# The 0-Rook Monoid and its Representation Theory

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**Abstract**

We show that a proper degeneracy at $q = 0$ of the $q$-deformed rook monoid of Solomon is the algebra of a monoid $R_n^0$ namely the 0-rook monoid, in the same vein as Norton's 0-Hecke algebra being the algebra of a monoid $H_n^0 := H^0(A_{n-1})$ (in Cartan type $A_{n-1}$). As expected, $R_n^0$ is closely related to the latter: it contains the $H^0(A_{n-1})$ monoid and is a quotient of $H^0(B_n)$. We give a presentation for this monoid as well as a combinatorial realization as functions acting on the classical rook monoid itself. On the way we get a Matsumoto theorem for the rook monoid a result which was conjectured by Solomon.

The 0-rook monoid shares many combinatorial properties with the Hecke monoid: its Green right preorder is an actual order, and moreover a lattice (analogous to the right weak order) which has some nice combinatorial, and geometrical features. In particular the 0-rook monoid is $\mathcal{J}$-trivial.

Following Denton-Hivert-Schilling-Thiéry, it allows us to describe its representation theory including the description of the simple and projective modules. We further show that $R_n^0$ is projective on $H_n^0$ and make explicit the restriction and induction functors along the inclusion map. We finally give a (partial) associative tower structures on the family of $(R_n^0)_{n\in\mathbb{N}}$ and we discuss its representation theory.

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# 1 Introduction

This article is the first of a series of two on the degeneracy at $q = 0$ of the $q$-rook and more generally $q$-Renner monoids and their representation theory [Gay and Hivert(2018)]. This first paper is focused on Cartan type $A$, that is only on the rook case. We start by recalling Iwahori's [Iwahori(1964)] construction of the Iwahori-Hecke algebra, and the importance of the $q = 0$ degeneracy.

## 1.1 Iwahori-Hecke algebra and its degeneracy at $q = 0$

Let $\mathbb{F}_q$ be the finite field with $q$ elements. Let $G := \mathbf{GL}_n(\mathbb{F}_q)$ be its general linear group of invertible $n \times n$ matrices, and $B \subset G$ its subgroup of upper triangular matrices. Both groups $G$ and $B$ are finite of respective cardinalities $|G| = (q^n - 1)(q^n - q)(q^n - q^2) \dots (q^n - q^{n-1})$ and $|B| = (q-1)^n q^{\binom{n}{2}}$. We denote $\mathfrak{S}_n$ the symmetric group acting on $\{1, \dots, n\}$ and identify a permutation with its associated permutation matrix. The Bruhat decomposition [Björner and Brenti(2005)] tells that for all $M \in G$ there is a unique permutation $\sigma \in \mathfrak{S}_n$ such that $M \in B\sigma B$, that is :

$$G = \bigsqcup_{\sigma \in \mathfrak{S}_n} B\sigma B. \tag{1.1}$$

For $\sigma \in \mathfrak{S}_n$, let $T_\sigma$ be the element of the group algebra $\mathbb{C}[G]$ defined by:

$$T_\sigma := \frac{1}{|B|} \sum_{x \in B\sigma B} x. \tag{1.2}$$

The Hecke ring $\mathcal{H}(G, B)$ is the $\mathbb{Z}$-ring spanned by the $T_w$. Its identity is $\varepsilon = T_{\mathrm{Id}} = \frac{1}{|B|}\sum_{b\in B} b$. Furthermore, $\mathcal{H}(G, B) = \varepsilon\, \mathbb{Z}[G]\, \varepsilon$. Let $S = \{s_1, \dots, s_{n-1}\}$ be the elementary transpositions which generate $\mathfrak{S}_n$ as a group. For $q \in \mathbb{C}$, let $\mathcal{H}_{\mathbb{Z}}(q)$ denote the $\mathbb{Z}$-algebra defined by generators and relations as follows:

$$\begin{aligned} T_i^2 &= q \cdot 1 + (q-1)T_i && 1 \le i \le n-1, && \text{(H1)}\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && 1 \le i \le n-2, && \text{(H2)}\\ T_iT_j &= T_jT_i && |i-j| \ge 2, . && \text{(H3)} \end{aligned}$$

If $q$ is the cardinality of a finite field, Iwahori proved that the maps $T_i \mapsto T_{s_i}$ extends to a full ring isomorphism from $\mathcal{H}_{\mathbb{Z}}(q)$ to $\mathcal{H}(G, B)$ and consequently, the equations above give a presentation of $\mathcal{H}_{\mathbb{Z}}(q)$. By extending the scalar to $\mathbb{C}$ we get a $\mathbb{C}$-algebra $\mathcal{H}_{\mathbb{C}}(q)$ which extends the definition of the Hecke ring outside of prime powers. It is well known that when $q$ is neither zero nor a root of the unity, the Iwahori-Hecke algebra is isomorphic to the complex group algebra $\mathbb{C}[\mathfrak{S}_n]$.

The degeneracy at $q = 0$ of the Iwahori-Hecke algebra has many interesting properties and applications. Its first appearance is perhaps in Demazure character formula [Demazure(1974)] through divided differences. Then, its central role in Schubert calculus was discovered by Lascoux [Lascoux(2001), Lascoux(2003), Lascoux(2003/04)], with further recent connection with $K$-theory through Grothendieck polynomials (see *e.g.* [Miller(2005), Lam et al.(2010)Lam, Schilling, and Shin Its representation theory was first studied by Norton [Norton(1979)] in type A and Carter [Carter(1986)] in the other types. In type $A$, Krob and Thibon [Krob and Thibon(1997)] explained how induction and restriction of these modules give an interpretation of the products and coproducts of the Hopf algebras of noncommutative symmetric functions and quasi-symmetric functions, giving thus analogue of the well know Frobenius isomorphism from the character ring of the symmetric groups to symmetric functions (See *e.g.* [Macdonald(1995)]). This was the main motivation for the present work at the beginning. Two other important steps were further made by Duchamp–Hivert–Thibon [Duchamp et al.(2002)Duchamp, Hivert, and Thibon] for type $A$ and Fayers [Fayers(2005)] for other types, using the Frobenius structure to get more results, including a description of the Ext-quiver. Denton [Denton(2010)] gave a family of minimal orthogonal idempotents.

This degeneracy is defined by putting $q = 0$ in the relation of the $q$-Iwahori-Hecke algebra:

$$\begin{aligned} T_i^2 &= -T_i && 1 \le i \le n-1, && (1.3)\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1} && 1 \le i \le n-2 && (1.4)\\ T_iT_j &= T_jT_i && |i-j| \ge 2. && (1.5) \end{aligned}$$

One interesting remark which as been discovered independently several times is that this is the algebra of a monoid [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry]. To see this, they are two possibilities: define either $\pi_i := -T_i$ or $\pi_i := T_i + 1$, and get the following presentation of the Hecke monoid at $q = 0$, which we denote $H_n^0$ (as opposite to its algebra denoted by $H_n(0)$):

$$\pi_i^2 = \pi_i \qquad 1 \leq i \leq n-1, \tag{M1}$$
$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} \qquad 1 \leq i \leq n-2, \tag{M2}$$
$$\pi_i\pi_j = \pi_j\pi_i \qquad |i-j| \geq 2. \tag{M3}$$

For a permutation $\sigma$, one defines $\pi_\sigma := \pi_{i_1}\dots\pi_{i_k}$ where $s_{i_1}\dots s_{i_k}$ is any reduced word (word of minimal length) for $\sigma$. Thanks to the braid relations M2,M3, and Matsumoto's theorem the result is independent of the choice of the reduced word. Then $H_n^0$ is nothing but the set $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ and therefore of cardinality $n!$.

In general, being the algebra of a monoid helps a lot understanding the representation theory. In this particular case, this is even more true since the monoid has a very specific property: it is $\mathcal{J}$-trivial. Those are the monoids which bears an order such that the product of $x$ and $y$ is smaller than both $x$ and $y$. According to Denton–Hivert–Schilling–Thiéry [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry], the representation theory of these kinds of monoids is entirely combinatorial (see section 2.2 for an overview of their properties). In particular, they showed that many of the previous works about the representation theory of $H_n^0$ such as [Norton(1979), Carter(1986), Duchamp et al.(2002)Duchamp, Hivert, and Thibon, Fayers(2005)] are just particular cases of the general theory for $\mathcal{J}$-trivial monoids.

### 1.2 Rook and $q$-rooks

In [Solomon(1990), Solomon(2004)], Solomon constructed an analogue of Iwahori's construction replacing the general linear group by its full matrix monoid $M := \mathbf{M}_n(\mathbb{F}_q)$. It goes as follows: recall that $B \subset M$ denotes the set of invertible upper triangular matrices. Then $M$ admits a Bruhat decomposition [Renner(1995)] too: the set of permutation matrices is replaced by the set $R_n$ of so-called *rook matrices* of size $n$, that is a $n \times n$ matrices with entries $\{0, 1\}$ and at most one nonzero entry in each row and column. Then

$$M = \bigsqcup_{r \in R_n} BrB\,. \tag{1.6}$$

The product of two rook matrices is still a rook matrix so that they form a submonoid $R_n$ of $M$. For any $r \in R_n$, Solomon defined as in Section 1.1 an element $T_r$ of the monoid algebra $\mathbb{C}[M]$ by

$$T_r := \frac{1}{|B|}\sum_{x \in BrB} x. \tag{1.7}$$

Those elements span a sub algebra $\mathcal{H}(M, B)$ which contains $\mathcal{H}(G, B)$ with the same identity $\varepsilon$, and can also be defined by $\mathcal{H}(M, B) = \varepsilon\,\mathbb{C}[M]\,\varepsilon$.

Halverson [Halverson(2004)] further got a presentation of this ring. It is generated by the two families $T_1, \dots, T_{n-1}$ and $P_1, \dots, P_n$ together with the relations of the Iwahori-Hecke algebra (Equations H1, H2, H3) and the following extra relations:

$$P_i^2 = P_i \qquad 1 \leq i \leq n, \tag{RH4}$$
$$P_iP_j = P_jP_i \qquad 1 \leq i, j \leq n, \tag{RH5}$$
$$P_iT_j = T_jP_i \qquad i < j \tag{RH6}$$

$$P_i T_j = T_j P_i = qP_i \qquad j < i \tag{RH7}$$
$$P_{i+1} = qP_i T_i^{-1} P_i \qquad 1 \le i < n. \tag{RH8}$$

Note that the last relation can also be reformulated using the first as

$$P_{i+1} = P_i T_i P_i - (q-1)P_i \tag{RH8a}$$

The question whether there exists a proper degeneracy at $q = 0$ of this ring and if it exists, if it is the monoid-ring of a monoid, is therefore very natural. The main goal of the present article is to construct such a monoid denoted $R_n^0$, show that it is, as $H_n^0$, a $\mathcal{J}$-trivial monoid, which allows us to analyze easily its representation theory.

