CONGRUENCES INVOLVING CENTRAL TRINOMIAL COEFFICIENTS

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Abstract. In this paper, we confirm some congruences conjectured by the second author. For example, we prove that for any prime $p > 3$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k \equiv \left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k} \equiv \frac{3+\left(\frac{p}{3}\right)}{2} - p\left(1+\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

where T_k is the coefficient of x^k in the expansion of $(1+x+x^2)^k$, $(-)$ denotes the Legendre symbol and $H_k := \sum_{0<j\le k} 1/j$ denotes the kth harmonic number.

1. Introduction

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the nth central trinomial coefficient

$$T_n = [x^n](1+x+x^2)^n$$

is the coefficient of x^n in the expansion of $(1+x+x^2)^n$. Note that $[x^n](1+x+x^2) = [x^0](1+x+x^{-1})$. By the multi-nomial theorem we have

$$T_n = \sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{k!k!(n-2k)!} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}. \tag{1.1}$$

T_n has many combinatorial interpretations (cf. [11]). For example, T_n is the number of lattice paths running from $(0,0)$ to $(n,0)$ with steps $(1,1)$, $(1,-1)$ and $(1,0)$. It is easy to see that T_n also has the following form

$$T_n = 3^n\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(-\frac{1}{3}\right)^k \tag{1.2}$$

which follows from [2, (3.136) and (3.137)]. The readers may refer to [11] for more identities involving T_n.

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In [13, 14], Z.-W. Sun systematically investigated congruences involving the generalized central trinomial coefficients

$$T_n(b,c) := [x^n](x^2+bx+c)^n, \quad b,c \in \mathbb{Z}$$

Clearly, $T_n = T_n(1,1)$. In [13], Sun determined the general sums

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(b,c)}{m^k}$$

modulo an odd prime p, where $b,c,m \in \mathbb{Z}$ and $p \nmid m$. In particular, letting $m = 12$ and $b = c = 1$ he obtained that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k}{12^k} \equiv \left(\frac{6}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{64^k} \equiv \left(\frac{p}{3}\right) \pmod{p}, \tag{1.3}$$

where $(-)$ denotes the Legendre symbol. In the same paper, as an extension of (1.3), Sun [13, Conjecture 2.1] conjectured the following congruence that we shall prove.

Theorem 1.1. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{12^k} T_k \equiv \left(\frac{p}{3}\right) \frac{3^{p-1}+3}{4} \pmod{p^2}. \tag{1.4}$$

Remark 1.1. For $p \neq 3$, by Fermat's little theorem (cf. [3]), we have $3^{p-1} \equiv 1 \pmod{p}$. Thus (1.4) implies (1.3).

Let n be a nonnegative integer. The nth harmonic number H_n is defined by

$$H_0 := 0 \quad \text{and} \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n = 1, 2, 3, \ldots).$$

In [14] Sun studied the sums involving $T_n(b,c)^2$ and products of $T_n(b,c)$ and other numbers (such as Motzkin numbers [10] and harmonic numbers). For example, he proved that

$$\sum_{k=0}^{n-1} (2k+1) T_k(b,c)^2 (b^2-4c)^{n-1-k} \equiv 0 \pmod{n^2}$$

for all $n = 1, 2, 3, \ldots$, and and

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}$$

for any odd prime p. Our next theorem confirms a conjecture posed by Sun in [14, Conjecture 1.1 (ii)].

Theorem 1.2. *Let p be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}\equiv\frac{3+\left(\frac{p}{3}\right)}{2}-p\left(1+\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

In this paper we also prove the following result which was conjectured by Sun in a recent paper [15].

Theorem 1.3. [15, Conjecture 33] *Let p be an odd prime and let $m\in\mathbb{Z}$ with $m\neq 1$ and $p\nmid m$. Then*

$$\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k\equiv p+2p\frac{1-\left(\frac{m}{p}\right)}{m-1}\pmod{p^2}. \tag{1.5}$$

Remark 1.2. One may easily prove (1.5) modulo p by exchanging the summation order and noting that $\binom{2k}{k}\equiv 0\pmod p$ for $k=(p+1)/2,\ldots,p-1$. However, it seems to be difficult to prove (1.5) entirely in this way. Here we use the Maple Package `APCI` (see [1]) to reduce the double sum on the left-hand side of (1.5).

