ON SOME PROPERTIES OF THE FUNCTION OF THE NUMBER OF RELATIVELY PRIME SUBSETS OF $\{1, 2, ..., n\}$

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Abstract. In the paper we solve few problems proposed by Prapanpong Pongsriiam. Let $f(n)$ denote the number of relatively prime subsets of $\{1, 2, 3, \ldots, n\}$ and $g(n)$ denote the number of subsets A of $\{1, 2, 3, \ldots, n\}$ such that $\gcd(A) > 1$ and $\gcd(A, n+1) = 1$. We show that $f_n^2 - f_{n-k}f_{n+k} > 0$ for $n \geq k+1 \quad (k \geq 2)$. We also show $\frac{g(6n-2)}{g(6n-4)} > \frac{g(6n)}{g(6n-2)} > \frac{g(6n+2)}{g(6n)} < \frac{g(6n+4)}{g(6n+2)}$ for large n.

1. Introduction

A finite set A is said to be relatively prime if $\gcd(A) = 1$.

Let $fX)$ denote the number of relatively prime subsets of X.

Let $f(n)$ be the number of relatively prime subsets of $\{1, 2, 3, \ldots, n\}$ in other words $f(n) = f([1, n])$. Sometimes we write f_n instead of $f(n)$.

Moreover, define function $g(n)$ by formula

$$g(n) = \sum_{\substack{\emptyset \neq A \subseteq [1,n] \\ \gcd(A)>1 \\ \gcd(A,n+1)=1}} 1. \tag{1.1}$$

We will use two inequalities

Lemma 1.1 ([1], Theorem 2)**.**

$$2^n - 2^{\lfloor \frac{n}{2} \rfloor} - n2^{\lfloor \frac{n}{3} \rfloor} \leq f(n) \leq 2^n - 2^{\lfloor \frac{n}{2} \rfloor}. \tag{1.2}$$

Moreover, we know that (Lemma 4 in [4])

$$g(n) = \sum_{\substack{2 \leq d \leq n \\ (d,n+1)=1}} f\left(\left\lfloor \frac{n}{d} \right\rfloor\right). \tag{1.3}$$

and

$$f(n) = \sum_{d \leq n} \mu(d) \left(2^{\lfloor \frac{n}{d} \rfloor} - 1\right). \tag{1.4}$$

More information on the function f_n can be found in the sequence A085945 in [5].

In paper [4] Pongsriiam proved that $f_n^2 - f_{n-1}f_{n+1}$ is positive for every odd number $n \geq 3$ and negative for every even number n.

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Recall that a sequence $(a_n)_{n\geq 0}$ is said to be log-concave if $a_n^2 - a_{n-1}a_{n+1} > 0$ for every $n > 1$ and is said to be log-convex if $a_n^2 - a_{n-1}a_{n+1} < 0$ for every $n > 1$. Stirling numbers, Bessel numbers are examples of log-concave sequences. Some sequences are not log-concave, but have similar properties. For example, if $(F_n)_{n\geq 0}$ is the Fibonacci sequence or $F_n = f_n$, then $F_n^2 - F_nF_{n+1} = (-1)^{n-1}$, which is positive for odd n and negative for even n (so called alternating sequence). In addition, the sequence $(f_n)_{n\geq 1}$ seems to have strong log-property ((Recall that $(a_n)_{n\geq 0}$ is said to be strong log-concave if $a_n^2 - a_{n-k}a_{n+k} > 0$ for every $k \geq 1$ and $n > k$)). For example, in the paper [4] Pongsriiam checked that $f_n^2 - f_{n-2}f_{n+2} > 0 (\text{for} 2 < n \leq 50)$, $f_n^2 - f_{n-3}f_{n+3} > 0 (\text{for} 3 < n \leq 50)$ and $f_n^2 - f_{n-4}f_{n+4} > 0 (\text{for} 4 < n \leq 50)$. In our paper we prove that these inequalities are true for all $n > 2, 3, 4$, respectively.

In this paper we prove that $f_n^2 - f_{n-k}f_{n+k} > 0$(for large $n \geq k + 1$ and $k \geq 2$)

We also propose new term :almost strong log-concave sequence if $a_n^2 - a_{n-k}a_{n+k} > 0$ for every $k \geq k_0$ and $n > k$) for some constant $k_0 \geq 2$.

