following [46, Theorem 3.2], one can understand all *pg*-critical algebras by 31 frames and 3 admissible operations. For simplicity, we only recall the admissible operations.

(1) Constructing the opposite algebra.
(2) Choice of arbitrary orientation in non-oriented edges.
(3) Replacing each subgraph

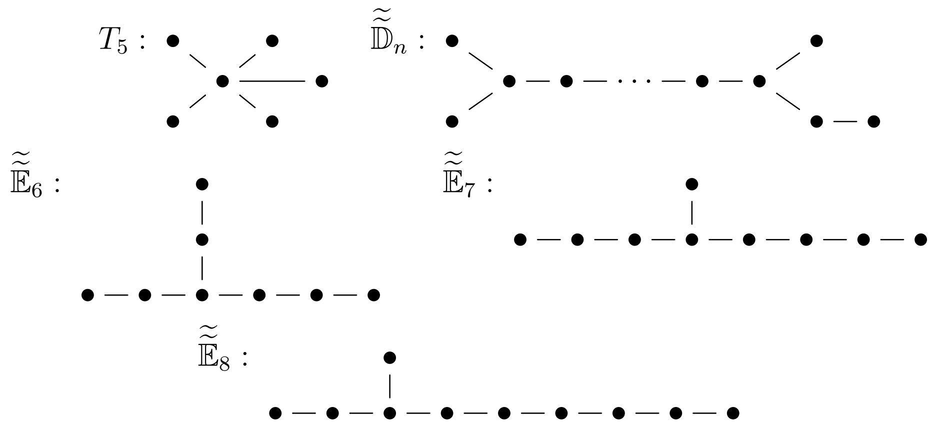


by

 or  .

Then, [46, Corollary 3.3] shows that *pg*-critical algebras are simply connected of global dimension 2. In particular, there are 16 frames among 31 frames which are strongly simply connected [45, Theorem 1].

3.7. **Hypercritical algebras.** Continuing to find the criteria for a simply connected algebra to be wild, Unger [59] (see also Lersch [41] and Wittman [61]) introduced the hypercritical algebras which are preprojective tilts of minimal wild hereditary tree algebras of the following types.



where in the case of $\widetilde{\widetilde{\mathbb{D}}}_n$ the number of vertices is $n+2, (4 \leqslant n \leqslant 8)$.

Similarly, one can understand hypercritical algebras by quivers and relations [59] and they are strongly simply connected. Actually, they are minimal wild strongly simply connected algebras as shown in Proposition 3.12.

3.8. **Strongly simply connected algebras.** Over the past several decades, the class of tame strongly simply connected algebras has been studied intensively. We recall the handy criteria of their representation type as follows.

Proposition 3.12. *Let A be a representation-infinite strongly simply connected algebra.*

(1) ([29, Corollary 1], [49, Theorem 2.2]) *A is tame if and only if q_A is weakly non-negative, or equivalently, A does not contain a hypercritical convex subcategory.*
(2) *A is tame minimal non-polynomial growth if and only if A is obtained from one of the frames in* [45, Theorem 1] *by admissible operations, or equivalently, A is obtained from one of the frames* (1)–(16) *in the list of pg-critical algebras in* [46, Theorem 3.2] *by admissible operations.*
(3) ([55, Theorem 4.1]) *A is of polynomial growth if and only if A does not contain a convex subcategory which is pg-critical or hypercritical.*
(4) ([55, Corollary 4.3]) *A is domestic if and only if A does not contain a convex subcategory which is tubular, or pg-critical, or hypercritical.*

Corollary 3.13. *All tubular algebras, pg-critical algebras and hypercritical algebras are τ-tilting infinite.*

Proof. This is immediate from Theorem 3.4 and Proposition 3.12. □

4. Multi-staircase Algebras

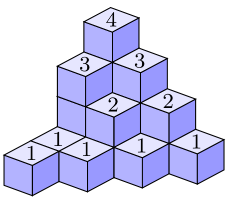
In this section, we deal with a subclass of strongly simply connected algebras, namely, the multi-staircase algebras. These algebras are introduced by Boos [27] and are parameterized by the so-called generalized Young diagrams.

Recall that a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of a positive integer n, is a non-increasing sequence satisfying $\sum_{i=1}^{\ell} \lambda_i = n$. We may merge same entries of λ by potencies, for example, $(3,3,2,1,1,1) = (3^2, 2, 1^3)$. It is well-known that we can visualize λ by the Young diagram $Y(\lambda)$, that is, a box-diagram of which the i-th row contains λ_i boxes.

4.1. **Generalized Young diagrams.** A plane partition is a Young diagram filled with positive integers such that all rows and columns are non-increasing. We can think of the numbers as representing the heights for stacks of blocks placed on each cell of the diagram. For example, let π be a plane partition

$$\pi = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline 3 & 2 & 1 & \\ \cline{1-3} 1 & 1 & & \\ \cline{1-2} 1 & & & \\ \cline{1-1} \end{array}$$

then it can be visualized as a 3-dimensional Young diagram $Y(\pi)$:



We can equip every cube in $Y(\pi)$ with a triple array $(\dot{x}, \dot{y}, \dot{z})$ of positive integers such that: *If $(\dot{x}, \dot{y}, \dot{z})$ corresponds a cube in $Y(\pi)$, then any $(\dot{a}, \dot{b}, \dot{c})$ also corresponds a cube in $Y(\pi)$ whenever $\dot{a} \leqslant \dot{x}, \dot{b} \leqslant \dot{y}$ and $\dot{c} \leqslant \dot{z}$.*

On the other hand, let $\Lambda \subseteq \mathbb{N}^3_{\geqslant 1}$ be a collection of all triple arrays that satisfy the condition: *If* $(\dot{x}, \dot{y}, \dot{z}) \in \Lambda$, *then* $(\dot{a}, \dot{b}, \dot{c}) \in \Lambda$ *whenever* $\dot{a} \leqslant \dot{x}, \dot{b} \leqslant \dot{y}$ *and* $\dot{c} \leqslant \dot{z}$. Then one easily find a bijection between Λ and the set of all 3-dimensional Young diagrams. This setup can be generalized to $m \in \mathbb{N}$ dimensions as follows.

Definition 4.1. ([27, Subsection 2.2]) For any $2 \leqslant m \in \mathbb{N}$, let $\Lambda \subseteq \mathbb{N}^m_{\geqslant 1}$ be a set of m-dimensional arrays such that

$$(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m) \in \Lambda \Rightarrow (\dot{a}_1, \dot{a}_2, \ldots, \dot{a}_m) \in \Lambda, \forall \dot{a}_i \leqslant \dot{x}_i, \forall i,$$

then we call Λ a m-dimensional Young diagram.

A 2-dimensional Young diagram is just a Young diagram. A 3-dimensional Young diagram Λ is said to be *flat* if one of the following holds:

- $\Lambda \subseteq \{(x, y, 1), (x, 1, z) \mid x \geqslant 1, y \geqslant 1, z \geqslant 1\}$.
- $\Lambda \subseteq \{(x, y, 1), (1, y, z) \mid x \geqslant 1, y \geqslant 1, z \geqslant 1\}$.
- $\Lambda \subseteq \{(x, 1, z), (1, y, z) \mid x \geqslant 1, y \geqslant 1, z \geqslant 1\}$.

