Sequences, q-multinomial Identities, Generalized Galois Numbers, and Integer Partitions with Kinds

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Abstract - Using sequences of finite length with positive integer entries and the inversion statistic on such sequences, a collection of binomial and multinomial identities are extended to their q-analog form via combinatorial proofs. Using the major index statistic on sequences, a connection between finite differences of the coefficients of generalized Galois numbers and integer partitions with kinds is established.

Keywords : q-analogs; inversion statistic; multinomial identities; generating functions; generalized Galois numbers; major index statistic; integer partitions with kinds.

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1 Introduction

This paper is the result of an investigation into a number of q-binomial and q-multinomial identities. To begin, we will establish essential definitions and ideas. In Section 2, we will concisely develop a robust collection of some classical and other less so classical binomial/multinomial identities in their q-analog form. In Section 3, we will demonstrate a connection between finite differences of the coefficients of generalized Galois numbers and integer partitions with kinds.

At the heart of this paper is a motivation to present proofs of results using combinatorial justification. Along the way, we will encounter a number of objects of regular study in discrete mathematics: the set $[m]$, namely the set $\{ 1, 2, \ldots, m \}$; the set $S_n^m(k_1, \ldots, k_m)$, namely the set of sequences of length n whose elements include k_1 1's, ..., k_m m's from the set $[m]$; the inversion and major index statistics on sequences; and partitions of a positive integer n , namely sums of nonincreasing positive integers that add to n .

1.1 Inversion Statistic

To begin, we will introduce the inversion statistic, which can be found in [7].

Definition 1.1 *Let n, m be nonnegative integers, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\mathrm{inv}(\sigma) := | \{ (a,b) \mid a < b \text{ and } \sigma_a > \sigma_b \} | .$$

If a particular σ_a is fixed, ordered pairs of the form (a,b) that are accounted for by $\mathrm{inv}(\sigma)$ *shall be referred to as the inversions induced by σ_a or simply* $\mathrm{i}(\sigma_a)$ *. Should a particular σ_b be fixed, ordered pairs of the form (a,b) that are accounted for by* $\mathrm{inv}(\sigma)$ *shall be referred to as the inversions received by σ_b or simply* $\mathrm{r}(\sigma_b)$ *.*

Figure 1 contains some examples.

2211 $\mathrm{inv}(\sigma) = 4$	2121 $\mathrm{inv}(\sigma) = 3$	2112 $\mathrm{inv}(\sigma) = 2$
1221 $\mathrm{inv}(\sigma) = 2$	1212 $\mathrm{inv}(\sigma) = 1$	1122 $\mathrm{inv}(\sigma) = 0$

Figure 1: All sequences of length 4 with two 2s and two 1s.

Proposition 1.2 *Let n, m be nonnegative integers, and let $\sigma = (\, \sigma_1, \, \ldots, \, \sigma_n \,)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\mathrm{inv}(\sigma) \;=\; \sum_{a\in[n]} \mathrm{i}\,(\sigma_a) \;=\; \sum_{b\in[n]} \mathrm{r}\,(\sigma_b) \;.$$

Proof. Observe the unions expressed below are disjoint.

$$\{\, (a,b) \;|\; a < b \,\} \;=\; \bigcup_{a\in[n]} \{\, (a,b) \;|\; a < b \,\} \;=\; \bigcup_{b\in[n]} \{\, (a,b) \;|\; a < b \,\} \,.$$

The result follows from the above statement of equality, and the definitions of: inversions, induced inversions, and received inversions. □

Corollary 1.3 *Let n, m be nonnegative integers, and let $\sigma = (\, \sigma_1, \, \ldots, \, \sigma_n \,)$ be a sequence whose elements are from the set $[m]$. Then,*

$$\mathrm{inv}(\sigma) \;=\; \sum_{\sigma_a \geq 2} \mathrm{i}\,(\sigma_a) \;=\; \sum_{\sigma_b \leq m-1} \mathrm{r}\,(\sigma_b) \;.$$

Proof. When σ_a is equal to 1 the value of $\mathrm{i}(\sigma_a)$ equals zero. Similarly, when σ_b is equal to m the value of $\mathrm{r}(\sigma_b)$ equals zero. □

1.2 q-binomial and q-multinomial Coefficients

The following definition, inspired by [4], is foundational.

Definition 1.4 *Let n, k be nonnegative integers such that $n \geq k$, and let q be an indeterminate. Then*

$$\binom{n}{k}_q := \sum_{\substack{E \subset [n] \\ |E| = k}} q^{\left(\sum_{i=1}^{k} (n-e_i)-(k-i) \right)} .$$

where $E = \{ e_1, \ldots, e_k \}$ with $e_i < e_{i+1}$ for every $1 \leq i \leq k-1$.

Noting that the number of subsets of $[n]$ of cardinality k is exactly $\binom{n}{k}$, one can see that letting $q = 1$ yields the corresponding standard binomial coefficient.

Figure 2 contains an example. Observe the parallelism between Figures 1 and 2.

$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
q^4	q^3	q^2
$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
q^2	q^1	q^0

Figure 2: The sets associated with the terms of $\binom{4}{2}_q = q^4 + q^3 + 2q^2 + q + 1$.

Proposition 1.5 *If n, k are nonnnegative integers such that $n \geq k$ and q is an indeterminate, then*

$$\binom{n}{k}_q = \sum_{\sigma \in S_n^2(k, n-k)} q^{\text{inv}(\sigma)} .$$

Proof. Let $E \subset [n]$ be of cardinality k, and let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be the sequence in $S_n^2(k, n-k)$ such that σ_a is 2 precisely when $a \in E$. Fix some $a \in E$ and consider σ_a. The ordered pairs (a, b) accounted for by $\text{inv}(\sigma)$ correspond to indices b such that σ_b is 1. Notice that $n - e_i$ equals $n - a$ and counts the number of indices j such that $j > a$. Also notice that $k - i$ counts the numbers of elements σ_j such that $j > a$ and σ_j is 2. Hence, $(n - e_i) - (k - i)$ counts all ordered pairs (a, b) of interest. The result follows from observing that every $\sigma \in S_n^2(k, n-k)$ can be attained similarly by some $E \subset [n]$. □

In other words, the polynomial $\binom{n}{k}_q$ is the generating function for the statistic of inversions on the set $S_n^2(k, n-k)$, a standard result which can be found in [7]. The following definition and proposition, also found in [7], provides a generalization.

