Power Partitions and Semi-m-Fibonacci Partitions

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Abstract

George Andrews recently proved a new identity between the cardinalities of the set of Semi-Fibonacci partitions and the set of partitions into powers of two with all parts appearing an odd number of times. This paper extends the identity to the set of Semi-m-Fibonacci partitions of n and the set of partitions of n into powers of m in which all parts appear with multiplicity not divisible by m. We also give a new characterization of Semi-m-Fibonacci partitions and some congruences satisfied by the associated number sequence.

Keywords: partition, bijection, congruence.

2010 Mathematics Subject Classification: 11P83, 11P84, 051A15.

1 Introduction

A partition λ of an integer $n > 0$ is a finite nonincreasing integer sequence whose sum is n. The terms of the sequence are called *parts* of λ. Thus a partition with k parts will generally be expressed as

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0, \tag{1}$$

or

$$\lambda = (\lambda_1^{v_1}, \lambda_2^{v_2}, \ldots, \lambda_t^{v_t}),\ \lambda_1 > \lambda_2 > \cdots > \lambda_t > 0,\ t \leq k, \tag{2}$$

where $\lambda_i^{v_i}$ indicates that λ_i occurs with multiplicity v_i, for each i, and $v_1 + \cdots + v_t = k$ [2].

In a recent paper paper Andrews [1] describes the set $SF(n)$ of semi-Fibonacci partitions as follows: $SF(1) = \{(1)\}$, $SF(2) = \{(2)\}$. If $n > 2$ and n is even then

$SF(n) = \{\lambda \mid \lambda \text{ is a partition of } \frac{n}{2} \text{ with each part doubled}\}$. If n is odd, then a member of $SF(n)$ is obtained by inserting 1 into each partition in $SF(n-1)$ or by adding 2 to the single odd part in a partition in $SF(n-2)$.

The cardinality $sf(n) = |SF(n)|$ satisfies the following recurrence relation for all $n > 0$ (with $sf(-1) = 0, sf(0) = 1$);

$$sf(n) = \begin{cases} sf(n/2), & \text{if } n \text{ is even;} \\ sf(n-1) + sf(n-2), & \text{if } n \text{ is odd.} \end{cases} \tag{3}$$

The semi-Fibonacci sequence $\{sf(n)\}_{n>0}$ occurs as sequence number A030067 in Sloane's database [5]. George Beck [3] has previously considered the properties of a set of polynomials related to the semi-Fibonacci partitions.

Andrews stated the following relation between the number of semi-Fibonacci partitions of n and the number $ob(n)$ of binary partitions of n in which every part occurs an odd number of times:

Theorem 1 ([1], Theorem 1)**.** *For each $n \geq 0$,*

$$sf(n) = ob(n), \tag{4}$$

Andrews gave a generating function proof and asked for a bijective proof.

The proof turns out to be remarkable simple. It goes as follows. Each part t of $\lambda \in SF(n)$ can be expressed as $t = 2^i \cdot h$, $i \geq 0$, where h is odd. Now transform t as

$$t = 2^i \cdot h \longmapsto 2^i, 2^i, \ldots, 2^i (h \text{ times}).$$

This gives a partition of n into powers of 2 in which every part has odd multiplicity. Conversely, consider $\beta \in OB(n)$. Since every part (a power of 2) has odd multiplicity we simply write β in the exponent notation $\beta = (\beta_1^{u_1}, \ldots, \beta_s^{u_s})$, $\beta_1 > \cdots > \beta_s$ with the u_i odd and positive. Since each $\beta_i^{u_i}$ has the form $(2^{j_i})^{u_i}$, $j_i \geq 0$, we apply the transformation:

$$\beta_i^{u_i} = (2^{j_i})^{u_i} \longmapsto 2^{j_i} u_i.$$

This gives a unique partition in $SF(n)$. Indeed the image may contain at most one odd part which occurs precisely when $j_i = 0$.