Let Λ be a flat 3-dimensional Young diagram, then it can be presented by two partitions. For example, if $\Lambda \subseteq \{(x, y, 1), (x, 1, z) \mid x \geqslant 1, y \geqslant 1, z \geqslant 1\}$, then $\{(x, y) \mid x \geqslant 1, y \geqslant 1\}$ determines a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\{(x, z) \mid x \geqslant 1, z \geqslant 1\}$ determines a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ such that

$$\lambda_1 = \mu_1 := \# \{(x, 1, 1) \mid x \geqslant 1, (x, 1, 1) \in \Lambda\}.$$

In this case, we denote Λ by $(\lambda, \mu) := (\lambda_k, \ldots, \lambda_2, \underline{\lambda_1 = \mu_1}, \mu_1, \ldots, \mu_\ell)$.

Moreover, let (λ, μ) and (λ', μ') be two bipartitions, we say $(\lambda, \mu) \preceq (\lambda', \mu')$ if λ is a subpartition of λ' and μ is a subpartition of μ'.

4.2. Multi-staircase algebras.

Definition 4.2. ([27, Definition 3.1]) Let Λ be a m-dimensional Young diagram and $A(\Lambda) := kQ_\Lambda / I_\Lambda$ such that

- the vertices of Q_Λ are given by the tuples appearing in Λ;
- there is an arrow $\varphi^{(i)}_{\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m} : (\dot{x}_1, \ldots, \dot{x}_m) \to (\dot{y}_1, \ldots, \dot{y}_m)$ if and only if there is exactly one index i such that $\dot{y}_i = \dot{x}_i - 1$ and $\dot{y}_j = \dot{x}_j$ for $j \neq i$;
- I_Λ is a two-sided ideal generated by all commutativity relations for all squares appearing in Q_Λ.

Then the bounded quiver algebra $A(\Lambda)$ is called a multi-staircase algebra. Besides, let

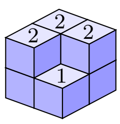
$$\Lambda_{\min} := \{(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_m) \mid \dot{x}_i = 2 \text{ for at most one } i\} \subset \{1, 2\}^m,$$

then $A(\Lambda)$ is said to be proper if $\Lambda_{\min} \subseteq \Lambda$.

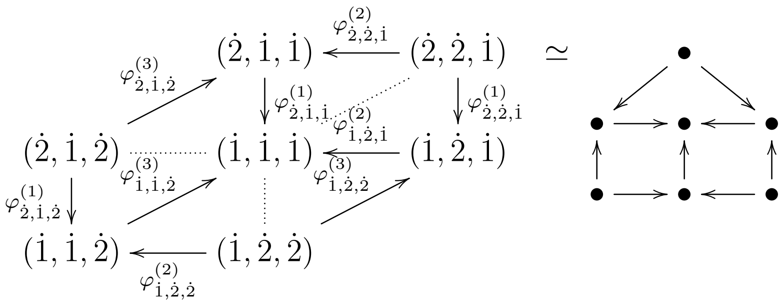
It is obvious that $A(\Lambda)$ is a basic, connected, triangular, finite dimensional k-algebra. In particular, $A(\Lambda)$ is called a staircase algebra if Λ is a 2-dimensional Young diagram and we shall give a detailed study on this case in the next section.

As already mentioned in the introduction, we give an example to show that multi-staircase algebras are generalizations of commutative ladders.

Example 4.3. Let Λ be the following 3-dimensional Young diagram



then the quiver Q_Λ is given by



Then, the corresponding multi-staircase algebra $A(\Lambda)$ is defined by

$A(\Lambda) = kQ_\Lambda / < \varphi^{(2)}_{\dot1,\dot2,\dot1}\varphi^{(1)}_{\dot2,\dot2,\dot1} - \varphi^{(1)}_{\dot2,\dot1,\dot1}\varphi^{(2)}_{\dot2,\dot2,\dot1}, \varphi^{(3)}_{\dot1,\dot1,\dot2}\varphi^{(1)}_{\dot2,\dot1,\dot2} - \varphi^{(1)}_{\dot2,\dot1,\dot1}\varphi^{(3)}_{\dot2,\dot1,\dot2}, \varphi^{(3)}_{\dot1,\dot1,\dot2}\varphi^{(2)}_{\dot1,\dot2,\dot2} - \varphi^{(2)}_{\dot1,\dot2,\dot1}\varphi^{(3)}_{\dot1,\dot2,\dot1} >.$

Proposition 4.4. ([27, Proposition 3.6]) *Let Λ be a m-dimensional Young diagram with $m \geqslant 2$, then $A(\Lambda)$ is strongly simply connected.*

Our main result Theorem 3.4 implies that a multi-staircase algebra is τ-tilting finite if and only if it is representation-finite. Therefore, we focus on the representation-finite multi-staircase algebras.

Proposition 4.5. ([27, Subsection 5.2]) *Let $\Lambda = (\lambda, \mu)$ be a flat 3-dimensional Young diagram and $A(\Lambda) := A(\lambda, \mu)$, then $A(\lambda, \mu)$ is Morita equivalent to $A(\mu, \lambda)$.*

We call $A(\Lambda)$ a tri-staircase algebra if Λ is a 3-dimensional Young diagram and $A(\Lambda)$ is flat if Λ is flat. Then we have the following classification.

Proposition 4.6. ([27, Theorem 5.2]) *A proper tri-staircase algebra $A(\Lambda)$ is*

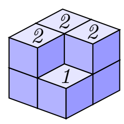
(1) *representation-finite if and only if $A(\Lambda) = A(\lambda, \mu)$ is flat and $(\lambda, \mu) \preceq (\lambda', \mu')$, where (λ', μ') comes up in the following list ($x \in \mathbb{N}$):*

$$(1, \underline{5}, 2), (4, \underline{4}, 1), (1, \underline{5}, 1^2), (2, \underline{2}, 2^2), (1, \underline{2}, 2^3), (3, \underline{3}, 1^3), (1, \underline{3}, 1^4),$$
$$(1, \underline{2}, 2, 1^3), (2^2, \underline{2}, 1^3), (1^2, \underline{2}, 1^3), (1^2, \underline{2}, 1^4), (1, \underline{x}, 1), (1, \underline{2}, 1^x), (2, \underline{2}, 1^x).$$

(2) *tame concealed if and only if $A(\Lambda) = A(\lambda, \mu)$ is flat and (λ, μ) comes up in the following list:*

$$(2, \underline{3}, 2), (3, \underline{5}, 1), (2, \underline{6}, 1), (1, \underline{3}, 2, 1), (2, \underline{4}, 1^2), (1, \underline{6}, 1^2), (1, \underline{4}, 1^3),$$
$$(1, 2, \underline{2}, 2, 1), (1^2, \underline{3}, 1^2), (1, 2^2, \underline{2}, 2), (1, \underline{2}, 2^2, 1^2), (1^2, \underline{2}, 2, 1^2),$$
$$(2, \underline{2}, 2, 1^3), (2, \underline{3}, 1^4), (1, \underline{2}, 2, 1^4), (1, 2, \underline{2}, 1^4), (1, \underline{3}, 1^5), (1^3, \underline{2}, 1^3), (1^2, \underline{2}, 1^5).$$

