

Generating Posets beyond N

Uli Fahrenberg¹, Christian Johansen², Georg Struth³, Ratan Badahur Thapa²

¹ École Polytechnique, Palaiseau, France

² University of Oslo, Norway

³ University of Sheffield, UK

Abstract. We introduce iposets—posets with interfaces—equipped with a novel gluing composition along interfaces and the standard parallel composition. We study their basic algebraic properties as well as the hierarchy of gluing-parallel posets generated from singletons by finitary applications of the two compositions. We show that not only series-parallel posets, but also interval orders, which seem more interesting for modelling concurrent and distributed systems, can be generated, but not all posets. Generating posets is also important for constructing free algebras for concurrent semirings and Kleene algebras that allow compositional reasoning about such systems.

1 Introduction

This work is inspired by Tony Hoare’s programme of building graph models of concurrent Kleene algebra (CKA) [12] for real-world applications. CKA extends the sequential compositions, nondeterministic choices and unbounded finite iterations of imperative programs modelled by Kleene algebra into concurrency, adding operations of parallel composition and iteration, and a weak interchange law for the sequential-parallel interaction. Such algebras have a long history in concurrency theory, dating back at least to Winkowski [35]. Commutative Kleene algebra—the parallel part of CKA—has been investigated by Pilling and Conway [2]. A double semiring with weak interchange—CKA without iteration—has been introduced by Gischer [8]; its free algebras have been studied by Bloom and Ésik [1]. CKA, like Gischer’s concurrent semiring, has both interleaving and true concurrency models, e.g. shuffle as well as pomset languages. Series-parallel pomset languages, which are generated from singletons by finitary applications of sequential and parallel compositions, form free algebras in this class [19, 22] (at least when parallel iteration is ignored). The inherent compositionality of algebra is thus balanced by the generative properties of this model. Yet despite this and other theoretical work, applications of CKA remain rare.

One reason is that series-parallel pomsets are not expressive enough for many real-world applications: even simple producer-consumer examples cannot be modelled [24]. *Tests*, which are needed for the control structure of concurrent programs and as assertions, are hard to capture in models of CKA (see [17] and its discussion in [18]). Finally, it remains unclear how modal operators could be

defined over graph models akin to pomset languages, which is desirable for concurrent dynamic algebras and logics beyond alternating nondeterminism [7, 28].

A natural approach to generating more expressive pomset languages is to “cut across” pomsets in more general ways when (de)composing them. This can be achieved by (de)composing along interfaces, and this idea can be traced back again to Winkowski [35]; see also [3, 4, 25] for interface-based compositions of graphs and posets, or [13, 26, 27] for recent interface-based graph models for CKA. As a side effect, interfaces may yield notions of tests or modalities. When they consist of events, cutting across them presumes that they extend in time and thus form intervals. Interval orders [5, 34] of events with duration have been applied widely in partial order semantics of concurrent and distributed systems [15, 20, 21, 30–33] and the verification of weak memory models [11], yet generating them remains an open problem [16].

Our main contribution lies in a new class and algebra of posets with interfaces (*iposets*) based on these ideas. We introduce a new gluing composition that acts like standard serial $\text{po}(m)$ set composition outside of interfaces, yet glues together interface events, thus composing events that did not end in one component with those that did not start in the other one. Our definitions are categorical so that isomorphism classes of posets are considered *ab initio*. Their decoration with labels is then trivial, so that we may focus on posets instead of pomsets.

Our main technical results concern the hierarchy of gluing-parallel posets generated by finitary applications of this gluing composition and the standard parallel composition of $\text{po}(m)$ sets, starting from singleton iposets.⁴ It is obvious that all series-parallel pomsets can be generated, but also all interval orders are captured at the second alternation level of the hierarchy. Beyond that, we show that the gluing-parallel hierarchy does not collapse and that posets with certain zigzag-shaped induced subposets are excluded. Yet a precise characterisation of the generated (i)posets remains open. Series-parallel posets, by comparison, exclude precisely those posets with induced N-shaped subposets; interval orders exclude precisely those with induced subposets $2+2$, which makes the two classes incomparable. Iposets thus retain at least the pleasant compositionality properties of series-parallel pomsets and the wide applicability of interval orders in concurrency and distributed computing.

In addition, we establish a bijection between isomorphism classes of interval orders and certain equivalence classes of interval sequences [30], and we study the basic algebraic properties of iposets, including weak interchange laws and a Levi lemma. The relationship between gluing-parallel ipo(m)set languages and CKA is left for another article.

2 Posets and Series-Parallel Posets

A *poset* (P, \leq) is a set P equipped with a *partial order* \leq ; a reflexive, transitive, antisymmetric relation \leq on P . A *morphism* of posets P and Q is an order-

⁴ There is only one singleton poset, but with interfaces, there are *four* singleton iposets.

preserving function $f : P \rightarrow Q$, that is, $x \leq_P y$ implies $f(x) \leq_Q f(y)$. Posets and their morphisms define the category \mathbf{Pos} .

A poset is *linear* if each pair of elements is comparable with respect to its order. We write $<$ for the strict part of \leq . We write $[n]$, for $n \geq 1$, for the *discrete n -poset* $(\{1, \dots, n\}, \leq)$, which satisfies $i \leq j \Leftrightarrow i = j$. Additionally, $[0] = \emptyset$.

The isomorphisms in \mathbf{Pos} are *order bijections*: bijective functions $f : P \rightarrow Q$ for which $x \leq_P y \Leftrightarrow f(x) \leq_Q f(y)$. We write $P \cong Q$ if posets P and Q are isomorphic. We generally consider posets up-to isomorphism and assume, moreover, that all posets are finite.

Concurrency theory often considers (isomorphism classes of) posets with points labelled by letters from some alphabet, which represent actions of some concurrent system. These are known as *partial words* or *pomsets*. As we are mainly interested in structural aspects of concurrency, we ignore such labels.

Series-parallel posets form a well investigated class that can be generated from the singleton poset by finitary applications of two compositions. Their labelled variants generalise rational languages into concurrency. For arbitrary posets, these compositions are defined as follows.

Definition 1. Let $P_1 = (P_1, \leq_1)$ and $P_2 = (P_2, \leq_2)$ be posets.

1. Their serial composition is the poset $P; Q = (P \sqcup Q, \leq_1 \cup \leq_2 \cup P_1 \times P_2)$.
2. Their parallel composition is the poset $P_1 \otimes P_2 = (P_1 \sqcup P_2, \leq_1 \cup \leq_2)$.

Here, \sqcup means disjoint union (coproduct) of sets. We generalise serial composition to a gluing composition in Section 4, after equipping posets with interfaces.

Serial and parallel compositions respect isomorphism, and $[n + m]$ is isomorphic to $[n] \otimes [m]$ with isomorphism $\varphi_{n,m} : [n + m] \rightarrow [n] \otimes [m]$ given by

$$\varphi_{n,m}(i) = \begin{cases} i_{[n]} & \text{if } i \leq n, \\ (i - n)_{[m]} & \text{if } i > n. \end{cases}$$

By definition, a poset is *series-parallel* (an *sp-poset*) if it is either empty or can be obtained from the singleton poset by applying the serial and parallel compositions a finite number of times. It is well known [10, 29] that a poset is series-parallel iff it does not contain the induced subposet $\mathbf{N} = \left(\begin{array}{c} \cdot \\ \nearrow \\ \cdot \\ \cdot \\ \searrow \\ \cdot \end{array} \right)$.⁵

Sp-po(m)sets form bi-monoids with respect to serial and parallel composition, and with the empty poset as shared unit—in fact the free algebras in this class. Compositionality of the recursive definition of sp-po(m)sets is thus reflected by the compositionality of their algebraic properties, which is often considered a desirable property of concurrent systems [33]. Yet sp-posets are, in fact, too compositional for many applications: even simple consumer-producer problems inevitably generate \mathbf{N} 's [24], as shown in Fig. 1 which contains the \mathbf{N} spanned by c_1 , c_2 , p_2 , and p_3 as an induced subposet among others.

⁵ This means that there is no injection f from \mathbf{N} satisfying $x \leq y \Leftrightarrow f(x) \leq f(y)$.

