

# Explicit formulas for $p$ -adic integrals: approach to $p$ -adic distributions and some families of special numbers and polynomials

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## Abstract

The main objective of this article is to give and classify new formulas of  $p$ -adic integrals and blend these formulas with previously well known formulas. Therefore, this article gives briefly the formulas of  $p$ -adic integrals which were found previously, as well as applying the integral equations to the generating functions and other special functions, giving proofs of the new interesting and novel formulas. The  $p$ -adic integral formulas provided in this article contain several important well-known families of special numbers and special polynomials such as the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums. In addition, we defined two new sequences containing the Bernoulli numbers and Euler numbers. These two sequences include central factorial numbers, Bernoulli numbers and Euler numbers. Some computation formulas and identities for these sequences are given. Finally we give further remarks, observations and comments related to content of this paper.

**Keywords:**  $p$ -adic  $q$ -integrals, Volkenborn integral, Generating function, Special functions, Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Lah numbers, Peters numbers and polynomials, Central factorial numbers, Daehee numbers and polynomials, Changhee numbers and polynomials, Harmonic numbers, Fubini numbers, Combinatorial numbers and sum.

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# 1 Introduction

Just before the turn of the 20th century, around the end of the 19th century, Kurt Hensel (1861–1941) constructed a new special number family, now called  $p$ -adic numbers. Although  $p$ -adic numbers have been known for nearly a hundred years, it is well-known that these numbers have recently been applied in the fields of physics, mathematics and other engineering, for example, coding theory and Diophantine equations. The mystery and development of  $p$ -adic numbers in science is still growing rapidly. In this way, many scientists continue to be the source of inspiration. Consequently, these numbers are used in several mathematical fields such as Number Theory, Algebraic Geometry, Algebraic Topology, Mathematical Analysis, Mathematical Physics, String Theory, Field Theory, Stochastic Differential Equations on real Banach Spaces and Manifolds, and other parts of the natural sciences. In addition to this rapid development of  $p$ -adic numbers, very important theories have been built in  $p$ -adic analysis and their applications. These include  $p$ -adic distributions and  $p$ -adic measure,  $p$ -adic integrals,  $p$ -adic  $L$ -function, and other generalized functions. In order to solve mathematical and physical problems,  $p$ -adic numbers are used. A connection between  $p$ -adic Analysis and Quantum Groups with Noncommutative Geometry,  $q$ -deformation of ordinary analysis has recently given (*cf.* [3], [49]-[75], [71]-[70], [83], [92], [94]-[121]). These major advances, especially the  $p$ -adic integral and applications of  $p$ -adic analysis, were greatly influenced. The  $p$ -adic integrals and their applications are of great importance to find solutions to special (differential) equations, solutions to real world problems in both mathematics, physics and engineering. In recent years, many books, scientific articles, theses and reports have been published on  $p$ -adic integrals and their applications. The  $p$ -adic integral and generating functions have been used in mathematics, in mathematical physics and in others sciences. From another perspective, the  $p$ -adic integral and  $p$ -adic numbers are also intensively used in the theory of ultrametric calculus, the  $p$ -adic quantum mechanics and the  $p$ -adic mechanics (*cf.* [3], [49]-[75], [71]-[70], [83], [92], [94]-[121]).

This paper is purpose to give comprehensive study of formulas and identities on theory of not only the Volkenborn integral, but also the fermionic  $p$ -adic integral and also ( $p$ -adic) distributions. Therefore, in addition to the well-known formulas including the Volkenborn integral and the fermionic  $p$ -adic integral, which we have found so far in various sources, we will give new and comprehensive formulas that we have found in this study. We hope and believe that these formulas have the enough quality and depth to be used in the fields of  $p$ -adic numbers and  $p$ -adic analysis we have mentioned above. The content of this paper is including a brief summary of generating functions for special numbers and polynomials,  $p$ -adic distribution, the Volkenborn integral and the fermionic  $p$ -adic integral. Our new and novel  $p$ -adic integral formulas involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini

numbers, combinatorial numbers and sums.

The following definitions relations and notations are used all sections of this article:

Let  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. Additionally, let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ .

Let  $n, k \in \mathbb{N}_0$ , the binomial coefficient  $\binom{n}{k}$  is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If  $k > n$  or  $k < 0$ , then we assume that

$$\binom{n}{k} = 0$$

(*cf.* [2]-[123]).

## 1.1 Rising and Falling Factorials

Let  $x \in \mathbb{R}$ . The rising factorial and the falling factorial are defined as follows, respectively:

$$x^{(n)} = \begin{cases} x(x+1)(x+2)\dots(x+n-1) & n \in \mathbb{N} \\ 1 & n = 0 \end{cases}, \quad (1)$$

and

$$x_{(n)} = \begin{cases} x(x-1)(x-2)\dots(x-n+1) & n \in \mathbb{N} \\ 1 & n = 0 \end{cases} \quad (2)$$

For  $n \in \mathbb{N}_0$ , we have

$$(-1)^n (-x)_{(n)} = (x+n-1)_{(n)} = x^{(n)} \quad (3)$$

(*cf.* [2]-[123]).

In order to define the central factorials of degree  $n$ , we need the following another type of falling factorial:

$$x^{[n]} = x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)\dots(x - \frac{n}{2} + 1) \quad (4)$$

and

$$x^{[0]} = 1$$

where  $n \in \mathbb{N}$ ,  $x \in \mathbb{R}$  (*cf.* [10], [13], [19], [20], [22], [116]).

Using (2), we have the following well-known identities and relations:

$$xx_{(n)} = x_{(n+1)} + nx_{(n)}, \quad (5)$$

$$x_{(n+1)} = x \sum_{k=0}^n (-1)^{n-k} n_{(n-k)} x_{(k)}, \quad (6)$$

(*cf.* [93, p. 58]). Additionally, we have

$$(x+1)_{(n+1)} = xx_{(n)} + x_{(n)}, \quad (7)$$

and

$$x_{(m)}x_{(n)} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! x_{(m+n-k)}, \quad (8)$$

(*cf.* [124]).

It is well-known that the coefficients of  $x_{(n+n-k)}$  are called connection coefficients and they have a combinatorial interpretation as the number of ways to identify  $k$  elements each from a set of size  $m$  and a set of size  $n$  (*cf.* [124]).

The well-known Chu-Vandermonde identity is defined as follows:

$$\sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} = \binom{x+y}{n}. \quad (9)$$

By (9), we have

$$\sum_{k=0}^n \binom{x}{k} \binom{m}{n-k} = \binom{x+m}{n}, \quad (10)$$

$$\Delta \binom{x}{n} = \binom{x}{n-1}$$

and

$$\Delta \binom{x}{n} = \binom{x+1}{n} - \binom{x}{n}.$$

Therefore

$$\binom{x+1}{n} = \binom{x}{n} + \binom{x}{n-1} \quad (11)$$

(*cf.* [42, p. 69, Eq-(7)]). By

$$\Delta x_{(n)} = (x+1)_{(n)} - x_{(n)}, \quad (12)$$

we have

$$(x+1)_{(n)} = x_{(n)} + nx_{(n-1)} \quad (13)$$

(*cf.* [93, p. 58]).

Thanks to the works [31] and [32] of Gould, it is also known that the following formulas hold true:

$$x \binom{x-2}{n-1} = \sum_{k=1}^n (-1)^{k-n} \binom{x}{k} k \quad (14)$$

(*cf.* [31, Vol. 3, Eq-(4.20)]),

$$\binom{n-x}{n} = \sum_{k=0}^n (-1)^{k-n} \binom{x}{k} \quad (15)$$

(cf. [31, Vol. 3, Eq-(4.19)]),

$$\binom{mx}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk - mj}{n} \quad (16)$$

(cf. [32, Eq-(2.65)]),

$$\binom{x}{n}^r = \sum_{k=0}^{nr} \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \quad (17)$$

(cf. [32, Eq-(2.66)]).

$$x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} = \sum_{k=0}^n (-1)^k \binom{x}{k} k^2, \quad (18)$$

where  $n \in \mathbb{N}$  with  $n > 1$  (cf. [32, Eq-(2.15)]),

$$\binom{x+n}{n} = \sum_{k=0}^n \binom{x}{k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n} \quad (19)$$

and

$$\binom{x+n}{n} = \sum_{k=0}^n x^k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!} \quad (20)$$

(cf. [32, Eq-(2.64) and Eq-(6.17)]),

$$\binom{x+n+\frac{1}{2}}{n} = (2n+1) \binom{2n}{n} \sum_{k=0}^n \binom{n}{k} \binom{x}{k} \frac{2^{2k-2n}}{(2k+1) \binom{2k}{k}} \quad (21)$$

(cf. [31, Vol. 3, Eq-(6.26)]),

$$x \binom{x-2}{n-1} = \sum_{k=1}^n (-1)^{k-n} \binom{x}{k} k \quad (22)$$

(cf. [31, Vol. 3, Eq-(4.20)] and

$$(-1)^n \binom{n-x}{n} = \sum_{k=1}^n (-1)^k \binom{x}{k} \quad (23)$$

(cf. [31, Vol. 3, Eq-(4.19)]).

## 1.2 Generating Functions for Special Numbers and Polynomials

Here, we give some well-known generating functions which are for special numbers and polynomials.

The Apostol-Bernoulli polynomials  $\mathcal{B}_n(x; \lambda)$  are defined by means of the following generating function:

$$F_A(t, x; \lambda) = \frac{t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{B}_n(x; \lambda) \frac{t^n}{n!}, \quad (24)$$

(cf. [6]).

Substituting  $x = 0$  into (24), we have

$$\lambda \mathcal{B}_1(1; \lambda) = 1 + \mathcal{B}_1(\lambda)$$

and for  $n \geq 2$ ,

$$\lambda \mathcal{B}_n(1; \lambda) = \mathcal{B}_n(\lambda),$$

(cf. [6]).

By using (24), we have

$$\mathcal{B}_n(x; \lambda) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{B}_j(\lambda) \quad (25)$$

(cf. [6]). By using (25), first few values of the Apostol-Bernoulli polynomials are given as follows:

$$\begin{aligned} \mathcal{B}_0(x; \lambda) &= 0, \\ \mathcal{B}_1(x; \lambda) &= \frac{1}{\lambda - 1}, \\ \mathcal{B}_2(x; \lambda) &= \frac{1}{\lambda - 1}x - \frac{2\lambda}{(\lambda - 1)^2}, \\ \mathcal{B}_3(x; \lambda) &= \frac{3}{\lambda - 1}x^2 - \frac{6\lambda}{(\lambda - 1)^2}x + \frac{3\lambda(\lambda + 1)}{(\lambda - 1)^3}, \\ \mathcal{B}_4(x; \lambda) &= \frac{4}{\lambda - 1}x^3 - \frac{12\lambda}{(\lambda - 1)^2}x^2 + \frac{12\lambda(\lambda + 1)}{(\lambda - 1)^3}x - \frac{4\lambda(\lambda^2 + 4\lambda + 1)}{(\lambda - 1)^4}, \\ \mathcal{B}_5(x; \lambda) &= \frac{5}{\lambda - 1}x^4 - \frac{20\lambda}{(\lambda - 1)^2}x^3 + \frac{30\lambda(\lambda + 1)}{(\lambda - 1)^3}x^2 - \frac{20t(\lambda^2 + 4\lambda + 1)}{(\lambda - 1)^4}x \\ &\quad + \frac{5\lambda(\lambda^3 + 11\lambda^2 + 11\lambda + 1)}{(\lambda - 1)^5}. \end{aligned}$$

Substituting  $x = 1$  into (24), we have the following Apostol-Bernoulli numbers:

$$\mathcal{B}_n(1, \lambda) = \sum_{j=0}^n \binom{n}{j} \mathcal{B}_j(\lambda),$$

where

$$\mathcal{B}_n(\lambda) = \mathcal{B}_n(0, \lambda)$$

and

$$\mathcal{B}_0(\lambda) = 0.$$

Since

$$\begin{aligned} \mathcal{B}_1(\lambda) &= \frac{1}{\lambda-1}, \\ \mathcal{B}_m(\lambda) &= \frac{\lambda}{1-\lambda} \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{B}_j(\lambda), \end{aligned} \quad (26)$$

we have the following few values of the Apostol-Bernoulli numbers:

$$\begin{aligned} \mathcal{B}_2(\lambda) &= \frac{-2\lambda}{(\lambda-1)^2}, \\ \mathcal{B}_3(\lambda) &= \frac{3\lambda(\lambda+1)}{(\lambda-1)^3}, \\ \mathcal{B}_4(\lambda) &= \frac{-4\lambda(\lambda^2+4\lambda+1)}{(\lambda-1)^4}, \\ \mathcal{B}_5(\lambda) &= \frac{5\lambda(\lambda^3+11\lambda^2+11\lambda+1)}{(\lambda-1)^5}, \\ \mathcal{B}_6(\lambda) &= \frac{-6\lambda(\lambda^4+26\lambda^3+66\lambda^2+26\lambda+1)}{(\lambda-1)^6}, \\ \mathcal{B}_7(\lambda) &= \frac{7\lambda(\lambda^5+57\lambda^4+302\lambda^3+302\lambda^2+57\lambda+1)}{(\lambda-1)^7}, \dots \end{aligned}$$

(cf. [6], for detail, see also [56], [36], [77], [115], [117]). When  $\lambda = 1$  in (24), we have the Bernoulli polynomials of the first kind

$$B_n(x) = \mathcal{B}_n(x; 1).$$

Hence, few values of the Bernoulli polynomials are given as follows:

$$\begin{aligned} B_0(x) &= 1, \\ B_1(x) &= x - \frac{1}{2}, \\ B_2(x) &= x^2 - x + \frac{1}{6}, \\ B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \\ B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30}, \\ B_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{1}{6}x, \\ B_6(x) &= x^6 - 3x^5 + \frac{5}{2}x^4 - \frac{1}{2}x^2 + \frac{1}{42}, \end{aligned}$$

Since  $B_n = B_n(0)$  denotes the Bernoulli numbers of the first kind, few of these numbers are given as follows:

$$\begin{aligned} B_0 &= 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \\ B_6 &= \frac{1}{42}, B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6}, \\ B_{16} &= -\frac{3617}{510}, B_{18} = \frac{43867}{798}, B_{20} = -\frac{174611}{330}, \dots \end{aligned}$$

with  $B_{2n+1} = 0$  for  $n \geq 2$  (*cf.* OEIS A000367, OEIS A002445; and also see [7]-[123]; see also the references cited in each of these works).

The  $\lambda$ -Bernoulli polynomials (Apostol-type Bernoulli) polynomials  $\mathfrak{B}_n(x; \lambda)$  are defined by means of the following generating function:

$$F_B(t, x; \lambda) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{tx} = \sum_{n=0}^{\infty} \mathfrak{B}_n(x; \lambda) \frac{t^n}{n!}, \quad (27)$$

(*cf.* [69]; see also [38], [117], [106]). For  $n > 1$ , combining (27) with (24), we have the following well-known identity:

$$\mathfrak{B}_{n-1}(x; \lambda) = \frac{\log \lambda}{n} \mathfrak{B}_n(x; \lambda) + \mathfrak{B}_{n-1}(x; \lambda).$$

The Apostol-Euler polynomials of the first kind  $\mathcal{E}_n(x, \lambda)$  are defined by means of the following generating function:

$$F_{P1}(t, x; k, \lambda) = \frac{2}{\lambda e^t + 1} e^{tx} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \lambda) \frac{t^n}{n!}, \quad (28)$$

and by using (28), we have

$$\mathcal{E}_n(x; \lambda) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{E}_j(\lambda) \quad (29)$$

(*cf.* [7]-[118]).