### 1.3 Outline of the paper

The paper is organized as follows: in Section 2, after some background on the rook matrices (or just *rooks*) and their one-line notations, we sketch out Denton-Hivert-Schilling-Thiéry work on representation theory of $\mathcal{J}$-trivial monoids and how it applies to 0-Hecke monoids. We also briefly review Krob-Thibon's work [Krob and Thibon(1997)] linking representation theory of 0-Hecke algebra to the Hopf algebras of noncommutative symmetric functions and quasi-symmetric functions.

In Section 3, we turn to the definition of the 0-rook monoid. We actually give two equivalent definitions: The first definition is by generators and relations (Subsection 3.1): We show that a suitable rewriting of Halverson's presentation when specialized at $q = 0$ is actually a monoid presentation (Definition 3.1). We then study some particular elements of this monoid which allows us to give a simpler equivalent presentation (Corollary 3.6).

The second definition is as operators acting on the rook monoid (Definition 3.8). To show that these two definitions are actually equivalent (Corollary 3.46), we choose to go a somewhat lengthy road, taking the following steps:

1. We first notice that the operators verify the relations of the presentation (Remark 3.9).
2. We generalize to rooks a variant of the notion of Lehmer code of permutations (Definition 3.12), building a bijection between rooks and the so-called $R$-code (Theorem 3.27).
3. After a little combinatorial detour (Section 3.2.2), we associate to each $R$-code $\mathsf{c}$, a canonical word $\pi_{\mathsf{c}}$ (Definition 3.34) and its corresponding $s_{\mathsf{c}}$ in the classical rook monoid such that (Proposition 3.36) for all rook $r \in R_n$ then $1_n \cdot \pi_{\mathrm{code}(r)} = 1_n \cdot s_{\mathrm{code}(r)} = r$.
4. We then translate on $R$-code $\mathsf{c}$ the action on rook (Definition 3.38), and prove that, for any generator $t$, the element $\pi_{\mathsf{c}} t$ is equivalent to $\pi_{\mathsf{c}\cdot t}$ modulo the relations of the presentation (Theorem 3.41).
5. By induction this shows that any word is equivalent to a word $\pi_{\mathsf{c}}$, but since there are as many $R$-codes as rooks we will conclude that the two definitions are equivalent (Corollary 3.46).

Note that we do not use the well-known presentation of the classical rook monoid or of the $q$-rook algebra, but prove them again from scratch. Though it is combinatorially technical, we argue that our approach has several advantages. First it is self contained and purely monoidal, providing arguments for monoid theory people which are not familiar with Coxeter group theory. Second, our approach is very explicit and algorithmic providing a canonical reduced word for all rooks or 0-rooks together with an explicit algorithm transforming any word in its equivalent canonical one. Moreover, the Lehmer code is central ingredient in the theory of Schubert polynomials whose modern combinatorial incarnation is the pipedream theory. We find interesting to provide such a combinatorial tool. Finally, this allows us to have a much finer understanding of the combinatorics of reduced words. In particular, we get an analogue of Matsumoto's theorem (Theorem 3.54), an ingredient which was noticed missing by Solomon [Solomon(2004)]. As a consequence, all the previous proof of presentation had to rely on some dimension argument so that they were only valid on a field. Notice that, if we had this theorem from the beginning, we could have worked only on reduced words as for the classical case of Hecke algebras.

Section 4 is devoted to the study of the analogue of the weak permutohedron order on rooks or equivalently to Green's $\mathcal{R}$-order of the 0-rook monoid. Using a generalization of the notion of inversion sets (Definition 4.6), we provide an algorithm to compare two rooks (Definition 4.10 and Theorem 4.16). A very important consequence in particular for the representation theory is that $R_n^0$ is $\mathcal{R}$-trivial, $\mathcal{L}$-trivial and thus $\mathcal{J}$-trivial (Corollary 4.17). We then show that the right order, as for permutations, is actually a lattice (Corollary 4.19), giving algorithms to compute the meet and the join (Theorem 4.18 and 4.22). We moreover provide a formula enumerating the meet irreducible (Proposition 4.28), give a bijection for a certain subposet with the subposet of singletons in the Tamari lattice (Section 4.3) and conclude this section by some geometric remarks.

Section 5 deals with the representation theory of the 0-rook monoid. It heavily uses the fact that $R_n^0$ is $\mathcal{J}$-trivial through the theory of Denton–Hivert–Schilling–Thiéry [Denton et al.(2010/11)Denton, Hiv We describe the set of idempotents and their lattice structure (Proposition 5.7 and 5.9). As for any $\mathcal{J}$-trivial monoids, we show that the simple modules are all 1-dimensional (Theorem 5.8), describe the indecomposable projective module as some kind of descent classes (Theorem 5.17) and describe the quiver (Theorem 5.19). We then study how the representation theory of $H_n^0$ and $R_n^0$ are related. The main result here is that the later is projective on the former (Theorem 5.24). We moreover give the decomposition functor (Theorem 5.27).

Finally Section 5.5, is devoted to the tower of monoids structure on the sequence of 0-rook monoids. Recall that Bergeron-Li [Bergeron and Li(2009)] gave some necessary condition to get Hopf algebra structure on the Grothendieck groups generalizing the algebras of symmetric [Macdonald(1995)], noncommutative symmetric and quasi-symmetric functions [Krob and Thibon(1997)]. This was the main motivation for this work, but unfortunately, it does not work as nicely as expected. We present such an associative structure but it does not fulfill all the requirement of Bergeron-Li. In particular, $R_{m+n}^0$ is not projective over $R_m^0 \times R_n^0$. We nevertheless explicit some structure and in particular the induction rule for simple modules (Theorem 5.46).

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# 2 Background

## 2.1 Rook monoids

We start by recalling some basic combinatorial facts about rooks.

**Definition 2.1.** *A rook matrix is a $n \times n$ matrix with entries $\{0, 1\}$ and at most one nonzero entry in each row and column.*

Enumeration of rook matrices has received a considerable research effort in the past (See *e.g.* [Riordan(2002), Butler et al.(2010)Butler, Haglund, Can, and Remmel] and the references therein) and has recently be renewed by connection with PASEP [Josuat-Vergès(2011)]. The product of two rook matrices is still a rook matrix. Thus the following definition:

**Definition 2.2.** *The rook monoid of size $n$ is the submonoid $R_n$ of the matrix monoid containing the rook matrices of size $n$.*

Identifying permutations with their matrices, we see that $\mathfrak{S}_n$ is a submonoid of $R_n$. To deal with rook matrices, it is easier to have an analogue of the so-called one line notation for permutations as in [Can and Renner(2012)]:

**Notation 2.3.** *We encode a rook matrix by its* rook vector *(or just* rook*) of size $n$ whose $i$-th coordinate is $0$ if there is no $1$ in the $i$-th column of $r$, and the index of the row containing the $1$ in the $i$-th column otherwise.*

**Example 2.4.** Here are two matrices with their associated rook vector:

$$\underset{0\,4\,2\,3\,1}{\begin{pmatrix} 0&0&0&0&1\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix}} \qquad \underset{0\,3\,0\,4\,1}{\begin{pmatrix} 0&0&0&0&1\\ 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&0&0 \end{pmatrix}}$$

In the sequel, we identify rooks matrices and rook vectors and speak about rooks when there is no ambiguity.

**Definition 2.5.** *In the monoid $R_n$, let $(s_i)_{i=1\dots n-1}$ denotes the rook matrices of the elementary transpositions $(i, i+1)$. Let $P_i$ also denote the diagonal $n \times n$ matrix with the $i$ first diagonal entries nul and the remaining one as $1$.*

For example with $n = 4$, here are the matrices of $s_1, s_2, s_3, P_1, P_2, P_3, P_4$ and their associated vectors

$$\underset{2\,1\,3\,4}{\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}} \quad \underset{1\,3\,2\,4}{\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&1&0&0\\ 0&0&0&1 \end{pmatrix}} \quad \underset{1\,2\,4\,3}{\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}} \quad \underset{0\,2\,3\,4}{\begin{pmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}} \quad \underset{0\,0\,3\,4}{\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}} \quad \underset{0\,0\,0\,4}{\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&1 \end{pmatrix}} \quad \underset{0\,0\,0\,0}{\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}}$$

It is well-known that the $(s_i)_i$ generate the symmetric group as a the group of permutation matrices and $(s_i)_i, P_1$ generate the rook monoid. We will later give a presentation (Remark 3.9).

## 2.2 $\mathcal{J}$-trivial monoids

We present here basic facts about monoids. We refer to [Pin(2010)] or [Steinberg(2016)] for more details. Through this paper, all monoids are supposed to be *finite*.

Recall that the left (resp. right, resp. bi-sided) ideal of $M$ generated by $x$ is the set $Mx := \{mx \mid m \in M\}$ (resp. $xM := \{xm \mid m \in M\}$, resp. $MxM := \{mxn \mid m, n \in M\}$). In 1951, Green [Green(1951)] introduced several preorders on monoids related to inclusion of ideals. The standard terminology is to write $\mathcal{R}$ for right ideal, $\mathcal{L}$ for left and $\mathcal{J}$ for bi-sided. Let $\mathcal{K} \in \{\mathcal{R}, \mathcal{L}, \mathcal{J}\}$ and $M$ be a monoid. For $x, y \in M$, we write $x \leq_{\mathcal{K}} y$ when the $\mathcal{K}$-ideal generated by $x$ is contained in the $\mathcal{K}$-ideal generated by $y$. For example, if $\mathcal{K} = \mathcal{L}$, this means that $x \leq_{\mathcal{L}} y$ if $Mx \subseteq My$ or equivalently if $x = uy$ for some $u \in M$. These relations are clearly preorders (reflexive and antisymmetric) and naturally give rise to equivalence relations denoted simply by $\mathcal{K}$: for example $x \,\mathcal{L}\, y$ if and only if $Mx = My$.

**Definition 2.6.** *A monoid $M$ is called $\mathcal{K}$-trivial if all $\mathcal{K}$-classes are of cardinality one, that is if the $\mathcal{K}$-preorder is antisymmetric and therefore an actual order. Specifically, $M$ is $\mathcal{J}$-trivial if $MxM = MyM$ implies $x = y$.*

For the reader which is more familiar with Cayley graph, this means that the $\mathcal{J}$-sided Cayley graphs has only trivial (i.e. singletons) strongly connected components. Examples of $\mathcal{J}$-trivial monoid of interest for this work include the 0-Hecke algebra for any Coxeter group [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry]. Beware that 1 is the largest element of those (pre)-orders. This is the usual convention in the semigroup community, but is the converse convention from the closely related notions of left and right weak order in a Coxeter group.