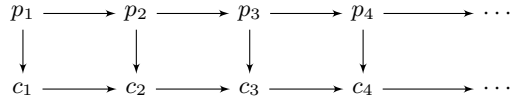


Fig. 1. The producer-consumer example

3 Interval orders and interval sequences

Interval orders [5, 34] form another class of posets that are ubiquitous in concurrent and distributed computing. Intuitively, they are isomorphic to sets of intervals on the real line that are ordered whenever they do not overlap.

Definition 2. An interval order is a relational structure $(P, <)$ with $<$ irreflexive such that $w < y$ and $x < z$ imply $w < z$ or $x < y$, for all $w, x, y, z \in P$.

Transitivity of $<$ follows. An alternative geometric characterisation is that interval orders are precisely those posets that do not contain the induced subposet $2+2 = \left(\begin{array}{c} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{array} \right)$.

The intuition is captured by Fishburn's theorem [5], which implies that a finite poset P is an interval order iff it has an *interval representation*: a pair of functions $b, e : P \rightarrow Q$ into some linear order $(Q, <_Q)$ such that $b(x) <_Q e(x)$, for all $x \in P$, and $x <_P y \Leftrightarrow e(x) <_Q b(y)$, for all $x, y \in P$. By the first condition, pairs $(b(x), e(x))$ correspond to intervals $I(x) = [b(x), e(x)]$ in Q ; by the second condition, $x <_P y$ iff $I(x)$ lies entirely before $I(y)$ in Q .

We write $\rho_I(P)$ for the set of interval representations of P . Each representation can be rearranged such that all endpoints of intervals are distinct ([9], Lemma 1.5). We henceforth assume that all interval presentations have this property. It then holds that $|Q| = 2|P|$, and we can fix Q as the target type of any interval representation of P .

Finally, with relation \sqsubset on the set of maximal antichains of poset P given by

$$A \sqsubset B \Leftrightarrow (\forall x \in A \setminus B. \forall y \in B \setminus A. x < y),$$

it has been shown that P is an interval order iff \sqsubset is a strict linear order [6].

Interval orders also occur implicitly in the ST-traces of Petri nets [30]. In a pure order-theoretic setting, these are *interval sequences*, that is, sequences of $b(x)$ and $e(x)$, with x from some finite set P , in which each $b(x)$ occurs exactly once and each $e(x)$ at most once and only after the corresponding $b(x)$. An interval sequence is *closed* if each $e(x)$ occurs exactly once [30, 33]. An *interval trace* [16] is an equivalence class of interval sequences modulo the relations $b(x)b(y) \approx b(y)b(x)$ and $e(x)e(y) \approx e(y)e(x)$ for all $x, y \in P$. We write \approx^* for the congruence generated by \approx on interval sequences. We identify interval sequences and interval traces with the Hasse diagrams of their linear orders over Q .

Lemma 3. Let P be an interval order and $(b, e) \in \rho_I(P)$. Then $(Q, <_Q)$ is a closed interval sequence.

Proof. Trivial. \square

We write $\sigma_{(b,e)}(P)$ for the interval sequence of interval order P and $(b, e) \in \rho_I(P)$, and $\Sigma(P)$ for the set of all interval sequences of interval representations of P .

Lemma 4. *If $\sigma \in \Sigma(P)$ and $\sigma \approx^* \sigma'$, then $\sigma' \in \Sigma(P)$.*

Proof (sketch). We show that $\sigma \in \Sigma(P)$ and $\sigma \approx \sigma'$ imply $\sigma' \in \Sigma(P)$. Suppose that $\sigma = \sigma_1 b(x) b(y) \sigma_2$ and $\sigma' = \sigma_1 b(y) b(x) \sigma_2$ and that $(b, e) \in \rho_I(P)$ generates σ . Then (b', e) with

$$b'(z) = \begin{cases} b(y), & \text{if } z = x, \\ b(x), & \text{if } z = y, \\ b(z), & \text{otherwise} \end{cases}$$

is in $\rho_I(P)$, as $b'(x) <_Q e(x)$, $b'(y) <_Q e(y)$ and, for all $v, w \in P$, $v <_P w \Leftrightarrow e(v) <_P b(w)$ still holds. In addition, (b', e) generates σ' . An analogous result for $\sigma = \sigma_1 e(x) e(y) \sigma_2$ and $\sigma' = \sigma_1 e(y) e(x) \sigma_2$ holds by opposition. The result for \approx^* follows by a simple induction. \square

Lemma 5. *Let P be an interval order. If $(b, e), (b', e') \in \rho_I(P)$ assign b and e to elements of P in interval sequences, then $\sigma_{(b,e)}(P) \approx^* \sigma_{(b',e')}(P)$.*

Proof (sketch). Let \prec_1 and \prec_2 be the orderings of the interval sequences for (b, e) and (b', e') in Q . Then $b(x) \prec_1 e(x)$ and $b(x) \prec_2 e(x)$ for all $x \in X$, and $e(x) \prec_1 b(y) \Leftrightarrow e(x) \prec_2 b(y)$ for all $x, y \in X$. It follows that there is no $b(z)$ in \prec_1 or \prec_2 between the positions of $e(x)$ in \prec_1 and \prec_2 and, by opposition, there is no $e(z)$ in \prec_1 or \prec_2 between the positions of $b(x)$ in \prec_1 and \prec_2 . But this means that the positions of $e(x)$ and $b(x)$ can be rearranged by \approx^* . \square

Proposition 6. *If P is an interval order and $(b, e) \in \rho_I(P)$, then $[\sigma_{(b,e)}(P)]_{\approx^*} = \Sigma(P)$. The mapping φ defined by $\varphi(P) = [\sigma_{(b,e)}(P)]_{\approx^*}$ is a bijection.*

Proof. By Lemma 4 and 5, and properties of interval representations. \square

4 Posets with interfaces

An element s of poset (P, \leq) is *minimal* (*maximal*) if $v \not\leq s$ ($v \not\geq s$) holds for all $v \in P$. We write P_{\min} (P_{\max}) for the sets of minimal (maximal) elements of P .

Definition 7. *A poset with interfaces (iposet) consists of a poset P together with two injective morphisms*

$$\begin{array}{ccc} [n] & \xrightarrow{s} & P \\ & & \xleftarrow{t} [m] \end{array}$$

such that $s[n] \subseteq P_{\min}$ and $t[m] \subseteq P_{\max}$.

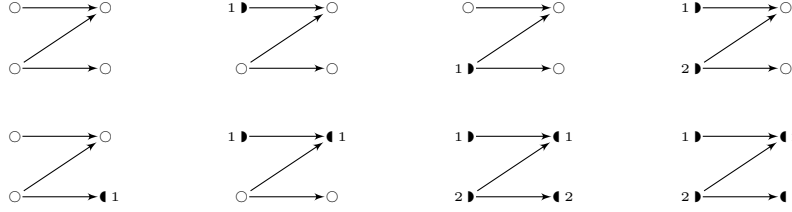


Fig. 2. Eight of 25 different iposets based on poset N.

Injection $s : [n] \rightarrow P$ represents the *source interface* of P and $t : [m] \rightarrow P$ its *target interface*. We write $(s, P, t) : n \rightarrow m$ for the iposet $s : [n] \rightarrow P \leftarrow [m] : t$.

Figure 2 shows some examples of iposets. Elements of source and target interfaces are depicted as filled half-circles to indicate the unfinished nature of the events they represent.

Next we define a sequential gluing composition on iposets whose interfaces agree and we adapt the standard parallel composition of posets to iposets.

Definition 8. Let $(s_1, P_1, t_1) : n \rightarrow m$ and $(s_2, P_2, t_2) : \ell \rightarrow k$ be iposets.

1. For $m = \ell$, their gluing composition is the iposet $(s_1, P_1 \triangleright P_2, t_2) : n \rightarrow k$ with $P_1 \triangleright P_2 = ((P_1 \sqcup P_2)_{/t_1(i)=s_2(i)}, \leq_1 \cup \leq_2 \cup (P_1 \setminus t_1[m]) \times (P_2 \setminus s_2[m]))$.
2. Their parallel composition is the iposet $(s, P_1 \otimes P_2, t) : n + \ell \rightarrow m + k$ with $s = (s_1 \otimes s_2) \circ \varphi_{n,\ell}$ and $t = (t_1 \otimes t_2) \circ \varphi_{m,k}$.