By combining (28) with (24), we have the following well-known relation:

$$\mathcal{E}_n(x; \lambda) = -\frac{2}{n+1} \mathfrak{B}_{n+1}(x; -\lambda) \quad (30)$$

(*cf.* [116]).

Substituting  $\lambda = 1$  into (28), we have the Euler polynomials of the first kind; that is

$$E_n(x) = \mathcal{E}_n(x; 1).$$



In the light of this thought and also with the help of the equation (29), few values of the Euler polynomials of the first kind are given as follows:

$$\begin{aligned}
E_0(x) &= 1, \\
E_1(x) &= x - \frac{1}{2}, \\
E_2(x) &= x^2 - x, \\
E_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \\
E_4(x) &= x^4 - 2x^3 + x, \\
E_5(x) &= x^5 - \frac{5}{2}x^4 + \frac{5}{2}x^2 - \frac{1}{2},
\end{aligned}$$

Substituting  $x = 0$  into (28), we have the Apostol-Euler numbers of the first kind:

$$\mathcal{E}_n(\lambda) = \mathcal{E}_n(0, \lambda).$$

Hence

$$\begin{aligned}
\mathcal{E}_0(\lambda) &= \frac{2}{\lambda + 1}, \\
\mathcal{E}_m(\lambda) &= -\frac{\lambda}{1 + \lambda} \sum_{j=0}^{m-1} \binom{m}{j} \mathcal{E}_j(\lambda). \tag{31}
\end{aligned}$$

Using (31), we have

$$\begin{aligned}
\mathcal{E}_1(\lambda) &= -\frac{2\lambda}{(\lambda + 1)^2}, \\
\mathcal{E}_2(\lambda) &= \frac{2\lambda(\lambda - 1)}{(\lambda + 1)^3}, \\
\mathcal{E}_3(\lambda) &= -\frac{2\lambda(\lambda^2 - 4\lambda + 1)}{(\lambda + 1)^4} \dots
\end{aligned}$$

Setting  $\lambda = 1$  into (28), we have the Euler numbers of the first kind:

$$E_n = \mathcal{E}_n^{(1)}(1) = E_n(0).$$

Hence few of values of the Euler numbers of the first kind are given as follows:

$$\begin{aligned}
E_0 &= 1, E_1 = -\frac{1}{2}, E_2 = 0, E_3 = \frac{1}{4}, \\
E_5 &= -\frac{1}{2}, E_7 = \frac{17}{8}, E_9 = -\frac{31}{2} \dots
\end{aligned}$$

with  $E_{2n} = 0$  for  $n \geq 1$  (cf. [7]-[118]; see also the references cited in each of these earlier works).

Let  $u$  be a complex numbers with  $u \neq 1$ . The Frobenius-Euler numbers  $H_n(u)$  are defined by means of the following generating function:

$$F_f(t, u) = \frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}. \quad (32)$$

By using (32), we have

$$H_n(u) = \begin{cases} 1 & \text{for } n = 0 \\ \frac{1}{u} \sum_{j=0}^n \binom{n}{j} H_j(u) & \text{for } n > 0. \end{cases}$$

By using the above formula, some values of the numbers  $H_n(u)$  are given as follows:

$$\begin{aligned} H_1(u) &= \frac{1}{u-1}, \\ H_2(u) &= \frac{u+1}{(u-1)^2}, \\ H_3(u) &= \frac{u^2+4u+1}{(u-1)^3}, \\ H_4(u) &= \frac{u^3+11u^2+11u+1}{(u-1)^4}, \dots \end{aligned}$$

Substituting  $u = -1$  into (32), we have

$$E_n = H_n(-1)$$

(*cf.* [45], [69, Theorem 1, p. 439], [117]; see also the references cited in each of these earlier works).

The Euler numbers of the second kind  $E_n^*$  are given by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!}.$$

Since

$$E_n^* = 2^n E_n \left( \frac{1}{2} \right),$$

and using the definition of the Euler polynomials the first kind  $E_n(x)$ , it is easy to give few values of the Euler numbers of the second kind  $E_n^*$  given as follows:

$$\begin{aligned} E_0^* &= 1, E_2^* = -1, E_4^* = 5, E_6^* = -61, E_8^* = 1385, E_{10}^* = -50521, \\ E_{12}^* &= 2702765, E_{14}^* = -199360981, E_{16}^* = 19391512145, \dots \end{aligned}$$

with  $E_{2n+1}^* = 0$  for  $n \geq 0$  (*cf.* [7]-[118]; see also the references cited in each of these earlier works).

The Bernstein basis functions are defined by means of the following generating functions:

$$\frac{(tx)^k}{k!} e^{(1-x)t} = \sum_{n=0}^{\infty} B_k^n(x)$$

where

$$B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad (33)$$

$n, k \in \mathbb{N}_0$  with  $0 \leq k \leq n$  (cf. [1], [63], [76], [112], [99]).

Note that there is one generating function for each value of  $k$ .

The Stirling numbers of the first kind  $S_1(n, k)$  are defined by means of the following generating function:

$$F_{S_1}(t, k) = \frac{(\log(1+t))^k}{k!} = \sum_{n=0}^{\infty} S_1(n, k) \frac{t^n}{n!}. \quad (34)$$

Some basic properties of these numbers are given as follows:

$$S_1(0, 0) = 1$$

and

$$S_1(0, k) = 0$$

if  $k > 0$ . Also

$$S_1(n, 0) = 0$$

if  $n > 0$  and

$$S_1(n, k) = 0$$

if  $k > n$ . A recurrence relation for these numbers is given by

$$S_1(n+1, k) = -nS_1(n, k) + S_1(n, k-1) \quad (35)$$

(cf. [19], [7], [17], [93], [98], [101]; and see also the references cited in each of these earlier works).

By using (35), few values of the Stirling numbers of the first kind  $S_1(n, k)$  are given by the following table:

$n \setminus k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	-1	1	0	0	0
3	0	2	-3	1	0	0
4	0	-6	11	-6	1	0
5	0	24	-50	35	-10	1

Another generating function for the Stirling numbers of the first kind is falling factorial function which is given as follows:

$$x_{(n)} = \sum_{k=0}^n S_1(n, k) x^k \quad (36)$$

(cf. [19], [20], [22], [116]).

Some well-known identities for the equation (36) are given as follows:

Multiplying both sides of the equation (36) by  $x^m$ , we have

$$x^m x_{(n)} = \sum_{k=0}^n S_1(n, k) x^{m+k}. \quad (37)$$

By combining (8) with (36), we have

$$x_{(m)} x_{(n)} = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! \sum_{l=0}^{m+n-k} S_1(m+n-k, l) x^l. \quad (38)$$

By using (36), we have

$$x_{(m)} x_{(n)} = \sum_{j=0}^n \sum_{l=0}^m S_1(n, k) S_1(m, l) x^{j+l}. \quad (39)$$

By combining (5) with (36), we have

$$x x_{(n)} = \sum_{k=0}^n (S_1(n+1, k) + n S_1(n, k)) x^k + x^{n+1}. \quad (40)$$

By combining the above equation with (35), and using  $S_1(n, k) = 0$  if  $k < 0$ , we have

$$x x_{(n)} = \sum_{k=1}^n S_1(n, k-1) x^k + x^{n+1}. \quad (41)$$

The unsigned Stirling numbers of the first kind are defined by

$$C(n, k) = |S_1(n, k)| = \left[ \begin{matrix} n \\ k \end{matrix} \right]$$

(cf. [19], [20], [22], [116]). The numbers  $C(n, k)$  are also defined as follows:

$$x^{(n)} = \sum_{k=0}^n C(n, k) x^k \quad (42)$$

(cf. [18]).

The Bernoulli polynomials of the second kind  $b_n(x)$  are defined by means of the following generating function:

$$F_{b_2}(t, x) = \frac{t}{\log(1+t)} (1+t)^x = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!} \quad (43)$$

(cf. [93, pp. 113-117]; see also the references cited in each of these earlier works).

The Bernoulli numbers of the second kind  $b_n(0)$  are defined by means of the following generating function:

$$F_{b_2}(t) = \frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} b_n(0) \frac{t^n}{n!}. \quad (44)$$

The Bernoulli polynomials of the second kind are defined by

$$b_n(x) = \int_x^{x+1} u_{(n)} du.$$

Substituting  $x = 0$  into the above equation, one has

$$b_n(0) = \int_0^1 u_{(n)} du. \quad (45)$$

The numbers  $b_n(0)$  are also so-called Cauchy numbers (i.e. Bernoulli numbers of the second kind) (*cf.* [93, p. 116], [50], [78], [104]; see also the references cited in each of these earlier works).

The  $\lambda$ -array polynomials  $S_k^n(x; \lambda)$  are defined by means of the following generating function:

$$F_A(t, x, k; \lambda) = \frac{(\lambda e^t - 1)^k}{k!} e^{tx} = \sum_{n=0}^{\infty} S_k^n(x; \lambda) \frac{t^n}{n!}, \quad (46)$$

where  $k \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$  (*cf.* [98], for detail, see also [7], [17], [101]).

By (46), we have

$$S_k^n(x; \lambda) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j (j+x)^n \quad (47)$$

(*cf.* [98]). Substituting  $x = 0$  into (46), we have the  $\lambda$ -Stirling numbers  $S_2(n, k; \lambda)$ , which are defined by the following generating function:

$$F_S(t, k; \lambda) = \frac{(\lambda e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k; \lambda) \frac{t^n}{n!}, \quad (48)$$

where  $k \in \mathbb{N}_0$  and  $\lambda \in \mathbb{C}$  (*cf.* [77], [115], see also [98]).

Substituting  $\lambda = 1$  into (48), then we get the Stirling numbers of the second kind, which is the number of partitions of a set of  $n$  elements into  $k$  nonempty subsets, as follows:

$$S_2(n, k) = S_2(n, k; 1) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} \lambda^j j^n. \quad (49)$$

The Stirling numbers of the second kind are also given by the following generating function including falling factorial:

$$x^n = \sum_{k=0}^n S_2(n, k)x_{(k)}, \quad (50)$$

(*cf.* [6]-[118]; see also the references cited in each of these earlier works).

By using (49), few values of the Stirling numbers of the second kind  $S_2(n, k)$  are given by the following table:

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	1	1	0	0	0
3	0	1	3	1	0	0
4	0	1	7	6	1	0
5	0	1	15	25	10	1

The Schlomilch formula, associated with the Stirling numbers of the first and the second kind, is given by

$$S_1(n, k) = \sum_{j=0}^{n-k} (-1)^j \binom{n+j-1}{k-1} \binom{2n-k}{n-k-j} S_2(n-k+j, j)$$

(*cf.* [18, p. 115], [19, p. 290, Eq-(8.21)]).

In [79], Osgood and Wu gave the following identity:

$$(xy)_{(k)} = \sum_{l,m=1}^k C_{l,m}^{(k)} x_{(l)} x_{(m)} \quad (51)$$

where

$$C_{l,m}^{(k)} = \sum_{j=1}^k (-1)^{k-j} S_1(k, j) S_2(j, l) S_2(j, m)$$

$$C_{l,m}^{(k)} = C_{m,l}^{(k)}, \quad C_{1,1}^{(1)} = 1, \quad C_{1,1}^{(2)} = 0, \quad C_{1,2}^{(3)} = 0 = C_{2,1}^{(3)}.$$

By using (50), we have

$$(xy)_{(k)} = \sum_{m=0}^k S_1(k, m) x^m y^m. \quad (52)$$

The Lah numbers are defined by means of the following generating function:

$$F_L(t, k) = \frac{1}{k!} \left( \frac{t}{1-t} \right)^k = \sum_{n=k}^{\infty} L(n, k) \frac{t^n}{n!} \quad (53)$$

(*cf.* [18], [91, p. 44], [8], [86], [19], [22], [86], [126], and the references cited therein).

Using (53), we have

$$L(n, k) = (-1)^n \frac{n!}{k!} \binom{n-1}{k-1}. \quad (54)$$

The unsigned Lah numbers are defined by

$$|L(n, k)| = \frac{n!}{k!} \binom{n-1}{k-1}, \quad (55)$$

where  $n, k \in \mathbb{N}$  with  $1 \leq k \leq n$ .

By using (55), few values of the unsigned Lah numbers  $|L(n, k)|$  are given by the following table:

$n \backslash k$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	0	1	0	0	0	0
2	0	2	1	0	0	0
3	0	6	6	1	0	0
4	0	24	36	12	1	0
5	0	120	240	120	20	1

By the help of the following the initial conditions

$$L(n, 0) = \delta_{n,0}$$

and

$$L(0, k) = \delta_{0,k},$$

for all  $k, n \in \mathbb{N}$ , we have recurrence relations for the Lah numbers given as follows:

$$L(n+1, k) = -(n+k)L(n, k) - L(n, k-1)$$

and

$$L(n, k) = \sum_{j=0}^n (-1)^j S_1(n, j) S_2(j, k)$$

(cf. [28], [91, p. 44], [86]).

Using (3), we have another definition of the Lah numbers including the falling factorial and the rising factorial:

$$(-x)_{(n)} = \sum_{k=1}^n L(n, k) x_{(k)} \quad (56)$$

so that

$$x_{(n)} = \sum_{k=1}^n L(n, k) (-x)_{(k)}$$

and

$$x^{(n)} = \sum_{k=1}^n |L(n, k)| x_{(k)}. \quad (57)$$

where  $n \in \mathbb{N}_0$  (cf. [18], [19], [22], [44], [28], [86], [91], [126]).

The equation (4) classification enables us to give the following central factorials of degree  $n$ ,  $t(n, k)$  and  $T(n, k)$  of the first and the second kind, respectively:

$$x^{[n]} = \sum_{k=0}^n t(n, k)x^k \quad (58)$$

and

$$x^n = \sum_{k=0}^n T(n, k)x^{[k]} \quad (59)$$

with

$$t(n, 0) = T(n, 0) = \delta_{0n}$$

where  $\delta_{mn}$  denotes the Kronecker delta and  $n \in \mathbb{N}_0$  (cf. [13]).

Observe that

$$D^j \{x^{[n]}\} = j! \sum_{k=j}^n \binom{k}{j} t(n, k)x^{k-j},$$

$$\delta^j \{x^n\} = j! \sum_{k=j}^n \binom{k}{j} T(n, k)x^{[k-j]},$$

$$\delta^j \{x^n\} |_{x=0} = j!T(n, j),$$

and

$$D^j \{x^{[n]}\} |_{x=0} = j!t(n, j)$$

where

$$\delta \{f(x)\} = f\left(x + \frac{1}{2}\right) - f\left(x - \frac{1}{2}\right)$$

and

$$D = \frac{d}{dx}$$

(cf. [13, Eq. (2.7), Eq. (2.9)], [61], [101]).

Applying the Cauchy's integral theorem to the function  $(\sinh(\frac{z}{2}))^m$ , we have the well-known integral representations for the numbers  $T(n, k)$  and  $t(n, k)$ , respectively, given as follows:

$$T(n, k) = \frac{k!}{m!2\pi i} \int_{|w|=r} \left(2 \sinh\left(\frac{z}{2}\right)\right)^m \frac{dz}{z^{k+1}}$$

and

$$t(n, k) = \frac{k!}{m!2\pi i} \int_{|w|=r} \left(2 \operatorname{area} \sinh\left(\frac{z}{2}\right)\right)^m \frac{dz}{z^{k+1}}$$

(cf. [13, Proposition 4.2.2]).