$SF(9)$	$\longrightarrow$	$OB(9)$
(8,1)	$\mapsto$	(8,1)
(4,3,2)	$\mapsto$	(4,2,1,1,1)
(6,3)	$\mapsto$	(2,2,2,1,1,1)
(5,4)	$\mapsto$	(4,1,1,1,1,1)
(7,2)	$\mapsto$	(2,1,1,1,1,1,1,1)
(9)	$\mapsto$	(1,1,1,1,1,1,1,1,1)

Table 1: The map $SF(n) \to OB(n)$ for $n = 9$.

We also consider the following congruence which Andrews proved with generating functions.

Theorem 2 ([1], Theorem 2)**.** *For each $n \geq 0$, $sf(n)$ is even if $3|n$ and odd otherwise.*

Proof. We give a combinatorial proof based on mathematical induction. The result holds for $n = 1, 2, 3$ since $sf(1) = 1 = sf(2)$ and $sf(3) = |\{(1,2),(3)\}| = 2$. Now let $n > 3$ and assume that the result holds for all integers less than n.

If $n \equiv 1 \pmod 3$, then $sf(n)$ is the sum of $sf(n-1)$ and $sf(n-2)$ which have opposite parities since, by the inductive hypothesis, $sf(n-1)$ is even (since $3|(n-1)$) and $sf(n-2)$ is odd.
If $n \equiv 2 \pmod 3$, then $sf(n)$ is the sum of $sf(n-1)$ which is odd (since $3 \nmid (n-1)$) and $sf(n-2)$ is even. Thus $sf(n)$ is odd.
If $3|n$ and n is even, then $sf(n) = sf(n/2)$. Since $3|\frac{n}{2}$, it follows that $sf(n/2)$ is even by the inductive hypothesis. Lastly, if $3|n$ and n is odd, then $sf(n) = sf(n-1) + s(n-2)$ which is even since $3 \nmid (n-1)$ and $3 \nmid (n-2)$.

This completes the proof. ∎

The following result is easily deduced from the definition of sets counted by $sf(n)$.

Corollary 1. Given a nonnegative integer v,

$$sf(2^v) = 1.$$

In Section 2 we define the semi-m-Fibonacci partitions by extending the previous construction using a fixed integer modulus $m > 1$. A generalized identity is then stated between the set of semi-m-Fibonacci partitions and the set of partitions into powers of m with multiplicities not divisible by m (Theorem 3). Then in Subsection 2.1 we give an independent characterization of the semi-m-Fibonacci partitions. Lastly, in Section 3 we discuss some arithmetic properties satisfied by the semi-m-Fibonacci sequence.

2 Generalization

We generalize the set of semi-Fibonacci Partitions to the set $SF(n, m)$ of semi-m-Fibonacci Partitions as follows:

$SF(n, m) = \{(n)\},\ n = 1, 2, \ldots, m$

If $n > m$ and n is a multiple of m, then

$SF(n, m) = \{\lambda \mid \lambda$ is a partition of $\frac{n}{m}$ with each part multiplied by $m\}$.

If n is not a multiple of m, that is, $n \equiv r \pmod{m}$, $1 \leq r \leq m-1$, then $SF(n, m)$ arises from two sources: first, partitions obtained by inserting r into each partition in $SF(n-r, m)$, and second, partitions obtained by adding m to the single part of each partition $\lambda \in SF(n-m, m)$ which is congruent to $r \pmod{m}$ (since λ contains exactly one part which is congruent to r modulo m, see Lemma 1 below).

Lemma 1. Let $\lambda \in SF(n, m)$.

If $m \mid n$, then every part of λ is a multiple of m.

If $n \equiv r \pmod{m}$, $1 \leq r < m$, then λ contains exactly one part $\equiv r \pmod{m}$.

Proof. If $m \mid n$, the parts of a partition in $SF(n, n)$ are clearly divisible by m by construction.