Parallel composition of iposets thus puts components “side by side”: it is the disjoint union of posets and interfaces. Gluing composition puts iposets “one after the other”, P_1 before P_2 , but glues their interfaces together (and adds arrows from all points in P_1 that are not in its target interface to all points in P_2 that are not in its source interface). Figures 3 and 4 show examples. The half-circles in source and target interfaces are glued to circles in the diagrams.

We define *identity iposets* $\text{id}_n = (\text{id}, [n], \text{id}) : n \rightarrow n$, for $n \geq 0$. For convenience, we generalise this notation to other singleton posets with interfaces: for $k, \ell \leq n$, we write ${}^k\text{id}_n^\ell$ for the iposet $(f_k^n, [n], f_\ell^n) : k \rightarrow \ell$, where $f_k^n : [k] \rightarrow [n]$ is the (identity) injection $x \mapsto x$ (similarly for f_ℓ^n). Hence $\text{id}_n = {}^n\text{id}_n^n$. We write $\mathcal{S} = \{{}^k\text{id}_1^\ell \mid k, \ell = 0, 1\}$ for the set of all singleton iposets.

Parallel composition need not be commutative, as the namings of interfaces in $P \otimes Q$ may differ from those in $Q \otimes P$. One can, however, rename interfaces using *symmetries*: iposets $(s, [n], t) : n \rightarrow n$ with s and t bijective. Figure 5 shows two parallel compositions where renaming of interfaces and gluing with another iposet yields non-isomorphic posets.

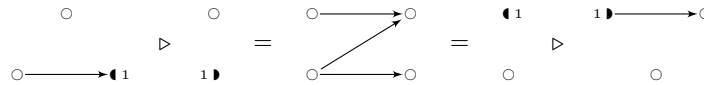


Fig. 3. Two different decompositions of the N.

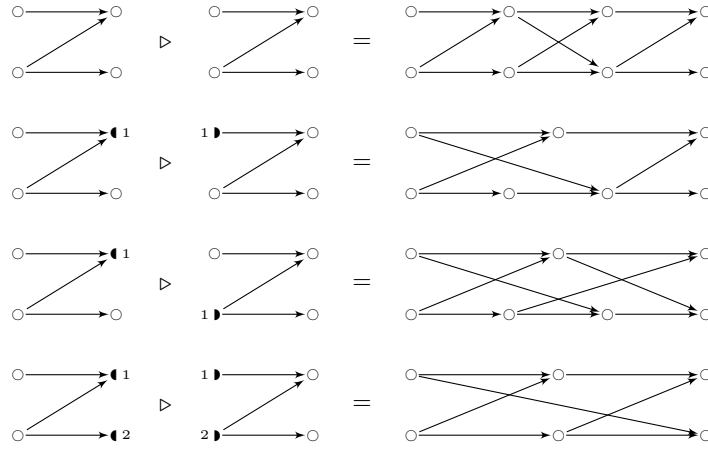


Fig. 4. Four gluings of different Ns with interfaces.

Also, gluing and parallel composition need not satisfy an interchange law:

$$({}^0\text{id}_1^0 \otimes {}^0\text{id}_1^0) \triangleright ({}^0\text{id}_1^0 \otimes {}^0\text{id}_1^0) = \left(\begin{array}{c} \circ \longrightarrow \circ \\ \circ \longrightarrow \circ \\ \circ \longrightarrow \circ \end{array} \right) \neq \left(\begin{array}{c} \circ \longrightarrow \circ \\ \circ \longrightarrow \circ \end{array} \right) = ({}^0\text{id}_1^0 \triangleright {}^0\text{id}_1^0) \otimes ({}^0\text{id}_1^0 \triangleright {}^0\text{id}_1^0).$$

Hence iposets do *not* form (strict) monoidal categories, or even PROPs, because \otimes is not a tensor. The situation differs from gluing compositions where interfaces of iposets are defined by *all* minimal and maximal elements [35], and also from sequential compositions of digraphs with “partial” interfaces similar to ours where interface points glue arrows together and disappear in these compositions [4]. Both of these give rise to a PROP.

Gluing composition, of course, is not commutative either:

$${}^0\text{id}_1^1 \triangleright {}^1\text{id}_1^0 = {}^0\text{id}_1^0 = (\cdot) \neq (\cdot \longrightarrow \cdot) = {}^1\text{id}_1^0 \triangleright {}^0\text{id}_1^1$$

Proposition 9. *Iposets form a small category with natural numbers as objects, iposets $(s, P, t) : n \rightarrow m$ as morphisms, \triangleright as composition, and identities id_n .*

Checking associativity of \triangleright and the existence of units is routine, as is the proof of the next proposition.

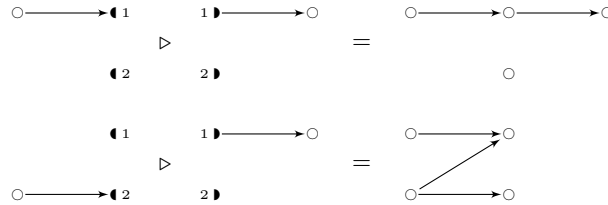


Fig. 5. Non-isomorphic gluings of symmetric parallel compositions.

Proposition 10. *Iposets form a monoid with composition \otimes and unit id_0 .*

A *morphism* of iposets is a commuting diagram

$$\begin{array}{ccccc} [n] & \xrightarrow{s} & P & \xleftarrow{t} & [m] \\ \nu \downarrow & & \downarrow f & & \downarrow \mu \\ [n'] & \xrightarrow{s'} & P' & \xleftarrow{t'} & [m'] \end{array} \quad (1)$$

where ν and μ are strictly order preserving with respect to $<_{\mathbb{N}}$ and f is an order morphism. Intuitively, iposet morphisms thus preserve interfaces and their order in \mathbb{N} . Let iPos denote the so-defined category.

An iposet morphism (ν, f, μ) is an *isomorphism* if ν , f and μ are order isomorphisms. Hence $n = n'$, $m = m'$, $\nu = \text{id} : n \rightarrow n$, and $\mu = \text{id} : m \rightarrow m$ in diagram (1). As a consequence, we note that iposets which are related by a symmetry $(s, [n], t) : n \rightarrow n$ need not be isomorphic.

We write $P \cong Q$ if there exists an isomorphism $\varphi : P \rightarrow Q$. The following lemma shows that the two compositions respect isomorphism.

Lemma 11. *Let P, P', Q, Q' be iposets. Then $P \cong P'$ and $Q \cong Q'$ imply $P \otimes Q \cong P' \otimes Q'$ and $P \triangleright Q \cong P' \triangleright Q'$.*

Proof. Let $\varphi : P \rightarrow P'$ and $\psi : Q \rightarrow Q'$ be (the poset components of) isomorphisms. Define the functions $\varphi \otimes \psi : P \sqcup Q \rightarrow P' \sqcup Q'$ and $\varphi \triangleright \psi : (P \sqcup Q)_{/t_P(i)=s_Q(i)} \rightarrow (P' \sqcup Q')_{/t_{P'}(i)=s_{Q'}(i)}$ as

$$(\varphi \square \psi)(x) = \begin{cases} \varphi(x) & \text{if } x \in P, \\ \psi(x) & \text{if } x \in Q, \end{cases}$$

for $\square \in \{\otimes, \triangleright\}$. First, $\varphi \otimes \psi$ is obviously an isomorphism. Second, $\varphi \triangleright \psi$ is well-defined because $\varphi \circ t_P(i) = \psi \circ s_Q(i)$ for all $i \in [m]$, and easily seen to be an isomorphism as well. \square

We write $P \preceq Q$ if there is a bijective (on points) morphism $\varphi : Q \rightarrow P$ between iposets P and Q . Intuitively, $P \preceq Q$ iff P has more arrows and is therefore less parallel than Q , while interfaces are preserved. Similar relations on posets and pomsets, sometimes called *subsumption*, are well studied [8, 10]. In particular, \preceq is a preorder on (finite) iposets and a partial order up to isomorphism.