The following tables give us the upper part of the matrices of central factorial numbers of the first and second kind, respectively (*cf.* [20, p. 13]; see also OEIS A036969):

$$(T(i, j))_{i, j=0}^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 1 & 0 & 0 & 0 \\ 0 & 1 & 21 & 14 & 1 & 0 & 0 \\ 0 & 1 & 85 & 147 & 30 & 1 & 0 \\ 0 & 1 & 341 & 1408 & 627 & 55 & 1 \end{bmatrix} \quad (60)$$

and

$$(t(i, j))_{i, j=0}^6 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 4 & -5 & 1 & 0 & 0 & 0 \\ 0 & -36 & 49 & -14 & 1 & 0 & 0 \\ 0 & 576 & -870 & 273 & -30 & 1 & 0 \\ 0 & -14400 & 21076 & -7645 & 1023 & -55 & 1 \end{bmatrix}. \quad (61)$$

The Daehee numbers of the first kind and the second kind are defined by means of the following generating functions, respectively:

$$\frac{\log(1+t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \quad (62)$$

and

$$\frac{(1+t)\log(1+t)}{t} = \sum_{n=0}^{\infty} \widehat{D}_n \frac{t^n}{n!},$$

(*cf.* [91, p. 45], [26], [46]). Using (62), we have

$$D_n = (-1)^n \frac{n!}{n+1}$$

(*cf.* [46], see also [90], [127], [35]).

The Changhee numbers of the first kind and the second kind are defined by means of the following generating functions, respectively:

$$\frac{2}{t+1} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \quad (63)$$

and

$$\frac{2(1+t)}{t+2} = \sum_{n=0}^{\infty} \widehat{Ch}_n \frac{t^n}{n!},$$

(*cf.* [49]). Using (63), we have

$$Ch_n = (-1)^n \frac{n!}{2^n}$$

and

$$Ch_n = \sum_{k=0}^n S_1(n, k) E_k \quad (64)$$

(*cf.* [49], see also [40], [41]).

The Peters polynomials  $s_k(x; \lambda, \mu)$  are defined by means of the following generating function:

$$\frac{1}{(1 + (1+t)^\lambda)^\mu} (1+t)^x = \sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!} \quad (65)$$

which, for  $x = 0$ , reduces to the Peters numbers  $s_n(0; \lambda, \mu) = s_n(\lambda, \mu)$  (*cf.* [93], [42]), [109]).

Substituting  $\mu = 1$  into (65), we have the Boole polynomials. If we substitute  $\lambda = 1$  and  $\mu = 1$  into (65), then we have the Changhee polynomials (*cf.* [49], [47]). We observe that recently, the Peters polynomials, the Boole polynomials, the Changhee polynomials, Daehee polynomials and combinatorial numbers and polynomials have been studied by many authors see, for details (*cf.* [25]-[27], [44]-[75]; see also many of the recent works cited in this paper).

By using (65), we have

$$\sum_{n=0}^{\infty} s_n(x; \lambda, \mu) \frac{t^n}{n!} = \sum_{n=0}^{\infty} s_n(\lambda, \mu) \frac{t^n}{n!} \sum_{n=0}^{\infty} x_{(n)} \frac{t^n}{n!}.$$

Therefore, we have

$$s_n(x; \lambda, \mu) = \sum_{v=0}^n \binom{n}{v} x_{(n-v)} s_v(\lambda, \mu) \quad (66)$$

(*cf.* [93], [42], [47], [109]).

In [107], we defined the generating function for the combinatorial numbers  $y_1(n, k; \lambda)$  as follows:

$$F_{y_1}(t, k; \lambda) = \frac{1}{k!} (\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n, k; \lambda) \frac{t^n}{n!}, \quad (67)$$

where  $k \in \mathbb{N}_0$ ,  $\lambda \in \mathbb{C}$  and

$$y_1(n, k; \lambda) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} j^n \lambda^j.$$

By (67), we have

$$B(n, k) = k! y_1(n, k; 1)$$

(*cf.* [29], [107]).

**Theorem 1 (cf. [111])** Let  $\mu \in \mathbb{N}$ . Then we have

$$x_{(n)} = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} s_{n-v}(x; \lambda, \mu). \quad (68)$$

**Theorem 2 (cf. [111])** Let  $\mu \in \mathbb{Z}^+$ . Then we have

$$x_{(n)} = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) S_1(v, k) s_{n-v}(x; \lambda, \mu). \quad (69)$$

In [111], we constructed the following generating function for combinatorial polynomials  $Y_{n,2}(x, \lambda)$ , which are member of the family of the Peters polynomials, as follows:

$$F_{Y_2}(t, x; \lambda) = \frac{2(1 + \lambda t)^x}{\lambda^2 t + 2(\lambda - 1)} = \sum_{n=0}^{\infty} Y_{n,2}(x; \lambda) \frac{t^n}{n!} \quad (70)$$

in which if we set  $x = 0$ , then we have combinatorial numbers  $Y_{n,2}(\lambda) = Y_{n,2}(0; \lambda)$ . By using (70), we have

$$Y_{n,2}(\lambda) = \frac{1}{2^{n+1}} Y_n(\lambda)$$

(cf. [108], [111]) so that the numbers  $Y_n(\lambda)$  are defined by the author in (cf. [108]).

Substituting  $x = 0$ ,  $\lambda = \mu = 1$  and  $t = \frac{\theta^2 u}{\theta - 1}$  into (65), we have

$$s_n(0; 1, 1) = \frac{(\theta - 1)^{n+1}}{2\theta^{2n}} Y_{n,2}(\theta)$$

(cf. [111]).

**Theorem 3 (cf. [111])** Let  $n \in \mathbb{N}$ . Then we have

$$\begin{aligned} s_n(x; \lambda, \mu) &= \frac{n}{2} \sum_{j=0}^{n-1} \binom{n-1}{j} \theta^{j+2-n} s_j(\lambda, \mu) Y_{n-1-j,2}(x, \theta) \\ &\quad + (\theta - 1) \sum_{j=0}^n \binom{n}{j} \theta^{j-n} s_j(\lambda, \mu) Y_{n-j,2}(x, \theta). \end{aligned}$$

**Theorem 4 (cf. [111])**

$$Y_{n,2}(x; \lambda) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) x_{(n-j)}. \quad (71)$$

**Lemma 5 (cf. [111])** Let  $n \in \mathbb{N}_0$ . Then we have

$$Y_{n,2}(\lambda) = 2(-1)^n n! \frac{\lambda^{2n}}{(2\lambda - 2)^{n+1}}. \quad (72)$$

Substituting (72) into (71), we get an explicit formula for the polynomials  $Y_{n,2}(x; \lambda)$  by the following theorem:

**Theorem 6 (cf. [111])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$Y_{n,2}(x; \lambda) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda - 2)^{j+1}} x_{(n-j)}. \quad (73)$$

This paper has exactly 9 main sections including introduction. We summarize as follows:

In Section 2, we give some definitions, notations and formulas for distributions and  $p$ -adic ( $q$ -) integrals on  $\mathbb{Z}_p$ .

In Section 3, we give some applications and formulas for the Volkenborn integral.

In Section 4, we give some applications and formulas for the fermionic  $p$ -adic integral.

In Section 5, we give many new formulas for the Volkenborn integral including the falling factorials, the raising factorials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 6, we give many new formulas for the fermionic  $p$ -adic integral including the falling factorials, the raising factorials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 7, by using formulas in Section 6 and in Section 7, we give various identities including the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

In Section 8, we give some questions and new sequences with their definitions and properties.

In Section 9, we give conclusion and observations on our results.

## 2 Distributions and $p$ -adic $q$ -integrals on $\mathbb{Z}_p$

In this section, we give brief introduction for  $p$ -adic distributions and  $p$ -adic ( $q$ -) integrals. For the fundamental properties of  $p$ -adic integrals and  $p$ -adic distributions, which are given briefly below, we may refer the references [3, 4, 5, 73, 54, 56, 80, 95, 120, 121]; and the references cited therein.

Some notations and definitions for  $p$ -adic integrals are given as follows:

Let  $p$  be an odd prime number. Let  $m \in \mathbb{N}$ . Let  $ord_p(m)$  denote the greatest integer  $k$  ( $k \in \mathbb{N}_0$ ) such that  $p^k$  divides  $m$  in  $\mathbb{Z}$ . If  $m = 0$ , then  $ord_p(m) = \infty$ . Let  $x \in \mathbb{Q}$ , the set of rational numbers, with  $x = \frac{a}{b}$  for  $a, b \in \mathbb{Z}$  with  $b \neq 0$ . Therefore,

$$ord_p(x) = ord_p\left(\frac{a}{b}\right) = ord_p(a) - ord_p(b).$$

Let  $|\cdot|_p$  is a map on  $\mathbb{Q}$ . This map, which is a norm over  $\mathbb{Q}$ , is defined by

$$|x|_p = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

For instance,  $x \in \mathbb{Q}$  with  $x = p^y \frac{x_1}{x_2}$  where  $y, x_1, x_2 \in \mathbb{Z}$  and  $x_1$  and  $x_2$  are not divisible by  $p$ . Hence,  $ord_p(x) = y$  and  $|x|_p = p^{-y}$ . The set  $\mathbb{Q}_p$  equipped with this norm  $|x|_p$  is a topological completion of of set  $\mathbb{Q}$ . Let  $\mathbb{C}_p$  be the field of  $p$ -adic completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{Z}_p$  be topological closure of  $\mathbb{Z}$ . Let  $\mathbb{Z}_p$  be a set of  $p$ -adic integers, which is related to the norm  $|x|_p$ , given as follows:

$$\mathbb{Z}_p = \left\{ x \in \mathbb{Q}_p : |x|_p \leq 1 \right\}$$

In order to define  $p$ -adic integral, we need the following definitions and formulas.

Let  $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ .  $f$  is called a uniformly differential function at a point  $a \in \mathbb{Z}_p$  if  $f$  satisfies the following conditions:

If the difference quotients  $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \rightarrow \mathbb{C}_p$  such that

$$\Phi_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a limit  $f'(z)$  as  $(x, y) \rightarrow (z, z)$  ( $x$  and  $y$  remaining distinct). A set of uniformly differential functions is briefly indicated by  $f \in UD(\mathbb{Z}_p)$  or  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{C}_p)$ . The additive cosets of  $\mathbb{Z}_p$  are given as follows:

$$p\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x|_p < 1 \right\}, 1 + p\mathbb{Z}_p, \dots, p-1 + p\mathbb{Z}_p,$$

where  $p\mathbb{Z}_p$  is a maximal ideal of  $\mathbb{Z}_p$  and for each  $j \in \{0, 1, \dots, p^n - 1\}$  we set

$$j + p^n\mathbb{Z}_p = \left\{ x \in \mathbb{Z}_p : |x - j|_p < p^{1-n} \right\}.$$

Thus, we have

$$\mathbb{Z}_p = \cup_{j=0}^{p-1} (j + p\mathbb{Z}_p).$$

By using the above coset, we give the following distributions on  $\mathbb{Z}_p$  for  $p$ -adic integrals:

Every map  $\mu$  from the set of intervals contained in  $X$  to  $\mathbb{Q}_p$  for which

$$\mu(x + p^n\mathbb{Z}_p) = \sum_{j=0}^{p-1} \mu(x + jp^n + p^{n+1}\mathbb{Z}_p)$$

whenever  $x + p^n \mathbb{Z}_p \subset X$ , exists uniquely to a  $p$ -adic distribution on  $X$  (cf. [3], [74], [95], [113], [120], [121]).

Some well-known examples for the  $p$ -adic distribution are given as follows:  
The Haar distribution is defined by

$$\mu_{Haar}(x + p^N \mathbb{Z}_p) = \mu_1(x + p^N \mathbb{Z}_p) = \frac{1}{p^N}, \quad (74)$$

which denotes by  $\mu_1(x + p^N \mathbb{Z}_p) = \mu_1(x)$ ;

The Dirac distribution is defined by

$$\mu_{Dirac}(x + p^N \mathbb{Z}_p) = \mu_\alpha(X) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise;} \end{cases}$$

The Mazur distribution is defined by

$$\mu_{Mazur}(x + p^N \mathbb{Z}_p) = \frac{a}{p^N} - \frac{1}{2},$$

where  $a \in \mathbb{Q}$  with  $0 \leq a \leq p - 1$ ;

The Bernoulli distribution is defined by

$$\mu_{B,k}(x + p^N \mathbb{Z}_p) = p^{N(k-1)} B_k \left( \frac{a}{p^N} \right),$$

(cf. [3], [53], [72], [74], [95], [113], [120], [121]).

Observe that special values of the Bernoulli distribution are related to Haar distribution and the Mazur distribution; that is

$$\mu_{B,0}(x + p^N \mathbb{Z}_p) = \mu_1(x + p^N \mathbb{Z}_p)$$

and

$$\mu_{B,1}(x + p^N \mathbb{Z}_p) = \mu_{Mazur}(x + p^N \mathbb{Z}_p)$$

(cf. [74, p.35]).

On the other hand, we have the following distribution  $\mu_{-1}(x + p^N \mathbb{Z}_p)$  on  $\mathbb{Z}_p$ :

$$\mu_{-1}(x + p^N \mathbb{Z}_p) = (-1)^x \quad (75)$$

which is denoted by  $\mu_{-1}(x + p^N \mathbb{Z}_p) = \mu_{-1}(x)$  (cf. [71], [70], [54], [56], [89]).

The Euler distribution is defined by

$$\mu_{\mathcal{E},k,q}(x + fp^N \mathbb{Z}_p) = (-1)^a (fp^N)^k \mathcal{E}_k \left( \frac{a}{fp^N}; q^{fp^N} \right),$$

where  $N, k, f \in \mathbb{N}$  and  $f$  is odd (cf. [89], [82], [81]).

Therefore

$$\begin{aligned} \lim_{q \rightarrow 1} \mu_{\mathcal{E},k,q}(x + fp^N \mathbb{Z}_p) &= \mu_{E,k}(x + fp^N \mathbb{Z}_p) \\ &= (-1)^x (fp^N)^k E_k \left( \frac{x}{fp^N} \right), \end{aligned} \quad (76)$$

(cf. [89]).

Since

$$|\mu_{E,k}(x + fp^N\mathbb{Z}_p)|_p \leq 1,$$

$\mu_{E,k}(x + fp^N\mathbb{Z}_p)$  is a measure on  $\mathbb{X}$  where

$$\mathbb{X} = \mathbb{X}_f = \varprojlim_{\mathbb{N}} \mathbb{Z}/fp^N\mathbb{Z} \text{ and } \mathbb{X}_1 = \mathbb{Z}_p$$

(cf. [89]; see also [72], [82], [81]).

Substituting  $f = 1$  and  $k = 0$  into (76), we have

$$\mu_{E,0}(x + fp^N\mathbb{Z}_p) = \mu_{-1}(x).$$

In order to define invariant  $p$ -adic  $z$ -integrals, Kim [55] gave the following distribution on  $\mathbb{Z}_p$ :

Let

$$\mu_z(a + p^N\mathbb{Z}_p) = \frac{z^a}{[p^N : z]},$$

where

$$[x : z] = \frac{1 - z^x}{1 - z}.$$

It well-known that  $\mu_z(x + p^N\mathbb{Z}_p) = \mu_z(x)$  is extended distribution on  $\mathbb{Z}_p$  (cf. [117, p. 244]).

For a compact-open subset  $\mathbb{X}$  of  $\mathbb{Q}_p$ , a  $p$ -adic distribution  $\mu$  on  $\mathbb{X}$  is a  $\mathbb{Q}_p$ -linear vector space homomorphism from the  $\mathbb{Q}_p$ -vector space of locally constant functions on  $\mathbb{X}$  to  $\mathbb{Q}_p$  (cf. [95]).

Let  $\mathbb{K}$  be a field with a complete valuation and  $C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$  be a set of functions which have continuous derivative (see, for detail, [95]).