For induction note that $SF(r, m) = \{(r)\}$, $r = 1, \ldots, m-1$, so the assertion holds trivially. Assume that the assertion holds for the partitions of all integers $< n$ and consider $\lambda \in SF(n, m)$ with $1 \leq r < m$. Then λ may be obtained by inserting r into a partition $\alpha \in SF(n-r, m)$. Since α consists of multiples of m (as $m|(n-r)$), λ contains exactly one part $\equiv r \pmod{m}$. Alternatively λ is obtained by adding m to the single part of a partition $\beta \in SF(n-m, m)$ which is $\equiv r \pmod{m}$. Indeed β contains exactly one such part by the inductive hypothesis. Hence the assertion is proved. ∎

As an illustration we have the following sets for small n when $m = 3$:

$SF(1, 3) = \{(1)\}$,
$SF(2, 3) = \{(2)\}$,
$SF(3, 3) = \{(3)\}$,
$SF(4, 3) = \{(4), (3, 1)\}$,
$SF(5, 3) = \{(5), (3, 2)\}$,
$SF(6, 3) = \{(6)\}$,
$SF(7, 3) = \{(7), (4, 3), (6, 1)\}$,
$SF(8, 3) = \{(8), (5, 3), (6, 2)\}$,
$SF(9, 3) = \{(9)\}$,
$SF(10, 3) = \{(10), (6, 4), (7, 3), (9, 1)\}$.

Thus if we define $sf(n, m) = |SF(n, m)|$, we obtain that

$sf(1, 3) = sf(2, 3) = sf(3, 3) = 1,\ sf(4, 3) = 2,\ sf(5, 3) = 2,\ sf(6, 3) = 1,$
$sf(7, 3) = 3,\ sf(8, 3) = 3,\ sf(9, 3) = 1,\ sf(10, 3) = 4.$

Therefore, for $m > 1$, we see that $sf(n, m) = 0$ if $n < 0$ and $sf(0, m) = 1$, and for $n > 0$,

$$sf(n, m) = \begin{cases} sf(n/m, m), & \text{if } n \equiv 0 \pmod{m}; \\ sf(n-r, m) + sf(n-m, m), & \text{if } n \equiv r \pmod{m}, 0 < r < m. \end{cases} \tag{5}$$

The case $m = 2$ gives the function considered by Andrews: $sf(n, 2) = sf(n)$.

Power partitions are partitions into powers of a positive integer m, also known as m-power partitions [4]. Let $nd(n, m)$ be the number of m-power partitions of n in which the multiplicity of each part is not divisible by m. Thus, for example, $nd(10, 3) = 4$, the enumerated partitions being $(9, 1), (3, 3, 1, 1, 1, 1), (3, 1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$.

Theorem 3. *For integers $n \geq 0, m > 1$,*

$$sf(n, m) = nd(n, m), \tag{6}$$

Proof. We give two proofs, one analytic one combinatorial.
First Proof. Let $|q| < 1$ and define

$$G_m(q) = \sum_{n\geq 0} sf(n,m)q^n. \tag{7}$$

Then we have

$$\begin{aligned}
G_m(q) &= \sum_{n\geq 0} sf(mn,m)q^{mn} + \sum_{n\geq 0} sf(mn+1,m)q^{mn+1} + \ldots + \sum_{n\geq 0} sf(mn+m-1,m)q^{mn+m-1} \\
&= \sum_{n\geq 0} sf(mn,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r,m)q^{mn+r} \qquad (8)\\
&= \sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0}(sf(mn,m) + sf(mn+r-m,m))q^{mn+r} \quad \text{(by (5))}\\
&= \sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0}(sf(n,m)q^{mn+r} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r-m,m)q^{mn+r}. \\
&= (1+\sum_{r=1}^{m-1} q^r)\sum_{n\geq 0} sf(n,m)q^{mn} + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(m(n-1)+r,m)q^{mn+r} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + \sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r,m)q^{mn+m+r} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + q^m\sum_{r=1}^{m-1}\sum_{n\geq 0} sf(mn+r,m)q^{mn+r} \\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r) + q^m\left(\sum_{n\geq 0} sf(n,m)q^n - \sum_{n\geq 0} sf(mn,m)q^{mn}\right) \quad \text{(by (8))}\\
&= G_m(q^m)\sum_{r=0}^{m-1} q^r + q^m(G_m(q) - G_m(q^m)) \\
&= \left(-q^m + \sum_{r=0}^{m-1} q^r\right) G_m(q^m) + q^m G_m(q). \qquad (9)
\end{aligned}$$