(3) *tame, but not tame concealed if and only if either Λ equals Λ_0:*



or $A(\Lambda) = A(\lambda, \mu)$ is flat and (λ, μ) comes up in the following list:

$$(2, \underline{3}, 3), (3, \underline{3}, 3), (4, \underline{5}, 1), (1, \underline{3}, 2^2), (1, \underline{3}, 3, 1), (3, \underline{4}, 1^2), (2, \underline{2}, 2^3),$$
$$(2^2, \underline{2}, 2, 1), (2^2, \underline{2}, 2^2), (1, \underline{2}, 2^3, 1), (1^2, \underline{2}, 2^2, 1), (3, \underline{3}, 1^4), (2^2, \underline{2}, 1^4).$$

Otherwise, $A(\Lambda)$ is of wild representation type.

Proposition 4.7. ([27, Theorem 6.1]) *Let $A(\Lambda)$ be a proper multi-staircase algebra with a m-dimensional Young diagram Λ, then $A(\Lambda)$ is representation-infinite if $m \geqslant 4$.*

Theorem 4.8. *Let $A(\Lambda)$ be a proper tri-staircase algebra with a 3-dimensional Young diagram Λ, then*

Λ	$(1, \underline{x}, 1)$	$(1, \underline{2}, 1^x)$
$\#\mathsf{s}\tau\text{-tilt } A(\Lambda)$	$\left[\begin{smallmatrix} 2x+3 \\ x+1 \end{smallmatrix}\right]$	$\left[\begin{smallmatrix} 2x+5 \\ x+2 \end{smallmatrix}\right]$

where $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \frac{x+y}{x}\binom{x}{y}$.

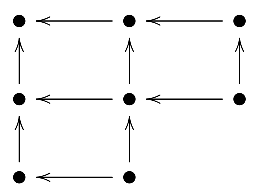
Proof. If $\Lambda = (1, \underline{x}, 1)$ or $(1, \underline{2}, 1^x)$, then $A(\Lambda)$ is a Dynkin algebra of type $\mathbb{D}$. The result follows Lemma 2.15. □

5. Staircase Algebras

In this section, we focus on the multi-staircase algebras with Young diagrams, which are called staircase algebras briefly. Since a Young diagram $Y(\lambda)$ corresponds to a partition $\lambda \vdash n$ of a positive integer n, we may denote a staircase algebra by $A(\lambda)$.

Example 5.1. Let $n = 8$ and $\lambda = (3, 3, 2)$, then its quiver Q_λ is given by

$$\begin{array}{ccccc} (\dot{1},\dot{1}) & \xleftarrow{\alpha_{\dot{2},\dot{1}}} & (\dot{2},\dot{1}) & \xleftarrow{\alpha_{\dot{3},\dot{1}}} & (\dot{3},\dot{1}) \\ \uparrow{\scriptstyle\beta_{\dot{1},\dot{2}}} & & \uparrow{\scriptstyle\beta_{\dot{2},\dot{2}}} & & \uparrow{\scriptstyle\beta_{\dot{3},\dot{2}}} \\ (\dot{1},\dot{2}) & \xleftarrow{\alpha_{\dot{2},\dot{2}}} & (\dot{2},\dot{2}) & \xleftarrow{\alpha_{\dot{3},\dot{2}}} & (\dot{3},\dot{2}) \\ \uparrow{\scriptstyle\beta_{\dot{1},\dot{3}}} & & \uparrow{\scriptstyle\beta_{\dot{2},\dot{3}}} & & \\ (\dot{1},\dot{3}) & \xleftarrow{\alpha_{\dot{2},\dot{3}}} & (\dot{2},\dot{3}) & & \end{array} \simeq$$



The corresponding staircase algebra $A(\lambda)$ is defined by

$$A(\lambda) := kQ_\lambda / < \beta_{\dot{2},\dot{2}}\alpha_{\dot{2},\dot{1}} - \alpha_{\dot{2},\dot{2}}\beta_{\dot{1},\dot{2}}, \beta_{\dot{3},\dot{2}}\alpha_{\dot{3},\dot{1}} - \alpha_{\dot{3},\dot{2}}\beta_{\dot{2},\dot{2}}, \beta_{\dot{2},\dot{3}}\alpha_{\dot{2},\dot{2}} - \alpha_{\dot{2},\dot{3}}\beta_{\dot{1},\dot{3}} >.$$

Let λ be a partition, we denote by λ^T the transposed partition given by the columns of the Young diagram (from right to left). Then, [26, Lemma 4.2] implies that $A(\lambda)$ is Morita equivalent to $A(\lambda^T)$.

We recall the classification of representation types of staircase algebras.

Proposition 5.2. ([26, Theorem 4.5]) *A staircase algebra $A(\lambda)$ with $\lambda \vdash n$ is*

(1) *representation-finite if and only if one of the following holds:*
  - $\lambda \in \{(n), (n-k, 1^k), (n-2, 2), (2^2, 1^{n-4})\}$ *for* $k \leqslant n$.
  - $n \leqslant 8$ *and* $\lambda \notin \{(4, 3, 1), (3^2, 2), (3, 2^2, 1), (4, 2, 1^2)\}$.

(2) *tame concealed if and only if λ comes up in the following list:*

$$(6, 3), (6, 2, 1), (5, 2^2), (4, 3, 1), (4, 2, 1^2), (3, 2^2, 1), (3^2, 1^3), (2^3, 1^3), (3, 2, 1^4).$$

(3) *tame, but not tame concealed if and only if λ comes up in the following list:*

$$(5^2), (5, 4), (4^2, 1), (3^3), (3^2, 2), (3, 2^3), (2^5), (2^4, 1).$$

Otherwise, $A(\lambda)$ is of wild representation type.

Theorem 5.3. *Let $A(\lambda)$ be a staircase algebra with $\lambda \vdash n$, then*

λ	(n)	$(n-k, 1^k)$
$\#\mathsf{s}\tau\text{-tilt } A(\lambda)$	$\frac{1}{n+2}\binom{2n+2}{n+1}$	

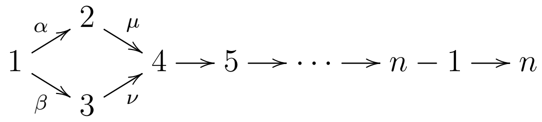
Proof. If $\lambda = (n)$ or $(n-k, 1^k)$, then $A(\lambda)$ is a Dynkin algebra of type $\mathbb{A}$. The result follows Lemma 2.15. □

6. Two Special Cases

Given a representation-finite multi-staircase algebra $A(\Lambda)$, we want to find the number of support τ-tilting $A(\Lambda)$-modules and we have showed some cases in Theorem 4.8 and 5.3. Among others, we are interested in $\Lambda = (2, \underline{2}, 1^x)$ and $\lambda = (n-2, 2)$ or $(2^2, 1^{n-4})$.