Kim [54] defined the  $p$ -adic  $q$ -integral as follows:

Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$  and  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ . Then we have

$$I_q(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (77)$$

where

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, & q \neq 1 \\ x, & q = 1 \end{cases}$$

and

$$\mu_q(x) = \mu_q(x + p^N\mathbb{Z}_p)$$

which denotes  $q$ -distribution on  $\mathbb{Z}_p$  and it is defined by

$$\mu_q(x + p^N\mathbb{Z}_p) = \frac{q^x}{[p^N]_q},$$

(cf. [54]).

Observe that

$$\lim_{q \rightarrow 1} \mu_q(x + p^N \mathbb{Z}_p) = \mu_{Haar}(x + p^N \mathbb{Z}_p) = \mu_1(x)$$

and

$$\lim_{q \rightarrow -1} \mu_q(x + p^N \mathbb{Z}_p) = \mu_{-1}(x).$$

Observe that if  $q \rightarrow 1$ , then (77) reduces to the following well-known Volkenborn integral (bosonic integral), which is denoted by  $I_1(f(x))$ :

$$\lim_{q \rightarrow 1} I_q(f(x)) = I_1(f(x)) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N \rightarrow \infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad (78)$$

where  $\mu_1(x)$  is given by the equation (74), that is

$$\mu_1(x) = \frac{1}{p^N}$$

(*cf.* [3], [73], [95], [120], [121]); see also the references cited in each of these earlier works).

The above integral has many applications not only in mathematics, but also in mathematical physics. By using this integral and its integral equations, various family of generating functions associated with Bernoulli-type numbers and polynomials have been constructed (*cf.* [6]-[123]).

Over and above, if  $q \rightarrow -1$ , then (77) reduces to the following well-known fermionic  $p$ -adic integral, which is denoted by  $I_{-1}(f(x))$ :

$$\begin{aligned} \lim_{q \rightarrow -1} I_q(f(x)) = I_{-1}(f(x)) &= \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ &= \lim_{N \rightarrow \infty} \sum_{x=0}^{p^N-1} (-1)^x f(x), \end{aligned} \quad (79)$$

where  $\mu_{-1}(x)$  is given by the equation (75), that is

$$\mu_{-1}(x) = (-1)^x$$

(*cf.* [56], see also [70]).

By using  $p$ -adic fermionic integral and its integral equations, various different generating functions, including Euler-type numbers and polynomials and Genocchi-type numbers and polynomials, have been constructed (*cf.* [6]-[123]).

We also note that  $p$ -adic  $q$ -integrals are related to the theory of the generating functions, ultrametric calculus, the quantum groups, cohomology groups,  $q$ -deformed oscillator and  $p$ -adic models (*cf.* [73, 120]).



## 2.1 Some Properties of the Volkenborn Integral

Here, we give some well-known properties of the Volkenborn integral (bosonic  $p$ -adic integral).

The Volkenborn integral is given in terms of the Mahler coefficients  $\binom{x}{n}$  as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} a_n,$$

where

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n} \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K}),$$

where

$$\binom{x}{n} = \frac{x^{(n)}}{n!},$$

$n \in \mathbb{N}_0$  (cf. [95, p. 168-Proposition 55.3]).

In [95], Schikhof gave the following integral formula:

$$\int_{\mathbb{Z}_p} f(x+n) d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{k=0}^{n-1} f'(k), \quad (80)$$

where

$$f'(x) = \frac{d}{dx} \{f(x)\}.$$

By substituting

$$f(x) = \binom{x}{n}$$

into (80), we get

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) + \frac{d}{dx} \left\{ \binom{x}{n} \right\} \Big|_{x=0},$$

where

$$\begin{aligned} \frac{d}{dx} \left\{ \binom{x}{n} \right\} \Big|_{x=0} &= \left\{ \frac{1}{n!} x^{(n)} \sum_{k=0}^{n-1} \frac{1}{x-k} \right\} \Big|_{x=0} \\ &= (-1)^{n-1} \frac{1}{n}. \end{aligned}$$

Therefore, we have

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \frac{(-1)^n}{n+1} + (-1)^{n-1} \frac{1}{n}. \quad (81)$$

Let  $f : \mathbb{Z}_p \rightarrow \mathbb{K}$  be an analytic function and

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

with  $x \in \mathbb{Z}_p$ .

The Volkenborn integral of this analytic function is given by

$$\int_{\mathbb{Z}_p} \left( \sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n \int_{\mathbb{Z}_p} x^n d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} \left( \sum_{n=0}^{\infty} a_n x^n \right) d\mu_1(x) = \sum_{n=0}^{\infty} a_n B_n,$$

(*cf.* [54], [56], [95]; see also the references cited in each of these earlier works).

Integral equation for the Volkenborn integral is given as follows:

$$\int_{\mathbb{Z}_p} E^m [f(x)] d\mu_1(x) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) + \sum_{j=0}^{m-1} \frac{d}{dx} \{f(x)\} |_{x=j} \quad (82)$$

where

$$E^m [f(x)] = f(x + m)$$

and

$$\frac{d}{dx} \{f(x)\} |_{x=j} = f'(j)$$

(*cf.* [54], [56], [95], [125]; see also the references cited in each of these earlier works).

Using (77), the following integral equation was given by Kim [58]:

$$q \int_{\mathbb{Z}_p} E [f(x)] d\mu_q(x) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) + \frac{q-1}{\log q} f'(0) + (q-1)f(0) \quad (83)$$

(*cf.* see also [49]-[67]).

As usual, exponential function is defined as follows:

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

The above series convergences in region  $\mathbb{E}$  which is a subset of field  $\mathbb{K}$  with  $\text{char}(\mathbb{K}) = 0$  (*cf.* [95, p. 70]). Let  $k$  be residue class field of  $\mathbb{K}$ . If  $\text{char}(k) = p$ , then

$$\mathbb{E} = \left\{ x \in \mathbb{K} : |x| < p^{\frac{1}{1-p}} \right\},$$

and if  $\text{char}(k) = 0$ , then

$$\mathbb{E} = \{x \in \mathbb{K} : |x| < 1\}.$$

Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ . Kim [58, Theorem 1] gave the following integral equation:

$$\begin{aligned} & q^n \int_{\mathbb{Z}_p} E^n [f(x)] d\mu_q(x) - \int_{\mathbb{Z}_p} f(x) d\mu_q(x) \\ &= \frac{q-1}{\log q} \left( \sum_{j=0}^{n-1} q^j f'(j) + \log q \sum_{j=0}^{n-1} q^j f(j) \right), \end{aligned} \quad (84)$$

where  $n$  is a positive integer.

Observe that substituting  $n = 1$  into (84), we arrive at (83).

**Theorem 7** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) = \frac{(-1)^n}{n+1}. \quad (85)$$

Note that Theorem 7 was proved by Schikhof [95].

Substituting  $m = 1$  and  $f(x) = (1+a)^x$  into (82), we have

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_1(x) = \frac{1}{t} \log(1+t).$$

Therefore,

$$\sum_{n=0}^{\infty} t^n \int_{\mathbb{Z}_p} \binom{x}{n} d\mu_1(x) = \frac{1}{t} \log(1+t).$$

Combining the above equation with (85), we have the following well-known relation:

$$\log(1+t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{n+1}$$

(cf. [95], [125]). We observe that

$$\int_{\mathbb{Z}_p} a^x d\mu_1(x) = \frac{1}{a-1} \log_p(a),$$

where  $a \in \mathbb{C}_p^+$  with  $a \neq 1$  (cf. [95, p. 170]).

Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ . Then we have

$$\int_{\mathbb{Z}_p} f(-x) d\mu_1(x) = \int_{\mathbb{Z}_p} f(1+x) d\mu_1(x)$$

and if  $f(-x) = -f(x)$ , which is geometrically symmetric about the origin, we have

$$\int_{\mathbb{Z}_p} f(x) d\mu_1(x) = -\frac{1}{2} f'(0) \quad (86)$$

(*cf.* [95, p. 169]). By using the above formulas, we give the following well-known examples:

$$\begin{aligned} \int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) &= \frac{a}{e^a - 1} \\ &= \sum_{n=0}^{\infty} B_n \frac{a^n}{n!}, \end{aligned} \quad (87)$$

where  $a \in \mathbb{E}$  with  $a \neq 0$  (*cf.* [95, p. 172]). Using Taylor series for  $e^{ax}$  in the left-hand side of the equation (87), we have the following well-known the Witt's formula for the Bernoulli numbers,  $B_n$ :

$$B_n = \int_{\mathbb{Z}_p} x^n d\mu_1(x), \quad (88)$$

where  $n \in \mathbb{N}_0$  (*cf.* [95]; see also [54], [56] and the references cited in each of these earlier works). It is well-known that the denominator of the Bernoulli numbers  $B_n$  is free of square. Consequently, for  $n \in \mathbb{N}_0$ ,  $x^n \in C^1(\mathbb{Z}_p \rightarrow \mathbb{Q})$ . Then we have

$$|B_n|_p = \left| \int_{\mathbb{Z}_p} x^n d\mu_1(x) \right|_p \leq p$$

(*cf.* [95, p. 172]). Similarly, we have  $p$ -adic representation for the Bernoulli polynomials as follows:

$$\int_{\mathbb{Z}_p} (z+x)^n d\mu_1(x) = B_n(z) \quad (89)$$

where  $n \in \mathbb{N}_0$  (*cf.* [54], [56], [95]; see also the references cited in each of these earlier works).

Let

$$P_n(x) = \sum_{j=0}^n a_j x^j$$

be a polynomial of degree  $n$  ( $n \in \mathbb{N}_0$ ). Substituting  $P_n(x)$  into (78), we have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_1(x) = \sum_{j=0}^n a_j \int_{\mathbb{Z}_p} x^j d\mu_1(x).$$

Since  $B_{2n+1} = 0$  for  $n \in \mathbb{N}$ , by combining the above equation with (88), we thus have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_1(x) = a_0 - \frac{1}{2}a_1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} a_{2j} B_{2j}.$$

Similarly, substituting

$$f(x, t; \lambda) = \lambda^x e^{tx}$$

into (78), we have

$$\int_{\mathbb{Z}_p} \lambda^x e^{t(x+y)} d\mu_1(x) = \frac{\log \lambda + t}{\lambda e^t - 1} e^{ty}, \quad (90)$$

where  $\lambda \in \mathbb{Z}_p$  (cf. [69]; see also [38], [117], [106]). Combining (90) with (27), we have

$$\int_{\mathbb{Z}_p} \lambda^x (x+y)^n d\mu_1(x) = \mathfrak{B}_n(y; \lambda)$$

According to [96], [122], [51] and [71], for each integer  $N \geq 0$ ;  $C_{p^N}$  denotes the multiplicative group of the primitive  $p^N$ th roots of unity in  $\mathbb{C}_p^* = \mathbb{C}_p \setminus \{0\}$ .

Let

$$\mathbb{T}_p = \left\{ \xi \in \mathbb{C}_p : \xi^{p^N} = 1, \text{ for } N \geq 0 \right\} = \cup_{N \geq 0} C_{p^N}$$

In the sense of the  $p$ -adic Pontrjagin duality, the dual of  $\mathbb{Z}_p$  is  $\mathbb{T}_p = C_{p^\infty}$ , the direct limit of cyclic groups  $C_{p^N}$  of order  $p^N$  with  $N \geq 0$ , with the discrete topology.  $\mathbb{T}_p$  accept a natural  $\mathbb{Z}_p$ -module structure which can be written briefly as  $\xi^x$  for  $\xi \in \mathbb{T}_p$  and  $x \in \mathbb{Z}_p$ .  $\mathbb{T}_p$  are embedded discretely in  $\mathbb{C}_p$  as the multiplicative  $p$ -torsion subgroup. If  $\xi \in \mathbb{T}_p$ , then  $\vartheta_\xi : (\mathbb{Z}_p, +) \rightarrow (\mathbb{C}_p, \cdot)$  is the locally constant character,  $x \rightarrow \xi^x$ , which is a locally analytic character if  $\xi \in \{\xi \in \mathbb{C}_p : \text{ord}_p(\xi - 1) > 0\}$ . Consequently, it is well-known that  $\vartheta_\xi$  has a continuation to a continuous group homomorphism from  $(\mathbb{Z}_p, +)$  to  $(\mathbb{C}_p, \cdot)$  (cf. [96], [122], [51], [71], [97]; see also the references cited in each of these earlier works).

We assume that  $\lambda \in \mathbb{T}_p$ . Then we have

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_1(x) = \mathcal{B}_n(\lambda) = \frac{nH_{n-1}(\lambda^{-1})}{\lambda - 1}$$

(cf. [69]).

The Volkenborn integral of some trigonometric functions are given as follows:

$$\begin{aligned} \int_{\mathbb{Z}_p} \cos(ax) d\mu_1(x) &= \frac{a \sin(a)}{2(1 - \cos(a))} \\ &= \frac{a}{2} \cot\left(\frac{a}{2}\right) \\ &= \sum_{n=0}^{\infty} (-1)^n B_{2n} \frac{a^{2n}}{(2n)!}, \end{aligned}$$

where  $a \in \mathbb{E}$  with  $a \neq 0$ ,  $p \neq 2$  (cf. [95, p. 172], [57]);

$$\int_{\mathbb{Z}_p} \sin(ax) d\mu_1(x) = -\frac{a}{2},$$

where  $a \in \mathbb{E}$  (cf. [95, p. 170], [57]); and also

$$\int_{\mathbb{Z}_p} \tan(ax) d\mu_1(x) = -\frac{a}{2}.$$

Note that

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = \frac{1}{2} \int_{\mathbb{Z}_p} e^{ax} d\mu_1(x) - \frac{1}{2} \int_{\mathbb{Z}_p} e^{-ax} d\mu_1(x).$$

Combining the above equation with (87), we have

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_1(x) = \frac{1}{2} \frac{a}{e^a - 1} + \frac{1}{2} \frac{a}{e^{-a} - 1} = -\frac{a}{2}.$$

## 2.2 $p$ -adic integral over subsets:

Let  $V$  be a compact open subset of  $\mathbb{Z}_p$ . Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ . Then we have

$$\int_V f(x) d\mu_1(x) = \int_{\mathbb{Z}_p} g(x) d\mu_1(x),$$

where

$$g(x) = \begin{cases} f(x) & \text{if } x \in V \\ 0 & \text{if } x \in \mathbb{Z}_p \setminus V \end{cases}$$

(cf. [95, p. 174]).

## 2.3 $p$ -adic integral over $j + p^n \mathbb{Z}_p$ :

Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ . Then we have

$$\int_{j+p^n \mathbb{Z}_p} f(x) d\mu_1(x) = \int_{p^n \mathbb{Z}_p} f(j+x) d\mu_1(x) = \frac{1}{p^n} \int_{\mathbb{Z}_p} f(j+p^n x) d\mu_1(x) \quad (91)$$

(cf. [95, p. 175]). Substituting  $f(x) = x^m$  with  $m \in \mathbb{N}$  into (91), we have

$$\int_{j+p^n \mathbb{Z}_p} x^m d\mu_1(x) = p^{n(m-1)} B_m \left( \frac{j}{p^n} \right)$$

(cf. [95, p. 175]).