In paper [4] Pongsriiam also asked is it true that

$$\frac{g(6n-2)}{g(6n-4)} > \frac{g(6n)}{g(6n-2)} > \frac{g(6n+2)}{g(6n)} < \frac{g(6n+4)}{g(6n+2)}. \tag{1.5}$$

In Section.3. we prove above inequalities for large n.

2. Sign of $f_n^2 - f_{n-k}f_{n+k} > 0$ for $n > k$ in general

First, using formula (1.4), we can write the following GP/PARI code :

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a(n) = sum(k = 1, n, moebius(k) * (2^floor(n/k) - 1))
for(n = 6, 50, print(a(n)^2 - a(n - l) * a(n + l))).
for(l = 2..8)
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We obtain that inequality $f_n^2 - f_{n-k}f_{n+k} > 0$ is true for $k = 2, 3, ..., 8$ and $n \leq 50$.

Using estimation (1.2) we get

$$\begin{aligned} f_n^2 &\geq \left(2^n - 2^{\lfloor \frac{n}{2} \rfloor} - n2^{\lfloor \frac{n}{3} \rfloor}\right)^2 \\ &= 2^{2n} - 2^{n+\lfloor \frac{n}{2} \rfloor+1} - n2^{n+\lfloor \frac{n}{3} \rfloor+1} + 2^{2\cdot\lfloor \frac{n}{2} \rfloor} + n2^{\lfloor \frac{n}{2} \rfloor+\lfloor \frac{n}{3} \rfloor+1} + n^2 2^{2\cdot\lfloor \frac{n}{3} \rfloor} \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} f_{n-k}f_{n+k} &\leq \left(2^{n-k} - 2^{\lfloor \frac{n-k}{2} \rfloor}\right)\left(2^{n+k} - 2^{\lfloor \frac{n+k}{2} \rfloor}\right) \\ &= 2^{2n} - 2^{n+\lfloor \frac{n+k}{2} \rfloor-k} - 2^{n+\lfloor \frac{n-k}{2} \rfloor+k} + 2^{\lfloor \frac{n-k}{2} \rfloor+\lfloor \frac{n+k}{2} \rfloor} \end{aligned} \tag{2.2}$$

So

$$\begin{aligned} f_n^2 - f_{n-k}f_{n+k} \geq\ & 2^{n+\lfloor \frac{n+k}{2} \rfloor} - 2^{n+\lfloor \frac{n}{2} \rfloor+1} + 2^{n-k+\lfloor \frac{n+k}{2} \rfloor} - n2^{n+\lfloor \frac{n}{3} \rfloor+1} \\ & + 2^{2\cdot\lfloor \frac{n}{2} \rfloor} - 2^{\lfloor \frac{n+k}{2} \rfloor+\lfloor \frac{n-k}{2} \rfloor} + n2^{\lfloor \frac{n}{2} \rfloor+\lfloor \frac{n}{3} \rfloor+1} + n^2 2^{2\cdot\lfloor \frac{n}{3} \rfloor} \end{aligned} \tag{2.3}$$

2.1. Case k=2.

$$\begin{aligned} f_{n-2}f_{n+2} &\leq \left(2^{n-2} - 2^{\lfloor \frac{n-2}{2} \rfloor}\right)\left(2^{n+2} - 2^{\lfloor \frac{n+2}{2} \rfloor}\right) \\ &= 2^{2n} - 2^{n+\lfloor \frac{n}{2} \rfloor+1} - 2^{n+\lfloor \frac{n}{2} \rfloor-1} + 2^{2\cdot\lfloor \frac{n}{2} \rfloor} \end{aligned} \tag{2.4}$$

We show that $f_n^2 - f_{n-2}f_{n+2}$ for $n \geq 51$.

(2.5)
$$f_n^2 - f_{n-2}f_{n+2} \geq 2^{n+\lfloor\frac{n}{2}\rfloor-1} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor}$$
$$> 2^{n+\lfloor\frac{n}{2}\rfloor-1} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} > 2^{n+\lfloor\frac{n}{2}\rfloor-1} - 2^{n+\lfloor\frac{n}{3}\rfloor+\log_2 n+1} > 0,$$

because $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - \log_2 n - 2 \geq \frac{n-1}{2} - \frac{n}{3} - \log_2 n - 2 = \frac{n-6\log_2 n-15}{6} > 0$ for $n \geq 51$. (Consider function $h(x) = x - 6\log_2 x - 15$, $h(51) = 36 - 6\log_2 55 > 0$, $h'(x) = 1 - \frac{6}{x\ln 2} > 0$ for $x \geq 51 \geq 6\ln 2$).