Definition 1.6 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then*

$$\binom{n}{k_1, \ldots, k_m}_q := \binom{n}{k_m}_q \binom{n-k_m}{k_{m-1}}_q \cdots \binom{n-k_m-\cdots-k_2}{k_1}_q .$$

Proposition 1.7 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then*

$$\binom{n}{k_1, \ldots, k_m}_q = \sum_{\sigma \in S_n^m(k_1,\ldots,k_m)} q^{\mathrm{inv}(\sigma)} .$$

Proof. Fix a sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$ in $S_n^m(k_1, \ldots, k_m)$. Note that the inversions induced by all σ_a for which σ_a equals m correspond to ordered pairs (a, b) such that σ_b is less than m. By Proposition 1.5, it follows that $\binom{n}{k_m}_q$ corresponds precisely to inversions induced by all σ_a equal to m.

Further observe that inversions induced by all σ_a for which σ_a equals $m-1$ correspond to ordered pairs (a, b) such that σ_b is less than $m-1$. In particular, no such (a, b) will correspond to a σ_b equal to m. As such, Proposition 1.5 applies to the subsequence of σ containing the $n - k_m$ elements of σ that do not equal m, and it follows that $\binom{n-k_m}{k_{m-1}}_q$ corresponds precisely to inversions induced by all σ_a equal to $m-1$.

A similar argument holds for the remaining elements of σ. □

1.3 Fundamental Sequences

We will introduce an additional definition that will be especially helpful in establishing the results of Section 3.

Definition 1.8 *If n, m are nonnegative integers, define S_n^m to be the set of all sequences of length n whose elements are in $[m]$. If σ is in S_n^m, define the fundamental sequence of σ to be*

$$F(\sigma) := (F_1, \ldots, F_m),$$

where each F_j is the multiset $\{ \mathrm{i}(\sigma_a) \mid a \in [n] \text{ and } \sigma_a = j \}$. Subsequently define the fundamental set of S_n^m to be the set

$$F_n^m := \{ F(\sigma) \mid \sigma \in S_n^m \} .$$

Figure 3 contains some examples.

2211	2121	2112
$(\{0,0\}, \{2,2\})$	$(\{0,0\}, \{2,1\})$	$(\{0,0\}, \{2,0\})$
1221	1212	1122
$(\{0,0\}, \{1,1\})$	$(\{0,0\}, \{1,0\})$	$(\{0,0\}, \{0,0\})$

Figure 3: The fundamental sequences of σ in $S_4^2(2,2)$.

Proposition 1.9 *If m, n are nonnegative integers, then*

$$|\, S_n^m \,| = |\, F_n^m \,| .$$

Proof. Define the function $\varphi\colon S_n^m \to F_n^m$ by the assignment $\sigma \mapsto F(\sigma)$. By the definition of fundamental set, φ is surjective.

Assume σ^1, σ^2 are sequences in S_n^m such that $F(\sigma^1)$ and $F(\sigma^2)$ are both equal to $(F_1, \ldots, F_m)$. Observe that the elements of F_m forces the set $\{\, a \in [n] \mid \sigma_a^i = m \,\}$ to be the equal for $i = 1, 2$. Subsequently observe that the elements of F_{m-1} forces the set $\{\, a \in [n] \mid \sigma_a^i = m-1 \,\}$ to be equal for $i = 1, 2$, and so on. Hence, φ is injective. $\square$

2 Binomial and Multinomial Identities

In this section, we will acquaint ourselves with the act of generalizing binomial and multinomial identities into their corresponding q-analogs.

2.1 Symmetry

We will begin with the q-analog to symmetry from [7], namely that $\binom{n}{k}$ equals $\binom{n}{n-k}$.

Proposition 2.1 *If n, k are nonnegative integers such that $n \geq k$, then*

$$\binom{n}{k}_q = \binom{n}{n-k}_q .$$

Proof. Let $S_n^2(k, n-k)$ be the set of sequences of length n whose elements are in $[2]$ with k 2's, and refer to an arbitrary sequence in $S_n^2(k, n-k)$ by $\sigma = (\, \sigma_1, \, \ldots, \, \sigma_n \,)$. For every $x \in [2]$, say that $\overline{x}$ equals 1 when x is 2 and $\overline{x}$ equals 2 when x is 1. Define a map

$$\varphi\colon S_n^2(k, n-k) \to S_n^2(n-k, k) \;\; by \;\; (\, \sigma_1, \, \ldots, \, \sigma_n \,) \mapsto (\, \overline{\sigma_n}, \, \ldots, \, \overline{\sigma_1} \,) .$$

Fix some $a \in [n]$ and consider σ_a. If σ_a is 2 and $\mathrm{i}(\sigma_a)$ is c, then the number of 1's that follow σ_a in σ must be c. By the definition of φ, notice the number of 2's preceding $\overline{\sigma_a}$ in $\varphi(\sigma)$ is also c. Hence, the numbers $\mathrm{i}(\sigma_a)$ and $\mathrm{r}(\overline{\sigma_a})$ are equal. Should σ_a be 1, observe that $\mathrm{i}(\sigma_a)$ and $\mathrm{r}(\overline{\sigma_a})$ are both zero. Further observing that φ is bijective, the desired result follows from Proposition 1.2. $\square$

We will now derive the multinomial generalization, also found in [7].

Proposition 2.2 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$ and π is a permutation of $[m]$, then*

$$\binom{n}{k_1, \, \ldots, \, k_m}_q = \binom{n}{k_{\pi(1)}, \, \ldots, \, k_{\pi(m)}}_q .$$

Proof. Refer to an arbitrary sequence in $S_n^m(k_1, \ldots, k_m)$ by $\sigma = (\, \sigma_1, \, \ldots, \, \sigma_n \,)$, and define a map

$$\theta\colon S_n^m(k_1, \ldots, k_i, k_{i+1}, \ldots, k_m) \to S_n^m(k_1, \ldots, k_{i+1}, k_i, \ldots, k_m)$$

such that $\theta(\sigma)_a$ equals σ_a when σ_a is neither i nor $i+1$. It follows that

$$\sum_{\sigma_a > i+1} \mathrm{i}(\sigma_a) \;=\; \sum_{\theta(\sigma)_a > i+1} \mathrm{i}\left(\theta(\sigma)_a\right) \,, \qquad\qquad \sum_{\sigma_a < i} \mathrm{i}(\sigma_a) \;=\; \sum_{\theta(\sigma)_a < i} \mathrm{i}\left(\theta(\sigma)_a\right) \,.$$

For the subsequence of σ for which σ_a is equal to i or $i+1$, let θ act on that subsequence analogously to φ in Proposition 2.1. It follows that

$$\sum_{\sigma_a \in \{i,i+1\}} \mathrm{i}(\sigma_a) \;=\; \sum_{\theta(\sigma)_a \in \{i,i+1\}} \mathrm{i}\left(\theta(\sigma)_a\right) \,.$$

By Proposition 1.2, we have that $\mathrm{inv}(\sigma)$ equals $\mathrm{inv}(\theta(\sigma))$.