Hence,

$$G_m(q) = \frac{1+q+q^2+q^3+\ldots+q^{m-1}-q^m}{1-q^m} G_m(q^m). \tag{10}$$

Equation (10) implies that

$$G_m(q) = \left(\frac{1+q+q^2+\ldots+q^{m-1}-q^m}{1-q^m}\right)\left(\frac{1+q+q^{2m}+\ldots+q^{(m-1)m}-q^{m^2}}{1-q^{m^2}}\right) G_m(q^{m^2}),$$

and continuing the iteration, we get

$$G_m(q) = \prod_{n=0}^{N}\left(\frac{1+q^{m^n}+q^{2m^n}+\ldots+q^{(m-1)m^n}-q^{m^{n+1}}}{1-q^{m^{n+1}}}\right) G_m(q^{m^{N+1}}).$$

Taking the limit as $N \to \infty$, we have $G_m(q^{m^{N+1}}) \to G_m(0) = 1$ (since $|q| < 1$) so that

$$
\begin{aligned}
G_m(q) &= \prod_{n=0}^{\infty} \left(\frac{1 + q^{m^n} + q^{2m^n} + \ldots + q^{(m-1)m^n} - q^{m^{n+1}}}{1 - q^{m^{n+1}}} \right) \\
&= \prod_{n=0}^{\infty} \left(\frac{q^{m^n} + q^{2m^n} + \ldots + q^{(m-1)m^n} + 1 - q^{m^{n+1}}}{1 - q^{m^{n+1}}} \right) \\
&= \prod_{n=0}^{\infty} \left(1 + \frac{q^{m^n} + q^{2m^n} + \ldots + q^{(m-1)m^n}}{1 - q^{m^{n+1}}} \right) \\
&= \prod_{n=0}^{\infty} \left(1 + (q^{m^n} + q^{2m^n} + \ldots + q^{(m-1)m^n}) \sum_{j=0}^{\infty} q^{j(m^{n+1})} \right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
G_m(q) &= \prod_{n=0}^{\infty} \left(1 + \sum_{j=0}^{\infty} q^{m^n(jm+1)} + \sum_{j=0}^{\infty} q^{m^n(jm+2)} + \sum_{j=0}^{\infty} q^{m^n(jm+3)} + \ldots + \sum_{j=0}^{\infty} q^{m^n(jm+m-1)} \right) \\
&= \sum_{n \geq 0} nd(n, m) q^n. \qquad (11)
\end{aligned}
$$

The assertion follows by comparing coefficients in (7) and (11).

Second Proof. Each part t of $\lambda \in SF(n, m)$ can be expressed as $t = m^i \cdot h$, $i \geq 0$, where m does not divide h. Now transform t as

$$t = m^i \cdot h \longmapsto m^i, m^i, \ldots, m^i (h \text{ times}).$$

This gives a partition of n into powers of m in which every part has multiplicity not divisible by m. Conversely, consider $\beta \in ND(n, m)$. Since every part (a power of m) has a non-multiple of m as multiplicity we simply write β in the exponent notation $\beta = (\beta_1^{u_1}, \ldots, \beta_s^{u_s})$, $\beta_1 > \cdots > \beta_s$ with the $u_i \not\equiv 0 \pmod m$. Since each $\beta_i^{u_i}$ has the form $(m^{j_i})^{u_i}$, we apply the transformation:

$$\beta_i^{u_i} = (m^{j_i})^{u_i} \longmapsto m^{j_i} u_i.$$

This gives a unique partition in $SF(n, m)$. If $m \mid n$, this image contains only multiples of m. If $n \equiv r \pmod m$, $1 \leq r < m$, the image consists of multiples of m and exactly one part $\equiv r$ (mod m) which occurs when $j_i = 0$. ∎