2.2. **Case k=3.**

$$f_n^2 - f_{n-3}f_{n+3} \geq 2^{n+\lfloor\frac{n+1}{2}\rfloor+1} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-2+\lfloor\frac{n+1}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1}$$
$$+ 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{n-1}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \quad (2.6)$$

We show that $f_n^2 - f_{n-3}f_{n+3}$ for $n \geq 51$.
First we see that

$$2^{n+\lfloor\frac{n+1}{2}\rfloor+1} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+1}{2}\rfloor+\lfloor\frac{n-1}{2}\rfloor} > 0 \quad (2.7)$$

So, it is enough to prove that

$$2^{n-2+\lfloor\frac{n+1}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} \geq 0 \quad (2.8)$$

for $n \geq 51$.
We have $\lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - 3 - \log_2 n \geq \frac{n}{2} - \frac{n}{3} - 3 - \log_2 n$.
Consider function $h(x) = \frac{x}{2} - \frac{x}{3} - 3 - \log_2 x$. We have $h(51) = \frac{33}{6} - \log_2 51 > 0$ and $h'(x) = \frac{1}{6} - \frac{1}{x\ln 2}, h'(x) > 0$ for $x > 51 > \frac{6}{\ln 2}$. So inequality (2.8) is true.

2.3. **Case k=4.**

$$f_n^2 - f_{n-4}f_{n+4} \geq 2^{n+\lfloor\frac{n}{2}\rfloor+2} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-2+\lfloor\frac{n}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1}$$
$$+ n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \quad (2.9)$$

$$f_n^2 - f_{n-4}f_{n+4} \geq 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-2+\lfloor\frac{n}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1}$$
$$+ n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \quad (2.10)$$

Now inequality (2.8) implies statement.

2.4. **Case k=5.**

$$f_n^2 - f_{n-5}f_{n+5} \geq 2^{n+\lfloor\frac{n+5}{2}\rfloor} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-5+\lfloor\frac{n+5}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1}$$
$$+ 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+5}{2}\rfloor+\lfloor\frac{n-5}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \quad (2.11)$$

We prove that for $n \geq 36$ above term is positive.
This term is great than

$$2^{n+\lfloor\frac{n+1}{2}\rfloor+1} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} = 2^{n+\lfloor\frac{n}{3}\rfloor+1}\left(2^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{n}{3}\rfloor} - 2^{\log_2 n}\right) \quad (2.12)$$

$\lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{n}{3}\rfloor - \log_2 n \geq \frac{n}{2} - \frac{n}{3} - \log_2 n \geq 0$, for $n \geq 36$.

2.5. Case k=6.

$$f_n^2 - f_{n-6}f_{n+6} \geq 2^{n+\lfloor\frac{n+6}{2}\rfloor} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-6+\lfloor\frac{n+6}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} + 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+6}{2}\rfloor+\lfloor\frac{n-6}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \tag{2.13}$$

We prove that for $n \geq 36$ above term is positive.
This term is great than

$$2^{n+\lfloor\frac{n}{2}\rfloor+2} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} = 2^{n+\lfloor\frac{n}{3}\rfloor+1}\left(2^{\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor+1} - 2^{\log_2 n}\right), \tag{2.14}$$

but $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor + 1 - \log_2 n \geq \frac{n-1}{2} - \frac{n}{3} + 1 - \log_2 n \geq 0$, for $n \geq 36$.

2.6. Case k=7.

$$f_n^2 - f_{n-7}f_{n+7} \geq 2^{n+\lfloor\frac{n+7}{2}\rfloor} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-7+\lfloor\frac{n+7}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} + 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+7}{2}\rfloor+\lfloor\frac{n-7}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \tag{2.15}$$

We prove that for $n \geq 36$ above term is positive.
This term is great than

$$2^{n+\lfloor\frac{n+1}{2}\rfloor+2} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} = 2^{n+\lfloor\frac{n}{3}\rfloor+1}\left(2^{\lfloor\frac{n+1}{2}\rfloor-\lfloor\frac{n}{3}\rfloor+1} - 2^{\log_2 n}\right), \tag{2.16}$$

but $\lfloor\frac{n+1}{2}\rfloor - \lfloor\frac{n}{3}\rfloor + 1 - \log_2 n \geq \frac{n}{2} - \frac{n}{3} + 1 - \log_2 n \geq 0$, for $n \geq 36$.