Observe that this Proposition has been shown for π that are of the form of a simple transposition. Given that any permutation is a composition of simple transpositions, we have our desired result for any permutation π. □

2.2 Pascal's Identity

We will now consider Pascal's Identity, which can be found in [6],

$$\binom{n}{k_1,\ldots,k_m} = \binom{n-1}{k_1-1,\ldots,k_m} + \binom{n-1}{k_1,k_2-1,\ldots,k_m} + \cdots + \binom{n-1}{k_1,\ldots,k_m-1} .$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $S_n^m(k_1,\ldots,k_m)$, then $\binom{n-1}{k_1-1,\ldots,k_m}$ counts such sequences that end in a 1, $\binom{n-1}{k_1,k_2-1,\ldots,k_m}$ counts such sequences that end in a 2, and so on.

Proposition 2.3 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1+\cdots+k_m = n$, then $\binom{n}{k_1,\ldots,k_m}_q$ is equal to*

$$q^{k_2+\cdots+k_m}\binom{n-1}{k_1-1,\ldots,k_m}_q + q^{k_3+\cdots+k_m}\binom{n-1}{k_1,k_2-1,\ldots,k_m}_q + \cdots + \binom{n-1}{k_1,\ldots,k_m-1}_q .$$

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $S_n^m(k_1,\ldots,k_m)$. For such sequences that end in a 1, note that $k_2+\cdots+k_m$ inversions will be received by that 1. Thus, the product of $q^{k_2+\cdots+k_m}$ and $\binom{n-1}{k_1-1,\ldots,k_m}_q$ accounts precisely for the inversions of sequences that end in a 1. The argument is similar for the remaining terms of our desired sum. □

Note that applying Proposition 2.2 to Proposition 2.3 yields $m!$ different articulations of the q-analog to Pascal's Identity. For the case $m=2$, Figure 4 contains the resulting $2!$ articulations.

$$q^{k_2}\binom{n-1}{k_1-1,k_2}_q + \binom{n-1}{k_1,k_2-1}_q \qquad\qquad \binom{n-1}{k_1-1,k_2}_q + q^{k_1}\binom{n-1}{k_1,k_2-1}_q$$

Figure 4: The two articulations of $\binom{n}{k_1,k_2}_q$ via the q-analog of Pascal's Identity.

2.3 Diagonal Sum Identity

We will now consider the Diagonal Sum Identity, which can be found in [6],

$$\binom{n}{k_1,\ldots,k_m} = \sum_{i=0}^{k_1}\sum_{j=2}^{m}\binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}.$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $S_n^m(k_1,\ldots,k_m)$, then the expression $\binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}$ counts such sequences that end in a j followed by i 1's.

Proposition 2.4 *If $m,n,k_1,\ldots,k_m$ are nonnegative integers such that $k_1+\cdots+k_m=n$, then*

$$\binom{n}{k_1,\ldots,k_m}_q = \sum_{i=0}^{k_1}\sum_{j=2}^{m} q^{\left((n-k_1)i+\sum\limits_{v=j+1}^{m}k_v\right)}\binom{n-i-1}{k_1-i,k_2,\ldots,k_j-1,\ldots,k_m}_q.$$

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $S_n^m(k_1,\ldots,k_m)$. Observe that for any such sequence σ, ordered pairs (a,b) associated with $\operatorname{inv}(\sigma)$ are of exactly one of the following forms: a,b are both less than $n-i$ in value; a is less than $n-i$ in value and b is at least $n-i$ in value; a,b are both at least $n-i$ in value.

Note that: $\binom{n-i-1}{k_1-i,\ldots,k_m}_q$ accounts for ordered pairs (a,b) associated with inversions such that a,b are both less than $n-i$ in value; there are $(n-k_1-1)i+\sum k_v$ ordered pairs (a,b) associated with inversions such that a is less than $n-i$ and b is at least $n-i$; and there are i ordered pairs (a,b) associated with inversions such that a,b are both at least $n-i$. □

2.4 Vandermonde's Identity

We will now consider Vandermonde's Identity, which can be found in [6],

$$\binom{n_1+n_2}{k_1,\ldots,k_m} = \sum_{\substack{r_1+\cdots+r_m=n_1\\ 0\le r_i\le k_i}}\binom{n_1}{r_1,\ldots,r_m}\binom{n_2}{k_1-r_1,\ldots,k_m-r_m}.$$

Interpreting $\binom{n_1+n_2}{k_1,\ldots,k_m}$ as the number of sequences in $S_{n_1+n_2}^m(k_1,\ldots,k_m)$, then each term of the sum accounts for the sequences whose first n_1 elements contains exactly r_1 1's, ..., r_m m's.

Proposition 2.5 *If $m, n_1, n_2, k_1, \ldots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m$ equals $n_1 + n_2$, then*

$$\binom{n_1+n_2}{k_1,\ldots,k_m}_q = \sum_{\substack{r_1+\cdots+r_m=n_1 \\ 0\le r_i\le k_i}} q^{\left(\sum\limits_{j\in[m]} f(r_j)\right)} \binom{n_1}{r_1,\ldots,r_m}_q \binom{n_2}{k_1-r_1,\ldots,k_m-r_m}_q$$

where $f(r_j) \;=\; r_j \sum\limits_{i\in[j-1]} (k_i - r_i)$ for every $j \in [m]$.