Lemma 12. *For iposets P, P', Q, Q' , the following lax interchange law holds:*

$$(P \otimes P') \triangleright (Q \otimes Q') \preceq (P \triangleright Q) \otimes (P' \triangleright Q')$$

Proof. Let $P_\ell = (P \otimes P') \triangleright (Q \otimes Q')$ and $P_r = (P \triangleright Q) \otimes (P' \triangleright Q')$. First, $P_\ell = (P \sqcup Q)_{/t_P \equiv s_Q} \sqcup (P' \sqcup Q')_{/t_{P'} \equiv s_{Q'}} = (P \sqcup Q \sqcup P' \sqcup Q')_{t_P \equiv s_Q, t_{P'} \equiv s_{Q'}} = P_r$, by definition of \otimes . Hence both posets have the same points, and we may choose

$\varphi : P_r \rightarrow P_\ell$ to be the identity. It remains to show that φ is order preserving, which means that every arrow in P_r must be in P_ℓ .

Hence suppose $x \leq_{P_r} y$, that is, $x \leq_{P \triangleright Q} y$ or $x \leq_{P' \triangleright Q'} y$. In the first case, if $x \leq_P y$ or $x \leq_Q y$, then $x \leq_{P \otimes P'} y$ or $x \leq_{Q \otimes Q'} y$ and therefore $x \leq_{P_\ell} y$; and if $x \in P \setminus t_P$ and $y \in Q \setminus s_Q$, then $x \in P \sqcup P' \setminus t_{P \otimes P'}$ and $y \in Q \sqcup Q' \setminus s_{Q \otimes Q'}$ and therefore $x \leq_{P_\ell} y$, too. The second case is symmetric. Thus, in any case, $x \leq_{P_\ell} y$. \square

In sum, the algebra of iposets is thus similar to concurrent monoids [12], but \triangleright is a partial operation with many units id_k . As \otimes is not a tensor, the categorical structure of iposets is somewhat unusual and deserves further exploration.

Proposition 13. *Pos embeds into iPos as iposets with both interfaces [0], and likewise for morphisms. The so-defined inclusion functor $J : \text{Pos} \rightarrow \text{iPos}$ is fully faithful and left adjoint to the forgetful functor $F : \text{iPos} \rightarrow \text{Pos}$ that maps (s, P, t) to P , hence Pos is coreflective in iPos . Under F , gluing composition of iposets becomes serial composition of posets, and parallel composition of iposets becomes that of posets (hence, commutative).*

Proof. It is clear that J is a functor. It is full because any morphism \tilde{f} from $P : 0 \rightarrow 0$ to $Q : 0 \rightarrow 0$ in iPos must have the form $(\emptyset, f, \emptyset) = Jf$ for some f in Pos . It is faithful because $Jf = (\emptyset, f, \emptyset) = (\emptyset, g, \emptyset) = Jg$ implies $f = g$. For $P \in \text{Pos}$ and $\tilde{Q} \in \text{iPos}$, J induces a natural bijection $J : \text{Pos}(P, F\tilde{Q}) \cong \text{iPos}(JP, \tilde{Q})$, hence J and F are indeed adjoint. The last claims about the operations are clear. \square

5 Further Properties of Iposets

We now derive additional algebraic properties of iposets, before turning to the set of iposets generated by gluing and parallel composition from singleton iposets.

For an iposet P with order relation \leq we write $\parallel = \not\leq \cap \not\geq$. Hence $x \parallel y$ iff x and y are unrelated and therefore *independent*.

In addition to the lax interchange in Lemma 12, we prove an equational interchange law that shows that the equational theory of iPos as given by the bimonoidal laws in Propositions 9 and 10 is not free. The lemmas further below then show that this law is the *only* non-trivial additional identity.

Lemma 14 (Interchange). *For all iposets P, Q and $k, \ell \in \{0, 1\}$,*

$$({}^k \text{id}_1^1 \otimes P) \triangleright ({}^1 \text{id}_1^\ell \otimes Q) = {}^k \text{id}_1^\ell \otimes (P \triangleright Q).$$

Proof (sketch). The interface between ${}^k \text{id}_1^1$ and ${}^1 \text{id}_1^\ell$ forces these iposets to be glued separately to the rest in the gluing composition $({}^k \text{id}_1^1 \otimes P) \triangleright ({}^1 \text{id}_1^\ell \otimes Q)$. \square

One the one hand, it follows that singleton iposets in \mathcal{S} do not interfere with compositions. On the other hand, Lemma 14 shows that decompositions need not be unique. The next lemma shows a kind of converse: if an iposet can be decomposed by \triangleright and also by \otimes , then all but one of the components must be in \mathcal{S} . Henceforth, let $\mathcal{C}_1 = \{P_1 \otimes \cdots \otimes P_n \mid P_1, \dots, P_n \in \mathcal{S}\}$ denote the set of multisets-with-interfaces, that is, iposets with discrete order.

Lemma 15 (Decomposition). *Let $P = P_1 \otimes P_2 = Q_1 \triangleright Q_2$ such that $P_1 \neq \text{id}_0$, $P_2 \neq \text{id}_0$, and $Q_1 \neq {}^k\text{id}_n$, $Q_2 \neq {}^n\text{id}_n^k$ for any $k \leq n$. Then $P_1 \in \mathcal{C}_1$ or $P_2 \in \mathcal{C}_1$.*

Proof. Suppose $P_1 \notin \mathcal{C}_1$ and $P_2 \notin \mathcal{C}_1$. Then P contains a 2+2: there are $w, x \in P_1$ and $y, z \in P_2$ for which $w <_P x$, $y <_P z$, $w \parallel_P y$, $w \parallel_P z$, $x \parallel_P y$, and $x \parallel_P z$.

If $w, y \notin Q_2$, then $w, y \in Q_1 \setminus t_{Q_1}$. As $Q_2 \neq {}^n\text{id}_n^k$ for any $k \leq n$, there must be an element $v \in Q_2 \setminus s_{Q_2}$. But then $w \leq_P v$ and $y \leq_P v$, which yields arrows between $w \in P_1$ and $y \in P_2$ that contradict $P = P_1 \otimes P_2$. A dual argument rules out that $x, z \notin Q_1$.

It follows that $w \in Q_2$ or $y \in Q_2$. Assume, without loss of generality, that $w \in Q_2$. Then $x \in Q_2 \setminus s_{Q_2}$ because $w \leq_{P_1} x$. Now if also $y \in Q_2$, then by the same argument, $z \in Q_2 \setminus s_{Q_2}$. Hence Q_2 contains two different points which are not in its starting interface; and as $Q_1 \setminus t_{Q_1}$ is non-empty, this again establishes a connection between $x \in P_1$ and $z \in P_2$ which cannot exist. Hence $y \notin Q_2$, but then $y \in Q_1 \setminus t_{Q_1}$, so that $y \leq_P x$, which contradicts $x \parallel_P y$. \square

The next lemma generalises Levi's lemma for words [23].

Lemma 16 (Levi property). *Let $P \square Q = U \square V$ for $\square \in \{\triangleright, \otimes\}$. Then there is an R so that either $P = U \square R$ and $R \square Q = V$, or $U = P \square R$ and $R \square V = Q$.*

Proof. The proof for \otimes is trivial: If $P \otimes Q = U \otimes V$, then this iposet is partitioned into three components according to $P \sqcup Q$ and $U \sqcup V$. If the decomposition of U and V happens within P , then there is an R such that $P = U \otimes R$ and $R \otimes Q = V$. Otherwise, if it happens within Q , then there exists an R such that $U = P \otimes R$ and $R \otimes V$. Finally, if $P = U$ and $Q = V$, there is nothing to show. The proof for \triangleright is similar, but more tedious due to gluing. \square

It is instructive to find the two cases in the decomposition of \mathbf{N} in Figure 3.

Levi's lemma is an interpolation property: every $P \square Q = U \square V$ has a common factorisation—either $U \square R \square Q$ or $P \square R \square V$. Hence sequential and gluing decompositions at top level are equal up-to associativity (and unit laws).