We now give some examples for the above formula:

Let

$$\mathbf{T}_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p$$

and  $f : \mathbf{T}_p \rightarrow \mathbb{Q}_p$  and a  $C^1$ -function and also  $f(-x) = -f(x)$  with  $x \in \mathbf{T}_p$ . Thus we have

$$\int_{\mathbf{T}_p} f(x) d\mu_1(x) = 0.$$

Therefore

$$\int_{\mathbf{T}_p} \frac{1}{x} d\mu_1(x) = \int_{\mathbf{T}_p} \frac{1}{x^3} d\mu_1(x) = \int_{\mathbf{T}_p} \frac{1}{x^5} d\mu_1(x) = \cdots = \int_{\mathbf{T}_p} \frac{1}{x^{2n+1}} d\mu_1(x) = 0,$$

where  $n \in \mathbb{N}$  (cf. [95, p. 175]) and

$$\int_{\mathbf{T}_p} x^j (x^{p-1})^s d\mu_1(x) = (j + (p-1)s) \zeta_{p,j}(s), \quad (92)$$

where  $\zeta_{p,j}(s)$  denotes the  $p$ -adic zeta function,  $|s|_p < p^{\frac{p-2}{p-1}}$ ,  $s \neq -\frac{j}{p-1}$  and  $j \in \{0, 1, \dots, p-2\}$ ,  $p \neq 2$  (cf. [95, p. 187], [117]).

Substituting  $s = n$  ( $n \in \mathbb{N}_0$ ) into (92), we have following values of the  $p$ -adic zeta function:

$$\int_{\mathbf{T}_p} (x^{p-1})^n d\mu_1(x) = (1 - p^{n(p-1)-1}) \frac{B_{n(p-1)}}{n(p-1)},$$

$$\int_{\mathbf{T}_p} x^j (x^{p-1})^n d\mu_1(x) = (1 - p^{j-1+n(p-1)}) \frac{B_{j+n(p-1)}}{j+n(p-1)}$$

whereas for  $n \in \{2, 4, 6, 8, \dots\}$ ,  $j = 0$  and  $p = 2$ ; and consequently we also have

$$\int_{\mathbf{T}_p} x^n d\mu_1(x) = (1 - 2^{n-1}) \frac{B_n}{n}$$

(cf. [95, p. 187], [117]).

## 2.4 $p$ -adic Integral of the Falling Factorial

Kim et al. [46] defined Witt-type identities for the Daehee numbers of the first kind by the following  $p$ -adic integral representation as follows:

$$D_n = \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x), \quad (93)$$

or equivalently

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \sum_{l=0}^n S_1(n, l) B_l, \quad (94)$$

where  $n \in \mathbb{N}_0$  (cf. [46]).

Kim et al. [46] defined the Daehee numbers of the second kind as follows:

$$\widehat{D}_n = \int_{\mathbb{Z}_p} t^{(n)} d\mu_1(t). \quad (95)$$

Kim et al. [46] defined the Daehee polynomials of the first and second kind, respectively, as follows:

$$D_n(x) = \int_{\mathbb{Z}_p} (x+t)_{(n)} d\mu_1(t) \quad (96)$$

and

$$\widehat{D}_n(x) = \int_{\mathbb{Z}_p} (x+t)^{(n)} d\mu_1(t), \quad (97)$$

where  $n \in \mathbb{N}_0$ .

Combining the following relation

$$x_{(n)} = n! \binom{x}{n},$$

and (85) with (93), we also have

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x) = \frac{(-1)^n n!}{n+1}, \quad (98)$$

where  $n \in \mathbb{N}_0$  (cf. [46]).

In [46], Kim et al. gave the following formula:

$$D_n = \frac{(-1)^n n!}{n+1}. \quad (99)$$

By using (85), we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_1(x) &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_1(x) \\ &= \sum_{m=1}^n (-1)^m \binom{n-1}{m-1} \frac{1}{m+1} \\ &= \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} \frac{1}{m+1} \end{aligned} \quad (100)$$



(cf. [49], [101], [46]). By using (100), we obtain

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_1(x) = n! \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} \frac{1}{m+1}. \quad (101)$$

### 3 Integral Formulas for the Volkenborn Integral

In [110], we gave the following interesting and new integral formulas for the Volkenborn integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Bernoulli numbers, the Stirling numbers and the Lah numbers.

Using (5) and (85), we [110] gave the following formula:

**Theorem 8 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} xx_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2 + 3n + 2}. \quad (102)$$

By using (57), we have the following formulas:

**Theorem 9 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} xx_{(n)} d\mu_1(x) = \sum_{k=1}^n S_1(n, k-1) B_k + B_{n+1}. \quad (103)$$

**Theorem 10 (cf. [110])** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} xx^{(n)} d\mu_1(x) = \sum_{k=1}^n (-1)^{k+1} \binom{n-1}{k-1} \frac{n!}{k^2 + 3k + 2}. \quad (104)$$

By using

$$xx^{(n)} = \sum_{k=1}^n C(n, k) x^{k+1} \quad (105)$$

and (88), we [110] gave the following formula:

**Theorem 11 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} xx^{(n)} d\mu_1(x) = \sum_{k=1}^n C(n, k) B_{k+1}. \quad (106)$$

**Theorem 12 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \frac{x^{(n+1)}}{x} d\mu_1(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{k+1}.$$

By applying the Volkenborn integral to (6), we have

$$\int_{\mathbb{Z}_p} x_{(n+1)} d\mu_1(x) = \sum_{k=0}^n (-1)^{n-k} n_{(n-k)} \int_{\mathbb{Z}_p} x x_{(k)} d\mu_1(x).$$

By combining (98), (112) and (102) with the above equation, we [110] have the following formula:

**Theorem 13 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\sum_{k=0}^n n_{(n-k)} \frac{k!}{k^2 + 3k + 2} = \frac{(n+1)!}{n+2}.$$

Applying the Volkenborn integral to (7), we have

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \int_{\mathbb{Z}_p} x x_{(n)} d\mu_1(x) + \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x).$$

Combining the above equation with (102) and (98), after some elementary calculations, we arrive at the following result:

**Corollary 14 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n+1)} d\mu_1(x) = \frac{(-1)^n}{n+2} n!.$$

By applying the Volkenborn integral to (10), we get the following formula:

$$\int_{\mathbb{Z}_p} \binom{x+m}{n} d\mu_1(x) = \sum_{m=0}^n (-1)^k \binom{m}{n-k} \frac{1}{k+1} \quad (107)$$

(cf. [110]).

By applying the Volkenborn integral with respect to  $x$  and  $y$  to (9), we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \sum_{k=0}^n \binom{x}{k} \binom{y}{n-k} d\mu_1(y) d\mu_1(y) \\ &= \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y) d\mu_1(y). \end{aligned} \quad (108)$$

By combining the following identity with the above equation:

$$\binom{x+y}{n} = \frac{1}{n!} (x+y)_{(n)} = \frac{1}{n!} \sum_{k=0}^n \binom{n}{k} x_{(k)} y_{(n-k)}, \quad (109)$$

and using (93) and (98), we also get the following lemma:

**Lemma 15 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y) d\mu_1(x) = \sum_{k=0}^n (-1)^k \frac{1}{(k+1)(n-k+1)}. \quad (110)$$

By combining (108) with the following identity:

$$\binom{x+y}{n} = \frac{1}{n!} (x+y)_{(n)} = \frac{1}{n!} \sum_{k=0}^n S_1(n, k) (x+y)^k,$$

and using (93) and (98), we also get the following lemma:

**Lemma 16 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \binom{x+y}{n} d\mu_1(y) d\mu_1(x) = \frac{1}{n!} \sum_{k=0}^n \sum_{j=0}^k \binom{k}{j} S_1(n, k) B_j B_{k-j}. \quad (111)$$

By using (81), we have the following formula:

**Lemma 17 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+1}{n} d\mu_1(x) = \frac{(-1)^{n+1}}{n^2 + n}. \quad (112)$$

**Theorem 18 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_1(x) = (-1)^{n+1} \frac{n!}{n^2 + n}.$$

**Corollary 19 (cf. [110])** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} \Delta x_{(n)} d\mu_1(x) = (-1)^{n+1} (n-1)!.$$

By applying the Volkenborn integral to the equation (56), we obtain

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=0}^n L(n, k) \int_{\mathbb{Z}_p} x_{(k)} d\mu_1(x),$$

where  $n \in \mathbb{N}_0$ .

By using (3), we get

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=0}^n (-1)^k \frac{k! L(n, k)}{k+1},$$

where  $n \in \mathbb{N}_0$ . Substituting (54) into the above equation, we arrive at the following theorem:

**Theorem 20 (cf. [110])** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} (-x)_{(n)} d\mu_1(x) = \sum_{k=1}^n (-1)^{k+n} \binom{n-1}{k-1} \frac{n!}{k+1}.$$

**Corollary 21 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+1}{n+1} d\mu_1(x) = \frac{(-1)^n}{n^2 + 3n + 2}. \quad (113)$$

By applying the Volkenborn integral to the equation (51) and (52), respectively, we have the following results:

**Lemma 22 (cf. [110])** *Let  $k \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_1(x) d\mu_1(y) = \sum_{l,m=1}^k D_l D_m C_{l,m}^{(k)}$$

and

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_1(x) d\mu_1(y) = \sum_{l,m=1}^k (-1)^{l+m} \frac{l!m!}{(l+1)(m+1)} C_{l,m}^{(k)}. \quad (114)$$

**Lemma 23 (cf. [110])** *Let  $k \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_1(x) d\mu_1(y) = \sum_{m=0}^k S_1(k, m) (B_m)^2. \quad (115)$$

By applying the Volkenborn integral to (14), and using (85), we arrive the following result:

**Theorem 24 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x \binom{x-2}{n-1} d\mu_1(x) = (-1)^n \sum_{k=1}^n \frac{k}{k+1}.$$

By applying the Volkenborn integral to (15) and using (85), we have the following result:

**Theorem 25 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_1(x) = (-1)^n H_n,$$

where  $H_n$  denotes the harmonic numbers given by

$$H_n = \sum_{k=0}^n \frac{1}{k+1}. \quad (116)$$

By applying the Volkenborn integral to (16), and using (85), we arrive at the following result:

**Theorem 26 (cf. [110])** *Let  $m \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_1(x) = \sum_{k=0}^n \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

By applying the Volkenborn integral to the above equation (17), and using (85), we arrive at the following theorem:

**Theorem 27 (cf. [110])** *Let  $n, r \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \quad (117)$$

**Remark 28** *Substituting  $r = 1$  into (117), since  $\binom{k-j}{n} = 0$  if  $k-j < n$ , we arrive at the equation (85).*

By applying the Volkenborn integral to (18), and using (85), we arrive at the following theorem:

**Theorem 29 (cf. [110])** *Let  $n \in \mathbb{N}$  with  $n > 1$ . Then we have*

$$\int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_1(x) = (-1)^n \sum_{k=0}^n \frac{k^2}{k+1}.$$

By applying the Volkenborn integral to the above equations (19) and (20), using (85) and (88), respectively, we arrive at the following theorem:

**Theorem 30 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_1(x) = \sum_{k=0}^n \frac{(-1)^k}{k+1} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n} \quad (118)$$

and

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_1(x) = \sum_{k=0}^n B_k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}. \quad (119)$$

By applying the Volkenborn integral to (21), and using (85), we arrive at the following theorem:

**Theorem 31 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_1(x) = \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{2k-2n} (2n+1)}{(k+1)(2k+1) \binom{2k}{k}}.$$

By applying the Volkenborn integral to (36), and using (88), we get the following result:

**Theorem 32 (cf. [110])** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_1(x) = \sum_{k=0}^n S_1(n, k) B_{k+m}. \quad (120)$$

By applying the Volkenborn integral to (39), and using (88), we get the following lemma:

**Lemma 33 (cf. [110])** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{j=0}^n \sum_{l=0}^m S_1(n, k) S_1(m, l) B_{j+l}. \quad (121)$$

**Theorem 34 (cf. [110])** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{k=0}^m (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{m+n-k+1}. \quad (122)$$

By applying the Volkenborn integral to (38), we get the following lemma:

**Lemma 35 (cf. [110])** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_1(x) = \sum_{k=0}^m \binom{m}{k} \binom{n}{k} k! \sum_{l=0}^{m+n-k} S_1(m+n-k, l) B_l. \quad (123)$$

Let  $z \in \mathbb{C}_p$ , we have

$$D_m(z : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_z(x) \quad (124)$$

(cf. [55]). If we take  $z = q$  in (124), then we see that  $D_m(q : q) = \beta_m(q)$ , Carlitz's  $q$ -Bernoulli numbers. In the case when  $z = u$  in (124),  $q$ -Daehee numbers and  $q$ -Daehee polynomials are defined, respectively, as follows:

$$D_m(u : q) = \int_{\mathbb{Z}_p} [x]^m d\mu_u(x) = H_m(u^{-1} : q) \quad (125)$$

and

$$D_m(z, x : q) = \int_{\mathbb{Z}_p} [x+t]^m d\mu_z(t),$$

where  $m \in \mathbb{N}_0$  and  $z \in \mathbb{C}_p$  (cf. [55], [117, Eq. (1.10)]).

**Theorem 36** (cf. [114, Theorem 1]) *Assume that  $a, b$  are integers with  $(a, b) = (p, b) = 1$ . Let*

$$S_q(a, b : n; q^l) = \sum_{M=1}^{k-1} \frac{[M]}{[b]} \int_{\mathbb{Z}_p} q^{-lx} \left[ x + \left\{ \frac{aM}{b} \right\} : q^l \right]^n d\mu_{q^l}(x).$$

Observe that

$$D_m(u : q) = H_m(u^{-1} : q)$$

(cf. [55, Eq. (5)]).

**Corollary 37** (cf. [114, Corollary 1]) *If  $z = \lambda$ , then we have*

$$B_q(d, c : 0, \lambda : m) = \frac{m}{[c^m]} \sum_{\lambda} \frac{1}{[\lambda - 1][\lambda^{-d} - 1]} \int_{\mathbb{Z}_p} [t]^m d\mu_{\lambda}(t). \quad (126)$$

**Remark 38** *If  $q \rightarrow 1$ , then (126) is reduced to (127). That is,*

$$\lim_{q \rightarrow 1} B_q(d, c : 0, \lambda : m) = S(d, c : m)$$

where

$$S(d, c : m) = \frac{m}{c^m} \sum_{\lambda} \frac{H_{m-1}(\lambda^{-1})}{(\lambda - 1)(\lambda^{-d} - 1)}, \quad (127)$$

where  $\lambda$  runs through the  $c$ th roots of unity distinct from 1 and  $H_m(\lambda)$  is the Frobenius-Euler numbers (cf. [114, Remark 1]).

Now, it is time to raise the following question:

*Is it possible to give any reciprocity law for the  $q$ -Dedekind type sum  $B_q(d, c : 0, \lambda : m)$ . That is, how can we calculate the following relation:*

$$B_q(d, c : 0, \lambda : m) + B_q(c, d : 0, \lambda : m) = ?$$

### 3.1 Some Properties of the Fermionic $p$ -adic Integral

Here, we give some well-known properties of the fermionic  $p$ -adic integral.

Let  $f \in C^1(\mathbb{Z}_p \rightarrow \mathbb{K})$ . Kim [57] gave the following integral equation for the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ :

$$\begin{aligned} & \int_{\mathbb{Z}_p} E^n[f(x)] d\mu_{-1}(x) + (-1)^{n+1} \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) \\ &= 2 \sum_{j=0}^{n-1} (-1)^{n-1-j} f(j), \end{aligned} \quad (128)$$

where  $n \in \mathbb{N}$ .

Substituting  $n = 1$  into (128), we have very useful integral equation, which is used to construct generating functions associated with Euler-type numbers and polynomials, given as follows:

$$\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0) \quad (129)$$

(*cf.* [57]).

By using (79) and (129), the well-known Witt's type formulas for the Euler numbers and polynomials are given as follows, respectively:

$$E_n = \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) \quad (130)$$

and

$$E_n(z) = \int_{\mathbb{Z}_p} (z+x)^n d\mu_{-1}(x), \quad (131)$$

where  $n \in \mathbb{N}_0$  (*cf.* [56], [36]; see also the references cited in each of these earlier works).

**Theorem 39** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n} d\mu_{-1}(x) = (-1)^n 2^{-n}. \quad (132)$$

Theorem 39 was proved by Kim et al. [49, Theorem 2.3].