Finally, for finite monoids, $\mathcal{R}, \mathcal{L}$ and $\mathcal{J}$ are related as follows:

**Lemma 2.7** ([Pin(2010)] V. Theorem 1.9)**.** *A* finite *monoid is $\mathcal{J}$-trivial if and only if it is both $\mathcal{R}$-trivial and $\mathcal{L}$-trivial.*

## 2.3 Representation theory of $\mathcal{J}$-trivial monoids

The representation theory of $\mathcal{J}$-trivial monoids has been well studied by Denton, Hivert, Schilling and Thiéry [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry]. It turns out that it is combinatorial: more precisely, one can compute the simple, projective modules, the Cartan matrix and even the quiver by computing only in the monoid, without requiring linear combinations. For example, the representation theory of any algebra $A$ is largely governed by its idempotents (elements such that $e^2 = e$). However, when dealing with a finite $\mathcal{J}$-trivial monoid $M$, it is sufficient to look for idempotents in the monoid $M$ itself rather than in its monoid algebra $\mathbb{C}[M]$.

In this subsection, $M$ will always by a finite $\mathcal{J}$-trivial monoid and we will denote by $E(M)$ the set of idempotents of $M$. They parameterize the simple $M$-modules:

**Theorem 2.8** ([Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Proposition 3.1 and 3.3])**.** *There are as many as (isomorphism classes of) simple modules $S_e$ as idempotents $e \in E(M)$, all of dimension 1. Their structure is as follows: $S_e$ is spanned by some vector $\epsilon_e$ with the action of any $m \in M$ given by*

$$m \cdot \epsilon_e = \begin{cases} \epsilon_e & \text{if } me = e, \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

We now describe the structure of the radical. Given $x \in M$, the sequence $(x^i)_{i\in\mathbb{N}}$ is decreasing for the $\mathcal{J}$-order, therefore it must eventually stabilize to an idempotent element which is usually denoted $x^\omega$.

**Theorem 2.9** ([Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Theorem 3.4 and 3.7])**.** *Define a product $\star$ on $E(M)$ by $x \star y := (xy)^\omega$. Then the restriction of $\leq_\mathcal{J}$ to $E(M)$ is a lower semi-lattice such that $x \wedge_\mathcal{J} y = x \star y$ where $x \wedge_\mathcal{J} y$ is the meet of $x$ and $y$. In particular, $(E(M), \star)$ is a commutative monoid.*

*Moreover $(\mathbb{C}[E(M)], \star)$ is isomorphic to $\mathbb{C}[M]/\operatorname{Rad}(\mathbb{C}[M])$ and the mapping $\phi : x \mapsto x^\omega$ is the canonical algebra morphism associated to this quotient.*

Finally, we also describe the projective module: Define

$$\operatorname{rfix}(x) := \min\{e \in E(M) \mid xe = x\}, \qquad \text{and} \qquad \operatorname{lfix}(x) := \min\{e \in E(M) \mid ex = x\}, \tag{2.2}$$

the min being taken for the $\mathcal{J}$-order (which exists according to [Denton et al.(2010/11)Denton, Hivert, Schillin Proposition 3.16]).

**Theorem 2.10** ([Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Theorem 3.23])**.** *For any idempotent $e$ denote by $L(e) := Me$, and set*

$$L_=(e) := \{x \in Me \mid \operatorname{rfix} x = e\} \qquad \textit{and} \qquad L_<(e) := \{x \in Me \mid \operatorname{rfix} x <_\mathcal{L} e\}\,. \tag{2.3}$$

*Then, the projective module $P_e$ associated to $S_e$ is isomorphic to $\mathbb{K}L(e)/\mathbb{K}L_<(e)$. In particular, taking as basis the image of $L_=(e)$ in the quotient, the action of $m \in M$ on $x \in L_=(e)$ is given by: $m \cdot x = mx$ if $\operatorname{rfix}(mx) = e$ and $0$ otherwise.*

Of course the corresponding statement holds on the right. Then [Denton et al.(2010/11)Denton, Hivert, Sc Theorem 3.20] further give a formula for the Cartan invariant matrix: for $i, j \in E(M)$ is given by:

$$c_{i,j} = |\{x \in M \mid i = \operatorname{lfix} x \text{ and } j = \operatorname{rfix} x\}|. \tag{2.4}$$

### 2.4 Descent sets, compositions and ribbons

Before applying the preceding theory to the 0-Hecke monoid, we recall some classical combinatorial ingredient: each subset $S$ of $[\![1, n-1]\!]$ of cardinality $p$ can be uniquely associated with a so called *composition of $n$* of length $p+1$ that is a tuple $I := (i_1, \dots, i_{p+1})$ of positive integers of sum $n$:

$$S = \{s_1 < s_2 < \cdots < s_p\} \longmapsto \mathrm{C}(S) := (s_1, s_2 - s_1, s_3 - s_2, \dots, n - s_p)\,. \tag{2.5}$$

The converse bijection, sending a composition to its *descent set*, is given by:

$$I = (i_1, \dots, i_p) \longmapsto \operatorname{Des}(I) = \{i_1 + \cdots + i_j \mid j = 1, \dots, p-1\}\,. \tag{2.6}$$

we write $I \vDash n$ when $I$ is a composition of $n$ and write $\ell(I)$ the length of $I$. We will sometimes extend this definition to subsets $J \subset [\![0, n-1]\!]$ by prepending a 0 to $\mathrm{C}(S)$ when $0 \in S$.

For instance, the composition $(3, 1, 2, 1, 2, 2) \vDash 11$ corresponds to the subset $\{3, 4, 6, 7, 9\}$ of $[\![0, 10]\!]$ and $(0, 3, 4, 1) \vDash 8$ corresponds to the subset $\{0, 3, 7\}$ of $[\![0, 7]\!]$.

Compositions can be pictured as a ribbon diagram, that is, a set of rows composed of square cells of respective lengths $i_j$, the first cell of each row being attached under the last cell of the previous one. $I$ is called the *shape* of the ribbon diagram. Recall also that the *descent set* $\mathrm{Des}(\sigma)$ of a permutation $\sigma$ is the set of $i$ such that $\sigma(i) > \sigma(i+1)$ (the *descents* of $\sigma$), and the *(right) descent composition* $\mathrm{C}(\sigma)$ of $\sigma$ is the unique composition $I$ of $n$ such that $\mathrm{Des}(I) = \mathrm{Des}(\sigma)$, that is the shape of a filled ribbon diagram whose row reading is $\sigma$ and whose rows are increasing and columns decreasing. For example, Figure 2.1 shows that the descent composition of $(3, 5, 4, 1, 2, 7, 6)$ is $I = (2, 1, 3, 1)$.

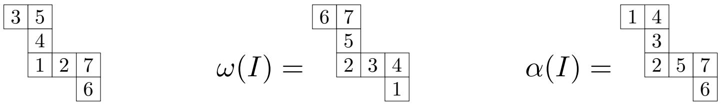



Figure 2.1: The ribbon diagram of the permutation 3541276.

Conversely, with a composition $I$, associate its *maximal permutation* $\sigma = \omega(I)$ as the permutation with descent composition $I$ and maximal inversion number. Similarly, the minimal permutation $\alpha(I)$ is the permutation with descent composition $I$ and minimal inversion number. It is well known that the set of permutations whose descent composition is $I$ is the weak order right interval $[\alpha(I), \omega(I)]$ (see *e.g.* [Krob and Thibon(1997), Lemma 5.2]). For example, if $I = (2, 1, 3, 1)$, $\omega(I) = 6752341$ and $\alpha(I) = 1432576$.

## 2.5 Representation theory of 0-Hecke monoids and algebras

We now shortly explain how the previous theory applies to $H_n^0$. First of all $H_n^0$ is $\mathcal{R}$-trivial, the corresponding order being defined as $\pi_\sigma \leq_{\mathcal{R}} \pi_\mu$ if and only if $\mu \leq_R \sigma$ where $\leq_R$ is the right weak order of the symmetric group. The same holds on the left, and actually $H_n^0$ is isomorphic to its own opposite. Thanks to Lemma 2.7, it is then $\mathcal{J}$-trivial.

For any composition $I = (i_1, \dots, i_p) \vDash n$, we consider the parabolic submonoid $H_I^0$ generated by $\{\pi_i \mid i \in \mathrm{Des}(I)\}$. It is isomorphic to the direct product $H_{i_1}^0 \times H_{i_2}^0 \times \cdots \times H_{i_p}^0$. Each parabolic submonoid contains a unique zero element $\pi_J = \pi_{\omega_J}$ where $\omega_J$ is the maximal element of the parabolic Coxeter subgroup $\mathfrak{S}_J$. The collection $\{\pi_J \mid J \vDash n\}$ is exactly the set of idempotents in $H_n^0$.

Recall that the length $\ell(\sigma)$ of a permutation $\sigma$ is the minimal length of a word in the $(s_i)_i$ whose product is $\sigma$. It is also equal to the number of inversions of $\sigma$. Recall also that such a minimal length word is called *reduced.* The *left and right descent sets* and *content* of $w \in \mathfrak{S}_n$ are respectively defined by:

$$D_L(w) = \{i \in I \mid \ell(s_i w) < \ell(w)\}, \qquad \text{and} \qquad D_R(w) = \{i \in I \mid \ell(w s_i) < \ell(w)\},$$
$$\mathrm{cont}(w) = \{i \in I \mid s_i \text{ appears in some reduced word for } w\},$$

Write $C_L$, $C_r$ and cont the associated compositions. In this last condition "some" may be replaced by "any". Moreover, the above conditions on $s_i w$ and $w s_i$ are respectively equivalent to $\pi_i \pi_w = \pi_w$ and $\pi_w \pi_i = \pi_w$. One has $\mathrm{cont}(\pi_J) = D_L(\omega_J)$, or equivalently $\mathrm{cont}(\pi_J) = D_R(\omega_J)$. Then, for any $\sigma \in \mathfrak{S}_n$, we have $\pi_\sigma^\omega = \pi_{\mathrm{cont}(\sigma)}$, $\mathrm{lfix}\, \pi_\sigma = \pi_{C_L(\sigma)}$, and $\mathrm{rfix}\, \pi_\sigma = \pi_{C_R(\sigma)}$.

The left projective module $P_J$ corresponding to the idempotent $\pi_J$ has its basis $b_w$ indexed by the elements of $w$ having $J$ as right descent composition. The action of $\pi_i$ coincides with the usual left action, except that $\pi_i \cdot b_w = 0$ if $\pi \cdot w$ has a different right descent composition than $w$.