$SF(11,3)$	$\longrightarrow$	$ND(11,3)$
(11)	$\mapsto$	(1,1,1,1,1,1,1,1,1,1,1)
(8,3)	$\mapsto$	(3,1,1,1,1,1,1,1,1)
(6,5)	$\mapsto$	(3,3,1,1,1,1,1)
(9,2)	$\mapsto$	(9,1,1)

Table 2: The map $SF(n, m) \to ND(n, m)$ for $n = 11$, $m = 3$.

2.1 A characterization of Semi-m-Fibonacci Partitions

Define the *max m-power* of an integer N as the largest power of m that divides N (not just the exponent of the power). Thus using the notation $x_m(N)$, we find that $N = u \cdot m^s$, $s \geq 0$, where $m \nmid u$ and $x_m(N) = m^s$. So $x_m(N) > 0$ for all N.

For example, $x_2(50) = 2$, $x_2(40) = 8$, $x_3(216) = 27$ and $x_5(216) = 1$.

Note that if the parts of a partition λ have distinct max m-powers, then the parts are distinct. For if $u \cdot m^s = \lambda_i = \lambda_j = v \cdot m^t \in \lambda$ with $m \nmid u, v$, and $s > t$, then $u \cdot m^{s-t} = v \implies m|v$ a contradiction.

We define three (reversible) operations on a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ with an integer $m > 1$:

(i) If the last part of λ is less than m, delete it: $\tau_1(\lambda) = (\lambda_1, \ldots, \lambda_{k-1})$;

(ii) If $m \nmid \lambda_t > m$, then $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k)$.

(iii) If λ consists of multiples of m, divide every part by m: $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m)$.

These operations are consistent with the recursive construction of the set $SF(n, m)$, where τ_3^{-1}, τ_1^{-1} and τ_2^{-1} correspond, respectively, to the three quantities in the recurrence (5).

Lemma 2. Let $B(n, m)$ denote the set of partitions of n in which the parts have distinct max m-powers and at most one non-multiple of m. Then if $\lambda \in B(n, m)$ and $\tau_i(\lambda) \neq \emptyset$, then $\tau_i(\lambda) \in B(N, m)$, $i = 1, 2, 3$, for some N.

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in B(n, m)$. If λ contains one part less than m, the part is λ_k. So $\tau_1(\lambda) \in B(n - \lambda_k, m)$ since the max m-powers remain distinct. It is obvious that the parity of λ is inherited by $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k) \in B(n - m, m)$. Lastly, since the parts of λ have distinct max m-powers $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m)$ may contain at most one non-multiple of m as a part. Hence $\tau_3(\lambda) \in B(n/m, m)$. ∎

We state an independent characterization of the Semi-m-Fibonacci Partitions.

Theorem 4. *A partition of n is a semi-m-Fibonacci partition if and only if the parts have distinct max m-powers and at most one non-multiple of m.*

Proof. We show that $SF(n, m) = B(n, m)$. Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in SF(n, m)$ such that $\lambda \notin B(n, m)$. Assume that there are $\lambda_i > \lambda_j$ satisfying $x_m(\lambda_i) = x_m(\lambda_j)$ and let $\lambda_i = u_i m^s$, $\lambda_j = u_j m^s$ with $m \nmid u_i, u_j$. Observe that τ_1 deletes a part less than m if it exists. So we can use repeated applications of τ_2 to reduce a non-multiple modulo m, followed by τ_1. This is tantamount to simply deleting the non-multiple of m, say λ_t, to obtain a member of $B(n - \lambda_t, m)$ from Lemma 2. By thus successively deleting non-multiples, and applying τ_3^c, $c > 0$, we obtain a partition $\beta = (\beta_1, \beta_2, \ldots)$ with $\beta_i = v_i m^w > \beta_j = v_j m^w$, where $m \nmid v_i, v_j$ and $w \leq s$. Then apply τ_3^w to obtain a partition γ with two non-multiples of m. Then by Lemma 1, $\gamma \notin SF(n, m)$. Therefore $\lambda \in SF(n, m) \implies \lambda \in B(n, m)$.