We will show Theorems 1.1–1.3 in Sections 2–4 respectively.

2. Proof of Theorem 1.1

In order to show Theorem 1.1, we need the following preliminary results.

Lemma 2.1. *Let $n,j\in\mathbb{N}$. Then we have the following identity*

$$\sum_{k=j}^{n}\frac{\binom{2k}{k}\binom{k}{j}}{4^k}=\frac{n+1}{2^{2n+1}(2j+1)}\cdot\binom{n}{j}\binom{2n+2}{n+1}. \tag{2.1}$$

Proof. This could be directly verified by induction on n. □

Lemma 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)3^k}\equiv\frac{1}{p}\cdot\left(\frac{(-1)^{(p-1)/2}}{4\cdot 3^{(p-1)/2}}+\frac{(-1)^{(p-1)/2}\cdot 3^{(p+1)/2}}{4}-\frac{(-1)^{(p-1)/2}\cdot 4^{p-1}}{3^{(p-1)/2}}\right)\pmod p. \tag{2.2}$$

Proof. In [8, Theorem 2], Kh. Hessami Pilehrood, T. Hessami Pilehrood and R. Tauraso obtained a general result involving

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}t^k}{2k+1}\pmod{p^3},$$

where p is an odd prime and t is a p-adic integer with $p\nmid t$. Substituting $t=1/3$ into their result, we immediately obtain (2.2). □

Proof of Theorem 1.1. By (1.2) we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k&=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{4^k}\sum_{j=0}^{k}\left(-\frac{1}{3}\right)^j\binom{k}{j}\binom{2j}{j}\\&=\sum_{j=0}^{p-1}\left(-\frac{1}{3}\right)^j\binom{2j}{j}\sum_{k=j}^{p-1}\frac{\binom{2k}{k}}{4^k}\binom{k}{j}.\end{aligned}$$

Replacing n with $p-1$ in Lemma 2.1 we arrive at

$$\sum_{k=j}^{p-1}\frac{\binom{2k}{k}}{4^k}\binom{k}{j}=\frac{p}{2^{2p-1}(2j+1)}\cdot\binom{p-1}{j}\binom{2p}{p}.$$

Noting that $\binom{2j}{j}\equiv 0\pmod p$ and $p\nmid 2j+1$ for $j\in\{(p+1)/2,\ldots,p-1\}$, we obtain

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k\equiv\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}\binom{p-1}{j}}{(2j+1)(-3)^j}\pmod{p^2}.$$

Clearly,

$$\binom{p-1}{j}=(-1)^j\prod_{k=1}^{j}\left(1-\frac{p}{k}\right)\equiv(-1)^j(1-pH_j)\pmod{p^2}.\tag{2.3}$$

Thus we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k\equiv\frac{\binom{2p}{p}\binom{p-1}{(p-1)/2}(1-pH_{(p-1)/2})}{3^{(p-1)/2}2^{2p-1}}+\frac{p\binom{2p}{p}}{2^{2p-1}}\sum_{j=0}^{(p-3)/2}\frac{\binom{2j}{j}}{(2j+1)3^j}\pmod{p^2}.\tag{2.4}$$

In 1862, J. Wolstenholme [17] showed that for all primes $p>3$

$$\binom{2p-1}{p-1}\equiv 1\pmod{p^3}.\tag{2.5}$$

From Morley's congruence [5] we have for any prime $p>3$

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1}\pmod{p^3}.\tag{2.6}$$

It is known [4] that

$$H_{(p-1)/2}\equiv -2q_p(2)\pmod p,\tag{2.7}$$

where $q_p(2):=(2^{p-1}-1)/p$ denotes the Fermat quotient. Now substituting (2.5)–(2.7) into (2.4) we deduce that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{12^k}T_k&\equiv\frac{(-1)^{(p-1)/2}}{3^{(p-1)/2}}(1+2pq_p(2))+\frac{(-1)^{(p-1)/2}}{4\cdot 3^{(p-1)/2}}+\frac{3}{4}\cdot(-3)^{(p-1)/2}-\frac{(-1)^{(p-1)/2(1+pq_p(2))^2}}{3^{(p-1)/2}}\\&\equiv\frac{(-1)^{(p-1)/2}}{4\cdot 3^{(p-1)/2}}+\frac{3}{4}\cdot(-3)^{(p-1)/2}\pmod{p^2}.\end{aligned}$$