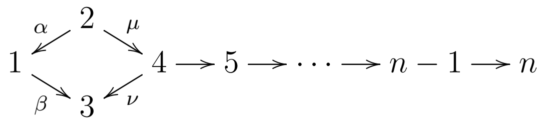
In this section, we will give detailed description on the number of support τ-tilting $A(\Lambda)$-modules for these cases.

(1) Let $\Lambda_n := kQ/ < \alpha\mu - \beta\nu >$ with $n \geqslant 4$ and Q:



This is the multi-staircase algebra $A(\Lambda)$ for $\Lambda = (2, \underline{2}, 1^x)$ with $x \geqslant 0$.

(2) Let $\Theta_n := kQ/ < \alpha\beta - \mu\nu >$ with $n \geqslant 4$ and Q:



This is the staircase algebra $A(\lambda)$ for $\lambda = (n-2, 2)$ or $(2^2, 1^{n-4})$ with $n \geqslant 4$.

Proposition 6.1. *Let $A = \Lambda_n$ or Θ_n, then the number of tilting A-modules is not equal to the number of τ-tilting A-modules.*

Proof. Since there exists a simple A-module S such that $\mathsf{gl.dim}\ S > 1$, then A is not hereditary. Note that [62, Theorem 2.1] implies that a triangular algebra A is hereditary if and only if every τ-tilting A-module is tilting. Then, the result follows. □

Let A be an algebra, a right A-module M is said to be support-rank s if there exists exactly s nonzero orthogonal idempotents $e_1, e_2, \dots, e_s$ such that $Me_i \neq 0$.

Proposition 6.2. *Let M be a support τ-tilting module with support-rank s, then we have $|M| = s$.*

Proof. Since there exists exactly s nonzero orthogonal idempotents $e_1, e_2, \dots, e_s$ such that $Me_i \neq 0$, we have $M(1 - \sum_{i=1}^{s} e_i) = 0$. Let $P := (1 - \sum_{i=1}^{s} e_i)A$, then

$$|M| = n - |P| = n - (n-s) = s$$

follows the fact that (M, P) becomes a support τ-tilting pair. □

Assume that $|A| = n$, let $a_s(A)$ be the number of support τ-tilting A-modules with support-rank s for any $0 \leqslant s \leqslant n$. Note that $a_n(A)$ is just the number of τ-tilting A-modules. Then, the number of all support τ-tilting A-modules is

$$a(A) := \textstyle\sum_{s=0}^{n} a_s(A).$$

As a beginning, we have $\Lambda_4 \simeq \Theta_4$ and the following result.

Proposition 6.3. *Let P_i be the indecomposable projective Λ_4-modules, then*

$$P_1 = \begin{smallmatrix} & 1 & \\ 2 & & 3 \\ & 4 & \end{smallmatrix}, P_2 = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}, P_3 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \text{ and } P_4 = 4.$$

Then, the Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_4)$ implies that

$$a_0(\Lambda_4) = 1, a_1(\Lambda_4) = 4, a_2(\Lambda_4) = 10, a_3(\Lambda_4) = 16 \text{ and } a_4(\Lambda_4) = 15.$$

and all support τ-tilting Λ_4-modules are shown in Appendix A.

From now on, let $\mathbb{A}_n$ and $\mathbb{D}_n$ be the path algebra of type $\mathbb{A}$ and type $\mathbb{D}$, respectively. We denote by $Q_s(A)$ the set of support τ-tilting A-modules with support-rank s.

6.1. The number of support τ-tilting Λ_n-modules.

Lemma 6.4. *For any $n \geqslant 4$ and $1 \leqslant s \leqslant n-3$, we have*

$$a_s(\Lambda_n) = a_s(\Lambda_{n-1}) + a_{s-1}(\Lambda_n).$$

Proof. Let e_n be the idempotent of Λ_n at vertex n. For any support τ-tilting Λ_n-module M satisfying $Me_n = 0$, it is obvious that M is a support τ-tilting Λ_{n-1}-module. Then, let $Q_s(\Lambda_n; e_n)$ be the set of the support τ-tilting Λ_n-modules with support-rank s and $Me_n \neq 0$. We show that there is a bijection

$$\mathfrak{q} : Q_s(\Lambda_n; e_n) \longleftrightarrow Q_{s-1}(\Lambda_n).$$

The result follows this bijection.

Let X be an indecomposable Λ_n-module with support-rank $t \leqslant n-3$ and $Xe_n \neq 0$. Then the radical series of X are pairwise non-isomorphic simple modules S_i and X is of the following form

$$\begin{smallmatrix} S_{n-t+1} \\ \vdots \\ S_{n-1} \\ S_n \end{smallmatrix}.$$

We denote X by $[n-t+1, n]$.

Let $T \in Q_s(\Lambda_n; e_n)$. There exists at least one indecomposable direct summand of T, say X, satisfies $Xe_n \neq 0$ and we choose $X = [n-t+1, n]$ of largest possible length t. Then, $Te_m = 0$ for any arrow $m \longrightarrow n-t+1$. In fact, if $t = n-3$, the statement is obvious since $s \leqslant n-3$. Now, assume that $t \leqslant n-4$ and there is an indecomposable direct summand Y of T satisfying $Ye_{n-t} \neq 0$, then $Ye_n = 0$ follows the maximality of X. It is enough to consider the following five types of Y:

$$\begin{smallmatrix} & S_1 & \\ S_2 & & S_3 \\ & S_4 & \\ & \vdots & \\ & S_{n-t} & \\ & \vdots & \\ & S_a & \end{smallmatrix}, \quad \begin{smallmatrix} S_2 & & S_3 \\ & S_4 & \\ & \vdots & \\ & S_{n-t} & \\ & \vdots & \\ & S_a & \end{smallmatrix}, \quad \begin{smallmatrix} S_2 \\ S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{smallmatrix}, \quad \begin{smallmatrix} S_3 \\ S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{smallmatrix}, \quad \begin{smallmatrix} S_4 \\ \vdots \\ S_{n-t} \\ \vdots \\ S_a \end{smallmatrix},$$

where $n-t \leqslant a \leqslant n-1$. One can check that $\mathsf{soc}(\tau(Y)) = S_{a+1}$ for any type above. Then, $a+1 \geqslant n-t+1$ implies $\mathsf{Hom}_{\Lambda_n}(X, \tau(Y)) \neq 0$. We get a contradiction.

Therefore, we can divide T into a direct sum $W \oplus Z$ such that

- the support of W is $\{e_{n-t+1}, \ldots, e_{n-1}, e_n\}$ and it is disjoint with the support of Z.
- the support of Z does not contain e_m with $m \longrightarrow n-t+1$.