## 2.6 Induction and restriction of $H_n^0$-modules

It is well known that character theory of the family of symmetric groups $(\mathfrak{S}_n)_n$ can be encoded into symmetric functions via the Frobenius isomorphism [Macdonald(1995)]. Under this morphism, irreducible characters $\chi_\lambda$ of $\mathfrak{S}_n$ are mapped to Schur functions $s_\lambda$ of degree $n$, induction and restriction along the natural inclusion $\mathfrak{S}_m \times \mathfrak{S}_n \longrightarrow \mathfrak{S}_{m+n}$ correspond respectively to product and coproduct (the so called Littlewood-Richardson rule) of the Hopf algebra **Sym** of symmetric function.

According to Krob-Thibon [Krob and Thibon(1997), yves Thibon(1998)], this construction has an analogue for the 0-Hecke monoids $(H_n^0)_n$. However, due to non semi-simplicity of $H_n^0$, the situation is a little more complicated. Note that the classical presentation deals with the algebra $H_n(0)$ rather than the monoid. First of all, the maps

$$\rho_{m,n} : \left\{ \begin{array}{ccc} H_m^0 \times H_n^0 & \longrightarrow & H_{m+n}^0 \\ (\pi_i, \pi_j) & \longmapsto & \pi_i \pi_{j+m} = \pi_{j+m} \pi_i \end{array} \right. \tag{2.7}$$

are injective monoid morphisms which moreover verify some associativity conditions endowing $(H_n^0)_n$ with a tower of monoid structure (see [Bergeron and Li(2009), Virmaux(2014)] for a precise definition). One can build two analogues of character rings, namely $\mathcal{G}_0 := \sum_n \mathbb{C}\mathcal{G}_0(H_n^0)$ the direct sum of the (complexified) Grothendieck groups of $H_n^0$-modules on one hand, and $\mathcal{K}_0 := \sum_n \mathbb{C}\mathcal{K}_0(H_n^0)$ the direct sum of the Grothendieck groups of projective $H_n^0$-modules on the other hand. Recall that $\mathcal{G}_0$ is the free $\mathbb{Z}$-module generated by simple module $S_I$, whereas $\mathcal{K}_0$ is the free $\mathbb{Z}$-module generated by the indecomposable projective modules $P_I$.

Now for two integers $m$ and $n$, we denote by $\mathrm{Res}_{m,n}$ the restriction functor from the category of $H_{m+n}^0$-modules to $H_m^0 \times H_n^0$-modules along the morphism $\rho_{m,n}$. It turns out that this defines proper co-products on $\mathcal{G}_0$ and $\mathcal{K}_0$. In particular, $H_{m+n}^0$ is projective over $H_m^0 \times H_n^0$. Dually, the induction $\mathrm{Ind}_{m,n}$ defines products on $\mathcal{G}_0$ and $\mathcal{K}_0$. These products and coproducts are compatible giving the structure of a Hopf algebra. The analogue of Frobenius isomorphism goes as follows: let **QSym** denote Gessel's [Gessel(1984)] Hopf algebra of quasi-symmetric functions, and **NCSF** denote the Hopf algebra of noncommutative symmetric functions [Gelfand et al.(1995)Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thibon]. Recall that these two dual Hopf algebras have their bases indexed by compositions. Then the map sending the simple module $S_I$ to the element $F_I$ of the fundamental basis is a Hopf algebra morphism from $\mathcal{G}_0$ to **QSym**. Dually, the map sending the indecomposable projective module $P_I$ to the so-called ribbon basis element $R_I$ [Gelfand et al.(1995)Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thib Krob and Thibon(1997)] is a Hopf algebra morphism from $\mathcal{K}_0$ to **NCSF**. The duality between **QSym** and **NCSF** mirrors Frobenius duality between $\mathcal{G}_0$ and $\mathcal{K}_0$, the commutative image $c : \mathbf{NCSF} \to \mathbf{QSym}$ being nothing but the Cartan map.

As an illustration, we give the induction rule of indecomposable projective $H_n^0$-modules. For any two compositions $I \vDash m$ and $J \vDash n$:

$$\mathrm{Ind}_{m,n}(P_I \otimes P_J) \simeq P_{I \cdot J} \oplus P_{I \triangleright J}\,, \tag{2.8}$$

where $I \cdot J$ is the concatenation of $I$ and $J$ and $I \triangleright J := (i_1, \dots i_{k-1}, i_k + j_1, j_2, \dots j_\ell)$. For example, $\mathrm{Ind}_{6,7}(P_{(3,1,2)} \otimes P_{(3,2,2)}) = P_{(3,1,2,3,2,2)} \oplus P_{(3,1,5,2,2)}$. This is the same rule as the multiplication rule of the ribbon basis of **NCSF** [Gelfand et al.(1995)Gelfand, Krob, Lascoux, Leclerc, Retakh, and Thil

As already said, the main motivation for the present paper was to understand how this picture translate to rook monoids. Unfortunately, it turns out that everything does not work as nicely as expected, but this may be because we did not choose the right tower of monoids structure.

## 3 The 0-rook monoid

### 3.1 Definition of $R_n^0$ by generators and relations

To define the 0-rook monoid, we take back Halverson's relations (Equations H1 to H3 and RH4 to RH8) and put $q = 0$. In order to get a monoid, we write Equation RH8 as

$$P_{i+1} = P_i T_i P_i + P_i = P_i T_i P_i + P_i P_i = P_i(T_i + 1)P_i\,. \tag{3.1}$$

Setting $\pi_i := T_i + 1$, we finally obtain:

**Definition 3.1.** *We denote by $G_n^0$ the monoid generated by the two families $\pi_1, \ldots, \pi_{n-1}$ and $P_1, \ldots, P_n$ together with relations*

$$\pi_i^2 = \pi_i \qquad 1 \le i \le n-1, \tag{R1}$$
$$\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1} \qquad 1 \le i \le n-2, \tag{R2}$$
$$\pi_i\pi_j = \pi_j\pi_i \qquad |i-j| \ge 2. \tag{R3}$$
$$P_i^2 = P_i \qquad 1 \le i \le n, \tag{R4}$$
$$P_iP_j = P_jP_i \qquad 1 \le i, j \le n, \tag{R5}$$
$$P_i\pi_j = \pi_jP_i \qquad i < j, \tag{R6}$$
$$P_i\pi_j = \pi_jP_i = P_i \qquad j < i, \tag{R7}$$
$$P_{i+1} = P_i\pi_iP_i \qquad 1 \le i < n. \tag{R8}$$

Using Relation R8 we note that it is generated only by $\pi_1, \ldots, \pi_{n-1}$ and $P_1$.

**Notation 3.2.** *To state that two words are equal in $G_n^0$, we rather write explicitely that they are equivalent modulo the relations above as $e \equiv_0 f$.*

We recall here the plan we introduced in the summary. Definition 3.1 introduces a monoid defined by generators and relations. The $G$ stands for "generators". We will later give a definition of the monoid $F_n^0$ (Definition 3.8) as a monoid of operators acting on rooks ($F$ stands for "functions"). We will actually prove in Corollary 3.46 that the two definitions actually coincide. We will then call this monoid the 0-rook monoid, and denote it by $R_n^0$.

We start by focusing on the monoid generated by the $(P_i)$:

**Lemma 3.3.** $P_iP_k \equiv_0 P_{\max(i,k)}$.

*Proof.* Thanks to Relation R5, we may assume that $k \ge i$. Relation R8 shows us that there is a word for $P_k$ beginning with $P_i$. Relation R4 says that $P_i$ is an idempotent. □

**Lemma 3.4.** *The element $P_n$ is the unique zero of the monoid $G_n^0$, that is for any $e \in G_n^0$ then $eP_n \equiv_0 P_ne \equiv_0 P_n$. Furthermore $P_n$ have the two following expressions:*

$$\begin{aligned} P_n &\equiv_0 P_1\pi_1P_1\pi_2\pi_1P_1\pi_3\pi_2\pi_1P_1 \ldots P_1\pi_{n-1}\pi_{n-2}\ldots\pi_1P_1 \\ &\equiv_0 P_1\pi_1\pi_2\ldots\pi_{n-2}\pi_{n-1}P_1 \ldots P_1\pi_1\pi_2\pi_3P_1\pi_1\pi_2P_1\pi_1P_1. \end{aligned} \tag{3.2}$$

*Proof.* We prove this by induction on $n \ge 1$. It is obvious that $P_2 \equiv_0 P_1\pi_1P_1$ by Relation R8. To show that $P_2$ is a zero, it is enough to prove that the generators $\pi_1$ et $P_1$ stabilize it. It is clear for $P_1$ which is idempotent, and $\pi_1P_1\pi_1P_1 \equiv_0 \pi_1P_2 \equiv_0 P_2$ by the Relation R7.

Assume that the result is proven for all $1 \leq k \leq n$. Let us prove it for $n+1$:

$$\begin{aligned} P_{n+1} \equiv_0 P_n\pi_nP_n &\equiv_0 P_n\pi_nP_{n-1}\pi_{n-1}\pi_{n-2}\dots\pi_3\pi_2\pi_1P_1 \text{ (by induction)} \\ &\equiv_0 P_nP_{n-1}\pi_n\pi_{n-1}\pi_{n-2}\dots\pi_3\pi_2\pi_1P_1 \text{ (by R6)} \\ &\equiv_0 P_n\pi_n\pi_{n-1}\pi_{n-2}\dots\pi_3\pi_2\pi_1P_1 \text{ (by Lemma 3.3).} \end{aligned}$$

Thus the result holds. Since all the relations are symmetric, we get the other formula.

To show that $P_{n+1}$ is a zero we prove that it is stabilized under multiplication by any generator among $\pi_1, \dots, \pi_n, P_1$. The stability by $P_1$ is obvious by Lemma 3.3. For all the others, we deduce from Relation R7 that $\pi_iP_n \equiv_0 P_n$ since $i \leq n-1$.

Finally, the uniqueness of the zero holds in any semigroup. □

**Corollary 3.5.** *In the presentation of $G_n^0$ one can replace the Relations R4, R5, R6 and R7 by the following three and still get the same monoid:*

$$P_1^2 = P_1\,, \tag{R4.1}$$

$$P_1\pi_j = \pi_jP_1 \qquad\qquad j \neq 1, \tag{R5.1}$$

$$\pi_1P_1\pi_1P_1 = P_1\pi_1P_1 = P_1\pi_1P_1\pi_1\,. \tag{R6.1}$$

*In particular the monoid $G_n^0$ is generated by $(\pi_i)_{1\leq i\leq n-1}$ and $P_1$ subject to Relations R1 to R3 and R4.1 to R6.1; Relation R8 being seen as a definition for $P_i$ for $i > 1$.*

*Proof.* Deducing Relations R5.1 and R6.1 from Relations R1 to R8 is obvious: Relation R6.1 is only Relation R7 applied with $i = 2$ and $j = 1$.