Proof. Interpret $\binom{n_1+n_2}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $S^m_{n_1+n_2}(k_1, ..., k_m)$. Observe that for any such sequence σ , ordered pairs (a, b) associated with $\mathrm{inv}(\sigma)$ are of exactly one of the following forms: a, b at most n_1 in value; a, b greater than n_1 in value; a at most n_1 in value and b greater than n_1 in value.

Note that: $\binom{n_1}{r_1,\ldots,r_m}_q$ accounts for ordered pairs (a, b) associated with inversions such that a, b are at most n_1 in value; $\binom{n_2}{k_1-r_1,\ldots,k_m-r_m}_q$ accounts for ordered pairs (a, b) associated with inversions such that a, b are greater than n_1 in value; $q^{\sum f(r_j)}$ accounts for ordered pairs (a, b) associated with inversions such that a is at most n_1 in value and b is greater than n_1 in value. □

We will now derive a generalization.

Proposition 2.6 *If $m, n_1, ..., n_s, k_1, ..., k_m$ are nonnegative integers such that $k_1+\cdots+k_m$ is equal to $n_1 + \cdots + n_s$, then*

$$\binom{n_1+\cdots+n_s}{k_1,\ldots,k_m}_q = \sum_{\substack{r_{i,1}+\cdots+r_{i,m}=n_i \\ r_{1,j}+\cdots+r_{s,j}=k_j \\ 0\le r_{i,j}}} q^{\left(\sum\limits_{(i,j)\in[s]\times[m]} f(r_{i,j})\right)} \binom{n_1}{r_{1,1},\ldots,r_{1,m}}_q \cdots \binom{n_s}{r_{s,1},\ldots,r_{s,m}}_q$$

where $f(r_{i,j}) \;=\; r_{i,j} \sum\limits_{v=1}^{j-1} \sum\limits_{u=i+1}^{s} r_{u,v}$ for every $(i, j) \in [s] \times [m]$.

Proof. Consider $S^m_{n_1+\cdots+n_s}(k_1, \ldots, k_m)$, and interpret $\binom{n_1+\cdots+n_s}{k_1,\ldots,k_m}_q$ as the generating function for inversions on this set of sequences. Let σ be such a sequence.

For every i in $[s]$, define X_i to be $\{\, x \in \mathbb{Z} \mid n_1 + \cdots + n_{i-1} + 1 \le x \le n_1 + \cdots + n_i \,\}$. Observe that ordered pairs (a, b) associated with $\mathrm{inv}(\sigma)$ are of exactly one of the following forms: a, b are both in X_i for some $i \in [s]$; a, b are not both in X_i for some $i \in [s]$.

Note that $\binom{n_i}{r_{i,1},\ldots,r_{i,m}}_q$ accounts for ordered pairs (a, b) associated with inversions such that (a, b) are both in X_i . Also note that $q^{\sum f(r_{i,j})}$ accounts for ordered pairs (a, b) associated with inversions such that a, b are not both in X_i for some $i \in [s]$. □

2.5 Chu Shih-Chieh (Zhu Shijie)'s Identity

We will now consider Chu Shih-Chieh's Identity, which can be found in [6],

$$\binom{n}{k_1,\ldots,k_m} = \sum_{r=0}^{n-k_1} \sum_{\substack{r_2+\cdots+r_m=r \\ 0\le r_j\le k_j}} \binom{r}{0,r_2,\ldots,r_m}\binom{n-r-1}{k_1-1,k_2-r_2,\ldots,k_m-r_m}.$$

Interpreting $\binom{n}{k_1,\ldots,k_m}$ as the number of sequences in $S_n^m(k_1,\ldots,k_m)$, then each term of the sum accounts for the sequences $(\sigma_1, \ldots, \sigma_n)$ such that σ_{r+1} equals 1 and $(\sigma_1, \ldots, \sigma_r)$ is a sequence with r_2 2's, …, r_m m's.

This can generalize as follows.

Proposition 2.7 *If $m, n, k_1, \ldots, k_m$ are nonnegative integers such that $k_1+\cdots+k_m$ is equal to n, then $\binom{n}{k_1,\ldots,k_m}_q$ is equal to*

$$\sum_{\substack{E\subset[n] \\ |E|=k_1}} \sum_{\substack{r_{i,2}+\cdots+r_{i,m}=n_i \\ r_{1,j}+\cdots+r_{s,j}=k_j \\ 0\le r_{i,j}}} q^{\left(\sum_{(i,j)\in[s]\times[m]} f(r_{i,j})\right)} \binom{n_1}{0,r_{1,2},\ldots,r_{1,m}}_q \cdots \binom{n_s}{0,r_{s,2},\ldots,r_{s,m}}_q$$

where $E = \{e_1, \ldots, e_{k_1}\}$ with $e_i < e_{i+1}$ for every $1 \le i \le k_1-1$; s is equal to k_1+1; n_1 equals e_1-1; n_i equals $e_i - e_{i-1} - 1$ for every $2 \le i \le k_1$; n_s equals $n - e_{k_1}$; and $f(r_{i,j}) = r_{i,j}\left(k_1 - i + 1 + \sum_{v=2}^{j-1}\sum_{u=i+1}^{s} r_{u,v}\right)$ for every $(i,j) \in [s]\times[m]$.

Proof. Interpret $\binom{n}{k_1,\ldots,k_m}_q$ as the generating function for inversions on $S_n^m(k_1,\ldots,k_m)$,. Given any such sequence $\sigma = (\sigma_1, \ldots, \sigma_n)$, let E be the set $\{i \in [n] \mid \sigma_i = 1\}$.

The remainder of the proof is similar to that of Proposition 2.6, with two exceptions. For every i in $[s]$, define X_i to be $\{x \in \mathbb{Z} \mid n_1+\cdots+n_{i-1}+i \le x \le n_1+\cdots+n_i+i-1\}$. Second, observe that the term k_1-i+1 in the expression of $f(r_{i,j})$ is to account for ordered pairs (a,b) such that a is in X_i and σ_b is equal to 1. □

2.6 "Apartment Complex" Identity

The following identity was adapted from an indentity contained in [8]. Consider a hypothetical scenario with an apartment complex whose buildings will contain exactly one unit per floor. Assume there are to be n_1 buildings, with n_2 of them receiving a second floor. Exactly k of the units will be rented.

$$\binom{n_1}{n_2}\binom{n_1+n_2}{k} = \sum_{k_1+k_2=k} \binom{n_1}{k_1}\binom{n_1}{n_1-n_2,k_2,n_2-k_2}.$$

The complex owner could first choose which n_2 of the n_1 buildings will receive a second floor, and then k tenants could choose which of the n_1+n_2 units to rent. Alternatively, for all k_1 in between 0 and k, the owner could first rent out k_1 of the n_1 first floor units, and then of the n_1 buildings: $n_1 - n_2$ buildings could receive no second floor; k_2 of them could receive a second floor that is rented; and $n_2 - k_1$ could receive a second floor that is unrented. This can generalize as follows.