The three lemmas in this section are helpful for characterising the iposets generated by \triangleright and \otimes from singletons. This is the subject of the next section.

6 Generating Iposets

Recall that \mathcal{S} is the set of *singleton* iposets. It contains the four iposets ${}^0\text{id}_1^0$, ${}^0\text{id}_1^1$, ${}^1\text{id}_1^0$ and ${}^1\text{id}_1^1$, that is,

$$[0] \rightarrow [1] \leftarrow [0], \quad [0] \rightarrow [1] \leftarrow [1], \quad [1] \rightarrow [1] \leftarrow [0], \quad [1] \rightarrow [1] \leftarrow [1],$$

with mappings uniquely determined. We are interested in the sets of iposets generated from singletons using \triangleright and \otimes . Note that strictly speaking, ${}^0\text{id}_1^0$ should not count as a generator, because by Lemma 14 it is equal to ${}^0\text{id}_1^1 \triangleright {}^1\text{id}_1^0$.

Definition 17. *The set of gluing-parallel iposets (gp-iposets) is the smallest set that contains the empty iposet id_0 and the singleton iposets in \mathcal{S} and is closed under gluing and parallel composition.*

Theorem 18. *The gp-ipo sets are generated freely by \mathcal{S} in the variety of algebras satisfying the equations of Propositions 9 and 10 and Lemma 14.*

Proof (sketch). Suppose $(A, \triangleright, \otimes, (1_i)_{i \geq 0})$ is any algebra satisfying the equations of Propositions 9 and 10 and Lemma 14 and let $\varphi : \mathcal{S} \rightarrow A$ be any function. We need to show that φ extends to a unique iposet morphism $\hat{\varphi}$.

We can generate any id_n as a parallel composition of id_1 . We map $\hat{\varphi}(\text{id}_i) \mapsto 1_i$ for any $i \geq 0$, and we map any other singleton $p \in \mathcal{S}$ as $\hat{\varphi}(p) = \varphi(p)$. For complex iposets we proceed by induction on the number of elements, assuming that homomorphism laws hold for iposets with n elements.

If the top composition of the size $n+1$ iposet is \triangleright , then we use Levi's lemma to factorise with respect to \triangleright and use associativity of \triangleright to establish the homomorphism property of $\hat{\varphi}$. For \otimes we proceed likewise. Finally, if the top composition is ambiguous, then the decomposition lemma forces the configuration in which the interchange lemma can be applied, yielding a parallel composition of the same size. Finally, this extension is unique, as it was forced by the construction. \square

Now we define hierarchies of iposets generated from \mathcal{S} . (If ${}^0\text{id}_1^0$ were removed from \mathcal{S} , the hierarchy would be different only for less than two alternations of \triangleright and \otimes .)

For any $\mathcal{Q} \subseteq \text{iPos}$ and $\square \in \{\otimes, \triangleright\}$, let

$$\mathcal{Q}^\square = \{P_1 \square \cdots \square P_n \mid n \in \mathbb{N}, P_1, \dots, P_n \in \mathcal{Q}\}.$$

Then define $\mathcal{C}_0 = \mathcal{D}_0 = \mathcal{S}$ and, for all $n \in \mathbb{N}$,

$$\mathcal{C}_{2n+1} = \mathcal{C}_{2n}^\otimes, \quad \mathcal{D}_{2n+1} = \mathcal{D}_{2n}^\triangleright, \quad \mathcal{C}_{2n+2} = \mathcal{C}_{2n+1}^\triangleright, \quad \mathcal{D}_{2n+2} = \mathcal{D}_{2n+1}^\otimes$$

(this agrees with the \mathcal{C}_1 notation used earlier). Finally, let

$$\bar{\mathcal{S}} \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \mathcal{C}_n = \bigcup_{n \geq 0} \mathcal{D}_n$$

be the set of all iposets generated from \mathcal{S} by application of \otimes and \triangleright .

Lemma 19. *For all $n \in \mathbb{N}$, $\mathcal{C}_n \cup \mathcal{D}_n \subseteq \mathcal{C}_{n+1} \cap \mathcal{D}_{n+1}$.*

Proof. We need to check the inclusions $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$, $\mathcal{D}_n \subseteq \mathcal{D}_{n+1}$, $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$ and $\mathcal{D}_0 \subseteq \mathcal{C}_1$. The first two are trivial by construction, plus $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$ and $\mathcal{D}_n \subseteq \mathcal{C}_{n+1}$. For the third one, note that $\mathcal{C}_0 \subseteq \mathcal{C}_0^\triangleright = \mathcal{S}^\triangleright = \mathcal{D}_0^\triangleright = \mathcal{D}_1$. Since \mathcal{C}_n is constructed from \mathcal{C}_0 by the same alternations of \otimes and \triangleright as \mathcal{D}_{n+1} is constructed from \mathcal{D}_1 , the inclusion holds. The proof of the fourth inclusion is similar. \square

Theorem 20. *An iposet is in \mathcal{C}_2 iff it is an interval order.*

Proof. Suppose $P \triangleright Q \in \mathcal{C}_2$. First it is clear that all elements of \mathcal{C}_1 are interval orders, so we will be done once we can show that the gluing composition of two interval orders is an interval order. This is precisely the proof of Lemma 15: if $P \triangleright Q$ contains a 2+2, then so do P or Q . Yet we also give a direct construction:

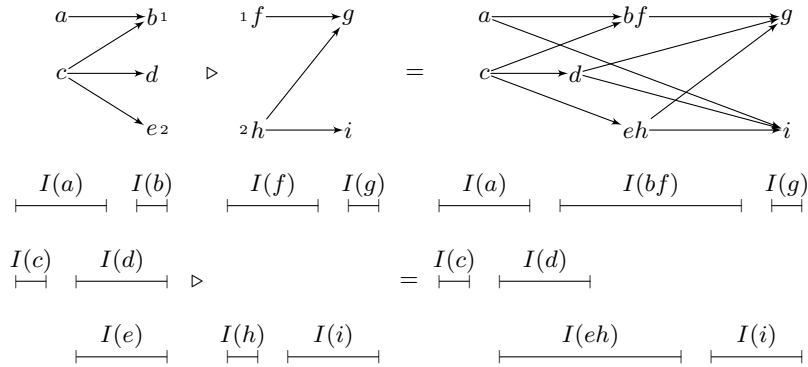


Fig. 6. Two interval orders and their concatenation: above as iposets, below using their interval representations. (Labels added for convenience.)

Let σ_P be the interval sequence for interval representation (b_P, e_P) of $P : n \rightarrow m$ and σ_Q the interval sequence for interval representation (b_Q, e_Q) of $Q : m \rightarrow k$. Then concatenate σ_P and σ_Q , rename b_P, b_Q as b and e_P, e_Q as e , delete $e(t_P(i))$, $b(s_Q(i))$ and replace $e(t_Q(i))$ with $e(t_P(i))$ for each $i \in [m]$. This yields the interval sequence for interval representation (b, e) of $P \triangleright Q$ and $P \triangleright Q$ is therefore an interval order. Figure 6 gives an example.

For the backward direction, let P be an interval order and A_P its set of maximal antichains. Then A_P is totally ordered by the relation \sqsubset defined in Section 3. Now write $A_P = \{P_1, \dots, P_k\}$ such that $P_i \sqsubset P_j$ for $i < j$. Then each P_i is an element of \mathcal{S}^\otimes . Write $s_1 : [n_1] \rightarrow P \leftarrow [n_{k+1}] : t_k$ for the sources and targets of P .