We denote by $Q_{[n-t+1,n]}$ the quiver with vertices $\{n-t+1, \ldots, n-1, n\}$ and let

$$\Lambda_{[n-t+1,n]} := kQ_{[n-t+1,n]}.$$

Then, W becomes a τ-tilting $\Lambda_{[n-t+1,n]}$-module and X is the unique indecomposable projective-injective $\Lambda_{[n-t+1,n]}$-module. Deleting X from W, we obtain a support τ-tilting $\Lambda_{[n-t+1,n]}$-module with support-rank $t-1$.

Now, we write $T = X \oplus U$, then U is the direct sum of Z and a support τ-tilting Λ_n-module by deleting X from W. Thus, U is a support τ-tilting Λ_n-module with support-rank $s-1$. The map from $Q_s(\Lambda_n; e_n)$ to $Q_{s-1}(\Lambda_n)$ is given by $\mathfrak{q}(T) = U$.

(1) (Injection) Since X is the indecomposable projective-injective $\Lambda_{[n-t+1,n]}$-module and the maximality of $X = [n-t+1, n]$, $\mathfrak{q}$ is injective.
(2) (Surjection) Let $U \in Q_{s-1}(\Lambda_n)$, then there are at least 4 idempotents outside of the support of U.
- If there are exactly 4 idempotents e_1, e_2, e_3 and e_i with $4 \leqslant i \leqslant n$ outside of the support of U, then $s = n-3$ and U becomes a support τ-tilting $\Lambda_{[4,n]}$-module with support-rank $n-4$. Let $P := [4, n]$ be the indecomposable projective $\Lambda_{[4,n]}$-module, then (U, P) becomes a support τ-tilting pair for $\Lambda_{[4,n]}$. Thus,
$$T := U \oplus P \in Q_{n-3}(\Lambda_n; e_n).$$
- Otherwise, there are at least two idempotents in $\{e_4, e_5, \dots, e_n\}$ outside of the support of U. Let $i < j$ be the smallest such numbers, then we can find an indecomposable projective Λ_n-module $P := [i+1, n]$ such that (U, P) becomes a support τ-tilting pair for Λ_n. Thus,
$$T := U \oplus P \in Q_s(\Lambda_n; e_n).$$

Therefore, $\mathfrak{q}$ is surjective.

□

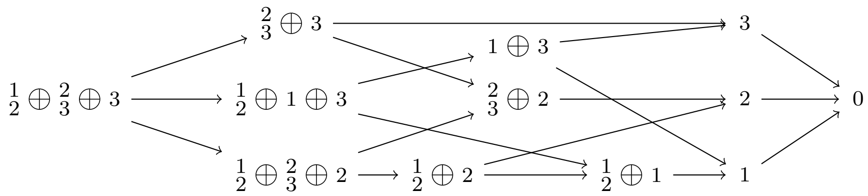
Lemma 6.5. *For any $n \geqslant 4$, we have*
$$a_{n-2}(\Lambda_n) = a_{n-2}(\Lambda_{n-1}) + a_{n-3}(\Lambda_n) + a_{n-3}(\mathbb{A}_{n-3}).$$

Proof. We construct a surjection $\mathfrak{q}$ from $Q_{n-2}(\Lambda_n; e_n)$ to $Q_{n-3}(\Lambda_n)$. Similar to the proof of Lemma 6.4, one can show that any module T in $Q_{n-2}(\Lambda_n; e_n)$ is of the form $T = X \oplus U$, where X is indecomposable, $Xe_n \neq 0$ and X is of maximal possible length. Then the support of X is contained either in $\{e_2, e_4, \dots, e_n\}$ or $\{e_3, e_4, \dots, e_n\}$ such that X is uniquely determined. Therefore, the map $\mathfrak{q}$ is defined by deleting X from T.

Note that $\mathfrak{q}$ is not an injection. Let X_1 be the indecomposable module with support $\{e_2, e_4, \dots, e_n\}$, and X_2 the indecomposable module with support $\{e_3, e_4, \dots, e_n\}$. Starting with a τ-tilting $\Lambda_{[4,n]}$-module U, we have
$$X_1 \oplus U, X_2 \oplus U \in Q_{n-2}(\Lambda_n; e_n).$$
But both of them are mapped to U. These are the $a_{n-3}(\Lambda_{[4,n]}) = a_{n-3}(\mathbb{A}_{n-3})$ pairs of elements of $Q_{n-2}(\Lambda_n; e_n)$, which are identified by $\mathfrak{q}$. □

In order to compute $a_{n-1}(\Lambda_n)$, we consider a factor algebra of $\mathbb{A}_n$. For any $n \geqslant 3$, we define $\mathbb{A}_n^1 := kQ/ < ab >$, where
$$Q: \ 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n \ .$$
Moreover, we assume $\mathbb{A}_2^1 := \mathbb{A}_2$.

Example 6.6. Let P_i be the indecomposable projective $\mathbb{A}_3^1$-modules, then
$$P_1 = {\scriptstyle \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}}, P_2 = {\scriptstyle \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}} \text{ and } P_3 = {\scriptstyle 3}.$$
The Hasse quiver of $\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathbb{A}_3^1$ is as follows.



Then, we can see that $a_0(\mathbb{A}_3^1) = 1, a_1(\mathbb{A}_3^1) = 3, a_2(\mathbb{A}_3^1) = 5$ and $a_3(\mathbb{A}_3^1) = 3$.

Theorem 6.7. *Assume that $\mathbb{A}_2^1 := \mathbb{A}_2$. For any $n \geqslant 4$, we have*

(1) $a_s(\mathbb{A}_n^1) = a_s(\mathbb{A}_{n-1}^1) + a_{s-1}(\mathbb{A}_n^1)$ *for any* $1 \leqslant s \leqslant n-2$.
(2) $a_{n-1}(\mathbb{A}_n^1) = a_{n-1}(\mathbb{A}_{n-1}^1) + a_{n-1}(\mathbb{A}_{n-1}) + a_{n-2}(\mathbb{A}_{n-2}) + \sum_{i=3}^{n-1} a_{i-1}(\mathbb{A}_{i-1}^1) \cdot a_{n-i}(\mathbb{A}_{n-i})$.
(3) $a_n(\mathbb{A}_n^1) = a_{n-1}(\mathbb{A}_{n-1}) + a_{n-2}(\mathbb{A}_{n-2})$.

Proof. (1) The proof is similar to the proof of Lemma 6.4.

(2) Let $T \in Q_{n-1}(\mathbb{A}_n^1)$, there exists exactly one idempotent e_i such that $Te_i = 0$.
- If $i = 1$, then T is a τ-tilting $\mathbb{A}_{n-1}$-module.
- If $i = 2$, then we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting $\mathbb{A}_1$-module and T_2 is a τ-tilting $\mathbb{A}_{n-2}$-module.
- If $3 \leqslant i \leqslant n-1$, then we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting $\mathbb{A}_{i-1}^1$-module and T_2 is a τ-tilting $\mathbb{A}_{n-i}$-module.
- If $i = n$, then T is a τ-tilting $\mathbb{A}_{n-1}^1$-module.

It is easy to see that the above is a complete classification.