From [3, Page 51], we know that $a^{(p-1)/2} \equiv (\frac{a}{p}) \pmod p$. Thus we may write $3^{(p-1)/2}$ as $(\frac{3}{p})(1+pt)$, where t is a p-adic integer. In view of this,

$$3^{p-1} = (3^{(p-1)/2})^2 \equiv 1 + 2pt \pmod{p^2}.$$

By the above and with the help of the law of quadratic reciprocity (cf. [3]), we get

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{12^k} T_k \equiv& \frac{1}{4}\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)(1-pt) + \frac{3}{4}\left(\frac{-1}{p}\right)\left(\frac{3}{p}\right)(1+pt) \\ =& \left(\frac{p}{3}\right)\frac{4+2pt}{4} \equiv \left(\frac{p}{3}\right)\frac{3^{p-1}+3}{4} \pmod{p^2}\end{aligned}$$

as desired.

The proof of Theorem 1.1 is now complete. □

3. Proof of Theorem 1.2

The following identity can be verified by induction.

Lemma 3.1. *Let n, j be nonnegative integers. Then we have*

$$\sum_{k=j}^{n} \binom{k}{j} H_k = \binom{n+1}{j+1}\left(H_{n+1} - \frac{1}{j+1}\right). \tag{3.1}$$

Lemma 3.2. [12, Corollary 1.1] *For any prime $p > 3$ we have*

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{3^k(k+1)} \equiv 3^{p-1} - 1 + \frac{(\frac{p}{3}) - 1}{2} \pmod{p^2}. \tag{3.2}$$

Lemma 3.3. *For $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \frac{\binom{n}{k} H_k}{k+1}\left(-\frac{4}{3}\right)^k = \frac{(-3+(-1/3)^n) H_n}{4(n+1)} - \frac{\sum_{k=1}^{n} \frac{(-3)^k}{k}}{4(-3)^n(n+1)} + \frac{3\sum_{k=1}^{n} \frac{1}{k(-3)^k}}{4(n+1)} \tag{3.3}$$

and

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+1)^2}\left(-\frac{4}{3}\right)^k = \frac{1}{n+1} + \frac{3\sum_{k=1}^{n} \frac{1}{k+1}}{4(n+1)} + \frac{\sum_{k=1}^{n} \frac{1}{(k+1)(-3)^k}}{4(n+1)}. \tag{3.4}$$

Proof. These two identities were found by `Sigma` (a Mathematica package to find identities, cf. [9]). Here we give a manual proof.

Denote the left-hand side of (3.3) by $F(n)$ and the right-hand side by $G(n)$. It is easy to check that $F(n)$ and $G(n)$ all satisfy the following recurrence relation:

$$(n+1)(n+2)F(n)+(n+2)(5n+13)F(n+1)+3(n+3)(n+4)F(n+2)-9(n+3)(n+4)F(n+3) = 12.$$

Then (3.3) can be proved by noting that $F(d) = G(d)$ for $d = 0, 1, 2$. We will not give the proof of (3.4) since its proof is analogous. □

Lemma 3.4. *For any prime $p>3$ we have*

$$\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k}\equiv -2q_p(2)\pmod{p}. \tag{3.5}$$

Proof. Clearly,

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k}&=\sum_{k=1}^{(p-1)/2}\frac{(1-4)^k}{k}=\sum_{k=1}^{(p-1)/2}\frac{1}{k}\sum_{j=0}^{k}\binom{k}{j}(-4)^j\\&=\sum_{j=1}^{(p-1)/2}(-4)^j\sum_{k=j}^{(p-1)/2}\frac{1}{k}\binom{k}{j}+H_{(p-1)/2}.\end{aligned}$$

By [2, (1.52)] we have

$$\sum_{k=j}^{(p-1)/2}\frac{1}{k}\binom{k}{j}=\frac{1}{j}\sum_{k=j-1}^{(p-3)/2}\binom{k}{j-1}=\frac{1}{j}\binom{\frac{p-1}{2}}{j}.$$