Substituting  $x_{(n)} = n! \binom{x}{n}$  into (132), we have the following well-known identity:

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = (-1)^n 2^{-n} n! \quad (133)$$

where  $n \in \mathbb{N}_0$  (*cf.* [49]).

Recently, by using the fermionic  $p$ -adic integral on  $\mathbb{Z}_p$ , Kim et al. [49] defined the Changhee numbers of the first and the second kind, respectively, as follows:

$$Ch_n = \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) \quad (134)$$

and

$$\widehat{Ch}_n = \int_{\mathbb{Z}_p} x^{(n)} d\mu_{-1}(x), \quad (135)$$

where  $n \in \mathbb{N}_0$ .



For  $n \in \mathbb{N}_0$ , Kim et al. [49] gave the following formula for the Changhee numbers of the first kind:

$$Ch_n = (-1)^n 2^{-n} n!. \quad (136)$$

Kim et al. [49] also defined the Changhee polynomials of the first and the second, respectively, as follows:

$$Ch_n(x) = \int_{\mathbb{Z}_p} (x+t)_{(n)} d\mu_{-1}(t) \quad (137)$$

and

$$\widehat{Ch}_n(x) = \int_{\mathbb{Z}_p} (x+t)^{(n)} d\mu_{-1}(t). \quad (138)$$

Therefore, by using Theorem 39, we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \binom{x+n-1}{n} d\mu_{-1}(x) &= \sum_{m=0}^n \binom{n-1}{n-m} \int_{\mathbb{Z}_p} \binom{x}{m} d\mu_{-1}(x) \\ &= \sum_{m=1}^n (-1)^m \binom{n-1}{m-1} 2^{-m} \\ &= \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} 2^{-m} \end{aligned} \quad (139)$$

(cf. [49], [101], [46]). By using (139), we have

$$\int_{\mathbb{Z}_p} (x+n-1)_{(n)} d\mu_{-1}(x) = n! \sum_{m=0}^n (-1)^m \binom{n-1}{n-m} 2^{-m}. \quad (140)$$

By using (128), Kim [58] modified (79). He gave the following integral equation:

$$q^d \int_{\mathbb{Z}_p} E^d f(x) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = [2] \sum_{j=0}^{d-1} (-1)^j q^j f(j), \quad (141)$$

where  $d$  is an positive odd integer.

Some examples for the fermionic  $p$ -adic integral are given as follows:

The Volkenborn integral of some trigonometric functions are given as follows:

$$\int_{\mathbb{Z}_p} \cos(ax) d\mu_{-1}(x) = 1,$$

where  $a \in \mathbb{E}$  with  $a \neq 0$ ,  $p \neq 2$  (cf. [57]);

$$\int_{\mathbb{Z}_p} \sin(a(x+1)) d\mu_{-1}(x) = - \int_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x),$$

where  $a \in \mathbb{E}$  (cf. [57]); and also

$$\int_{\mathbb{Z}_p} \sin(ax) d\mu_{-1}(x) = -\frac{\sin(a)}{\cos(a) + 1}.$$

Note that

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = \frac{1}{2} \int_{\mathbb{Z}_p} e^{ax} d\mu_{-1}(x) - \frac{1}{2} \int_{\mathbb{Z}_p} e^{-ax} d\mu_{-1}(x).$$

Combining the above equation with (129), we have

$$\int_{\mathbb{Z}_p} \sinh(ax) d\mu_{-1}(x) = \frac{1}{e^a + 1} + \frac{1}{e^{-a} + 1} = 1.$$

Let

$$P_n(x) = \sum_{j=0}^n a_j x^j$$

be a polynomial of degree  $n$  ( $n \in \mathbb{N}_0$ ). Substituting  $P_n(x)$  into (79), we have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = \sum_{j=0}^n a_j \int_{\mathbb{Z}_p} x^j d\mu_{-1}(x).$$

Since  $E_{2n} = 0$  for  $n \in \mathbb{N}$ , by combining the above equation with (130), we thus have

$$\int_{\mathbb{Z}_p} P_n(x) d\mu_{-1}(x) = 1 + \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} a_{2j+1} E_{2j+1}.$$

By using (76), we have

$$\int_{\mathbb{X}} d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = \mathcal{E}_k(\lambda), \quad (142)$$

where  $\lambda \in \mathbb{Z}_p$  (cf. [82], [81]).

Substituting

$$g(x, t; \lambda) = \lambda^x e^{tx}$$

into (129), we have the following well-known formula:

$$\int_{\mathbb{Z}_p} g(x, t; \lambda) d\mu_{-1}(x) = \frac{2}{\lambda e^t + 1}.$$

Combining the above equation with (28), we have

$$\int_{\mathbb{Z}_p} \lambda^x x^n d\mu_{-1}(x) = \mathcal{E}_n(\lambda). \quad (143)$$

By assuming that  $\chi$  is the primitive Dirichlet's character with odd conductor  $f$ , Rim and Kim [89] gave the following formula:

$$\int_{\mathbb{X}} \chi(x) d\mu_{E,k}(x + fp^N \mathbb{Z}_p) = E_{k,\chi}$$

where

$$\frac{2}{e^{ft} + 1} \sum_{j=0}^{f-1} (-1)^j \chi(j) e^{tj} = \sum_{n=0}^{\infty} E_{n,\chi} \frac{t^n}{n!}.$$

Combining (142) with (143), we have the following well-known relation:

$$d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = \lambda^x x^n d\mu_{-1}(x + p^N \mathbb{Z}_p). \quad (144)$$

Setting  $\lambda = 1$  in (144), we have

$$d\mu_{E,k}(x + fp^N \mathbb{Z}_p) = x^n d\mu_{-1}(x + p^N \mathbb{Z}_p)$$

or equivalently

$$d\mu_{E,k}(x) = x^n d\mu_{-1}(x)$$

(cf. [89]).

Therefore, combining (30) with (144), we have

$$d\mu_{\mathcal{E},k,\lambda}(x + fp^N \mathbb{Z}_p) = -\frac{2}{k+1} d\mu_{\mathcal{B},k,-\lambda}(x + fp^N \mathbb{Z}_p).$$

## 4 Integral Formulas for the Fermionic $p$ -adic Integral

In [110], we gave the following interesting and new integral formulas for the fermionic  $p$ -adic integral including the falling factorial and the rising factorial with their identities and relations, the combinatorial sums, the special numbers such as the Euler numbers, the Stirling numbers and the Lah numbers.

By applying the  $p$ -adic fermionic integral to the both sides of equation (5) and using (133), we have the following theorem:

**Theorem 40 (cf. [110])** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} x x_{(n)} d\mu_{-1}(x) = (-1)^n \frac{(n-1)}{2^{n+1}} n!. \quad (145)$$

By applying the  $p$ -adic fermionic integral to the both sides of equation (??), and using (145) and (55), we have the following theorem:

**Theorem 41 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x x^{(n)} d\mu_{-1}(x) = \sum_{k=1}^n (-1)^k \binom{n-1}{k-1} \frac{(k-1)}{2^{k+1}} n!. \quad (146)$$

By applying the  $p$ -adic fermionic integral to (13) and using (133), we have the following theorem:

**Theorem 42 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} (x+1)_{(n)} d\mu_{-1}(x) = (-1)^{n+1} \frac{1}{2^n} n!. \quad (147)$$

By applying the  $p$ -adic fermionic integral to equation (6), and using (133), we arrive at the following theorem:

**Theorem 43 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \frac{x_{(n+1)}}{x} d\mu_{-1}(x) = \sum_{k=0}^n (-1)^n n_{(n-k)} \frac{k!}{2^k}.$$

**Lemma 44 (cf. [110])** *Let  $k \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x) d\mu_{-1}(y) = \sum_{l,m=1}^k (-1)^{l+m} 2^{-m-l} l! m! C_{l,m}^{(k)}. \quad (148)$$

**Lemma 45 (cf. [110])** *Let  $k \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} (xy)_{(k)} d\mu_{-1}(x) d\mu_{-1}(y) = \sum_{m=0}^k S_1(k, m) (E_m)^2. \quad (149)$$

**Theorem 46 (cf. [110])** *Let  $n \in \mathbb{N}$  with  $n > 1$ . Then we have*

$$\int_{\mathbb{Z}_p} \left\{ x \binom{x-2}{n-1} + x(x-1) \binom{n-3}{n-2} \right\} d\mu_{-1}(x) = (-1)^n \sum_{k=0}^n \frac{k^2}{2^k}.$$

**Theorem 47 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j+n}{n} \quad (150)$$

and

$$\int_{\mathbb{Z}_p} \binom{x+n}{n} d\mu_{-1}(x) = \sum_{k=0}^n E_k \sum_{j=0}^n \binom{n}{j} \frac{S_1(j, k)}{j!}. \quad (151)$$

**Theorem 48 (cf. [110])** *Let  $m \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{mx}{n} d\mu_{-1}(x) = \sum_{k=0}^n \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{mk-mj}{n}.$$

**Theorem 49 (cf. [110])** *Let  $n, r \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \frac{(-1)^k}{2^k} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r. \quad (152)$$

**Theorem 50 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) = (-1)^n \sum_{k=1}^n 2^{-k}.$$

By applying the  $p$ -adic fermionic integral to (21), and using (132), we arrive at the following theorem:

**Theorem 51 (cf. [110])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} \binom{x+n+\frac{1}{2}}{n} d\mu_{-1}(x) = (2n+1) \binom{2n}{n} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2^{k-2n}}{(2k+1) \binom{2k}{k}}.$$

By using (128), Kim et al. [63, Theorem 2.1] proved the following theorem:

**Theorem 52 (cf. [63])** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} (1-x)^n d\mu_{-1}(x) = 2 + \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x). \quad (153)$$

By using (153), Kim et al. [63, Theorem 2.1] proved the following theorem:

**Theorem 53 (cf. [63])** *Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . If  $k = 0$ , we have*

$$\int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = 2 + E_n \quad (154)$$

and if  $k > 0$ , we have

$$\int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} E_{n-j}. \quad (155)$$

In [66], Kim et al. gave the following formula:

$$\int_{\mathbb{Z}_p} (-y^w (e^t - 1)^w)^x d\mu_{-1}(x) = \frac{2}{1 - y^w (e^t - 1)^w}, \quad (156)$$

where  $w \in \mathbb{N}$ . By using the above formula, they defined so-called  $w$ -torsion Fubini polynomials. If  $w = y = 1$ , right-hand side of the equation (156) reduces to generating function for the Fubini numbers (cf. [43]).

## 5 New Integral Formulas Involving Volkenborn Integral

In this section, we give some new integral formulas for the Volkenborn integral. These new formulas are related to some special functions, special numbers and polynomials such as rising factorial and the falling factorial, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

**Theorem 54** *Let  $m, n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = (-1)^{m+n} \frac{(m+n)!}{m+n+1}. \quad (157)$$

**Proof.** By applying the Volkenborn integral to following well-known identity:

$$x_{(m+n)} = x_{(m)} (x - m)_{(n)}, \quad (158)$$

we get

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = \int_{\mathbb{Z}_p} x_{(m+n)} d\mu_1(x).$$

Combining the above equation with (98), we get the desired result. ■

**Theorem 55** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k. \quad (159)$$

**Proof.** By applying the Volkenborn integral to the equation (58), we get

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) = \sum_{k=0}^n t(n, k) \int_{\mathbb{Z}_p} x^k d\mu_1(x).$$

Combining the above equation with (88), we arrive at the desired result. ■

**Theorem 56** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ . Then we have*

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_1(x) = \sum_{k=0}^n t(n, k) B_k + \left(\frac{n-2}{2}\right)^2 \sum_{k=0}^{n-2} t(n-2, k) B_k.$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^{[n]} = \left( x^2 - \left( \frac{n-2}{2} \right)^2 \right) x^{[n-2]} \quad (160)$$

(cf. [13, p. 11]), we get

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[n]} d\mu_1(x) + \left( \frac{n-2}{2} \right)^2 \int_{\mathbb{Z}_p} x^{[n-2]} d\mu_1(x).$$

Combining the above equation with (159), we arrive at the desired result. ■

**Theorem 57** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \sum_{k=0}^{2n} t(2n, k) B_{2k}.$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^{[2n]} = x^2 \prod_{k=1}^{n-1} (x^2 - k^2), \quad (161)$$

which is an even function (cf. [13, Eq. (2.1)]), we get

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[2n]} d\mu_1(x).$$

Combining right-hand side of the above equation with (159), we arrive at the desired result. ■

**Theorem 58** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x \prod_{k=1}^n \left( x^2 - \frac{(2k-1)^2}{4} \right) d\mu_1(x) = -\frac{1}{2} \frac{d}{dx} \left\{ x^{[2n+1]} \right\} \Big|_{x=0}$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^{[2n+1]} = x \prod_{k=1}^n \left( x^2 - \frac{(2k-1)^2}{4} \right), \quad (162)$$

which is an odd function (cf. [13, Eq. (2.2)]), we get

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_1(x) = \int_{\mathbb{Z}_p} x^{[2n+1]} d\mu_1(x). \quad (163)$$

Since the function  $x^{[2n+1]}$  is an odd function, combining right-hand side of the equation (163) with (86), we arrive at the desired result. ■

**Remark 59** By combining (157) with (93) and (94), we get the following identities:

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = D_{m+n}$$

and

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_1(x) = \sum_{k=0}^{n+m} S_1(m+n, k) B_k.$$

**Theorem 60** Let  $n \in \mathbb{N}$ . Then we have

$$\int_{\mathbb{Z}_p} x \binom{x-2}{n-1} d\mu_1(x) = (-1)^{-n} \sum_{k=1}^n \frac{k}{k+1}.$$

**Proof.** By applying the Volkenborn integral to (22), and using (85), we get the desired result. ■

**Theorem 61** Let  $n \in \mathbb{N}_0$ . Then we have

$$\begin{aligned} \int_{\mathbb{Z}_p} \binom{n-x}{n} d\mu_{-1}(x) &= (-1)^n \sum_{k=1}^n \frac{1}{k+1} \\ &= (-1)^n (H_n - H_0). \end{aligned}$$

**Proof.** By applying the Volkenborn integral to the above integral, and using (85) and (116), we get the desired result. ■

**Theorem 62** Let  $n, r \in \mathbb{N}_0$ . Then we have

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l) B_{v+l}}{k!}.$$

**Proof.** By applying the Volkenborn integral to (17), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} x^v \binom{x}{k} d\mu_1(x) \\ &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} \frac{x^v}{k!} x_{(k)} d\mu_1(x). \end{aligned}$$

Combining the above equation with (36), we obtain

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_1(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l)}{k!} \int_{\mathbb{Z}_p} x^{v+l} d\mu_1(x).$$

Combining the above equation with (88), we arrive at the desired result. ■



**Theorem 63** Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have

$$\sum_{k=0}^n (-1)^{k-n} \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} B_{n-j}.$$

**Proof.** By applying the Volkenborn integral to following well-known identity:

$$\sum_{k=0}^n (-1)^k B_k^n(x) = (1-2x)^n \quad (164)$$

(cf. [102, Theorem 3.4]), we get

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) &= \int_{\mathbb{Z}_p} (1-2x)^n d\mu_1(x) \\ &= \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \int_{\mathbb{Z}_p} x^{n-j} d\mu_1(x). \end{aligned} \quad (165)$$

By combining (165) with (88), we get the desired result. ■

Combining (165) with (50), we obtain

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) \\ = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) \int_{\mathbb{Z}_p} x_{(m)} d\mu_1(x). \end{aligned} \quad (166)$$

Combining (166) with (98), we arrive at the following theorem:

**Theorem 64** Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) \\ = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{n+m-j} 2^{n-j} S_2(n-j, m) \frac{m!}{m+1}. \end{aligned} \quad (167)$$

Combining (166) with (93), we arrive at the following result:

**Corollary 65** Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have

$$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_1(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) D_m. \quad (168)$$

By applying the Volkenborn integral to (66) and (68), using (85), (93) (98) and (94), we arrive at the following results:

**Theorem 66** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) D_{n-v}. \quad (169)$$

**Theorem 67**

$$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \frac{s_v(\lambda, \mu)(n-v+1)!}{n-v+1}. \quad (170)$$

**Theorem 68**

$$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_1(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) \sum_{l=0}^{n-v} S_1(n-v, l) B_l. \quad (171)$$

Combining (171) with (98), we also have the following theorem:

**Theorem 69**

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_1(x) = (-1)^n \frac{n!}{n+1}. \quad (172)$$

**Theorem 70** *Let  $H_k \in \mathbb{H}$ , the set of harmonic numbers. Let  $1 \leq n \leq k$ . Then we have*

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1+jx) d\mu_1(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} B_n,$$

where  $\binom{H_k}{n}_{\mathbb{H}}$  denotes the harmonic binomial coefficient.