2.7. Case k=8.

$$f_n^2 - f_{n-8}f_{n+8} \geq 2^{n+\lfloor\frac{n+8}{2}\rfloor} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-8+\lfloor\frac{n+8}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} + 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+8}{2}\rfloor+\lfloor\frac{n-8}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \tag{2.17}$$

We prove that for $n \geq 36$ above term is positive.
This term is great than

$$2^{n+\lfloor\frac{n}{2}\rfloor+3} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} = 2^{n+\lfloor\frac{n}{3}\rfloor+1}\left(2^{\lfloor\frac{n}{2}\rfloor-\lfloor\frac{n}{3}\rfloor+2} - 2^{\log_2 n}\right), \tag{2.18}$$

but $\lfloor\frac{n}{2}\rfloor - \lfloor\frac{n}{3}\rfloor + 2 - \log_2 n \geq \frac{n-1}{2} - \frac{n}{3} + 2 - \log_2 n \geq 0$, for $n \geq 36$.

2.8. Case $k \geq 9$.

$$f_n^2 - f_{n-k}f_{n+k} \geq 2^{n+\lfloor\frac{n+k}{2}\rfloor} - 2^{n+\lfloor\frac{n}{2}\rfloor+1} + 2^{n-k+\lfloor\frac{n+k}{2}\rfloor} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} + 2^{2\cdot\lfloor\frac{n}{2}\rfloor} - 2^{\lfloor\frac{n+k}{2}\rfloor+\lfloor\frac{n-k}{2}\rfloor} + n2^{\lfloor\frac{n}{2}\rfloor+\lfloor\frac{n}{3}\rfloor+1} + n^2 2^{2\cdot\lfloor\frac{n}{3}\rfloor} \tag{2.19}$$

We prove that for $n \geq k+1$ above term is positive.
This term is great than

$$2^{n+\lfloor\frac{n+k}{2}\rfloor-1} - n2^{n+\lfloor\frac{n}{3}\rfloor+1} \tag{2.20}$$

It is enough to show that

$$(n + \left\lfloor\frac{n+k}{2}\right\rfloor - 1) - (n + \left\lfloor\frac{n}{3}\right\rfloor + 1 + \log_2 n) \geq 0, \tag{2.21}$$

but

(2.22)

$$(n+\left\lfloor\frac{n+k}{2}\right\rfloor-1)-(n+\left\lfloor\frac{n}{3}\right\rfloor+1+\log_2 n)\geq\frac{n+k-1}{2}-1-\frac{n}{3}-1-\log_2 n$$
$$=\frac{n}{6}+\frac{k-5}{2}-\log_2 n$$

Let $i(x)=\frac{x}{6}+\frac{k-5}{2}-\log_2 x$, $i(k+1)=\frac{4k-14}{6}-\log_2(k+1)$. Function i(x) is increasing for $x\geq 8$ and $i(k+1)>0$ for $k\geq 9$.

3. Proof that $\frac{g(6n-2)}{g(6n-4)}>\frac{g(6n)}{g(6n-2)}>\frac{g(6n+2)}{g(6n)}<\frac{g(6n+4)}{g(6n+2)}$ for large n.

3.1. $\frac{g(6n-2)}{g(6n-4)}>\frac{g(6n)}{g(6n-2)}$. Above inequality is equivalent to inequality

$$[g(6n-2)]^2>g(6n)g(6n-4). \tag{3.1}$$

Using (1.3) we get estimation

$$\begin{aligned} g(6n-2)=&\sum_{\substack{2\leq d\leq 6n-2\\(d,6n-1)=1}} f\left(\left\lfloor\frac{6n-2}{d}\right\rfloor\right)=f(3n-1)+f(2n-1)\\ &+f\left(\left\lfloor\frac{3n-1}{2}\right\rfloor\right)+\chi_5(6n-1)f\left(\left\lfloor\frac{6n-2}{5}\right\rfloor\right)+C_1(n), \end{aligned} \tag{3.2}$$

where $C_1(n)=\sum_{\substack{6\leq d\leq 6n-2\\(d,6n-1)=1}} f\left(\left\lfloor\frac{6n-2}{d}\right\rfloor\right)\leq 6(n-1)f(n-1)\leq 6(n-1)2^{n-1}$.