Proposition 2.8 *If $n_1, \dots, n_j, k$ are nonnegative integers such that $n_j \le \cdots \le n_1$ and $k \le n_1 + \cdots + n_j$, then*

$$\left(\prod_{i=2}^{j} \binom{n_{i-1}}{n_i} \right) \binom{n_1 + \cdots + n_j}{k} = \sum_{k_1 + \cdots + k_j = k} \binom{n_1}{k_1} \prod_{i=2}^{j} \binom{n_{i-1}}{n_{i-1} - n_i, n_i - k_i, k_i} .$$

Proof. For every $2 \le i \le j$, let S_{i-1} be the set $S^2_{n_{i-1}}(n_i, n_{i-1} - n_i)$. Also let S_j be the set $S^2_{n_1 + \cdots + n_j}(k, n_1 + \cdots + n_j - k)$. In addition, let T_1 be the set $S^2_{n_1}(k_1, n_1 - k_1)$. For every $2 \le i \le j$, let T_i be the set $S^3_{n_{i-1}}(n_{i-1} - n_i, n_i - k_i, k_i)$.

Define

$$\varphi\colon \prod_{i=1}^{j} S_i \to \prod_{i=1}^{j} T_j \;\text{ via }\; (\sigma^1, \dots, \sigma^j) \mapsto (\tau^1, \dots, \tau^j)$$

in the following way. For every $1 \le i \le j-1$, let $C_i = \{ s \in [n_i] \mid \sigma^i_s = 2 \}$. Express C_i as $\{ c_{i,1}, \dots, c_{i,n_{i+1}} \}$ where $c_{i,p} < c_{i,p+1}$ for every $1 \le p \le n_{i+1} - 1$. Further, let N_i be equal to $n_1 + \cdots + n_i$. Finally, for every $1 \le i \le j-1$, let

$$\tau^1_s = \sigma^j_s ,$$

$$\tau^{i+1}_s = \begin{cases} 1 \text{ if } \sigma^i_s = 1 , \\ 2 \text{ if } \sigma^i_s = 2 \text{ and } \sigma^j_{N_i + p} = 1 \text{ where } s = c_{i,p} , \\ 3 \text{ if } \sigma^i_s = 2 \text{ and } \sigma^j_{N_i + p} = 2 \text{ where } s = c_{i,p} . \end{cases}$$

The desired result follows from observing that φ is bijective. □

Proposition 2.9 *If $n_1, \dots, n_j, k$ are nonnegative integers such that $n_j \le \cdots \le n_1$ and $k \le n_1 + ... + n_j$, then*

$$\left(\prod_{i=2}^{j} \binom{n_{i-1}}{n_i}_q \right) \binom{n_1 + \cdots + n_j}{k}_q = \sum_{k_1 + \cdots + k_j = k} q^{f(K)} \binom{n_1}{k_1}_q \prod_{i=2}^{j} \binom{n_{i-1}}{n_{i-1} - n_i, n_i - k_i, k_i}_q$$

where $f(K) = \sum_{i=1}^{j-1} k_i \left(\sum_{u=i+1}^{j} n_u - k_u \right)$ for every K equal to $(k_1, \dots, k_j)$.

Proof. We will utilize the notation of Proposition 2.8 and interpret the q-analogs within this identity as generating functions for the inversion statistic on sequences.

We will begin by accounting for the inversions associated with $\binom{n_1 + \cdots + n_j}{k}_q$. For every i in $[j]$, define X_i to be $\{ x \in \mathbb{Z} \mid n_0 + \cdots + n_{i-1} + 1 \le x \le n_1 + \cdots + n_i \}$ where n_0 is equal to zero. Observe that ordered pairs (a,b) associated with $\text{inv}(\sigma^j)$ are of exactly one of the following forms: a,b are both in X_i for some i in $[j]$; a,b are not both in X_i for some i in $[j]$.

Note that for every i in $[j]$, the ordered pairs (a,b) associated with inversions of $\text{inv}(\sigma^j)$ such that a,b are both in X_i is accounted for by

$$\text{inv}\left(\tau^1\right)\ ,\ \text{when}\ \ i=1\,;$$
$$\sum_{\tau_s^i=2}\text{r}\left(\tau_s^r\right)\ ,\ \text{when}\ \ i\geq 2\,.$$

Also note that $q^{f(K)}$ accounts for ordered pairs (a,b) associated with inversions such that a,b are not both in X_i for some i in $[j]$.

We will now account for inversions associated with $\prod\binom{n_{i-1}}{n_i}_q$. Observe that for every $2\leq i\leq j$,

$$\text{inv}\left(\sigma^{i-1}\right)\ =\sum_{\sigma_s^{i-1}=1}\text{r}\left(\sigma_s^{i-1}\right)\ =\sum_{\tau_s^i=1}\text{r}\left(\tau_s^i\right)\ .$$

The desired result follows as an application of Corollary 1.3. □

Notice that developing a complete enumerative understanding of the original "apartment complex" identity in terms of sequences enabled us to develop the corresponding q-analog. It is the viewpoint of the authors that a deep grasp of the enumerative combinatorics of any binomial or multinomial identity enables the development of a q-analog generalization.

3 Galois Numbers and Integer Partitions

In this section, we will investigate a connection between the coefficients of generalized Galois numbers and integer partitions with kinds.

3.1 Major Index Statistic

To support our investigation, we will require a different statistic on sequences from [5].

Definition 3.1 *If m,n are nonnegative integers and $\sigma=(\sigma_1,\ldots,\sigma_n)$ is a sequence whose elements are in $[m]$, then*

$$\text{maj}(\sigma)\ :=\sum_{\substack{a\in[n-1]\\ \sigma_a>\sigma_{a+1}}}a\,.$$

The value of $\text{maj}(\sigma)$ shall be referred to as the major index of σ .