**Proof.** In [12, Theorem 3.17], Brigham II defined the following identity:

$$\prod_{j=1}^k (1+jx) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} x^n, \quad (173)$$

where  $\binom{H_k}{n}_{\mathbb{H}}$  denotes the harmonic binomial coefficient, which given in [12, Lemma 3.2] as follows:

$$\binom{H_k}{n}_{\mathbb{H}} = \frac{1}{k!} \left[ \begin{matrix} k+1 \\ n+1 \end{matrix} \right]$$

in which  $\left[ \begin{matrix} k \\ n \end{matrix} \right]$  denotes the unsigned Stirling numbers of the first kind. By applying the Volkenborn integral to (173), we get

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1+jx) d\mu_1(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} \int_{\mathbb{Z}_p} x^n d\mu_1(x).$$

Combining the above equation with (88), we arrive at the desired result. ■

## 6 New Integral Formulas Involving Fermionic $p$ -adic Integral

In this section, we give some new integral formulas for the fermionic  $p$ -adic integral. These new formulas are related to some special functions, special numbers and polynomials such as rising factorial and the falling factorial, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

By applying the fermionic  $p$ -adic integral to (36), and using (130), we get the following theorem:

**Theorem 71** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x^m x_{(n)} d\mu_{-1}(x) = \sum_{k=0}^n S_1(n, k) E_{k+m}. \quad (174)$$

**Theorem 72** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k. \quad (175)$$

**Proof.** By applying the fermionic  $p$ -adic integral to the equation (58), we get

$$\int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x).$$

Combining the above equation with (130), we arrive at the desired result. ■

**Theorem 73** *Let  $n \in \mathbb{N}$  with  $n \geq 2$ . Then we have*

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_{-1}(x) = \sum_{k=0}^n t(n, k) E_k + \left(\frac{n-2}{2}\right)^2 \sum_{k=0}^{n-2} t(n-2, k) E_k.$$

**Proof.** By applying the fermionic  $p$ -adic integral to the equation (160), we get

$$\int_{\mathbb{Z}_p} x^2 x^{[n-2]} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x^{[n]} d\mu_{-1}(x) + \left(\frac{n-2}{2}\right)^2 \int_{\mathbb{Z}_p} x^{[n-2]} d\mu_{-1}(x).$$

Combining the above equation with (175), we arrive at the desired result. ■

**Corollary 74** *Let  $n \in \mathbb{N}$ . Then we have*

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_{-1}(x) = \sum_{k=0}^{2n} t(2n, k) E_{2k}.$$

**Proof.** By applying the Volkenborn integral to the equation (161), we get

$$\int_{\mathbb{Z}_p} x^2 \prod_{k=1}^{n-1} (x^2 - k^2) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x^{[2n]} d\mu_{-1}(x).$$

Combining right-hand side of the above equation with (175), we arrive at the desired result. ■

**Theorem 75** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = (-1)^{m+n} \frac{(m+n)!}{2^{m+n}}. \quad (176)$$

**Proof.** By applying the fermionic  $p$ -integral to (158), we obtain

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x_{(m+n)} d\mu_{-1}(x).$$

Combining the above equation with (133), we get the desired result. ■

**Remark 76** *Combining (176) with (134) and (64), we arrive at the following identities:*

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = Ch_{m+n}$$

and

$$\int_{\mathbb{Z}_p} x_{(m)} (x - m)_{(n)} d\mu_{-1}(x) = \sum_{k=0}^{n+m} S_1(n+m, k) E_k.$$

By applying the fermionic  $p$ -adic integral to (8), and using (133), we get the following theorem:

**Theorem 77** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x_{(m)} d\mu_{-1}(x) = \sum_{k=0}^m (-1)^{m+n-k} \binom{m}{k} \binom{n}{k} \frac{k!(m+n-k)!}{2^{m+n-k}}. \quad (177)$$

**Theorem 78** *Let  $n, r \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \sum_{j=0}^k \sum_{l=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \frac{S_1(k, l) E_{v+l}}{k!}.$$

**Proof.** By applying the fermionic  $p$ -adic integral to (17), we get

$$\begin{aligned} \int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} x^v \binom{x}{k} d\mu_{-1}(x) \\ &= \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \int_{\mathbb{Z}_p} \frac{x^v}{k!} x_{(k)} d\mu_{-1}(x). \end{aligned}$$

Combining the above equation with (36), we obtain

$$\int_{\mathbb{Z}_p} x^v \binom{x}{n}^r d\mu_{-1}(x) = \sum_{k=0}^{nr} \sum_{j=0}^k (-1)^j \binom{k}{j} \binom{k-j}{n}^r \sum_{l=0}^k \frac{S_1(k, l)}{k!} \int_{\mathbb{Z}_p} x^{v+l} d\mu_{-1}(x).$$

Combining the above equation with (130), we arrive at the desired result. ■

**Theorem 79** Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have

$$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} E_{n-j}. \quad (178)$$

**Proof.** By applying the fermionic  $p$ -integral to (164), we obtain

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} (1-2x)^n d\mu_{-1}(x) \\ &= \sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \int_{\mathbb{Z}_p} x^{n-j} d\mu_{-1}(x). \end{aligned} \quad (179)$$

Combining the above equation with (130), we arrive at the desired result. ■

**Theorem 80** Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have

$$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{m+n-j} 2^{n-j-m} S_2(n-j, m) m!.$$

**Proof.** Combining (179) with (50), we have

$$\begin{aligned} \sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) &= \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) \int_{\mathbb{Z}_p} x_{(m)} d\mu_{-1}(x). \end{aligned} \quad (180)$$

Combining (180) with (133), we get the desired result. ■

Combining (180) with (134), we arrive at the following corollary:

**Corollary 81** *Let  $k, n \in \mathbb{N}_0$  with  $0 \leq k \leq n$ . Then we have*

$$\sum_{k=0}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-2)^{n-j} S_2(n-j, m) Ch_m. \quad (181)$$

By applying the fermionic  $p$ -integral to (66) and (68), using (133) and (134), we arrive at the following theorems, respectively:

**Theorem 82** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_{-1}(x) = \sum_{v=0}^n \binom{n}{v} s_v(\lambda, \mu) Ch_{n-v}. \quad (182)$$

**Theorem 83** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} s_n(x; \lambda, \mu) d\mu_{-1}(x) = \sum_{v=0}^n (-1)^{n-v} \binom{n}{v} \frac{s_v(\lambda, \mu)(n-v)!}{2^{n-v}}. \quad (183)$$

**Theorem 84** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x) = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x). \quad (184)$$

**Theorem 85** *Let  $n, \mu \in \mathbb{N}_0$ . Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = (-1)^n \frac{n!}{2^n}. \quad (185)$$

By applying the fermionic  $p$ -integral to (69), we have

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (133) and (134), we obtain the following results:

**Theorem 86** *Let  $n, v \in \mathbb{N}_0$ . Then we have*

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = Ch_n. \quad (186)$$

**Theorem 87** *Let  $n, v \in \mathbb{N}_0$ . Then we have*

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_{-1}(x) = (-1)^n \frac{n!}{2^n}. \quad (187)$$

**Theorem 88** Let  $H_k \in \mathbb{H}$ , the set of harmonic numbers. Let  $1 \leq n \leq k$ . Then we have

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1 + jx) d\mu_{-1}(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} E_n,$$

where  $\binom{H_k}{n}_{\mathbb{H}}$  denotes the harmonic binomial coefficient.

**Proof.** By applying the fermionic  $p$ -adic integral to (173), we get

$$\int_{\mathbb{Z}_p} \prod_{j=1}^k (1 + jx) d\mu_{-1}(x) = \sum_{n=0}^k k! \binom{H_k}{k-n}_{\mathbb{H}} \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x).$$

Combining the above equation with (130), we arrive at the desired result. ■

## 7 Identities and Relations

By using the results obtained in the previous sections, we give some new formulas and relations in this section. These formulas and relations are involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, the Lah numbers, the Peters numbers and polynomials, the central factorial numbers, the Daehee numbers and polynomials, the Changhee numbers and polynomials, the Harmonic numbers, the Fubini numbers, combinatorial numbers and sums.

**Theorem 89** Let  $l \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ . Then we have

$$\sum_{j=0}^n \binom{n}{j} \frac{B_{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left( \frac{B_{n+k}(1) - B_0}{n+k} \right). \quad (188)$$

**Proof.** By applying the Volkenborn integral to the following well-known combinatorial series identity:

$$\sum_{j=0}^n \binom{n}{j} \frac{x^{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left( \frac{(1+x)^{n+k} - 1}{n+k} \right), \quad (189)$$

(cf. [21, Eq. (2.4)]), we obtain

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} \frac{1}{j+l} \int_{\mathbb{Z}_p} x^{j+l} d\mu_1(x) \\ &= \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \frac{1}{n+k} \int_{\mathbb{Z}_p} \left( (1+x)^{n+k} - 1 \right) d\mu_1(x). \end{aligned}$$

Combining the above equation with (88) and (89), we arrive at the desired result. ■

For  $l = 1$ , (188) coincides with the following corollary:

**Corollary 90** Let  $n \in \mathbb{N}_0$ . Then we have

$$\sum_{j=0}^n \binom{n}{j} \frac{B_{j+1}}{j+1} = \frac{B_{n+1}(1) - B_0}{n+1}.$$

**Theorem 91** Let  $l \in \mathbb{N}$  and  $n \in \mathbb{N}_0$ . Then we have

$$\sum_{j=0}^n \binom{n}{j} \frac{E_{j+l}}{j+l} = \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \left( \frac{E_{n+k}(1) - E_0}{n+k} \right). \quad (190)$$

**Proof.** That is, by applying the fermionic  $p$ -adic integral (189), we obtain

$$\begin{aligned} & \sum_{j=0}^n \binom{n}{j} \frac{1}{j+l} \int_{\mathbb{Z}_p} x^{j+l} d\mu_{-1}(x) \\ &= \sum_{k=1}^l (-1)^{l-k} \binom{l-1}{l-k} \frac{1}{n+k} \int_{\mathbb{Z}_p} \left( (1+x)^{n+k} - 1 \right) d\mu_{-1}(x). \end{aligned}$$

Combining the above equation with (130) and (131), we arrive at the desired result. ■

For  $l = 1$ , (190) coincides with the following corollary:

**Corollary 92** Let  $n \in \mathbb{N}_0$ . Then we have

$$\sum_{j=0}^n \binom{n}{j} \frac{E_{j+1}}{j+1} = \frac{E_{n+1}(1) - E_0}{n+1}.$$

**Remark 93** By using (189), Choi and Srivastava [21, Lemma 1] gave the following summation formulas involving harmonic numbers and combinatorial series identity:

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \frac{1}{j+l} = \frac{1}{n+1},$$

where  $n \in \mathbb{N}_0$  and

$$\sum_{j=0}^n (-1)^{j+1} \binom{n}{j} \frac{H_j}{j+l} = \frac{H_n}{n+1},$$

where  $n \in \mathbb{N}_0$ .

**Theorem 94** Let  $n, k \in \mathbb{N}_0$ . Then we have

$$B_n = \sum_{k=0}^n \sum_{j=0}^k T(n, k) t(j, k) B_j. \quad (191)$$



**Proof.** By applying the Volkenborn integral to the equation (59), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_1(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^{[k]} d\mu_1(x).$$

Combining the above equation with (88) and (159), we arrive at the desired result. ■

**Theorem 95** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$E_n = \sum_{k=0}^n \sum_{j=0}^k T(n, k) t(j, k) E_j. \quad (192)$$

**Proof.** By applying the Volkenborn integral to the equation (59), we get

$$\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^{[k]} d\mu_{-1}(x).$$

Combining the above equation with (130) and (175), we arrive at the desired result. ■

By using (178), we have

$$\int_{\mathbb{Z}_p} B_0^n(x) d\mu_{-1}(x) + \sum_{k=1}^n (-1)^k \int_{\mathbb{Z}_p} B_k^n(x) d\mu_{-1}(x) = \sum_{j=0}^n (-2)^{n-j} E_{n-j}.$$

Combining the above equation with (154) and (155), we arrive at the following theorem:

**Theorem 96** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$E_n = \sum_{j=0}^n (-2)^{n-j} E_{n-j} - \sum_{k=1}^n (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{n-k-j} \binom{n-k}{j} E_{n-j} - 2.$$

Combining (178) with (181), we get the following result:

**Theorem 97** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\sum_{j=0}^n \binom{n}{j} (-2)^{n-j} \left( E_{n-j} - \sum_{m=0}^{n-j} S_2(n-j, m) Ch_m \right) = 0.$$

Combining (167) with (168) and (94), we arrive at the following results:

**Theorem 98** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\begin{aligned} & \sum_{j=0}^n \sum_{m=0}^{n-j} \binom{n}{j} (-1)^{n+m-j} \frac{2^{n-j} S_2(n-j, m) m!}{m+1} \\ &= \sum_{j=0}^n \sum_{m=0}^{n-j} \sum_{l=0}^m \binom{n}{j} (-2)^{n-j} S_2(n-j, m) S_1(m, l) B_l. \end{aligned}$$

By applying the Volkenborn integral to (69), we have

$$\sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \int_{\mathbb{Z}_p} s_{n-v}(x; \lambda, \mu) d\mu_1(x) = \int_{\mathbb{Z}_p} x_{(n)} d\mu_1(x).$$

Combining the above equation with (85), (93), (98), (99) and (94), we arrive at the following theorems, respectively:

**Theorem 99** *Let  $n, v \in \mathbb{N}_0$ . Then we have*

$$D_n = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l.$$

**Theorem 100** *Let  $n, v \in \mathbb{N}_0$ . Then we have*

$$\begin{aligned} & \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l \\ &= (-1)^n \frac{n!}{n+1}. \end{aligned}$$

**Theorem 101** *Let  $n, v \in \mathbb{N}_0$ . Then we have*

$$\begin{aligned} & \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) \sum_{l=0}^{n-v-m} S_1(n-v-m, l) B_l \\ &= \sum_{v=0}^n S_1(n, v) B_l. \end{aligned}$$

By combining (169), (170), (171) and (172) with (98), (99) and (94), we get the following results:

**Theorem 102** *Let  $n, \mu \in \mathbb{N}_0$ . Then we have*

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} \binom{n-v}{l} s_l(\lambda, \mu) D_{n-v-l} = (-1)^n \frac{n!}{n+1}.$$