For $i = 2, \dots, k$, let $[n_i] = P_{i-1} \cap P_i$ be the overlap and $s_i : [n_i] \hookrightarrow P_i$, $t_{i-1} : [n_i] \hookrightarrow P_{i-1}$ the inclusions. Together with s_1 and t_k this defines iposets $s_i : [n_i] \rightarrow P_i \leftarrow [n_{i+1}] : t_i$. (Note that $s_1 : [n_1] \rightarrow P_1$ because P_1 is the minimal element in A_P ; similarly for $t_k : [n_{k+1}] \rightarrow P_k$.) It is clear that $P = P_1 \triangleright \dots \triangleright P_k$; see also [14, Prop. 2]. \square

In order to compare with series-parallel posets, we construct a similar hierarchy for these. Let $\mathcal{T}_0 = \mathcal{U}_0 = \mathcal{S}_0 = \{\text{id}_1^0\}$ and, for all $n \in \mathbb{N}$,

$$\mathcal{T}_{2n+1} = \mathcal{T}_{2n}^\otimes, \quad \mathcal{U}_{2n+1} = \mathcal{U}_{2n}^\triangleright, \quad \mathcal{T}_{2n+2} = \mathcal{T}_{2n+1}^\triangleright, \quad \mathcal{U}_{2n+2} = \mathcal{U}_{2n+1}^\otimes.$$

Then, noting that any element of any \mathcal{T}_n or \mathcal{U}_n has empty interfaces and that for iposets with empty interfaces, \triangleright is serial composition, we see that

$$\bar{\mathcal{S}}_0 \stackrel{\text{def}}{=} \bigcup_{n \geq 0} \mathcal{T}_n = \bigcup_{n \geq 0} \mathcal{U}_n$$

is the set of series-parallel posets. Note that $\mathcal{T}_n \subseteq \mathcal{C}_n$ and $\mathcal{U}_n \subseteq \mathcal{D}_n$ for all n , hence also $\bar{\mathcal{S}}_0 \subseteq \bar{\mathcal{S}}$. Now $\bar{\mathcal{S}}_0$ contains precisely the \mathbb{N} -free posets whereas \mathbb{N} is an interval order. Hence $\mathbb{N} \in \mathcal{C}_2$, implying the next lemma. On the other hand, we will see below that $\bar{\mathcal{S}}_0 \not\subseteq \mathcal{C}_n$ for any n .

Lemma 21. $\mathcal{C}_2 \not\subseteq \bar{\mathcal{S}}_0$.

Lemma 22. $\mathcal{C}_1 \cup \mathcal{D}_1 \subsetneq \mathcal{C}_2 \cap \mathcal{D}_2$, i.e., there is an iposet with two non-trivial different decompositions.

Proof. Directly from Lemma 14. \square

Next we show that the \mathcal{C}_n hierarchy is infinite, by exposing a sequence of witnesses for $\mathcal{C}_{2n-1} \subsetneq \mathcal{C}_{2n}$ for all $n \geq 1$.

Let $Q = {}^0\text{id}_1^0$, $P_1 = Q \triangleright Q$, and for $n \geq 1$, $P_{n+1} = Q \triangleright (P_n \otimes P_n)$. Note that all these are series-parallel posets. Graphically:

$$P_1 = (\cdot \longrightarrow \cdot) \quad P_2 = \left(\begin{array}{c} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{array} \right) \quad P_3 = \left(\begin{array}{c} \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \\ \cdot \longrightarrow \cdot \end{array} \right) \quad \dots$$

Lemma 23. $P_n \in \mathcal{C}_{2n} \setminus \mathcal{C}_{2n-1}$ for all $n \geq 1$.

Proof. By induction. For $n = 1$, $P_1 \notin \mathcal{C}_1$, but $Q \in \mathcal{C}_0 \subseteq \mathcal{C}_1$ and hence $P_1 = Q \triangleright Q \in \mathcal{C}_2 = \mathcal{C}_1^\triangleright$.

Now for $n \geq 1$, suppose $\mathcal{C}_{2n-1} \not\ni P_n \in \mathcal{C}_{2n}$. We use Lemma 15 to show that $P_n \otimes P_n \in \mathcal{C}_{2n+1} \setminus \mathcal{C}_{2n}$: Obviously $P_n \otimes P_n \in \mathcal{C}_{2n+1} = \mathcal{C}_{2n}^\otimes$. If $P_n \otimes P_n \in \mathcal{C}_{2n} = \mathcal{C}_{2n-1}^\triangleright$, then $P_n \otimes P_n = Q_1 \triangleright \dots \triangleright Q_k$ for some $Q_1, \dots, Q_k \in \mathcal{C}_{2n-1}$. Yet $P_n \notin \mathcal{C}_1$, which contradicts Lemma 15.

Now to $P_{n+1} = Q \triangleright (P_n \otimes P_n)$. Trivially, $P_{n+1} \in \mathcal{C}_{2n+2} = \mathcal{C}_{2n+1}^\triangleright$. Suppose $P_{n+1} \in \mathcal{C}_{2n+1} = \mathcal{C}_{2n}^\otimes$. P_{n+1} is connected, hence not a parallel product, so that P_{n+1} must already be in $\mathcal{C}_{2n} = \mathcal{C}_{2n-1}^\triangleright$ and therefore $P_{n+1} = R_1 \triangleright R_2$. Then, by Levi's lemma, there is an iposet S such that either $Q = R_1 \triangleright S$ and $S \triangleright (P_n \otimes P_n) = R_2$ or $R_1 = Q \triangleright S$ and $S \triangleright R_2 = P_n \otimes P_n$. In the second case, $S \triangleright R_2 = P_n \otimes P_n$, which again contradicts Lemma 15; in the first case, both R_1 and S must be single points (with suitable interfaces), so that either $R_1 = {}^0\text{id}_1^1$ and $R_2 = P_{n+1}$ (with an extra starting interface) or $R_1 = Q$ and $R_2 = P_n \otimes P_n$. This shows that $P_{n+1} = Q \triangleright (P_n \otimes P_n)$ is the only non-trivial \triangleright -decomposition of P_{n+1} . Thus $P_n \in \mathcal{C}_{2n-1}$, a contradiction, and therefore $P_{n+1} \notin \mathcal{C}_{2n+1}$. \square

Corollary 24. $\mathcal{C}_{2n-1} \subsetneq \mathcal{C}_{2n}$ for all $n \geq 1$, hence the \mathcal{C}_n hierarchy does not collapse, and neither does the \mathcal{D}_n hierarchy.

Proof. The last statement follows from $\mathcal{D}_{2n-2} \subseteq \mathcal{C}_{2n-1} \subsetneq \mathcal{C}_{2n} \subseteq \mathcal{D}_{2n+1}$. \square

Corollary 25. For all $n \in \mathbb{N}$, $\bar{\mathcal{S}}_0 \not\subseteq \mathcal{C}_n$ and $\bar{\mathcal{S}}_0 \not\subseteq \mathcal{D}_n$.

Proof. As we have already noted above, $P_n \in \bar{\mathcal{S}}_0$ for all n , which together with Lemma 23 implies the first statement. The second follows from $\mathcal{C}_n \subseteq \mathcal{D}_{n+1}$. \square

We have seen that the \mathcal{C}_n and \mathcal{D}_n hierarchies are properly infinite and that they contain the set of sp-posets only in the limit $\bar{\mathcal{S}} = \bigcup_{n \geq 0} \mathcal{C}_n = \bigcup_{n \geq 0} \mathcal{D}_n$.

Finally, we turn to the question of characterising this limit $\bar{\mathcal{S}}$ geometrically. Recalling that a poset is series-parallel iff if it does not contain an induced subposet isomorphic to \mathbb{N} , we would like a similar characterisation using forbidden

subposets for the gp-(i)posets. We expose five such forbidden subposets, but leave the question of whether there are others to future work.

Define the following five posets on six points:

$$\begin{aligned} \mathbb{N} &= \left(\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right) & \quad \mathbb{M} &= \left(\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right) & \quad \mathbb{W} &= \left(\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right) \\ 3\mathbb{C} &= \left(\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right) & \quad \mathbb{LN} &= \left(\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \\ \cdot & \xrightarrow{\quad} & \cdot \end{array} \right) \end{aligned}$$

Proposition 26. *If $P \in \bar{\mathcal{S}}$, then P does not contain \mathbb{N} , \mathbb{M} , \mathbb{W} , $3\mathbb{C}$, or \mathbb{LN} as induced subposets.*

Proof. We only show the proof for \mathbb{N} ; the others are very similar and are left to the reader. We can assume that P is connected. We use structural induction, noting that all $P \in \mathcal{S}$ are \mathbb{N} -free, so it remains to show that $P \triangleright Q$ is \mathbb{N} -free whenever P and Q are.