Let us prove the converse: Relations R1 to R8 can be deduced from Relations R1 to R4, R5.1, R6.1 and R8 seen as a definition. We will now prove that Lemma 3.3 and Lemma 3.4 (and Relation R4) are still true. We prove simultaneously by induction on $n$ the following statements

- for all $k \leq n$, the element $P_k$ is given by the relation of Lemma 3.4.
- for all $i, k \leq n$, then $P_k^2 \equiv_0 P_k$ and $P_iP_k \equiv_0 P_{\max(i,k)}$.

The case $n = 1$ is obvious with Relation R4.

We now assume the statements for $n \geq 1$. We only have to prove that two words for $P_{n+1}$ are given by Lemma 3.4, that $P_{n+1}^2 \equiv_0 P_{n+1}$ and that $\forall i \leq n+1$, $P_{n+1}P_i \equiv_0 P_{n+1}$.

Regarding the words for $P_{n+1}$, a close look to the proof of Lemma 3.4 shows that we use only Relation R6.1 (for the basis step), Relation R6 when $i < j \leq n$, Relation R4 when $i \leq n$ and Lemma 3.3 for $i, k \leq n$. But all these relations have already been proved by induction. Consequently we have the two expressions for $P_{n+1}$.

From there, the relation $P_iP_{n+1} \equiv_0 P_{n+1}P_i \equiv_0 P_{n+1}$ for $i \leq n$ is clear using these words and the fact that $P_i^2 = P_i$. It remains only to prove that $P_{n+1}$ is idempotent.

$$\begin{aligned} P_{n+1}^2 &\equiv_0 P_1\pi_1\pi_2\dots\pi_{n-1}\pi_nP_1\dots P_1\pi_1\pi_2P_1\pi_1P_1 \cdot P_1\pi_1P_1\pi_2\pi_1P_1\dots P_1\pi_n\pi_{n-1}\dots\pi_1P_1 \\ &\equiv_0 P_1\pi_1\pi_2\dots\pi_{n-1}\pi_nP_1P_nP_n\pi_n\pi_{n-1}\dots\pi_2\pi_1P_1 \\ &\equiv_0 P_1\pi_1\pi_2\dots\pi_{n-1}\pi_nP_1P_n\pi_n\pi_{n-1}\dots\pi_2\pi_1P_1\,, \end{aligned}$$

by induction. Now using R3 and R5.1:

$$P_{n+1}^2 \equiv_0 P_1\pi_1\pi_2\dots\pi_{n-1}\pi_nP_1\pi_1\pi_2\dots\pi_{n-1}\pi_nP_1\dots P_1\pi_1\pi_2\pi_3P_1\pi_1\pi_2P_1\pi_1P_1\,. \tag{*}$$

Now, call $\rho$, the first part of the previous calculation:

$$\rho := P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_n P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_n\,.$$

Then

$$\begin{aligned}
\rho &\equiv_0 P_1\pi_1\pi_2P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_n\pi_2\pi_3\ldots\pi_{n-2}\pi_{n-1} && \text{(by R2, R3 and R5.1)}\\
&\equiv_0 P_1\pi_1P_1\pi_2\pi_1\pi_2\ldots\pi_{n-1}\pi_n\pi_2\pi_3\ldots\pi_{n-2}\pi_{n-1} && \text{(by R5.1)}\\
&\equiv_0 P_1\pi_1P_1\pi_1\pi_2\pi_1\pi_3\ldots\pi_{n-1}\pi_n\pi_2\pi_3\ldots\pi_{n-2}\pi_{n-1} && \text{(by R2)}\\
&\equiv_0 P_1\pi_1P_1\pi_1\pi_2\pi_3\ldots\pi_{n-1}\pi_n\pi_1\pi_2\pi_3\ldots\pi_{n-2}\pi_{n-1} && \text{(by R3)}\\
&\equiv_0 P_1\pi_1P_1\pi_2\pi_3\ldots\pi_{n-1}\pi_n\pi_1\pi_2\pi_3\ldots\pi_{n-2}\pi_{n-1} && \text{(by R6.1)}\\
&\equiv_0 P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_nP_1\pi_1\pi_2\ldots\pi_{n-2}\pi_{n-1} && \text{(by R5.1)}.
\end{aligned}$$

Taking back Relation (*) we thus have:

$$\begin{aligned}
P_{n+1}^2 \equiv_0\ & P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_n\\
&\quad P_1\pi_1\pi_2\ldots\pi_{n-2}\pi_{n-1}P_1\pi_1\pi_2\ldots\pi_{n-2}\pi_{n-1}P_1\ldots P_1\pi_1\pi_2\pi_3P_1\pi_1\pi_2P_1\pi_1P_1\,.
\end{aligned}$$

We recognize the end of the left term to be Equation * for $n$ instead of $n+1$. Thus:

$$P_{n+1}^2 \equiv_0 P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_nP_nP_n \equiv_0 P_1\pi_1\pi_2\ldots\pi_{n-1}\pi_nP_n \equiv_0 P_{n+1}\,.$$

Finally we have proved that the statement holds for $n+1$: indeed, we have thus Relations R1 to R4 and the two Lemmas 3.3 and 3.4. Relation R5 follows directly from Lemma 3.3, and Relation R6 can be deduced from Lemma 3.4 using R5.1 and R3.

It remains to prove R7 using only R6.1 and Lemma 3.4. Since Lemma 3.4 and all the relations are symmetric, we only need to show that $\pi_jP_i \equiv_0 P_i$ for $j < i$, the proof of the other case could be conducted the same way.

For $j = 1$ and $i = 2$ it is exactly Relation R6.1. For $j = 1$ without condition on $i$, it comes from the fact that, because of Lemma 3.4, a word for $P_i$ begin with $P_1\pi_1P_1$, and we conclude with R6.1.

Otherwise, for $j \geq 2$ and $i > j$, we get:

$$\pi_jP_i \equiv_0 \pi_jP_1\pi_1P_1\pi_2\pi_1P_1\ldots P_1\pi_{j-1}\pi_{j-2}\ldots\pi_2\pi_1P_1\pi_j\pi_{j-1}\ldots\pi_2\pi_1P_1\ldots P_1\pi_{i-1}\pi_{i-2}\ldots\pi_1P_1$$

with R3 and R5.1:

$$\begin{aligned}
&\equiv_0 P_1\pi_1P_1\pi_2\pi_1P_1\ldots P_1\pi_j\pi_{j-1}\pi_{j-2}\ldots\pi_2\pi_1P_1\pi_j\pi_{j-1}\ldots\pi_2\pi_1P_1\ldots P_1\pi_{i-1}\pi_{i-2}\ldots\pi_1P_1\\
&\equiv_0 P_1\pi_1P_1\pi_2\pi_1P_1\ldots P_1\pi_{i-1}\pi_{i-2}\ldots\pi_1P_1 = P_i \text{ (with } \rho).
\end{aligned}$$

Hence the result. □

We finally get a new shorter presentation for $G_n^0$, by setting $\pi_0 := P_1$.

**Corollary 3.6.** *The monoid $G_n^0$ is generated by $\pi_0,\ldots,\pi_{n-1}$ subject to the relations:*

$$\begin{aligned}
\pi_i^2 &= \pi_i && 0\leq i\leq n-1, && \text{(RB1)}\\
\pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} && 1\leq i\leq n-2, && \text{(RB2)}\\
\pi_1\pi_0\pi_1\pi_0 &= \pi_0\pi_1\pi_0 = \pi_0\pi_1\pi_0\pi_1\,, && && \text{(RB3)}\\
\pi_i\pi_j &= \pi_j\pi_i && 0\leq i,j\leq n-1, \quad |i-j|\geq 2. && \text{(RB4)}
\end{aligned}$$

*Proof.* It is obvious from Corollary 3.5 by letting $\pi_0 = P_1$. □

**Remark 3.7.** We can see that $G_n^0$ is a quotient of the Hecke monoid of type $B_n$ at $q = 0$ (see [Fayers(2005)] for a study of the representation theory of it).

### 3.2 Definition by action and $R$-codes

The goal of this section is to construct a bijection between $R_n$ and $R_n^0$ which generalizes the bijection between $\mathfrak{S}_n$ and $H_n^0$. In the case of permutations, one argues using Matsumoto's theorem: recall that it says that two reduced words (words of minimal length) generate the same permutation if and only if they are congruent using only braid Relations M2, M3 and not the quadratic one. Then, for a permutation $\sigma$, one defines $\pi_\sigma := \pi_{i_1} \dots \pi_{i_k}$ where $s_{i_1} \dots s_{i_k}$ is any reduced word for $\sigma$. Thanks to Matsumoto's theorem the result is independent of the choice of the reduced word. One concludes that $H_n^0$ is nothing but the set $\{\pi_\sigma \mid \sigma \in \mathfrak{S}_n\}$ and therefore of cardinality $n!$. The same argument is in fact valid for the algebras $H_n(q)$ and is often found in this case in the literature.

Unfortunately, as noticed by Solomon [Solomon(2004), p. 209, bottom of the middle paragraph], such a theorem is not known for the rook monoid. So we choose a different path (see the discussion in the outline) effectively ending up proving the generalization of Matsumoto's theorem. We introduce another monoid defined in term of a faithful action of $R_n^0$ on $R_n$. It will turns out (Corollary 3.49) that this action is nothing but the right multiplication.

**Definition 3.8.** *We denote $F_n^1$ the submonoid of the monoid of functions on $R_n$ generated by $s_1, \dots, s_{n-1}, P_1$ acting on $R_n$ by right multiplication of matrices. Namely, if $(r_1, \dots, r_n)$ is a rook then:*

$$(r_1 \dots r_n) \cdot s_k = r_1 r_2 \dots r_{k-1} r_{k+1} r_k r_{k+2} \dots r_n \,, \tag{3.3}$$
$$(r_1 \dots r_n) \cdot P_1 = 0\, r_2 \dots r_n \,. \tag{3.4}$$

*We denote $F_n^0$ the submonoid of the monoid of functions generated by $\pi_1, \dots, \pi_{n-1}, P_1$ acting on $R_n$ by the action:*

$$(r_1 \dots r_n) \cdot \pi_k := \begin{cases} (r_1 \dots r_n) \cdot s_k & \textit{if } r_k < r_{k+1}, \\ (r_1 \dots r_n) & \textit{otherwise.} \end{cases} \tag{3.5}$$

**Remark 3.9.** A simple calculation shows that the generators of $F_n^0$ satisfy the Relations R1 to R3 and R4.1 to R6.1. Similarly, the generators of $F_n^1$ satisfy

$$s_i^2 = 1 \qquad 1 \le i \le n-1, \tag{Rs1}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad 1 \le i \le n-2, \tag{Rs2}$$
$$s_i s_j = s_j s_i \qquad |i-j| \ge 2. \tag{Rs3}$$
$$P_1^2 = P_1, \tag{Rs4.1}$$
$$P_1 s_j = s_j P_1 \qquad j \ne 1 \tag{Rs5.1}$$
$$s_1 P_1 s_1 P_1 = P_1 s_1 P_1 = P_1 s_1 P_1 s_1. \tag{Rs6.1}$$

We denote by $G_n^1$ the monoid generated by $\{s_1, \dots, s_{n-1}, P_1\}$ with the relations above. We can rephrase Remark 3.9 as follows: there are two surjective morphisms of monoids:

$$\Phi_1 : G_n^1 \twoheadrightarrow F_n^1 \qquad \text{and} \qquad \Phi_0 : G_n^0 \twoheadrightarrow F_n^0. \tag{3.6}$$

Furthermore, these two morphisms give us an action of $G_n^1$ and $G_n^0$ over $R_n$.