(3) Let $T \in Q_n(\mathbb{A}_n^1)$.
- If $T = \frac{1}{2} \oplus U$ with $Ue_1 = 0$, then U is a τ-tilting $\mathbb{A}_{n-1}$-module.
- If $T = \frac{1}{2} \oplus U$ with $Ue_1 \neq 0$, then $U = 1 \oplus V$ with $Ve_1 = 0$. Since $\tau(1) = S_2$, then $Ve_2 = 0$. Thus, V is a τ-tilting $\mathbb{A}_{n-2}$-module.

□

Remark 6.8. For any $n \geqslant 2$ and $0 \leqslant s \leqslant n$, the $a_s(\mathbb{A}_n^1)$ is as follows.

n \ s	0	1	2	3	4	5	6	7	8	9	10	$a(\mathbb{A}_n^1)$
2	1	2	2									5
3	1	3	5	3								12
4	1	4	9	12	7							33
5	1	5	14	26	33	19						98
6	1	6	20	46	79	98	56					306
7	1	7	27	73	152	250	306	174				990
8	1	8	35	108	260	510	816	990	561			3289
9	1	9	44	152	412	922	1738	2728	3289	1859		11154
10	1	10	54	206	618	1540	3278	6006	9295	11154	6292	38454

Based on the above data, we can see that

(1) $a(\mathbb{A}_n^1) = a_n(\mathbb{A}_{n+1}^1)$.
(2) $a_{n-1}(\mathbb{A}_n^1) = a_{n-1}(\mathbb{A}_{n-1}^1) + a_{n-2}(\mathbb{A}_n^1)$.
(3) $a_n(\mathbb{A}_n^1)$ is the sequence A005807 in Sloane's OEIS [58].

Example 6.9. Let $\mathbb{A}_4^{1\text{op}}$ be the opposite algebra of $\mathbb{A}_4^1$ and P_i the indecomposable projective $\mathbb{A}_4^{1\text{op}}$-modules, then

$$P_1 = \begin{smallmatrix}1\\2\\3\end{smallmatrix}, P_2 = \begin{smallmatrix}2\\3\end{smallmatrix}, P_3 = \begin{smallmatrix}3\\4\end{smallmatrix} \text{ and } P_4 = 4.$$

The Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathbb{A}_4^{1\text{op}})$ can verify the formulas in Theorem 6.7 and all support τ-tilting $\mathbb{A}_4^{1\text{op}}$-modules are shown in Appendix B.

Lemma 6.10. *Assume that $\Lambda_3 := \mathbb{A}_3$. For any $n \geqslant 5$, we have*

$$a_{n-1}(\Lambda_n) = a_{n-1}(\Lambda_{n-1}) + a_{n-1}(\mathbb{D}_{n-1}) + 2a_{n-1}(\mathbb{A}_{n-1}^1) + \sum_{i=4}^{n-1} a_{i-1}(\Lambda_{i-1}) \cdot a_{n-i}(\mathbb{A}_{n-i}).$$

Proof. Let $T \in Q_{n-1}(\Lambda_n)$, there exists exactly one idempotent e_i such that $Te_i = 0$.

- If $i = 1$, then T is a τ-tilting $\mathbb{D}_{n-1}$-module.
- If $i = 2$ or 3, then $\beta\nu = 0$ or $\alpha\mu = 0$, and T becomes a τ-tilting $\mathbb{A}_{n-1}^1$-module.
- If $4 \leqslant i \leqslant n-1$, then we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting Λ_{i-1}-module and T_2 is a τ-tilting $\mathbb{A}_{n-i}$-module.
- If $i = n$, then T is a τ-tilting Λ_{n-1}-module.

It is easy to see that the above is a complete classification. □

Lemma 6.11. *Let $n \geqslant 4$, then $a_n(\Lambda_n) = a_{n-1}(\Lambda_n) - 1$.*

Proof. We explain the relation between $Q_n(\Lambda_n)$ and $Q_{n-1}(\Lambda_n)$. Let T be a τ-tilting Λ_n-module and P_1 the indecomposable projective module at vertex 1. Starting with Λ_n,

(1) If $T = P_1 \oplus U$ with $Ue_1 = 0$, then $U \in Q_{n-1}(\Lambda_n)$.
(2) If $T = P_1 \oplus U$ with $Ue_1 \neq 0, Ue_2 = 0, Ue_3 \neq 0$ or $Ue_1 \neq 0, Ue_2 \neq 0, Ue_3 = 0$, then $U \in Q_{n-1}(\Lambda_n)$.
(3) If $T = P_1 \oplus U$ with $Ue_1 \neq 0, Ue_2 \neq 0, Ue_3 \neq 0$, then
 (a) if $T = P_1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with $Ve_1 = 0$, then $Ve_2 = Ve_3 = 0$ since $\tau(\begin{smallmatrix}1\\2\end{smallmatrix}) = S_3$ and $\tau(\begin{smallmatrix}1\\3\end{smallmatrix}) = S_2$. Thus, V is a τ-tilting $\Lambda_{[4,n]}$-module. Furthermore, we have
 $$T \longrightarrow 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \in \mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \Lambda_n).$$
 Note that any left mutation of the latter one goes to $Q_{n-1}(\Lambda_n)$.
 (b) Otherwise, $U \in Q_{n-1}(\Lambda_n)$.

On the other hand, there is a surjection from $Q_{n-1}(\Lambda_n)$ to $Q_n(\Lambda_n)/\mathcal{S}$, where

$$\mathcal{S} = \left\{P_1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V \mid V \text{ is a } \tau\text{-tilting } \Lambda_{[4,n]}\text{-module}\right\}.$$

Let $U \in Q_{n-1}(\Lambda_n)$, then U does not contain P_1 as a direct summand.

- If $U = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus V$ or $1 \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$ with a τ-tilting $\Lambda_{[4,n]}$-module V, then U is mapped to $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$.
- If $U = 1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus Z$ with $Z \in Q_{n-3}(\Lambda_{[4,n]})$ and Z is a left mutation of V, then U is mapped to $1 \oplus \begin{smallmatrix}1\\2\end{smallmatrix} \oplus \begin{smallmatrix}1\\3\end{smallmatrix} \oplus V$.
- Otherwise, U is mapped to $P_1 \oplus U$.

Therefore,

$$\begin{aligned} a_n(\Lambda_n) =& a_{n-1}(\Lambda_n) - (a_{n-4}(\Lambda_{[4,n]}) + 1) + a_{n-3}(\Lambda_{[4,n]}) \\ =& a_{n-1}(\Lambda_n) - (a_{n-4}(\mathbb{A}_{n-3}) + 1) + a_{n-3}(\mathbb{A}_{n-3}). \end{aligned}$$

Note that $a_{n-4}(\mathbb{A}_{n-3}) = a_{n-3}(\mathbb{A}_{n-3})$, then the result follows. □

Finally, we can determine the number of support τ-tilting Λ_n-modules.