By contraposition, suppose $P \triangleright Q$ contains the induced sub- \mathbb{N} $\left(\begin{array}{ccc} a & \xrightarrow{\quad} & b \\ c & \xrightarrow{\quad} & d \\ e & \xrightarrow{\quad} & f \end{array} \right)$. Then we show that either P or Q also have an induced sub- \mathbb{N} .

Assume first that $a \in Q$. Then $a \leq_Q b$, hence also $b \in Q$, but $b \notin Q_{\min}$, that is, $b \notin s_Q$. Now $e \not\leq_{P \triangleright Q} b$, which forces $e \in t_P$ and therefore in $e \in Q$. This in turn implies that $d, f \in Q$ and in particular $e \leq_Q f$. Thus $f \notin Q_{\min}$ and therefore $f \notin s_Q$, which forces $c \in t_P$ and therefore $c \in Q$. This shows that \mathbb{N} lies entirely in Q .

Finally assume that $a \notin Q$. Then $a \in P \setminus t_P$, and as $a \not\leq_{P \triangleright Q} d$ and $a \not\leq_{P \triangleright Q} f$, we must have $d, f \in s_Q$ and therefore $d, f \in P$. This forces $c, e \in P$ and in particular $e \leq_P f$. Thus $e \notin P_{\min}$, whence $e \notin t_P$. This in turn forces $b \in s_Q$ and therefore $b \in P$. This shows that \mathbb{N} lies entirely in P . \square

7 Experiments

We have encoded most of the constructions in this paper with Python to experiment with gluing-parallel (i)posets. Notably, Proposition 26 is, in part, a result of these experiments.⁶ Our prototype is rather inefficient, which explains why some numbers are “n.a.”, *i.e.*, not available, in Table 1.

Using procedures to generate non-isomorphic posets of different types, we have used our software to verify that

1. all posets on five points are in $\bar{\mathcal{S}}$, *i.e.*, gp-posets;
2. \mathbb{N} , \mathbb{M} , \mathbb{W} , $3\mathbb{C}$, and \mathbb{LN} are the only six-point posets that are not in $\bar{\mathcal{S}}$.

We provide tables of gluing-parallel decompositions of posets in appendix to prove these claims.

We have also used our software to count non-isomorphic posets and iposets of different types, see Table 1. We note that P and SP are sequences no. A000112

Table 1. Different types of posets with n points: all posets; sp-posets; gp-posets; (weakly) connected gp-posets; iposets with starting interfaces only; iposets; gp-iposets.

n	$P(n)$	$SP(n)$	$GP(n)$	$GPC(n)$	$SIP(n)$	$IP(n)$	$GPI(n)$
0	1	1	1	1	1	1	1
1	1	1	1	1	2	4	4
2	2	2	2	1	5	17	16
3	5	5	5	3	16	86	74
4	16	15	16	10	66	532	419
5	63	48	63	44	350	n.a.	2980
6	318	167	313	233	n.a.	n.a.	26566

and A003430, respectively, in the On-Line Encyclopedia of Integer Sequences (OEIS).⁷ Sequences GPC, SIP, IP, and GPI are unknown to the OEIS.

The single iposet on two points which is not gluing-parallel is the symmetry $[2] : 2 \rightarrow 2$ with $s(1) = 1$, $s(2) = 2$, $t(1) = 2$, and $t(2) = 1$. The prefix of GP we were able to compute equals the corresponding prefix of sequence no. A079566 in the OEIS,⁷ which counts the number of connected (undirected) graphs which have no induced 4-cycle C_4 . We leave it to the reader to ponder upon the relation between gp-posets and C_4 -free connected graphs.

References

1. S. L. Bloom and Z. Ésik. Free shuffle algebras in language varieties. *Theor. Comput. Sci.*, 163(1&2):55–98, 1996.
2. J. H. Conway. *Regular Algebra and Finite Machines*. Chapman and Hall, 1971.
3. B. Courcelle and J. Engelfriet. *Graph Structure and Monadic Second-Order Logic - A Language-Theoretic Approach*. Cambridge University Press, 2012.
4. M. P. Fiore and M. D. Campos. The algebra of directed acyclic graphs. In *Computation, Logic, Games, and Quantum Foundations*, vol. 7860 of *LNCS*. Springer, 2013.
5. P. C. Fishburn. Intransitive indifference with unequal indifference intervals. *J. Math. Psych.*, 7(1):144–149, 1970.
6. P. C. Fishburn. *Interval Orders and Interval Graphs: A Study of Partially Ordered Sets*. Wiley, 1985.
7. H. Furusawa and G. Struth. Concurrent dynamic algebra. *ACM Trans. Comput. Log.*, 16(4):30:1–30:38, 2015.
8. J. L. Gischer. The equational theory of pomsets. *Theor. Comput. Sci.*, 61:199–224, 1988.
9. M. C. Golumbic and A. N. Trenk. *Tolerance Graphs*. Cambridge University Press, 2004.
10. J. Grabowski. On partial languages. *Fund. Inf.*, 4(2):427, 1981.
11. M. Herlihy and J. M. Wing. Linearizability: A correctness condition for concurrent objects. *ACM Trans. Program. Lang. Syst.*, 12(3):463–492, 1990.

⁶ Our software is available at <http://www.lix.polytechnique.fr/~uli/posets/>

⁷ See <http://oeis.org/A000112>, oeis.org/A003430, and oeis.org/A079566.

12. T. Hoare, B. Möller, G. Struth, and I. Wehrman. Concurrent Kleene algebra and its foundations. *J. Log. Algebr. Program.*, 80(6):266–296, 2011.
13. T. Hoare, S. van Staden, B. Möller, G. Struth, J. Villard, H. Zhu, and P. W. O’Hearn. Developments in concurrent Kleene algebra. In *RAMiCS 2014*, vol. 8428 of *LNCS*. Springer, 2014.
14. R. Janicki. Modeling operational semantics with interval orders represented by sequences of antichains. In *PETRI NETS 2018*, vol. 10877 of *LNCS*. Springer, 2018.
15. R. Janicki and M. Koutny. Structure of concurrency. *Theor. Comput. Sci.*, 112(1):5–52, 1993.
16. R. Janicki and X. Yin. Modeling concurrency with interval traces. *Inf. Comput.*, 253:78–108, 2017.
17. P. Jipsen and M. A. Moshier. Concurrent Kleene algebra with tests and branching automata. *J. Log. Algebr. Meth. Program.*, 85(4):637–652, 2016.
18. T. Kappé, P. Brunet, J. Rot, A. Silva, J. Wagemaker, and F. Zanasi. Kleene algebra with observations. In *CONCUR 2019*, vol. 140 of *LIPICs*. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2019.
19. T. Kappé, P. Brunet, A. Silva, and F. Zanasi. Concurrent Kleene algebra: Free model and completeness. In *ESOP 2018*, vol. 10801 of *LNCS*. Springer, 2018.
20. L. Lamport. The mutual exclusion problem: Part I - a theory of interprocess communication. *J. ACM*, 33(2):313–326, 1986.
21. L. Lamport. On interprocess communication. Part I: basic formalism. *Distributed Computing*, 1(2):77–85, 1986.
22. M. R. Laurence and G. Struth. Completeness theorems for pomset languages and concurrent Kleene algebras. *CoRR*, abs/1705.05896, 2017.
23. F. W. Levi. On semigroups. *Bull. Calcutta Math. Soc.*, 36:141–146, 1944.
24. K. Lodaya and P. Weil. Series-parallel languages and the bounded-width property. *Theor. Comput. Sci.*, 237(1-2):347–380, 2000.
25. S. Mimram. Presenting finite posets. In *TERMGRAPH 2014*, vol. 183 of *EPTCS*, 2014.
26. B. Möller and T. Hoare. Exploring an interface model for CKA. In *MPC 2015*, vol. 9129 of *LNCS*. Springer, 2015.
27. B. Möller, T. Hoare, M. E. Müller, and G. Struth. A discrete geometric model of concurrent program execution. In *UTP 2016*, vol. 10134 of *LNCS*. Springer, 2016.
28. D. Peleg. Concurrent dynamic logic. *J. ACM*, 34(2):450–479, 1987.
29. J. Valdes, R. E. Tarjan, and E. L. Lawler. The recognition of series parallel digraphs. *SIAM J. Comput.*, 11(2):298–313, 1982.
30. R. J. van Glabbeek. The refinement theorem for ST-bisimulation semantics. In *IFIP TC2 Working Conf. Programming Concepts and Methods*. North-Holland, 1990.
31. R. J. van Glabbeek and F. W. Vaandrager. Petri net models for algebraic theories of concurrency. In *PARLE (2)*, vol. 259 of *LNCS*. Springer, 1987.
32. W. Vogler. Failures semantics based on interval semiwords is a congruence for refinement. *Distributed Computing*, 4:139–162, 1991.
33. W. Vogler. *Modular Construction and Partial Order Semantics of Petri Nets*, vol. 625 of *Lecture Notes in Computer Science*. Springer, 1992.
34. N. Wiener. A contribution to the theory of relative position. *Proc. Camb. Philos. Soc.*, 17:441–449, 1914.
35. J. Winkowski. An algebraic characterization of the behaviour of non-sequential systems. *Inf. Process. Lett.*, 6(4):105–109, 1977.