**Theorem 103** *Let  $n, \mu \in \mathbb{N}_0$ . Then we have*

$$\begin{aligned} & \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} \binom{n-v}{l} s_l(\lambda, \mu) \\ & \times \sum_{m=0}^{n-v-l} \binom{n-v-l}{m} s_m(\lambda, \mu) \sum_{k=0}^m S_1(m, k) B_k \\ &= (-1)^n \frac{n!}{n+1}. \end{aligned}$$

**Theorem 104** Let  $n, \mu \in \mathbb{N}_0$ . Then we have

$$\begin{aligned} & \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)(n-v)!}{n-v-l+1} \\ &= (-1)^n \frac{n!}{n+1}. \end{aligned}$$

Combining (186) and (187), we arrive at the following results:

**Theorem 105** Let  $n, v \in \mathbb{N}_0$ . Then we have

$$Ch_n = \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \sum_{m=0}^{n-v} \binom{n-v}{m} s_m(\lambda, \mu) Ch_{n-v-m}. \quad (193)$$

Combining (136) with (193), we arrive at the following corollary:

**Corollary 106** Let  $n, v \in \mathbb{N}_0$ . Then we have

$$\begin{aligned} & \sum_{v=0}^n \sum_{k=0}^v \binom{n}{v} \lambda^k B(k, \mu) s(v, k) \sum_{m=0}^{n-v} (-1)^{n-v-m} \binom{n-v}{m} \frac{s_m(\lambda, \mu)(n-v-m)!}{2^{n-v-m}} \\ &= (-1)^n \frac{n!}{2^n}. \end{aligned}$$

After comparing and combining the equation (182) with the equation (185), and making the necessary algebraic operations, we obtain the following results, respectively:

**Theorem 107** Let  $n, \mu \in \mathbb{N}_0$ . Then we have

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} \binom{n-v}{l} s_l(\lambda, \mu) Ch_{n-v-l} = (-1)^n \frac{n!}{2^n}.$$

**Theorem 108** Let  $n, \mu \in \mathbb{N}_0$ . Then we have

$$Ch_n = \sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)}{2^{n-v-l}}. \quad (194)$$

Combining (136) with (194), we arrive at the following corollary:

**Corollary 109** Let  $n, \mu \in \mathbb{N}_0$ . Then we have

$$\sum_{v=0}^n \sum_{j=0}^{\mu} \binom{\mu}{j} \binom{n}{v} (\lambda j)_{(v)} \sum_{l=0}^{n-v} (-1)^{n-v-l} \binom{n-v}{l} \frac{s_l(\lambda, \mu)}{2^{n-v-l}} = (-1)^n \frac{n!}{2^n}.$$

**Theorem 110** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) Ch_{n-j} = \sum_{j=0}^n (-1)^n j!(n-j)! \binom{n}{j} \frac{\lambda^{n+j}}{2^n (\lambda-1)^{j+1}}.$$

**Proof.** By applying the fermionic  $p$ -adic integral to (71) and (73), we have the following relations, respectively:

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_{-1}(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_{-1}(x)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_{-1}(x) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda-2)^{j+1}} \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_{-1}(x).$$

Combining the above equations with (134) and (133), we get:

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) Ch_{n-j} \quad (195)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n (-1)^n j!(n-j)! \binom{n}{j} \frac{\lambda^{n+j}}{2^n (\lambda-1)^{j+1}}. \quad (196)$$

Combining (195) with (196), we arrive at the desired result. ■

**Theorem 111** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) D_{n-j} = 2 \sum_{j=0}^n \sum_{l=0}^{n-j} (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j} S_1(n-j, l) B_l}{(2\lambda-2)^{j+1}}.$$

**Proof.** By applying the Volkenborn integral to (71) and (73), we have the following relations, respectively:

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_1(x)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = 2 \sum_{j=0}^n (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j}}{(2\lambda-2)^{j+1}} \int_{\mathbb{Z}_p} x_{(n-j)} d\mu_1(x).$$

Combining the above equations with (93) and (94), we get

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = \sum_{j=0}^n \binom{n}{j} \lambda^{n-j} Y_{j,2}(\lambda) D_{n-j} \quad (197)$$

and

$$\int_{\mathbb{Z}_p} Y_{n,2}(x; \lambda) d\mu_1(x) = 2 \sum_{j=0}^n \sum_{l=0}^{n-j} (-1)^j j! \binom{n}{j} \frac{\lambda^{n+j} S_1(n-j, l) B_l}{(2\lambda - 2)^{j+1}}. \quad (198)$$

Combining (197) with (198), we arrive at the desired result. ■

## 8 New Sequences Containing Bernoulli Numbers and Euler Numbers

In this section, we examine  $p$ -adic integrals of the function

$$J(x) = x_{(n)} x^{(m)}.$$

Moreover, we give some applications of these integrals. With the help of the integrals of this special function  $J(x)$ , we define two new sequences containing the Bernoulli numbers of the first kind and the Euler numbers of the first kind, respectively. We give some properties of these two sequences. We also prove that the general term of these sequences can be written in terms of the central factorial numbers. We also give some identities and relations involving the Bernoulli numbers, the Euler numbers, the Stirling numbers, the Lah numbers, and the central factorial numbers.

Let's start this section with the following questions:

How can we compute the following integrals:

Question 1:

$$\int_{\mathbb{Z}_p} J(x) d\mu_1(x) = ?$$

Question 2:

$$\int_{\mathbb{Z}_p} J(x) d\mu_{-1}(x) = ?$$

By using (57), we have the following identity:

$$x_{(n)} x^{(m)} = \sum_{k=1}^m |L(m, k)| x_{(k)} x_{(n)}. \quad (199)$$

By applying the Volkenborn integral to (199), we get

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_1(x) = \sum_{k=1}^m |L(m, k)| \int_{\mathbb{Z}_p} x_{(k)} x_{(n)} d\mu_1(x).$$

Combining the above equation with (177), we arrive at the following theorem. The result of the following theorem gives us the solution of the Question 1.

**Theorem 112** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_1(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{k+n-j} \binom{m}{j} \binom{k}{j} \frac{j!(n+k-j)! |L(m, k)|}{m+k-j+1}.$$

By applying the fermionic  $p$ -adic integral to (199), we get

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_{-1}(x) = \sum_{k=1}^m |L(m, k)| \int_{\mathbb{Z}_p} x_{(k)} x_{(n)} d\mu_{-1}(x).$$

Combining the above equation with (122), we arrive at the following theorem. The result of the following theorem gives us the solution of the Question 2.

**Theorem 113** *Let  $m, n \in \mathbb{N}_0$ . Then we have*

$$\int_{\mathbb{Z}_p} x_{(n)} x^{(m)} d\mu_{-1}(x) = \sum_{k=1}^m \sum_{j=0}^n (-1)^{n+k-j} \binom{n}{j} \binom{k}{j} \frac{j!(n+k-j)! |L(m, k)|}{2^{n+k-j}}.$$

Substituting  $m = n$  into Question 1 and Question 2, we define the following sequences containing the Bernoulli numbers of the first kind and the Euler numbers of the first kind, respectively:

$$\begin{aligned} \mathcal{Y}(n, B) &= \int_{\mathbb{Z}_p} x_{(n)} x^{(n)} d\mu_1(x) \\ &= \int_{\mathbb{Z}_p} x^2 (x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n-1)^2) d\mu_1(x) \end{aligned} \quad (200)$$

and

$$\begin{aligned} \mathcal{Y}(n, E) &= \int_{\mathbb{Z}_p} x_{(n)} x^{(n)} d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} x^2 (x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n-1)^2) d\mu_{-1}(x). \end{aligned} \quad (201)$$

By using (88) and (130), we compute few values of the sequences given by (200) and (201), respectively, as follows:

$$\begin{aligned} \mathcal{Y}(0, B) &= B_0, \\ \mathcal{Y}(1, B) &= B_2, \\ \mathcal{Y}(2, B) &= B_4 - B_2, \\ \mathcal{Y}(3, B) &= B_6 - 5B_4 + 4B_2, \\ \mathcal{Y}(4, B) &= B_8 - 14B_6 + 49B_4 - 36B_2, \\ \mathcal{Y}(5, B) &= B_{10} - 30B_8 + 273B_6 - 870B_4 + 576B_2, \\ \mathcal{Y}(6, B) &= B_{12} - 55B_{10} + 1023B_8 - 7645B_6 + 21076B_4 - 14400B_2, \dots \end{aligned}$$

and

$$\begin{aligned}
\mathcal{Y}(0, E) &= E_0, \\
\mathcal{Y}(1, E) &= E_2, \\
\mathcal{Y}(2, E) &= E_4 - E_2, \\
\mathcal{Y}(3, E) &= E_6 - 5E_4 + 4E_2, \\
\mathcal{Y}(4, E) &= E_8 - 14E_6 + 49E_4 - 36E_2, \\
\mathcal{Y}(5, E) &= E_{10} - 30E_8 + 273E_6 - 870E_4 + 576E_2 \\
\mathcal{Y}(6, E) &= E_{12} - 55E_{10} + 1023E_8 - 7645E_6 + 21076E_4 - 14400E_2, \dots
\end{aligned}$$

When the integrals, given by (200) and (201), are calculated for the special values of the number  $n$ , we can observe that the row numbers given in the matrix representation of the central factorial numbers  $t(i, j)$  in the equation (61) and the coefficients of the Bernoulli numbers of the first kind and the Euler of the first kind are equal. Therefore, we arrive at the following theorems:

**Theorem 114** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\mathcal{Y}(n, B) = \sum_{k=1}^n t(2n, 2k)B_{2k}.$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^2(x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) = \sum_{k=1}^n t(2n, 2k)x^{2k} \quad (202)$$

(cf. [13, p. 430], [61]), we get

$$\int_{\mathbb{Z}_p} x^2(x^2 - 1)(x - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) d\mu_1(x) = \sum_{k=1}^n t(2n, 2k) \int_{\mathbb{Z}_p} x^{2k} d\mu_1(x).$$

Combining the above equation with (88), we arrive at the desired result. ■

**Theorem 115** *Let  $n \in \mathbb{N}$ . Then we have*

$$\begin{aligned}
\mathcal{Y}(n, E) &= \sum_{k=1}^{2n} t(2n, 2k)E_{2k} \\
&= 0.
\end{aligned}$$

**Proof.** By applying the fermionic  $p$ -integral to the equation (202), we get

$$\int_{\mathbb{Z}_p} x^2(x^2 - 1)(x^2 - 2^2)(x^2 - 3^2) \cdots (x^2 - (n - 1)^2) d\mu_{-1}(x) = \sum_{k=1}^n t(2n, 2k) \int_{\mathbb{Z}_p} x^{2k} d\mu_{-1}(x),$$

(cf. [13, p. 430], [61]). Combining the above equation with (130), we arrive at the desired result. ■

**Remark 116** In [110], we defined two other kinds of sequences including Bernoulli numbers and polynomials and Euler numbers and polynomials. Let's briefly give information about two of them: The sequence  $(Y_1(n : B))$  is associated with the Bernoulli numbers. That is,  $Y_1(0 : B) = B_0 = 1$ ,  $Y_1(1 : B) = B_1 = -\frac{1}{2}$ ,  $Y_1(2 : B) = B_2 - B_1$ ,  $Y_1(3 : B) = B_3 - 3B_2 + 2B_1, \dots$ . If we continue to calculate the terms of the sequence  $(Y_1(n : B))$  in this way, the general term of this sequence is given by the following formula including the Daehee numbers:

$$Y_1(n : B) = D_n. \quad (203)$$

The sequence  $(y_2(n : E))$  is associated with the Euler numbers of the first kind. That is,  $y_1(0 : E) = y_2(0 : E) = E_0 = 1$  and  $y_1(1 : E) = y_2(1 : E) = E_1 = -\frac{1}{2}$ ,  $y_1(2 : E) = E_2 - E_1$ ,  $y_1(3 : E) = E_3 - 3E_2 + 2E_1, \dots$ . Similarly, if we continue to calculate the terms of the sequence  $(y_2(n : E))$  in this way, the general term of this sequence is given by the following formula including the Changhee numbers:

$$y_1(n : E) = Ch_n.$$

In this paper, we do not consider whether there is any relationship between the sequences  $Y_1(n : B)$  and the sequence  $y_1(n : E)$  and the newly defined the sequence  $\mathcal{Y}(n, B)$  and the sequence  $\mathcal{Y}(n, E)$ . Perhaps the sequence  $\mathcal{Y}(n, B)$  and the sequence  $\mathcal{Y}(n, E)$  may be subsequences of the sequences  $Y_1(n : B)$  and the sequence  $y_1(n : E)$ , respectively.

The following theorem gives us that Bernoulli numbers of the first kind can be computed with the help of the central factorial numbers of the second kind  $T(n, k)$  and the sequence  $\mathcal{Y}(k, B)$ .

**Theorem 117** Let  $n \in \mathbb{N}_0$ . Then we have

$$B_{2n} = \sum_{k=0}^n T(n, k) \mathcal{Y}(k, B).$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^n = \sum_{k=0}^n T(n, k) x(x-1)(x-2^2)(x-3^2) \cdots (x-(n-1)^2), \quad (204)$$

(cf. [13, p. 430], [61]). By replacing  $x$  by  $x^2$  into (204), we have

$$x^{2n} = \sum_{k=0}^n T(n, k) x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2).$$

By applying the Volkenborn integral to the above equation, we get

$$\int_{\mathbb{Z}_p} x^{2n} d\mu_1(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2) d\mu_1(x).$$



Combining the above equation with (88) and (200), we arrive at the desired result.

■

The following theorem gives us that Euler numbers of the first kind can be computed with the help of the central factorial numbers of the second kind  $T(n, k)$  and the sequence  $\mathcal{Y}(k, E)$ .

**Theorem 118** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$E_{2n} = \sum_{k=0}^n T(n, k) \mathcal{Y}(k, E).$$

**Proof.** By applying the Volkenborn integral to the following well-known equation

$$x^n = \sum_{k=0}^n T(n, k) x(x-1)(x-2^2)(x-3^2) \cdots (x-(n-1)^2), \quad (205)$$

By replacing  $x$  by  $x^2$  into (205), we have

$$x^{2n} = \sum_{k=0}^n T(n, k) x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2).$$

By applying the fermionic  $p$ -adic integral to the above equation, we get

$$\int_{\mathbb{Z}_p} x^{2n} d\mu_{-1}(x) = \sum_{k=0}^n T(n, k) \int_{\mathbb{Z}_p} x^2(x^2-1)(x^2-2^2)(x^2-3^2) \cdots (x^2-(n-1)^2) d\mu_{-1}(x).$$

Combining the above equation with (130) and (201), we arrive at the desired result.

■

We now give another solutions of Question 1 and Question 2 in theorems stated below.

Combining (200) with (36) and (57), we get

$$\mathcal{Y}(n, B) = \sum_{j=0}^n \sum_{k=1}^n S_1(n, j) |L(n, k)| \int_{\mathbb{Z}_p} x^j x_{(k)} d\mu_1(x).$$

Combining the above equation with (120), we arrive at the following theorem:

**Theorem 119** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\mathcal{Y}(n, B) = \sum_{j=0}^n \sum_{k=1}^n \sum_{m=0}^k S_1(n, j) S_1(k, m) B_{j+m} |L(n, k)|.$$

Combining (201) with (36) and (57), we get

$$\mathcal{Y}(n, E) = \sum_{j=0}^n \sum_{k=1}^n S_1(n, j) |L(n, k)| \int_{\mathbb{Z}_p} x^j x_{(k)} d\mu_{-1}(x).$$

Combining the above equation with (174), we arrive at the following theorem:

**Theorem 120** *Let  $n \in \mathbb{N}_0$ . Then we have*

$$\mathcal{Y}(n, E) = \sum_{j=0}^n \sum_{k=1}^n \sum_{m=0}^k S_1(n, j) S_1(k, m) E_{j+m} |L(n, k)|.$$

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