Subsequently

$$\begin{aligned} g(6n-4)=&\sum_{\substack{2\leq d\leq 6n-4\\(d,6n-3)=1}} f\left(\left\lfloor\frac{6n-4}{d}\right\rfloor\right)=f(3n-2)+f\left(\left\lfloor\frac{3n}{2}\right\rfloor-1\right)\\ &+\chi_5(6n-3)f\left(\left\lfloor\frac{6n-4}{5}\right\rfloor\right)+C_2(n), \end{aligned} \tag{3.3}$$

where $C_2(n)=\sum_{\substack{6\leq d\leq 6n-4\\(d,6n-3)=1}} f\left(\left\lfloor\frac{6n-4}{d}\right\rfloor\right)\leq 6(n-1)f(n-1)\leq 6(n-1)2^{n-1}$.

Similarly

$$\begin{aligned} g(6n)=&\sum_{\substack{2\leq d\leq 6n\\(d,6n+1)=1}} f\left(\left\lfloor\frac{6n}{d}\right\rfloor\right)=f(3n)+f(2n)+f\left(\left\lfloor\frac{3n}{2}\right\rfloor\right)\\ &+\chi_5(6n+1)f\left(\left\lfloor\frac{6n}{5}\right\rfloor\right)+f(n)+C_3(n), \end{aligned} \tag{3.4}$$

where $C_3(n)=\sum_{\substack{7\leq d\leq 6n\\(d,6n+1)=1}} f\left(\left\lfloor\frac{6n}{d}\right\rfloor\right)\leq 6(n-1)f(n-1)\leq 6(n-1)2^{n-1}$.

Now, (3.1) is equivalent to inequality

(3.5)

$$\left[f(3n-1)+f(2n-1)+f\left(\left\lfloor\frac{3n-1}{2}\right\rfloor\right)+\chi_5(6n-1)f\left(\left\lfloor\frac{6n-2}{5}\right\rfloor\right)+C_1(n)\right]^2$$
$$>\left[f(3n)+f(2n)+f\left(\left\lfloor\frac{3n}{2}\right\rfloor\right)+\chi_5(6n+1)f\left(\left\lfloor\frac{6n}{5}\right\rfloor\right)+f(n)+C_3(n)\right]$$
$$\cdot\left[f(3n-2)+f\left(\left\lfloor\frac{3n}{2}\right\rfloor-1\right)+\chi_5(6n-3)f\left(\left\lfloor\frac{6n-4}{5}\right\rfloor\right)+C_2(n)\right]$$

Above inequality after calculation, cancellation summands which are $O\left(2^{\frac{9}{2}n}\right)$, leads to inequality

$$[f(3n-1)]^2+2f(3n-1)f(2n-1)>f(3n)f(3n-2)+f(3n-2)f(2n), \tag{3.6}$$

which is true for large n. So the inequality $\frac{g(6n-2)}{g(6n-4)}>\frac{g(6n)}{g(6n-2)}$ is true for large n.

3.2. $\frac{g(6n)}{g(6n-2)}>\frac{g(6n+2)}{g(6n)}$. Above inequality is equivalent to inequality

$$[g(6n)]^2>g(6n-2)g(6n+2). \tag{3.7}$$

Above inequality after calculation, cancellation summands which are $O\left(2^{\frac{9}{2}n}\right)$, leads to inequality

$$[f(3n)]^2+2f(3n)f(2n)>f(3n-1)f(3n+1)+f(2n-1)f(3n+1), \tag{3.8}$$

which is true for large n.

3.3. $\frac{g(6n+2)}{g(6n)}<\frac{g(6n+4)}{g(6n+2)}$. Above inequality is equivalent to inequality

$$[g(6n+2)]^2>g(6n)g(6n+4). \tag{3.9}$$

Above inequality after calculation, cancellation summands which are $O\left(2^{\frac{9}{2}n}\right)$, leads to inequality

$$[f(3n+1)]^2<f(3n)f(2n+1)+f(2n)f(3n+2), \tag{3.10}$$

Using inequalities from Lemma.1.1. we can prove that above inequality is true for large n.

We have proved that exists natural number n_0 such that inequality from the title of the section is true for $n\geq n_0$. Exact value n_0 needs more careful calculation.

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