Conversely let $\lambda = (\lambda_1, \ldots, \lambda_k) \in B(n, m)$. If $\lambda = (t)$, $1 \leq t \leq m$, then $\lambda \in SF(t, m)$. If $m|\lambda_i$ for all i, then $\tau_3(\lambda) = (\lambda_1/m, \ldots, \lambda_k/m) \in B(n/m, m)$ contains at most one part $\not\equiv 0 \pmod{m}$, so $\lambda \in SF(n, m)$. Lastly assume that $n \equiv r \not\equiv 0 \pmod{m}$. Then $r \in \lambda$ or $\lambda_t \equiv r \pmod{m}$ for exactly one index t. Thus $\tau_1(\lambda) = (\lambda_1, \ldots, \lambda_{k-1})$ consists of multiples of m while $\tau_2(\lambda) = (\lambda_1, \ldots, \lambda_{t-1}, \lambda_t - m, \lambda_{t+1}, \ldots, \lambda_k)$ still contains one part $\not\equiv 0 \pmod{m}$. In either case $\lambda \in SF(n, m)$. Hence $B(n, m) \subseteq SF(n, m)$. The the two sets are identical. ∎

Remark. Notice that Theorem 4 certifies the second (bijective) proof of Theorem 3. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in SF(n, m)$ but $\lambda \notin B(n, m)$ on account of having two parts λ_i, λ_j such that $\lambda_i = u_i m^s > \lambda_j = u_j m^s$ with $m \nmid u_i, u_j$, then it cannot have an inverse image. Assume that λ maps to $\beta \in ND(n, m)$ which then includes the parts $m^{u_i + u_j}$ ($u_i + u_j$ copies of m). Then $u_i + u_j$ may be a multiple of m (for example, when $u_i = 1, u_j = m - 1$) which implies that $\beta \notin ND(n, m)$, a contradiction. Alternatively the pre-image of β would include the part $m(u_i + u_j)$ and so cannot be λ.

3 Arithmetic Properties

We prove several congruence properties of the numbers $sf(n, m)$.

Theorem 5. *Let n, m be integers with $n \geq 0$, $m > 1$. Then*

$$sf(nm+1, m) = sf(nm+2, m) = \cdots = sf(nm+m-1, m) = \sum_{j=0}^{n} sf(j, m).$$

Proof. Let $J_{r,m}(q) = \sum_{n\geq 0} sf(nm+r, m)q^n$ where $r = 1, 2, 3, \ldots m-1$. Then

$$\begin{aligned} J_{r,m}(q) &= \sum_{n\geq 0} sf(nm, m)q^n + \sum_{n\geq 0} sf(mn+r-m, m)q^n \ \text{ (by (5))} \\ &= \sum_{n\geq 0} sf(n, m)q^n + \sum_{n\geq 0} sf(mn+r, m)q^{n+1} \\ &= G_m(q) + q\sum_{n\geq 0} sf(mn+r, m)q^n \\ &= G_m(q) + qJ_{r,m}(q) \end{aligned}$$

so that

$$J_{r,m}(q) = \frac{G_m(q)}{1-q}. \tag{12}$$

Since the right hand side of (12) is independent of r, we must have $J_{1,m}(q) = J_{2,m}(q) = \ldots = J_{m-1,m}(q)$ so that $sf(nm+1, m) = sf(nm+2, m) = \cdots = sf(nm+m-1, m)$. Furthermore, from (12), we observe that