**Remark 3.10.** The map $(r_1 r_2) \mapsto (r_1 0)$ is equal to the composition $s_1 P_1 s_1$ and therefore belongs to $F_2^1$. However, it can be checked that it does not belong to $F_2^0$, neither to its algebra $\mathbb{C}[F_2^0]$. More generally, in $F_n^0$, for any subset $I \subset [\![1, n]\!]$ which is not of the form $[\![1, k]\!]$ the maps replacing the letter in position $i$ by 0, does not belong to $F_n^0$ or $\mathbb{C}[F_n^0]$.

Our goal is now to show that $\Phi_1$ and $\Phi_0$ are actually isomorphisms.

### 3.2.1 $R$-code and rooks

In this subsection, we build a combinatorial tool, namely the *R-code*, which allows us to define for any rook a canonical reduced word. A classical way to do that for permutations is to proceed by induction along the chain of inclusions $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots \subset \mathfrak{S}_{n-1} \subset \mathfrak{S}_n \subset \dots$ noticing that the number of cosets in $\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n$ is exactly $n$. One can for example take $\{1, s_{n-1}, s_{n-1}s_{n-2}, s_{n-1}s_{n-2}s_{n-3}, \dots\}$ as a cross-section. In a more combinatorial setting, this is equivalent to say that given a permutation $\sigma \in \mathfrak{S}_{n-1}$ there are exactly $n$ permutations which give back $\sigma$ when erasing the letter $n$. Therefore any permutation can be encoded by a sequence $\mathsf{c} = (\mathsf{c}_1 \dots \mathsf{c}_n)$ satisfying $0 \le \mathsf{c}_i < i$. This can be done by the *Lehmer code* ([Lothaire(2002), Page 330]) of the permutation, or a variant of thereof. See Remark 3.15 for a definition of the Lehmer code and how it relates to our generalized $R$-code.

The case of rooks is more involved because some times $n$ does not appear in the rook vector and to go from $R_n$ to $R_{n-1}$ one has to erase a 0. It turns out that the right choice to minimize the number of moves (since we are looking for a reduced word) is to remove the *first* 0. However, this means that, given a rook $r$ of size $n-1$, the number of rook of size $n$ which give back $r$ depends on $r$ and more precisely on the position of its first 0. We now unravel the corresponding combinatorics, starting with some notations:

**Notation 3.11** (Word and Letter)**.** *The length of a word $w$ is denoted by $\ell(w)$. The empty word (the only word of length 0) will be denoted by $\varepsilon$. When we need to distinguish between words and letters (for example when matching a word), we use the convention that words will be underlined as in $\underline{w}$, while $i$ will rather be a single letter. If the letter $i \in \mathbb{Z}$ appears in the word $\underline{w}$ we write it $i \in \underline{w}$; it means for example that $\underline{w}$ can be written as $\underline{w} = \underline{a}i\underline{b}$.*

**Definition 3.12.** *For a rook $r$ of length $n$, we call the* code of $r$ *and denote* $\mathrm{code}(r)$ *the word on $\mathbb{Z}$ of length $n$ defined recursively by:*

1. *If $n = 0$ then $\mathrm{code}(\varepsilon) := \varepsilon$.*
2. *Otherwise, if $n \in r$, then $r$ can be written uniquely $r = \underline{b}n\underline{e}$. Let $r' := \underline{be}$ (the subword of $r$ where the unique occurrence of $n$ is removed). Then $\mathrm{code}(r) := \mathrm{code}(r') \cdot (\ell(\underline{b}) + 1)$.*
3. *Otherwise, $n \notin r$, and therefore $r$ can be written uniquely $r = \underline{b}0\underline{e}$ with $0 \notin \underline{b}$. Let $r' := \underline{be}$ (the subword of $r$ where the first 0 is removed). Then $\mathrm{code}(r) := \mathrm{code}(r') \cdot (-\ell(\underline{b}))$.*

**Notation 3.13.** *When writing a code, $\bar{i}$ stands for $-i$ for $i \in \mathbb{N}$.*

**Example 3.14.** Let $r = 02401$. Then:

$$\mathrm{code}(02401) = \mathrm{code}(2401)\mathsf{0} = \mathrm{code}(201)\mathsf{20} = \mathrm{code}(21)\bar{\mathsf{1}}\mathsf{20} = \mathrm{code}(1)\mathsf{1}\bar{\mathsf{1}}\mathsf{20} = \mathsf{11}\bar{\mathsf{1}}\mathsf{20}.$$

An easy remark is that $r$ is a permutation if and only if its code contains only positive letters.

**Remark 3.15.** Recall that the Lehmer code [Lothaire(2002), Page 330] of a permutation is defined by

$$\mathrm{Lehmer}(\sigma) = \mathsf{c}_1 \dots \mathsf{c}_n \qquad \text{with} \qquad \mathsf{c}_i := |\{j > i \mid \sigma(i) > \sigma(j)\}| \,. \tag{3.7}$$

When $r$ is actually a permutation $\sigma$, the codes are related as follows: write the code as $\mathrm{code}(\sigma) = r_1 \dots r_n$ and the Lehmer code as $\mathrm{Lehmer}(\sigma) = \mathsf{c}_1 \dots \mathsf{c}_n$. Then $\mathsf{c}_i = \sigma(i) - r_{\sigma(i)}$. For example taking $\sigma = 516432$, then $\mathrm{code}(\sigma) = 122213$ and $\mathrm{Lehmer}(\sigma) = 403210$.

We now describe a subset $C_n$ of $\mathbb{Z}^n$ that we call the set of $R$-codes. We will see in Proposition 3.22 and Theorem 3.27 that it is exactly the set of codes of a rook.

**Definition 3.16.** *To each word $\underline{w}$ over $\mathbb{Z}$, we associate a nonnegative number $m(\underline{w})$ defined recursively by: $m(\varepsilon) = 0$ and for any word $\underline{w}$ and any letter $d$,*

$$m(\underline{w}d) := \begin{cases} -d & \text{if } d \le 0\,, \\ m(\underline{w}) + 1 & \text{if } 0 < d \le m(\underline{w}) + 1\,, \\ m(\underline{w}) & \text{if } d > m(\underline{w}) + 1\,. \end{cases} \tag{3.8}$$

*A word on $\mathbb{Z}$ is an $R$-*code *if it can be obtained by the following recursive construction: the empty word $\varepsilon$ is a code, and $\underline{w}d$ is a code if $\underline{w}$ is a code and $-m(\underline{w}) \le d \le n$. We denote by $C_n$ the set of $R$-codes of size $n$.*

**Notation 3.17.** *In order to make the difference between the rook* 1234 *and the code* $\mathsf{1234}$*, we make the convention to write codes in sans-serif font.*

**Example 3.18.** $m(\mathsf{12836427}) = 5$: there is no negative letter, thus it only increments on integers 1, 2, 3, 4 and 2 in this order. $m(\mathsf{364\bar{4}294\bar{3}52538}) = 6$. Indeed, the last negative letter is $-3$, thus $m(\mathsf{364\bar{4}294\bar{3}}) = 3$ and it increments on letters 2, 5 and 3 in this order. Similarly, $m(\mathsf{021\bar{1}1254}) = 4$.

**Example 3.19.** Here are the first $R$-codes: $C_1 = \{\mathsf{0}, \mathsf{1}\}$, $\mathsf{C}_2 = \{\mathsf{00}, \mathsf{01}, \mathsf{02}, \mathsf{1\bar{1}}, \mathsf{10}, \mathsf{11}, \mathsf{12}\}$ and

$$\begin{aligned} C_3 = \{&\mathsf{000}, \mathsf{001}, \mathsf{002}, \mathsf{003}, \mathsf{01\bar{1}}, \mathsf{010}, \mathsf{011}, \mathsf{012}, \mathsf{013}, \mathsf{020}, \mathsf{021}, \mathsf{022}, \mathsf{023}, \mathsf{1\bar{1}\bar{1}}, \mathsf{1\bar{1}0}, \mathsf{1\bar{1}1}, \mathsf{1\bar{1}2}, \\ &\mathsf{1\bar{1}3}, \mathsf{100}, \mathsf{101}, \mathsf{102}, \mathsf{103}, \mathsf{11\bar{2}}, \mathsf{11\bar{1}}, \mathsf{110}, \mathsf{111}, \mathsf{112}, \mathsf{113}, \mathsf{12\bar{2}}, \mathsf{12\bar{1}}, \mathsf{120}, \mathsf{121}, \mathsf{122}, \mathsf{123}\}\,. \end{aligned}$$

The $R$-codes of $C_9$ with prefix $\mathsf{021\bar{1}1254}$ are $\mathsf{021\bar{1}1254\bar{4}}$ , $\mathsf{021\bar{1}1254\bar{3}}$ , $\dots$, $\mathsf{021\bar{1}12549}$.

**Remark 3.20.** If $\mathsf{c} \in C_n$, then necessarily we have $m(\mathsf{c}) \le \ell(\mathsf{c})$.

**Definition 3.21.** *We note* FZ *(standing for First Zero) the function defined for any rook $r = r_1 \dots r_n$ by*

$$\mathrm{FZ}(r) := \min\{j \le n \,|\, r_j = 0\} - 1\,, \tag{3.9}$$

*with the convention that if there is no zero among the $r_j$ (that is $r$ is in fact a permutation), we set $\mathrm{FZ}(r) = n$.*

We now show that code is a bijection between $R$-codes and rook vectors of the same length.