Figure 5 contains some examples, and the following two lemmas and corollary will develop additional familiarity with the major index statistic while also proving useful in a later theorem.

2211	2121	2112
$\mathrm{maj}(\sigma) = 2$	$\mathrm{maj}(\sigma) = 4$	$\mathrm{maj}(\sigma) = 1$
1221	1212	1122
$\mathrm{maj}(\sigma) = 3$	$\mathrm{maj}(\sigma) = 2$	$\mathrm{maj}(\sigma) = 0$

Figure 5: All sequences of length 4 with two 2s and two 1s.

Lemma 3.2 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$,*

$$\begin{aligned} \mathcal{M}_n^{m+1}(k) &:= \{ \sigma \in S_n^{m+1} \mid \mathrm{maj}(\sigma) = k \}, \\ A_i &= \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_{n-i} = \sigma_{n-i+1} \} \text{ when } 1 \leq i \leq m-1, \\ A_m &= \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_n = m+1 \}. \end{aligned}$$

Then,

$$\mathcal{M}_n^{m+1}(k) \setminus \bigcup_{i \in [m]} A_i = \{ \sigma \in \mathcal{M}_n^{m+1}(k) \mid \sigma_{k+1} = 1 \text{ and } \omega = (1, 2, \ldots, m) \},$$

where $\omega = (\sigma_{n-m+1}, \ldots, \sigma_n)$ is the subsequence of σ containing its last m elements.

Proof. Let σ be in $\mathcal{M}_n^{m+1}(k) \setminus \cup A_i$. Since $n - m + 1$ must be at least $k + 1$ in value and $\mathrm{maj}(\sigma)$ is equal to k, the subsequence ω must be nondecreasing. In addition, since σ is not in $\cup A_i$, the subsequence ω must be strictly increasing and not end in $m + 1$. Given that the length of ω is m, it is forced that $\omega = (1, 2, \ldots, m)$. The desired inclusion follows from observing that for every $k + 1 \leq j \leq n - m + 1$, the value of σ_j must be 1 or else the major index of σ would be greater than k.

The reverse inclusion follows by the definitions of the A_i's and $\mathcal{M}_n^{m+1}(k)$. □

Corollary 3.3 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$. Also let $A_1, \ldots, A_m$ be as in Lemma 3.2. Then,*

$$\left| \mathcal{M}_n^{m+1}(k) \setminus \bigcup_{i \in [m]} A_i \right| = \left| \{ \sigma \in \mathcal{M}_{k+1}^{m+1}(k) \mid \sigma_{k+1} = 1 \} \right|.$$

Proof. The result follows from observing that for every σ in $\mathcal{M}_n^{m+1}(k) \setminus \cup A_i$, the value of elements $\sigma_{k+2}, \ldots, \sigma_n$ are fixed and can be removed without affecting $\mathrm{maj}(\sigma)$. □

Lemma 3.4 *Let m, n, k be nonnegative integers such that $n - m + 1 \geq k + 1$. Also let $A_1, \ldots, A_m$ and ω be as in Lemma 3.2. If J is a subset of $[m]$ with $|J| = i$, then*

$$\left| \bigcap_{j \in J} A_j \right| = \left| \mathcal{M}_{n-i}^{m+1}(k) \right|.$$

Proof. Let $\varphi\colon \cap A_j \to \mathcal{M}_{n-i}^{m+1}(k)$ via $\sigma \mapsto \overline{\sigma}$, where σ is $(\sigma_1, \ldots, \sigma_{n-m}, \omega_1, \ldots, \omega_m)$ and $\overline{\sigma}$ is the subsequence of σ with ω_j removed for every $j \in J$. Note that $\overline{\sigma}$ is of the proper length for the expressed codomain of φ. Also note that the elements of σ whose indices are accounted for by $\mathrm{maj}(\sigma)$ are unaffected by φ: when $|J| < m$, the values of $n-m$ is at least k; when $|J| = m$, every ω_i equals $m+1$. As such, the values of $\mathrm{maj}(\sigma)$ and $\mathrm{maj}(\overline{\sigma})$ are equal. Hence, the image of φ is contained within the desired codomain.

To show surjectivity, observe that each A_j in $\cap A_j$ induces a loss of one degree of freedom in the expression of any σ from $\mathcal{M}_n^{m+1}(j)$. Viewing this loss as being induced on the element ω_j, the map φ results in $\overline{\sigma}$ being free from the adjacent element equality that is forced by the A_j's.

To show injectivity, consider σ^1, σ^2 in $\cap A_j$ such that σ^1 and σ^2 are unequal. Let a be the largest index of element such that σ_a^1 differs from σ_a^2. If a is greater than $n-m$, the result follows from observing that σ_a^1 and σ_a^2 are necessarily not among the ω_j removed by φ. Should a be at most $n-m$, the result follows given that such σ_a^1 and σ_a^2 are unaffected by φ. □

It is encouraged to take a moment to observe the parallelism that exists between Figure 1 and Figure 5. This parallelism is in fact not a coincidence. MacMahon showed in [5] that when considering the set of sequences $S_n^m(k_1, \ldots, k_m)$, the generating function for major index and the generating function for inversions are equal. Stated precisely, if $m, n, k_1, \cdots, k_m$ are nonnegative integers such that $k_1 + \cdots + k_m = n$, then

$$\binom{n}{k_1, \ldots, k_m}_q = \sum_{\sigma \in S_n^m(k_1,\ldots,k_m)} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n^m(k_1,\ldots,k_m)} q^{\mathrm{maj}(\sigma)}. \qquad (1)$$

3.2 The Insertion Method

We now want to describe a construction that uses the major index statistic to form a unique sequence of length n whose entries are in $[m]$ from a given fundamental sequence in F_n^m. This construction, called The Insertion Method, was first developed by Carlitz [1] and later was clarified by Wilson [11].

Let m, n be nonnegative integers, and let $(F_1, \ldots, F_m)$ be a fundamental sequence in F_n^m. For every v in $[m]$, list the elements of F_v in nonincreasing order, labeling them as $f_{v,1} \geq \cdots \geq f_{v,k_v}$ where k_v equals $|F_v|$. The sequence $(f_{1,1}, f_{1,2}, \ldots, f_{m,k_m})$ will be referred to as $\tau = (\tau_1, \ldots, \tau_n)$. Also define the value function $v\colon [n] \to [m]$ such that $v(i)$ equals j, where τ_i corresponds to its respective $f_{j,k}$. We will build a sequence σ in S_n^m inductively using τ and v.