Thus we obtain

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-3)^k}{k}&=\sum_{k=1}^{(p-1)/2}\frac{(-4)^j}{j}\binom{\frac{p-1}{2}}{j}+H_{(p-1)/2}\\&\equiv\sum_{k=1}^{(p-1)/2}\frac{\binom{2j}{j}}{j}+H_{(p-1)/2}\pmod{p}.\end{aligned}$$

In 2006, H. Pan and Sun [7] proved that for any prime $p>3$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p}.$$

Thus (3.5) follows from (2.7). □

Proof of Theorem 1.2. By (1.2) and Lemma 3.1 we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}=&\sum_{j=0}^{p-1}\frac{\binom{2j}{j}}{(-3)^j}\sum_{k=j}^{p-1}\binom{k}{j}H_k\\=&p\sum_{j=0}^{p-1}\frac{\binom{2j}{j}}{(-3)^j(j+1)}\binom{p-1}{j}\left(H_{p-1}+\frac{1}{p}-\frac{1}{j+1}\right)\\\equiv&p\sum_{j=0}^{p-1}\frac{\binom{2j}{j}(1-pH_j)}{3^j(j+1)}\left(\frac{1}{p}-\frac{1}{j+1}\right)\pmod{p^2},\end{aligned}$$

where the last step follows from (2.3) and the fact $H_{p-1}\equiv 0 \pmod{p^2}$ (cf. [17]). Noting that $\binom{2j}{j}\equiv 0 \pmod p$ for $j\in\{(p+1)/2,\ldots,p-1\}$ we arrive at

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}\equiv \Sigma_1-p\Sigma_2-p\Sigma_3 \pmod{p^2}, \tag{3.6}$$

where

$$\Sigma_1:=\sum_{j=0}^{p-2}\frac{\binom{2j}{j}}{3^j(j+1)},\quad \Sigma_2:=\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}H_j}{3^j(j+1)},\quad \Sigma_3:=\sum_{j=0}^{(p-1)/2}\frac{\binom{2j}{j}H_j}{3^j(j+1)}.$$

In view of (2.5),

$$\binom{2p-2}{p-1}=\frac{p}{2p-1}\binom{2p-1}{p-1}\equiv -p-2p^2 \pmod{p^3}.$$

Thus by Lemma 3.2 we get that

$$\Sigma_1\equiv 3^{p-1}+\frac{\left(\frac{p}{3}\right)-1}{2}-\frac{\binom{2p-2}{p-1}}{3^{p-1}p}\equiv 3^{p-1}+\frac{\left(\frac{p}{3}\right)-1}{2}+\frac{2p+1}{3^{p-1}} \pmod{p^2}. \tag{3.7}$$

Substituting $n=(p-1)/2$ into (3.3) and in view of (2.7) and Lemma 3.4 we deduce that

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}H_k}{(k+1)3^k}&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}H_k}{k+1}\left(-\frac{4}{3}\right)^k\\ &\equiv-\left(-3+\left(-\frac{1}{3}\right)^{(p-1)/2}\right)q_p(2)+q_p(2)\left(-\frac{1}{3}\right)^{(p-1)/2}\\ &\quad+\frac{3}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k(-3)^k} \pmod p.\end{aligned} \tag{3.8}$$

Also, letting $n=(p-1)/2$ in (3.4) we obtain that

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(k+1)^23^k}&\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}}{(k+1)^2}\left(-\frac{4}{3}\right)^k\\ &\equiv 3-3q_p(2)-\frac{3}{2}\sum_{k=1}^{(p-1)/2}\frac{1}{k(-3)^k}+\frac{1}{(-3)^{(p-1)/2}} \pmod p.\end{aligned} \tag{3.9}$$

Now combining (3.6)–(3.9) we arrive at

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}\equiv 3^{p-1}+\frac{\left(\frac{p}{3}\right)-1}{2}+\frac{2p}{3^{p-1}}+\frac{1}{3^{p-1}}-3p-\frac{p}{(-3)^{(p-1)/2}} \pmod{p^2}.$$

As in the proof or Theorem 1.1, we write $3^{(p-1)/2}$ as $(\frac{3}{p})(1+pt)$. By Fermat's little theorem and the law of quadratic reciprocity we finally obtain

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}\equiv&(1+2pt)+\frac{(\frac{p}{3})-1}{2}+2p+1-2pt-3p-p\left(\frac{p}{3}\right)\\ \equiv&\frac{(\frac{p}{3})+3}{2}-p\left(1+\left(\frac{p}{3}\right)\right)\pmod{p^2}\end{aligned}$$

as desired. We are done. □

4. Proof of Theorem 1.3

To show Theorem 1.3 we need a telescoping method for double summations developed by W.Y.C. Chen, Q.-H. Hou and Y.-P. Mu [1]. To learn how to use the telescoping method one may refer to [6, 16].