Appendix

The following tables show gluing-parallel decompositions of all (weakly) connected posets on four points, all connected posets on five points, and all connected posets on six points except for the five posets NN, M, W, 3C, and LN which are not gluing-parallel.

Given that disconnected posets can be decomposed into posets with fewer points using \otimes and that all posets on fewer than four points are series-parallel, hence gluing-parallel, these tables show the claims in Section 7: All posets on five points are gluing-parallel, as are all but the five exceptional posets NN, M, W, 3C, and LN on six points.

Table 2: Gluing-Parallel decompositions of connected posets on four points

no.	Poset	Decomposition	
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			

Table 3: Gluing-parallel decompositions of connected posets on five points

no.	Poset	Decomposition	
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			

Table 3: Gluing-parallel decompositions of connected posets on five points

no.	Poset	Decomposition	
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			

Table 3: Gluing-parallel decompositions of connected posets on five points

no.	Poset	Decomposition	
29			
30			
31			
32			
33			
34			
35			
36			
37			
38			
39			

Table 3: Gluing-parallel decompositions of connected posets on five points

no.	Poset	Decomposition	
40		○ ● 1 ● 2	1 ► ○ 2 ► ○
41		○ ○ ○ ○	○
42		○ ► ○ ○ ► ○	○
43		○ ► ○ ○ ► ● 1	○ 1 ►
44		○ ► ● 1 ○ ► ● 2	○ 1 ► 2 ►

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
1		○ ► ○ ► ○	○ ► ○ ► ○
2		○ ► ○ ► ○	
3		○ ► ○ ► ○	
4		○ ► ○ ► ○	○
5		○ ► ○ ► ○	○ ○ ○
6		○ ► ○	○ ► ○ ► ○
7		○ ► ○	

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
22			
23			
24			
25			
26			
27			
28			
29			
30			
31			
32			
33			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
34			
35			
36			
37			
38			
39			
40			
41			
42			
43			
44			
45			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
46			
47			
48			
49			
50			
51			
52			
53			
54			
55			
56			
57			
58			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
59			
60			
61			
62			
63			
64			
65			
66			
67			
68			
69			
70			
71			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
72			
73			
74			
75			
76			
77			
78			
79			
80			
81			
82			
83			
84			
85			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
86			
87			
88			
89			
90			
91			
92			
93			
94			
95			
96			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
97			
98			
99			
100			
101			
102			
103			
104			
105			
106			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
107			
108			
109			
110			
111			
112			
113			
114			
115			
116			
117			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
118			
119			
120			
121			
122			
123			
124			
125			
126			
127			
128			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
129			
130			
131			
132			
133			
134			
135			
136			
137			
138			
139			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
140			
141			
142			
143			
144			
145			
146			
147			
148			
149			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
150			
151			
152			
153			
154			
155			
156			
157			
158			
159			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
160			
161			
162			
163			
164			
165			
166			
167			
168			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

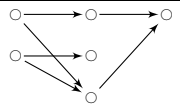
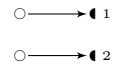
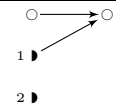
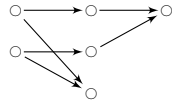
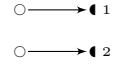
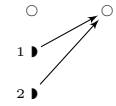
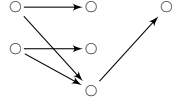
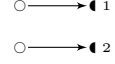
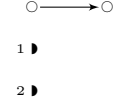
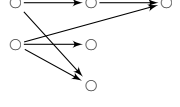
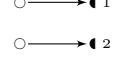
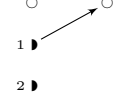
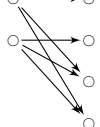

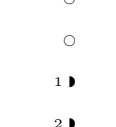
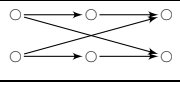

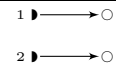
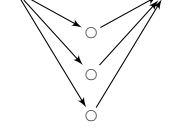

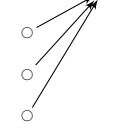
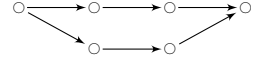

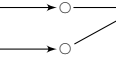
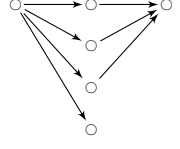

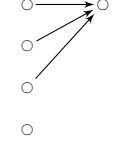
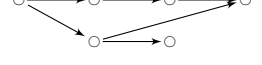

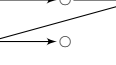
no.	Poset	Decomposition	
169			
170			
171			
172			
173			
174			
175			
176			
177			
178			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

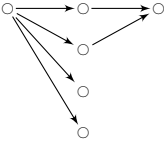
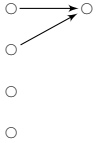
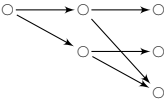
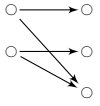
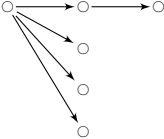
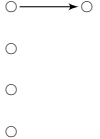
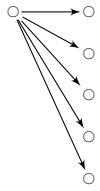

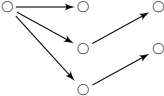
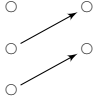
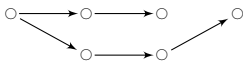
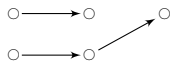
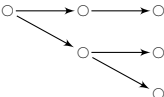
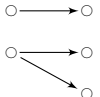
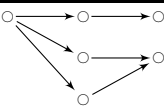
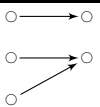
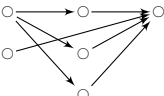
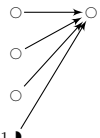
no.	Poset	Decomposition
179		
180		
181		
182		
183		
184		
185		
186		
187		

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
188			
189			
190			
191			
192			
193			
194			
195			
196			
197			

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
198		○	
199		○	
200		○	
201		○	
202		○	
203		○	
204		○	
205		○	
206		○	
207		○	

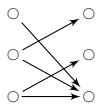
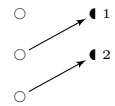

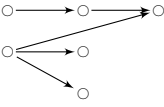
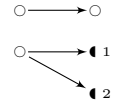

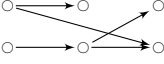
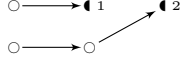

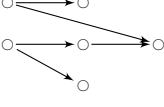
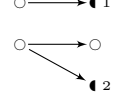

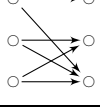
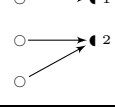

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
208		○ ● 1 ● 2	
209		○ ● 1 ● 2	
210		○ ● 1 ● 2	
211		○ ● 1 ● 2	
212		○ ● 1 ● 2	
213		○ ● 1 ● 2 ● 3	
214		○ ● 1 ● 2 ● 3	
215		○ ● 1 ● 2 ● 3	
216		○ ○ ○ ○	

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition
217		
218		
219		
220		
221		
222		
223		
224		
225		
226		
227		

Table 4: Gluing-parallel decompositions of connected gp-posets on six points

no.	Poset	Decomposition	
228			
229			
230			
231			
232			
233	