Let $\sigma^1 = (v(1))$. For every $2 \leq i \leq n$, there is some $a \in [i]$ such that σ_a^i equals $v(i)$. Moreover, the sequence σ^i shall be of the form

$$\sigma_b^i = \begin{cases} v(i) & \text{when } b = a, \\ \sigma_b^{i-1} & \text{when } 1 \leq b < a, \\ \sigma_{b-1}^{i-1} & \text{when } a < b \leq i. \end{cases}$$

The value a shall be determined by the following process:

1. Label σ_i^i with a zero.
2. Working greatest to least among j in $[i-2]$, for every $\sigma_j^{i-1} > \sigma_{j+1}^{i-1}$ label σ_{j+1}^i with successively increasing positive integers $1, 2, 3, \ldots, d$.
3. Working least to greatest among j in $[i-1]$, if σ_j^i is currently unlabeled, label σ_j^i with successively positive integers $d+1, d+2, \ldots, i-1$.
4. Find the σ_j^i labeled with a τ_i, and let a equal j.

Example 3.5 Consider the fundamental sequence

$$\big(F_1, F_2, F_3, F_4 \big) = \big(\{0,0\}, \{1\}, \{2,3\}, \{1,5\} \big).$$

Note the contents of Figure 6.

i	τ_i	$v(i)$	Labeling for σ^i	σ^i	$\mathrm{maj}(\sigma^i)$
1	0	1		(1)	0
2	0	1	$(1,0)$	$(1,1)$	0
3	1	2	$(1,2,0)$	$(2,1,1)$	1
4	3	3	$(2,1,3,0)$	$(2,1,3,1)$	4
5	2	3	$(3,2,4,1,0)$	$(2,3,1,3,1)$	6
6	5	4	$(3,4,2,5,1,0)$	$(2,3,1,4,3,1)$	11
7	1	4	$(4,5,3,6,2,1,0)$	$(2,3,1,4,3,4,1)$	12

Figure 6: The construction of σ for Example 3.5

The proof of the fact that The Insertion Method provides a bijection from F_n^m to S_n^m is omitted here as it is contained in [1]. Additional consequences of [1] include: $\mathrm{maj}(\sigma^i) = \mathrm{maj}(\sigma^{i-1}) + \tau_i$, which will be an essential observation for the two propositions that follow.

Proposition 3.6 *Let m, n be nonnegative integers, let σ be in S_n^m, and let $F(\sigma)$ be the fundamental sequence of σ. Then, σ_n equals 1 if and only if all elements of the multisets $F_2, \ldots, F_m$ are nonzero.*

Proof. The desired result follows from: step 1 in The Insertion Method, namely that σ_i^i is labeled with a zero; and the fact that $\mathrm{maj}(\sigma^i) = \mathrm{maj}(\sigma^{i-1}) + \tau_i$ □

Our observations can be further clarified through the following.

Proposition 3.7 *Let m, n, k be nonnegative integers. Then*

$$F_n^m(k) \coloneqq \{\, F(\sigma) \mid \sigma \in S_n^m \text{ and } \mathrm{inv}(\sigma) = k \,\} = \{\, F(\sigma) \mid \sigma \in S_n^m \text{ and } \mathrm{maj}(\sigma) = k \,\}.$$

Proof. The desired result follows from: the definition of $F(\sigma)$; and the fact that $\mathrm{maj}(\sigma^i) = \mathrm{maj}(\sigma^{i-1}) + \tau_i$. □

3.3 Integer Partitions with Kinds

We will now define the notion of an integer partition with kinds, which can be found in [3].

Definition 3.8 *Let k, m be nonnegative integers. An integer partition of k with m kinds is a composition of k whose parts are positive integers of the form*

$$k \;=\; k_1^1 + \cdots + k_1^{i_1} + k_2^1 + \cdots + k_2^{i_2} + \cdots + k_m^1 + \cdots + k_m^{i_m}\,,$$

where $i_1, \ldots, i_m$ are nonnegative integers, and when i_a is nonzero $k_a^j \geq k_a^{j+1}$ for all j in the set $[i_a - 1]$. The set of all integer partitions of k with m kinds shall be referred to as P_k^m.

Figure 7 contains some examples.

3_1	3_2	$2_1 + 1_1$	$2_1 + 1_2$	$1_1 + 2_2$
$2_2 + 1_2$	$1_1 + 1_1 + 1_1$	$1_1 + 1_1 + 1_2$	$1_1 + 1_2 + 1_2$	$1_2 + 1_2 + 1_2$

Figure 7: The integer partitions of 3 with 2 kinds.

Proposition 3.9 *Let m, n, k be nonnegative integers such that $k \leq n$. Then*

$$|\, P_k^m \,| \;=\; \Big|\, \{\, (F_1, \ldots, F_{m+1}) \in F_n^{m+1}(k) \;\mid\; 0 \notin F_2\,, \ldots,\, 0 \notin F_{m+1} \,\} \,\Big|\,.$$

Proof. Define $\varphi\colon P_k^m \to \{\, (F_1, \ldots, F_{m+1}) \in F_n^{m+1}(k) \;\mid\; 0 \notin F_2\,, \ldots,\, 0 \notin F_{m+1} \,\}$ via

$$k_1^1 + \cdots + k_1^{i_1} + k_2^1 + \cdots + k_2^{i_2} + \cdots + k_m^1 + \cdots + k_m^{i_m} \;\mapsto\; (F_1, \ldots, F_{m+1})\,,$$

where: for all $2 \leq j \leq m{+}1$, the multiset F_j equals $\{\, k_{j-1}^1, \ldots, k_{j-1}^{i_j} \,\}$; and F_1 is a multiset of cardinality $n - i_1 - \cdots - i_m$ containing only zeros. Since each k_j^i is positive: the value of $i_1 + \cdots + i_n$ is at most k and hence $|F_1|$ is nonnegative; and all elements of the multisets $F_2, \ldots, F_{m+1}$ are nonzero.