$$\begin{aligned} \sum_{n\geq 0} sf(mn+r, m)q^n &= \sum_{n\geq 0} q^n \sum_{n\geq 0} sf(n, m)q^n \\ &= \sum_{n\geq 0}\sum_{j=0}^{n} sf(j, m)q^n \end{aligned}$$

which implies that $sf(mn+r, m) = \sum_{j=0}^{n} sf(j, m)$. ∎

Corollary 2. Given integers $m \geq 2$, then for any $j \geq 0$ and a fixed $v \in \{0, 1, \ldots, m\}$,

$$sf(m^j(mv+r), m) = v+1, \ 1 \leq r \leq m-1.$$

Proof. By applying (5) several times (the case when $m \mid n$), it is clear that for any $j \geq 0$, $sf(m^j(mv+r), m) = sf(m^{j-1}(mv+r), m) = sf(m^{j-2}(mv+r), m) = \ldots = sf(mv+r, m)$. By the last equality in Theorem 5, we have

$$sf(mv+r, m) = \sum_{i=0}^{v} sf(i, m) = 1 + \sum_{i=1}^{v} sf(i, m), \ v \geq 0, \ 1 \leq r < m.$$

If $1 \leq v < m$, then $\sum_{i=1}^{v} sf(i, m) = \sum_{i=1}^{v}(sf(i-i, m) + sf(i-m, m))$ (by (5)). Since $0 < i \leq v < m$, we have $sf(mv+r, m) = 1 + \sum_{i=1}^{v}(1+0) = 1+v$.

If $v = m$, then $\sum_{i=1}^{v} sf(i, m) = \sum_{i=1}^{m-1} sf(i, m) + sf(m, m) = m-1+sf(1, m) = m-1+1 = m$; thus $sf(mv+r) = v+1$ is true in this case. Finally, if $v = 0$, it is not difficult to see that $sf(r, m) = 1$. ∎

We note a few interesting special cases of Corollary 2 below.

Corollary 3. We have the following for any integer $m \geq 2$:
(i) $sf(m^i, m) = 1,\ i \geq 0$.
(ii) $sf(m^i h, m) = 1,\ 1 \leq h \leq m-1, i \geq 0$.
(iii) Given an integer $n \geq 0$, then for each $n \in \{0, 1, \ldots, m\}$,

$$sf(nm+1, m) = sf(nm+2, m) = \cdots = sf((n+1)m-1, m) = v+1.$$

Proof. Part (i) is the case $h = 1$ of part (ii). Parts (ii) and (iii) are obtained by setting $v = 0$ and $j = 0$,respectively, in Corollary 2. ∎

Note that part (i) of Corollary 3 implies Corollary 1. Also when $m = 2$, part (iii) gives just the three values $sf(1) = 1$, $sf(3) = 2$ and $sf(5) = 3$, the parities of which are consistent with Theorem 2. Part (iii) is a stronger version of Theorem 5 since the restriction of n to the set $\{0, 1, \ldots, m\}$ specifies a common value.

Theorem 6. *For any* $j \geq 0$,

$$\sum_{r=0}^{2j+1} sf(r, 3) \equiv 0 \pmod{2}.$$

Consequently,

$$sf(3j+4, 3) = sf(3j+5, 3) \equiv 0 \pmod{2} \text{ where } j \equiv 0 \pmod{2}, \tag{13}$$

$$sf(3^r j+4, 3) = sf(3^r j+5, 3) \equiv 0 \pmod{2} \text{ for all } j \geq 0, r \geq 2. \tag{14}$$

Proof. Note the following identity

$$\frac{1}{1-q} = \prod_{n=0}^{\infty} (1 + q^{3^n} + q^{2\cdot 3^n}). \tag{15}$$