Theorem 6.12. *Assume that $\Lambda_3 := \mathbb{A}_3$ and $n \geqslant 5$, then*

(1) $a_n(\Lambda_n) = a_{n-1}(\Lambda_n) - 1$.
(2) $a_{n-1}(\Lambda_n) = a_{n-1}(\Lambda_{n-1}) + \frac{3n-7}{2n-4}\binom{2n-4}{n-3} + 2(\frac{1}{n-1}\binom{2n-4}{n-2} + \frac{1}{n-2}\binom{2n-6}{n-3}) + \sum_{i=4}^{n-1} a_{i-1}(\Lambda_{i-1}) \cdot \frac{1}{n-i+1}\binom{2(n-i)}{n-i}$.
(3) $a_{n-2}(\Lambda_n) = a_{n-2}(\Lambda_{n-1}) + a_{n-3}(\Lambda_n) + \frac{1}{n-2}\binom{2n-6}{n-3}$.
(4) $a_s(\Lambda_n) = a_s(\Lambda_{n-1}) + a_{s-1}(\Lambda_n)$ *for any* $1 \leqslant s \leqslant n-3$.

Proof. It follows Lemma 2.15, 6.4, 6.5, 6.7, 6.10 and 6.11. □

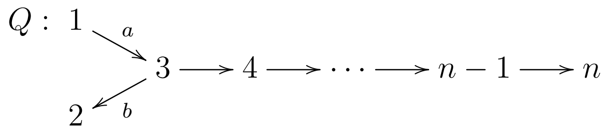
Remark 6.13. Let $n \geqslant 3$ and $0 \leqslant s \leqslant n$, then $a_s(\Lambda_n)$ is as follows.

n \ s	0	1	2	3	4	5	6	7	8	9	10	$a(\Lambda_n)$
3	1	3	5	5								14
4	1	4	10	16	15							46
5	1	5	15	33	54	53						161
6	1	6	21	54	113	193	192					580
7	1	7	28	82	195	402	706	705				2126
8	1	8	36	118	313	715	1463	2618	2617			7889
9	1	9	45	163	476	1191	2654	5404	9803	9802		29548
10	1	10	55	218	694	1885	4539	9943	20175	36984	36983	111487

6.2. **The number of support τ-tilting Θ_n-modules.** The Auslander-Reiten quiver Γ_{Θ_n} of Θ_n is given by Boos [26, Appendix A.2], one can observe that

(1) Θ_n is a representation-finite tiled algebra of type $\mathbb{D}$.
(2) (The number of vertices in $\Gamma_{\mathbb{D}_n}$) $-$ (The number of vertices in Γ_{Θ_n}) $= 1$, and the extra one is a sincere $\mathbb{D}_n$-module.
(3) Every indecomposable Θ_n-module M is a brick, that is, $\mathrm{End}_{\Theta_n} M = k$.

For any $n \geqslant 4$, we define $\mathbb{D}_n^1 := kQ/ < ab >$, where



Moreover, we assume $\mathbb{D}_3^1 := \mathbb{A}_3^1$.

Example 6.14. Let P_i be the indecomposable projective $\mathbb{D}_4^1$-modules, then

$$P_1 = \begin{smallmatrix}1\\3\\4\end{smallmatrix}, P_2 = \begin{smallmatrix}2\end{smallmatrix}, P_3 = \begin{smallmatrix}&3&\\2&&4\end{smallmatrix} \text{ and } P_4 = \begin{smallmatrix}4\end{smallmatrix}.$$

The Hasse quiver $\mathcal{H}(\mathsf{s}\tau\text{-}\mathsf{tilt}\ \mathbb{D}_4^1)$ implies that

$$a_0(\mathbb{D}_4^1) = 1, a_1(\mathbb{D}_4^1) = 4, a_2(\mathbb{D}_4^1) = 9, a_3(\mathbb{D}_4^1) = 14 \text{ and } a_4(\mathbb{D}_4^1) = 9.$$

Then, all support τ-tilting $\mathbb{D}_4^1$-modules are shown in Appendix C.

Theorem 6.15. *For any $n \geqslant 5$, we have*

(1) $a_s(\mathbb{D}_n^1) = a_s(\mathbb{D}_{n-1}^1) + a_{s-1}(\mathbb{D}_n^1)$ *for any* $1 \leqslant s \leqslant n-2$.
(2) $a_{n-1}(\mathbb{D}_n^1) = a_{n-1}(\mathbb{D}_{n-1}^1) + a_{n-2}(\mathbb{D}_n^1) + a_{n-2}(\mathbb{A}_{n-2})$.
(3) $a_n(\mathbb{D}_n^1) = a_{n-1}(\mathbb{D}_{n-1}^1) + a_{n-1}(\mathbb{A}_{n-1}) + a_{n-3}(\mathbb{A}_{n-3}) + \sum_{i=4}^{n-1} a_{i-1}(\mathbb{D}_{i-1}^1) \cdot a_{n-i}(\mathbb{A}_{n-i})$.

Proof. The first statement and the second statement are similar to Lemma 6.4 and 6.5.

Let T be a τ-tilting $\mathbb{D}_n^1$-module and P_1 the indecomposable projective module at vertex 1, then T is of form $P_1 \oplus U$ with $Ue_2 \neq 0$ and $U \in Q_{n-1}(\mathbb{D}_n^1)$.

- If $Ue_1 = 0$, then U is a τ-tilting $\mathbb{A}_{n-1}$-module.
- If $Ue_3 = 0$, then $U = 1 \oplus 2 \oplus V$, where V is a τ-tilting $\mathbb{A}_{n-3}$-module.
- If $Ue_i = 0$ with $4 \leqslant i \leqslant n-1$, then we can divide U into a direct sum $U_1 \oplus U_2$ such that U_1 is a τ-tilting $\mathbb{D}_{i-1}^1$-module and U_2 is a τ-tilting $\mathbb{A}_{n-i}$-module.
- If $Ue_n = 0$, then U is a τ-tilting $\mathbb{D}_{n-1}^1$-module.

Then, we get a complete classification. □

Remark 6.16. For any $n \geqslant 3$ and $0 \leqslant s \leqslant n$, the $a_s(\mathbb{D}_n^1)$ is as follows.

$n \backslash s$	0	1	2	3	4	5	6	7	8	9	10	$a(\mathbb{D}_n^1)$
3	1	3	5	3								12
4	1	4	9	14	9							37
5	1	5	14	28	42	28						118
6	1	6	20	48	90	132	90					387
7	1	7	27	75	165	297	429	297				1298
8	1	8	35	110	275	572	1001	1430	1001			4433
9	1	9	44	154	429	1001	2002	3432	4862	3432		15366
10	1	10	54	208	637	1638	3640	7072	11934	16796	11934	53924

Based on the above data, we can observe that

(1) $a(\mathbb{D}_n^1)$ is the sequence A280891 in Sloane's OEIS [58].
(2) $a_n(\mathbb{D}_n^1) = a_{n-2}(\mathbb{D}_n^1) = a_{n-2}(\mathbb{A}_n)$ is the sequence A000245.
(3) $a_{n-1}(\mathbb{D}_n^1) = a_n(\mathbb{A}_n)$ is just the Catalan number sequence.