Observing that Definition 1.8 implies F_1 must contain only zeros for any fundamental sequence, the desired bijectivity of φ follows naturally from its rule of assignment. $\square$

3.4 Generalized Galois Numbers

We will begin by defining a generalized Galois number, which can be found in [10].

Definition 3.10 *If m, n are nonnnegative integers, then*

$$G_n^m \;:=\; \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m}_q\,.$$

This polynomial is sometimes referred to as the generalized Galois number of (m, n).

$$G_2^3 = 3q + 6 \qquad G_3^3 = q^3 + 8q^2 + 8q + 10$$

$$G_4^3 = 3q^5 + 9q^4 + 18q^3 + \cdots + 15 \qquad G_5^3 = 3q^8 + \cdots + 48q^4 + 45q^3 + \cdots + 21$$

$$G_6^3 = q^{12} + \cdots + 107q^4 + 82q^3 + \cdots + 28 \qquad G_7^3 = 3q^{16} + \cdots + 186q^4 + 129q^3 + \cdots + 36$$

Figure 8: Generalized Galois numbers $G_2^3, \ldots, G_7^3$.

Figure 8 contains examples of generalized Galois numbers of $(3, n)$ that were calculated using a recursive relation from [10].

Proposition 3.11 *If m, n are nonnegative integers, then*

$$G_n^m = \sum_{\sigma \in S_n^m} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma \in S_n^m} q^{\mathrm{maj}(\sigma)}.$$

Proof. The result follows from Definition 3.10 and Equation (1). □

One final definition, from [2], is needed to concisely state the theorem that follows.

Definition 3.12 *Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function, and define the finite difference of f to be*

$$\nabla f : \mathbb{Z} \to \mathbb{Z} \quad \text{via} \quad n \mapsto f(n) - f(n-1).$$

Inductively defining the m^{th}-finite difference of f to be $\nabla^m f := \nabla(\nabla^{m-1} f)$ for any positive integers $m \geq 2$, a standard result that can be found in [2] follows

$$\nabla^m f(n) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(n-i). \tag{2}$$

Letting $f_k^3(n)$ be the coefficient of q^k in the simplified polynomial G_n^3, Figure 9 contains some example finite difference computations.

$$\nabla^2 f_3^3(4) = 16 \qquad \nabla^2 f_3^3(5) = 10 \qquad \nabla^2 f_3^3(6) = 10 \qquad \nabla^2 f_3^3(7) = 10$$

$$\nabla^2 f_4^3(4) = 9 \qquad \nabla^2 f_4^3(5) = 30 \qquad \nabla^2 f_4^3(6) = 20 \qquad \nabla^2 f_4^3(7) = 20$$

Figure 9: Sample finite difference computations using $f_k^3(n)$.

Observe that $\nabla^2 f_3^3(5)$, $\nabla^2 f_3^3(6)$, $\nabla^2 f_3^3(7)$ are equal to the number of integer partitions of 3 with 2 kinds (from Figure 7).

Theorem 3.13 *Let m, n, k be nonnegative integers such that $n \geq m + k$. Then,*

$$\nabla^m f_k^{m+1}(n) \;=\; |\, P_k^m \,| \;,$$

where $f_k^{m+1}(n)$ evaluates to the coefficient of q^k in the simplified polynomial G_n^{m+1} .

Proof. By the definition of $\mathcal{M}_n^{m+1}(k)$ in Lemma 3.2, observe that $f_k^{m+1}(n-i)$ is equal to $\left|\, \mathcal{M}_{n-i}^{m+1}(k) \,\right|$. Applying this observation and Equation 2, we have that

$$\nabla^m f_k^{m+1}(n) \;=\; \sum_{i=0}^{m} (-1)^i \binom{m}{i} \,|\, \mathcal{M}_{n-i}^{m+1}(k) \,| \;.$$

Note that the assumed relation $n \geq m + k$ satisfies the similar assumption of Lemma 3.2 and Lemma 3.4. Applying these two lemmas and the Principle of Inclusion and Exclusion from [2], the following equality is yielded

$$\sum_{i=0}^{m} (-1)^i \binom{m}{i} \,|\, \mathcal{M}_{n-i}^{m+1}(k) \,| \;=\; \left| \mathcal{M}_n^{m+1}(k) \;\setminus\; \bigcup_{i \in [m]} A_i \right| ,$$

where $A_1, \ldots, A_m$ are as defined in Lemma 3.2. Letting $\mathcal{T}$ be $\{\, \sigma \in \mathcal{M}_{k+1}^{m+1}(k) \;|\; \sigma_{k+1} = 1 \,\}$ and applying Corollary 3.3, it follows that

$$\nabla^m f_k^{m+1}(n) \;=\; |\, \mathcal{T} \,| \;.$$

By restricting the domain of φ from Proposition 1.9, we have that

$$|\, \mathcal{T} \,| \;=\; |\, \{\, F(\sigma) \;|\; \sigma \in \mathcal{T} \,\} \,| \;.$$

Since the rightmost element of every sequence in $\mathcal{T}$ is 1, Proposition 3.6 applies to $\mathcal{T}$ and it follows that

$$\nabla^m f_k^{m+1}(n) \;=\; \left|\, \{\, (F_1, \ldots, F_{m+1}) \in F_{k+1}^{m+1}(k) \;|\; 0 \notin F_2 \,, \ldots ,\, 0 \notin F_{m+1} \,\} \,\right| \;.$$

Further applying Proposition 3.9, the desired result is achieved. □

Stated explicitly, Theorem 3.13 expresses that as n grows the m^{th} finite difference of $f_k^{m+1}(n)$ is eventually constant, and the resulting constant is precisely the number of integer partitions of k with m kinds. Reflecting back to Figure 9, observe that the sample computations of $\nabla^2 f_k^3(n)$ become constant when n is at least $k + 2$ in value.

Corollary 3.14 *If n, k are nonnegative integers such that $n \geq k$, then*

$$\frac{d^k}{dq^k} \left(\frac{G_{n+1}^2 - G_n^2}{k!} \right) \Bigg|_{q=0} \;=\; \operatorname{part}(k) \,,$$

where $\frac{d}{dq}$ is the derivative operator on polynomials and $\operatorname{part}(k)$ *is the number of integer partitions of k with 1 kind.*

Proof. Follows directly from Theorem 3.13 and Taylor's Theorem. □

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