**Proposition 3.22.** *If $r \in R_n$ then* $\mathrm{code}(r) \in C_n$ *and* $\mathrm{FZ}(r) = m(\mathrm{code}(r))$.

*Proof.* We show the result by induction on $n$: it is trivial for $n = 0$. We now show the induction step, assuming that it holds for $n-1$. Let $r \in R_n$. Let us first prove the case $n \in r$. We then write $r = \underline{b}n\underline{e}$ and $r' = \underline{b}\underline{e}$. By induction $\mathrm{code}(r') \in C_{n-1}$ and $\mathrm{code}(r) = \mathrm{code}(r') \cdot (\ell(\underline{b}) + 1)$ with $(\ell(\underline{b}) + 1) \in [\![1, n]\!] \subset [\![-m(\mathrm{code}(r')), n]\!]$ so that $r \in R_n$.

The only remaining case is $n \notin r$. We write $r = \underline{b}0\underline{e}$ with $0 \notin \underline{b}$, $r' = \underline{b}\underline{e}$. By induction $\mathrm{code}(r') \in C_{n-1}$ and $\mathrm{code}(r) = \mathrm{code}(r') \cdot -\ell(\underline{b})$. By definition of FZ we have $\ell(\underline{b}) = \mathrm{FZ}(r')$, and $\mathrm{FZ}(r') = m(\mathrm{code}(r'))$ by induction. So $-\ell(\underline{b}) \in [\![-m(\mathrm{code}(r')), 0]\!] \subset [\![-m(\mathrm{code}(r')), n]\!]$ and so $r \in R_n$.

We have proven the first part of the statement in every case. Let us now focus on the second part. First of all, if $0 \notin r$, then $r$ is a permutation and its code $\mathsf{c}_1 \ldots \mathsf{c}_n$ is such that $0 < \mathsf{c}_i \leq i$. As a consequence $m(\mathrm{code}(r)) = n = \mathrm{FZ}(r)$.

We finally need to prove that when $0 \in r$ then $\mathrm{FZ}(r) = m(\mathrm{code}(r))$, knowing by induction that $\mathrm{FZ}(r') = m(\mathrm{code}(r'))$. We distinguish the two nontrivial cases:

- If $n \in r$ then $r = \underline{b}n\underline{e}$ and $r' = \underline{b}\underline{e}$. The number of 0 of $r$ is the same that $r'$. We have two possibilities:
  - If $0 \notin \underline{b}$ then the first zero of $r'$ is in $\underline{e}$. Thus $\mathrm{FZ}(r) = \mathrm{FZ}(r') + 1$. But also $\mathrm{code}(r) = \mathrm{code}(r') \cdot (\ell(\underline{b}) + 1)$ with $\ell(\underline{b}) + 1 \leq m(\mathrm{code}(r')) = \mathrm{FZ}(r')$. So, by definition of $m$, $m(\mathrm{code}(r)) = m(\mathrm{code}(r')) + 1$. Hence the equality.
  - If $0 \in \underline{b}$ then $\mathrm{FZ}(r) = \mathrm{FZ}(r')$. Furthermore $m(\mathrm{code}(r)) = m(\mathrm{code}(r'))$ by definition of $m$. So that we get $\mathrm{FZ}(r) = \mathrm{FZ}(r') = m(\mathrm{code}(r')) = m(\mathrm{code}(r))$.
- If $n \notin r$, then $r = \underline{b}0\underline{e}$ with $0 \notin \underline{b}$, $r' = \underline{b}\underline{e}$ and $\mathrm{code}(r) = \mathrm{code}(r') \cdot -\ell(\underline{b})$. Since $0 \notin \underline{b}$ we have $\mathrm{FZ}(r) = \ell(\underline{b})$. We write $\mathrm{code}(r) = \mathsf{c}_1 \ldots \mathsf{c}_n$ then $\mathrm{FZ}(r) = -\mathsf{c}_n$ by definition of code. Furthermore $m(\mathrm{code}(r)) = -\mathsf{c}_n$ so that $\mathrm{FZ}(r) = m(\mathrm{code}(r))$. □

We now define a candidate for the converse bijection.

**Definition 3.23.** *For $\mathsf{c} = \mathsf{c}_1 \ldots \mathsf{c}_n \in C_n$, we define inductively a vector* $\mathrm{decode}(\mathsf{c})$ *as follows: first, set* $\mathrm{decode}(\varepsilon) := \varepsilon$. *Then, let* $r' := \mathrm{decode}(\mathsf{c}_1 \ldots \mathsf{c}_{n-1})$. *If $\mathsf{c}_n$ is nonnegative, insert the letter $n$ in $r'$ at the position $\mathsf{c}_n$. Otherwise insert* $0$ *at* $-\mathsf{c}_n + 1$.

**Proposition 3.24.** *If* $\mathsf{c} \in C_n$ *then* $\mathrm{decode}(\mathsf{c}) \in R_n$.

*Proof.* It is clear that we get a rook, since only 0 can be repeated. The size is also clear. □

**Example 3.25.** Let $\mathsf{c} = 11\bar{1}20$. Then $\mathrm{decode}(1) = 1$. $\mathrm{decode}(11) = 21$. $\mathrm{decode}(11\bar{1}) = 201$. $\mathrm{decode}(11\bar{1}2) = 2401$. Finally $\mathrm{decode}(11\bar{1}20) = 02401$.

**Proposition 3.26.** *Let* $\mathsf{c} = \mathsf{c}_1 \ldots \mathsf{c}_n \in C_n$. *Then* $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = m(\mathsf{c})$. *In particular, if* $\mathsf{c}_n \leq 0$, $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = -\mathsf{c}_n$.

*Proof.* We prove it by induction on $n$. The assertion is clear for words of length 0. Otherwise, assume that we have proved the result for all words of length strictly less than $n$. Let $\mathsf{b} := \mathsf{c}_1 \ldots \mathsf{c}_{n-1}$.

- If $\mathsf{c}_n > 0$: by induction $\mathrm{FZ}(\mathrm{decode}(\mathsf{b})) = m(\mathsf{b})$. But $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = \mathrm{FZ}(\mathrm{decode}(\mathsf{b})) + 1$ if $\mathsf{c}_n \leq \mathrm{FZ}(\mathrm{decode}(\mathsf{b})) + 1$ and $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = \mathrm{FZ}(\mathrm{decode}(\mathsf{b}))$ otherwise. By definition of function $m$ we get $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = m(\mathsf{c})$.

- If $\mathsf{c}_n \leq 0$ we have two possibilities:
  - If $\forall i \leq n-1$, $\mathsf{c}_i > 0$ then $0 \notin \mathrm{decode}(\mathsf{b})$ by definition, and so $\mathrm{decode}(\mathsf{c})$ has a single zero which is the one inserted between $\mathrm{decode}(\mathsf{b})$ and $\mathrm{decode}(\mathsf{c})$, and is thus at position $(-\mathsf{c}_n + 1) - 1 = m(\mathsf{c})$.
  - Otherwise, by induction $m(\mathsf{b}) = \mathrm{FZ}(\mathrm{decode}(\mathsf{b}))$. By definition of $m$, $m(\mathsf{c}) = -\mathsf{c}_n$. By definition of $R$-codes we get $-\mathsf{c}_n \leq m(\mathsf{b}) = \mathrm{FZ}(\mathrm{decode}(\mathsf{b}))$. Thus the zero inserted at position $-\mathsf{c}_n + 1$ is left to the former first zero.
    Finally $\mathrm{FZ}(\mathrm{decode}(\mathsf{c})) = -\mathsf{c}_n = m(\mathsf{c})$. □

**Theorem 3.27.** *The functions* code *and* decode *are inverse one from the other: for all* $\mathsf{c} \in C_n$ *and* $r \in R_n$ *then*

$$\mathrm{code}(\mathrm{decode}(\mathsf{c})) = \mathsf{c} \qquad \text{and} \qquad \mathrm{decode}(\mathrm{code}(r)) = r. \tag{3.10}$$

*Proof.* We proceed by induction on the size $n$ of $r$ and $\mathsf{c}$. The result is clear if $n = 0$. Assume now that we have proved the result up to $n-1$. We begin with rooks. Let $r \in R_n$.

- If $n \in r$, write $r = \underline{b}n\underline{e}$ and $r' = \underline{b}\underline{e}$ with $\mathrm{decode}(\mathrm{code}(r')) = r'$ by induction. Since $\mathrm{code}(r) = \mathrm{code}(r') \cdot (\ell(\underline{b}) + 1)$, $\mathrm{code}(r)$ is the word $\mathrm{code}(r')$ with the position of $n$ as final letter. Since $\mathrm{decode}(\mathrm{code}(r))$ inserts in $\mathrm{decode}(\mathrm{code}(r')) = r'$ the $n$ at this position, we have the result.

- Otherwise $\mathrm{code}(r)$ is the word $\mathrm{code}(r')$ with at the end the opposite of the position minus 1 of the first zero of $r$. But $\mathrm{decode}(\mathrm{code}(r))$ insert a zero in $\mathrm{decode}(\mathrm{code}(r')) = r'$ at this position.

We now do the proof for $R$-codes in a similar way: Let $\mathsf{c} = \mathsf{c}_1 \dots \mathsf{c}_n \in C_n$ and $\mathsf{c}' = \mathsf{c}_1 \dots \mathsf{c}_{n-1}$, and assume that $\mathrm{code}(\mathrm{decode}(\mathsf{c}')) = \mathsf{c}'$.

- If $c_n > 0$ then $\mathrm{decode}(\mathsf{c})$ inserts in $\mathrm{decode}(\mathsf{c}')$ a letter $n$ at position $c_n$. Computing further $\mathrm{code}(\mathrm{decode}(\mathsf{c}))$ adds at the end of $\mathrm{code}(\mathrm{decode}(\mathsf{c}')) = \mathsf{c}'$ this position.