Lemma 4.1. *For any nonnegative integer n and $t\neq 0$ we have*

$$\sum_{k=0}^{n}\frac{\binom{n}{k}t^{k+1}}{k+1}=\frac{(1+t)^{n+1}-1}{n+1}.$$

Proof. It is easy to see that

$$(n+1)\sum_{k=0}^{n}\frac{\binom{n}{k}t^{k+1}}{k+1}=\sum_{k=0}^{n}\binom{n+1}{k+1}t^{k+1}=\sum_{k=1}^{n+1}\binom{n+1}{k}t^k=(1+t)^{n+1}-1.$$

This proves Lemma 4.1. □

Proof of Theorem 1.3. Set

$$F(n,k)=\frac{1}{m^n}\binom{n}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k.$$

Via `APCI` we find

$$G_1(n,k)=\frac{2kn+k+n}{m^n(k+1)}\binom{n}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k$$

and

$$G_2(n,k)=\frac{2k}{m^{n+1}}\binom{n+1}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k$$

so that

$$F(n,k)=(G_1(n+1,k)-G_1(n,k))+(G_2(n,k+1)-G_2(n,k)).$$

Therefore

$$\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k=\sum_{n=0}^{p-1}\sum_{k=0}^{n}F(n,k)$$

$$=\sum_{k=0}^{p-1}(G_1(p,k)-G_1(k,k))+\sum_{n=0}^{p-1}(G_2(n,n+1)-G_2(n,0))=\Sigma_1-\Sigma_2+\Sigma_3,$$

where

$$\Sigma_1:=\sum_{k=0}^{p-1}G_1(p,k),\quad \Sigma_2:=\sum_{k=0}^{p-1}G_1(k,k),\quad \Sigma_3:=\sum_{n=0}^{p-1}G_2(n,n+1).$$

If $m-1\not\equiv 0 \pmod p$, by (2.5) and Lemma 4.1 we have

$$\begin{aligned}\Sigma_1=&\sum_{k=0}^{p-1}\frac{2pk+k+p}{m^p(k+1)}\binom{p}{k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k\\ \equiv&\sum_{k=1}^{p-2}\frac{2pk+k+p}{m^p(k+1)}\cdot\frac{p}{k}\cdot\binom{p-1}{k-1}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k+\frac{2p}{m}\\ \equiv&\sum_{k=1}^{p-2}\frac{p(-1)^{k-1}}{k+1}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k+\frac{2p}{m}\\ \equiv&-\frac{p}{m}\sum_{k=0}^{(p-1)/2}\frac{\binom{(p-1)/2}{k}}{k+1}(m-1)^k+\frac{3p}{m}\\ \equiv&\frac{p}{m}+2p\frac{1-(\frac{m}{p})}{m-1}\pmod{p^2}.\end{aligned}$$

Also,

$$\begin{aligned}\Sigma_3-\Sigma_2=&\sum_{k=1}^{p}\frac{2k}{m^k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k-\sum_{k=0}^{p-1}\frac{2k}{m^k}\binom{2k}{k}\left(\frac{m-1}{4}\right)^k\\ =&\frac{2p}{m}\binom{2p}{p}\left(\frac{m-1}{4}\right)^p\equiv p-\frac{p}{m}\pmod{p^2}.\end{aligned}$$

Combining the above we obtain (1.5) immediately.

If $m-1\equiv 0 \pmod p$, by Lemma 4.1 it is easy to check that

$$\Sigma_1\equiv p\pmod p\quad\text{and}\quad \Sigma_3-\Sigma_2\equiv 0\pmod p.$$

Thus (1.5) holds again. We are done. □

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