Recall that

$$\begin{aligned}
\sum_{n\geq 0} sf(n, 3)q^n &= \prod_{n=0}^{\infty} \left(\frac{1 + q^{3^n} + q^{2\cdot 3^n} - q^{3\cdot 3^n}}{1 - q^{3\cdot 3^n}} \right) \\
&\equiv \prod_{n=0}^{\infty} \left(\frac{1 + q^{3^n} + q^{2\cdot 3^n} + q^{3\cdot 3^n}}{1 + q^{3\cdot 3^n}} \right) \pmod{2} \\
&= \prod_{n=0}^{\infty} \frac{(1 + q^{3^n})(1 + q^{2\cdot 3^n})}{1 + q^{3\cdot 3^n}} \\
&= \prod_{n=0}^{\infty} \left(\frac{1 + q^{2\cdot 3^n}}{1 + q^{3^n} + q^{2\cdot 3^n}} \right) \\
&= (1-q) \prod_{n=0}^{\infty} (1 + q^{2\cdot 3^n}) \ \text{(by (15))}.
\end{aligned}$$

Thus

$$\frac{1}{1-q} \sum_{n\geq 0} sf(n, 3)q^n \equiv \prod_{n=0}^{\infty} (1 + q^{2\cdot 3^n}) \pmod{2},$$

i.e.

$$\sum_{n\geq 0} \sum_{r=0}^{n} sf(r, 3)q^n \equiv \prod_{n=0}^{\infty} (1 + q^{2\cdot 3^n}) \pmod{2}.$$

Since the series expansion of the right-hand side of the preceeding equation has even exponents, the result follows.

To prove (13), we have

$$\begin{aligned} sf(3j+4,3) &= sf(3(j+1)+1,3) \\ &= sf(3(j+1)+2,3) \ \text{(by Theorem 5)} \\ &= \sum_{r=0}^{j+1} sf(r,3) \ \text{(by Theorem 5)} \\ &\equiv 0 \pmod 2 \ \text{(since } j+1 \text{ is odd).} \end{aligned}$$

Furthermore, for (14), observe that

$$3^{r-1}j+1 \equiv \begin{cases} 0, & \text{if } j \equiv 1 \pmod 2; \\ 1, & \text{otherwise.} \end{cases}$$

Now, if j is odd, then

$$\begin{aligned} sf(3^r j+4,3) &= sf(3(3^{r-1}j+1)+1,3) \\ &= sf(3(3^{r-1}j+1)+2,3) \\ &= \sum_{r=0}^{3^{r-1}j+1} sf(r,3) \ \text{(by Theorem 5)} \\ &= sf(3^{r-1}j+1,3) + \sum_{r=0}^{3^{r-1}j} sf(r,3) \\ &\equiv sf(3^{r-1}j+1,3) \pmod 2 \ \text{(since } 3^{r-1}j \text{ is odd)} \\ &= \sum_{r=0}^{3^{r-2}j} sf(r,3) \\ &\equiv 0 \pmod 2 \ \text{(since } 3^{r-2}j \text{ is odd).} \end{aligned}$$

■

On the other hand, if j is even, use (13).

Theorem 7. *Let $k \equiv m+r \pmod{2m}$ and $k \le m^2+r$ for $1 \le r \le m-1$. If $n \ge 0$, $m \ge 2$ and $n = m^i k$ for $i \ge 0$, then $sf(n,m)$ is even.*

Proof. $k \equiv m+r \pmod{2m}$ and $k \le m^2+r$ for $1 \le r \le m-1$ imply that $k = m(2t+1)+r \le m^2+r \Rightarrow 2t+1 \le m$, for some positive integer t. Then from Corollary 2, we have

$$\begin{aligned} sf(m^i k,m) &= sf(m^i(m(2t+1)+r),m) \\ &= sf(m(2t+1)+r,m) \ \text{(by (5))} \\ &= 2t+1+1 \ \text{(by Corollary 2 and since } 2t+1 \le m) \\ &= 2t+2. \end{aligned}$$

Remark. When $m=3$, Theorem 7 reduces to Theorem 6 without the restriction $k \le m^2+r$.

■

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