Theorem 6.17. *Assume that* $\Theta_3 := \mathbb{A}_3^1$ *and* $n \geqslant 5$, *then*

(1) $a_s(\Theta_n) = a_s(\Theta_{n-1}) + a_{s-1}(\Theta_n)$ *for any* $1 \leqslant s \leqslant n-3$.
(2) $a_{n-2}(\Theta_n) = a_{n-2}(\Theta_{n-1}) + a_{n-3}(\Theta_n) + a_{n-3}(\mathbb{A}_{n-3})$.
(3) $a_{n-1}(\Theta_n) = a_{n-1}(\Theta_{n-1}) + a_{n-1}(\mathbb{D}_{n-1}^1) + 2a_{n-1}(\mathbb{A}_{n-1}) + \sum_{i=4}^{n-1} a_{i-1}(\Theta_{i-1}) \cdot a_{n-i}(\mathbb{A}_{n-i})$.

Proof. The first statement and the second statement are similar to Lemma 6.4 and 6.5.

Let $T \in Q_{n-1}(\Theta_n)$, there exists exactly one idempotent e_i such that $Te_i = 0$.

- If $i = 1$, then $\mu\nu = 0$ and T is a τ-tilting $\mathbb{D}_{n-1}^1$-module.
- If $i = 2$ or 3, then T becomes a τ-tilting $\mathbb{A}_{n-1}$-module.
- If $4 \leqslant i \leqslant n-1$, then we can divide T into a direct sum $T_1 \oplus T_2$ such that T_1 is a τ-tilting Θ_{i-1}-module and T_2 is a τ-tilting $\mathbb{A}_{n-i}$-module.
- If $i = n$, then T is a τ-tilting Θ_{n-1}-module.

The above is a complete classification. □

We don't have a good method to calculate $a_n(\Theta_n)$, but with the structure of Γ_{Θ_n} mentioned at the beginning of this subsection, we have the following conjecture.

Conjecture 6.18. For any $n \geqslant 5$, we have $a_n(\Theta_n) = a_n(\mathbb{D}_n) - a_{n-1}(\mathbb{A}_{n-1})$.

In fact, Example 6.3 and Theorem 6.17 imply that $a_3(\Theta_3) = a_3(\mathbb{D}_3) - a_2(\mathbb{A}_2)$ and $a_4(\Theta_4) = a_4(\mathbb{D}_4) - a_3(\mathbb{A}_3)$. Similarly, we have $a_3(\Theta_4) = a_3(\mathbb{D}_4)$ and $a_4(\Theta_5) = a_4(\mathbb{D}_5)$, then it is natural to conjecture that $a_{n-1}(\Theta_n) = a_{n-1}(\mathbb{D}_n)$ for any $n \geqslant 6$.

Moreover, one can easily check by Theorem 6.17 (3) that for any $n \geqslant 5$,

$$a_{n-1}(\Theta_n) = a_{n-1}(\mathbb{D}_n)$$

if and only if

$$a_n(\Theta_n) = a_n(\mathbb{D}_n) - a_{n-1}(\mathbb{A}_{n-1}).$$

Remark 6.19. Assume that $n \geqslant 3$, $0 \leqslant s \leqslant n$ and Conjecture 6.18 is true, the $a_s(\Theta_n)$ is as follows.

n \ s	0	1	2	3	4	5	6	7	8	9	10	$a(\Theta_n)$
3	1	3	5	3								12
4	1	4	10	16	15							46
5	1	5	15	33	55	63						172
6	1	6	21	54	114	196	252					644
7	1	7	28	82	196	406	714	990				2424
8	1	8	36	118	314	720	1476	2640	3861			9174
9	1	9	45	163	477	1197	2673	5445	9867	15015		34892
10	1	10	55	218	695	1892	4565	10010	20306	37180	58344	133276

Appendix A.

(1) All τ-tilting Λ_4-modules are

$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2$
$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$
$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$
$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 3\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}&1&\\2&&3\\&4&\end{smallmatrix}\oplus 2\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	

(2) All support τ-tilting Λ_4-modules with support-rank 3 are

$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$2\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$3\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 4$
$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$3\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$1\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 4$
$2\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus\begin{smallmatrix}&1&\\2&&3\end{smallmatrix}$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}2&&3\\&4&\end{smallmatrix}$	$1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$		

(3) All support τ-tilting Λ_4-modules with support-rank $0 \leqslant s \leqslant 2$ are

$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3$	$1\oplus 4$	1	4	0
$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}2\\4\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 1$	$2\oplus 3$	2	3	

Appendix B.

(1) All τ-tilting $\mathbb{A}_4^{1\mathrm{op}}$-modules are

$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 1\oplus 4$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2\oplus 4$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2\oplus 4$
$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 1\oplus 3$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1\oplus 4$	

(2) All support τ-tilting $\mathbb{A}_4^{1\mathrm{op}}$-modules with support-rank 3 are

$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\oplus 1\oplus 3$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2\oplus 4$
$4\oplus\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3$	$1\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 1\oplus 3$	$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2\oplus 4$

(3) All support τ-tilting $\mathbb{A}_4^{1\mathrm{op}}$-modules with support-rank $0 \leqslant s \leqslant 2$ are

$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 2$	$1\oplus 3$	$2\oplus 4$	1	2
$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}2\\3\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\2\end{smallmatrix}\oplus 1$	$1\oplus 4$	0	3	4

Appendix C.

(1) All τ-tilting $\mathbb{D}_4^{1}$-modules are

$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus 2\oplus 4$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 2\oplus 1\oplus 4$
$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 2\oplus 1\oplus\begin{smallmatrix}1\\3\end{smallmatrix}$
$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}3\\2\end{smallmatrix}$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 3\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}$

(2) All support τ-tilting $\mathbb{D}_4^{1}$-modules with support-rank 3 are

$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus 1\oplus 4$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}1\\3\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$
$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus 2\oplus 4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus 2\oplus\begin{smallmatrix}3\\2\end{smallmatrix}$	$2\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 1$	$\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}3\\2\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$
$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}&3&\\2&&4\end{smallmatrix}\oplus\begin{smallmatrix}3\\4\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}$	$2\oplus\begin{smallmatrix}1\\3\end{smallmatrix}\oplus\begin{smallmatrix}3\\2\end{smallmatrix}$	$1\oplus 2\oplus 4$	

(3) All support τ-tilting $\mathbb{D}_4^{1}$-modules with support-rank $0 \leqslant s \leqslant 2$ are

$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 4$	$\begin{smallmatrix}3\\2\end{smallmatrix}\oplus 2$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 3$	$1\oplus 2$	$2\oplus 4$	1	2
$\begin{smallmatrix}3\\4\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}3\\2\end{smallmatrix}\oplus 3$	$\begin{smallmatrix}1\\3\end{smallmatrix}\oplus 1$	$1\oplus 4$	0	3	4

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Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, 1-5 Yamadaoka, Suita, Osaka 565-0871, Japan
E-mail address: q.wang@ist.osaka-u.ac.jp