- Otherwise, $\mathrm{decode}(\mathsf{c})$ insert in $\mathrm{decode}(\mathsf{c}')$ a letter 0 in position $-c_n + 1$. Since it is the first zero of $\mathrm{decode}(\mathsf{c})$ by Proposition 3.26, $\mathrm{code}(\mathrm{decode}(\mathsf{c}))$ add $c_n$ at the end of $\mathrm{code}(\mathrm{decode}(\mathsf{c}')) = \mathsf{c}'$. □

In particular, there are as many $R$-codes of size $n$ as rooks:

**Corollary 3.28.** *For all* $n$*:* $|C_n| = |R_n|$.

### 3.2.2 Counting rook according to the position of the first 0

This subsection is a little detour through enumerative combinatorics and permutations statistics. It is interesting to count rooks of size $n$ according to the position of the first zero. We denote $R(n,k) := \{r \in R_n \mid \mathrm{FZ}(r) = k\}$ and $r(n,k) := |R(n,k)|$. Here are the first values:

| $n/k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | |
| 1 | 1 | 1 | | | | | | |
| 2 | 3 | 2 | 2 | | | | | |
| 3 | 13 | 9 | 6 | 6 | | | | |
| 4 | 73 | 52 | 36 | 24 | 24 | | | |
| 5 | 501 | 365 | 260 | 180 | 120 | 120 | | |
| 6 | 4051 | 3006 | 2190 | 1560 | 1080 | 720 | 720 | |
| 7 | 37633 | 28357 | 21042 | 15330 | 10920 | 7560 | 5040 | 5040 |

For example, here are the rooks of size 2 sorted according to their first zero:

$$R(2,0) = \{00, 01, 02\}, \qquad R(2,1) = \{10, 20\}, \qquad R(2,2) = \{12, 21\}$$

**Lemma 3.29.** *The sequence $r(n,k)$ verifies the following recurrence relation for $n > 0$:*

$$r(n,k) = k\, r(n-1,k-1) + (n-k-1)r(n-1,k) + \sum_{i=k}^{n} r(n-1,i)\,, \tag{3.11}$$

*with the convention that $r(n,k) = 0$ if $k < 0$ or $k > n$.*

*Proof.* To get the set of rooks of size $n$ from the set of rooks of size $n-1$, one has either to insert $n$ or to insert a 0. To make sure to get each rook only once, one has to insert 0 only before the first zero. According to the definition of FZ, in what follows, positions are counted starting with 0. Then

- $k \cdot r(n-1,k-1)$ is the number of rooks where $n$ is (and therefore was inserted) before position FZ.
- $k(n-k-1)r(n-1,k)$ is the number of rooks where $n$ is after the first 0.
- $\sum_{i=k}^{n} r(n-1,i)$ is the number of rooks where $n$ does not appear. They are obtained by inserting a 0 in position $k$, in a rook $r$ such that $i := \mathrm{FZ}(r) \geq k$. □

One recognizes the triangle A206703 of [Sloane(2015)]. It is defined as the number $C(n,k)$ of the injective partial function on $[\![1,n]\!]$ where the union the cycle supports has cardinality $k$. Recall that a rook vector $r = (r_1, \dots, r_n)$ can been seen as an injective partial function by setting $r(i) = r_i$ if $r_i \neq 0$ and $r(i)$ is undefined otherwise. We consider the generalization of the notion of cycle of permutations to rooks (See [Flajolet and Sedgewick(2009), Example II.21, page 132]), this combinatorics was studied in details in [Ganyushkin and Mazorchuk(2006)]): the sequence of the iterated images $(r^n(i))_{n\in\mathbb{N}}$ of some integer $i$ under $r$ can have one of the two following behaviors:

- Either for some $n \geq 1$ one has $r^n(i) = i$ (the sequence must be periodic and not only ultimately periodic because of injectivity). We say that $i$ belongs to a *cycle* of $r$.
- Or starting from some $n \geq 1$ the iterated image $r^n(i)$ stops being defined; we say that $i$ belongs to a *chain* of $r$.

Rooks can therefore be decomposed as two sets: the set of its cycles (counting fixed points) and the set of its maximal chains, that is maximal finite sequences $(c_1, \dots, c_k)$ such that $r(c_i) = c_{i+1}$ if $i < k$ and undefined otherwise. Clearly, the *supports* of the cycles and the chains of the rook $r$ form a partition of $[\![1, n]\!]$.

**Example 3.30.** Consider the rook vector $r = 205109706$, it corresponds to the function

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & \perp & 5 & 1 & \perp & 9 & 7 & \perp & 6 \end{pmatrix},$$

where $\perp$ means undefined. It has two cycles $(6, 9)$ and $(7)$ and three maximal chains $(4, 1, 2)$, $(3, 5)$ and $(8)$.

**Proposition 3.31.** *Let $C(n, k)$ be the set of rooks of size $n$ where the union of cycle supports has cardinality $k$, and denote by $c(n, k)$ its cardinality. Then $c(n, k) = r(n, k)$ for all $k$ and $n$.*

We show here the rooks of size 2 sorted according to their number of points in a cycle:

$$C(2, 0) = \{00, 01, 20\}, \qquad C(2, 1) = \{10, 02\}, \qquad C(2, 2) = \{12, 21\}.$$

*Proof.* We define a bijection $\Phi$ from $C(n, k)$ to $R(n, k)$. It is an adaptation of Foata fundamental transformation (See [Lothaire(2002), Chapter. 10]). For $r \in C(n, k)$, write its cycles starting from the smallest elements and sort the set of cycles according to their smallest element in decreasing order. By concatenating those words one obtains a first word $\mathrm{CycleW}(r)$. Second, write the maximal chain backward replacing the last element of the chain (now the first of the word) by a 0 and sort the chains according to their last element in increasing order. By concatenating those words one obtains a second word $\mathrm{ChainW}(r)$. Now define $\Phi(r) := \mathrm{CycleW}(r) \cdot \mathrm{ChainW}(r)$. Then $\Phi(r)$ is a rook of size $n$ whose first zero is in position $k$, so that $\Phi(r) \in R(n, k)$.

We now explain how to recover $r$ from $s := \Phi(r)$, that is the converse bijection: cut $s$ at the places just before the zeros replacing those zeros by the values missing in $s$ in increasing order. The various words obtained except the first one are the (reversed) chains of $r$. One recover the cycle of $r$ by cutting the first word before the *lower records* (elements that are only preceded by larger ones) and interpret each part as a cycle. Knowing all the chains and cycles of $r$ is sufficient to recover $r$. □

**Example 3.32.** We get back to Example 3.30. The rook vector $r = 205109706$ has cycles $(6, 9)$ and $(7)$ and chains $(4, 1, 2)$, $(3, 5)$ and $(8)$. Therefore $\mathrm{CycleW}(r) = 769$ and $\mathrm{ChainW}(r) = 014030$, so that $\Phi(r) = 769014030$.

To demonstrate the computation of the inverse, we start with 769014030. The missing numbers are $\{2, 5, 8\}$. Replacing the zeros by them and cutting gives $769|214|53|8$. So that we already got the chains $(4, 1, 2)$, $(3, 5)$ and $(8)$. Now the word 769 is cut as $7|69$ recovering the cycles.

Using the so-called symbolic method (See [Flajolet and Sedgewick(2009), Example II.21, page 132]), the decomposition by cycles and chains shows that the generating series is given by

$$\sum_{n,k} r(n, k) \, \frac{x^n y^k}{n!} = \frac{\exp(x/(1-x))}{1 - xy}. \tag{3.12}$$

### 3.3 Equivalence of the definitions of $R_n^0$

We now get back to the 0-rook monoid. Thanks to the previously defined $R$-code, we are now in position to define the canonical reduced word $\pi_{\mathsf{c}}$ associated to a $R$-code and thus to a rook. To define $\pi_{\mathsf{c}}$, the following notation is handy:

**Notation 3.33.** *For $i, n \in \mathbb{N}$ we write (with $\pi_0 := P_1$):*

$$\boxed{\begin{smallmatrix} n \\ \vdots \\ i \end{smallmatrix}} := \begin{cases} 1 & \text{if } i > n, \\ \pi_n \dots \pi_i & \text{if } 0 \le i \le n, \\ \pi_n \dots \pi_1 \pi_0 \pi_1 \dots \pi_i & \text{if } i < 0, \end{cases} \quad \text{and} \quad \boxed{\boxed{\begin{smallmatrix} n \\ \vdots \\ i \end{smallmatrix}}} := \begin{cases} 1 & \text{if } i > n, \\ s_n \dots s_i & \text{if } 0 \le i \le n, \\ s_n \dots s_1 \pi_0 s_1 \dots s_i & \text{if } i < 0. \end{cases}$$

*A priori* $\boxed{\begin{smallmatrix} n \\ \vdots \\ i \end{smallmatrix}} \in G_n^0$ and $\boxed{\boxed{\begin{smallmatrix} n \\ \vdots \\ i \end{smallmatrix}}} \in G_n^1$. Using $\Phi_0$ and $\Phi_1$ of Remark 3.9 we will sometimes see them as elements of $F_n^0$ or $F_n^1$.

**Definition 3.34.** *For any $R$-code $\mathsf{c} = \mathsf{c}_1 \dots \mathsf{c}_n \in C_n$, we define $\pi_{\mathsf{c}} \in G_n^0$ and $s_{\mathsf{c}} \in G_n^1$ by*

$$\pi_{\mathsf{c}} := \boxed{\begin{smallmatrix} 0 \\ \vdots \\ \mathsf{c}_1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 1 \\ \vdots \\ \mathsf{c}_2 \end{smallmatrix}} \cdot \dots \cdot \boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}, \qquad \text{and} \qquad s_{\mathsf{c}} := \boxed{\boxed{\begin{smallmatrix} 0 \\ \vdots \\ \mathsf{c}_1 \end{smallmatrix}}} \cdot \boxed{\boxed{\begin{smallmatrix} 1 \\ \vdots \\ \mathsf{c}_2 \end{smallmatrix}}} \cdot \dots \cdot \boxed{\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}}. \tag{3.13}$$

**Example 3.35.** Let $\mathsf{c} = 11\bar{1}20$. Then:

$$\pi_{\mathsf{c}} = \boxed{\begin{smallmatrix} 0 \\ \vdots \\ 1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 1 \\ \vdots \\ 1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 2 \\ \vdots \\ -1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} = 1 \cdot \pi_1 \cdot \pi_2\pi_1\pi_0\pi_1 \cdot \pi_3\pi_2 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0\,.$$

Going further, let us show how $\pi_{\mathsf{c}}$ acts on the identity rook 12345:

$$\begin{aligned} 12345 \cdot \pi_c = 12345 \cdot 1 \cdot \pi_1 \cdot \boxed{\begin{smallmatrix} 2 \\ \vdots \\ -1 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} &= 21\mathbf{3}45 \cdot \pi_2\pi_1\pi_0\pi_1 \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} = \\ \mathbf{2}3145 \cdot \pi_1\pi_0\pi_1 \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} = \mathbf{3}2145 \cdot \pi_0\pi_1 \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} &= \mathbf{0}2145 \cdot \pi_1 \cdot \boxed{\begin{smallmatrix} 3 \\ \vdots \\ 2 \end{smallmatrix}} \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} = \\ 2\mathbf{0}145 \cdot \pi_3\pi_2 \cdot \boxed{\begin{smallmatrix} 4 \\ \vdots \\ 0 \end{smallmatrix}} &= 24015 \cdot \pi_4\pi_3\pi_2\pi_1\pi_0 = 02401 = \operatorname{decode}(\mathsf{c})\,. \end{aligned}$$

We see that the $i$-th column of $\pi_{\mathsf{c}}$ places the letter $i$ (or the corresponding zero), at its place, effectively decoding $\mathsf{c}$. This is actually a general fact and it is also true replacing $\pi_i$ by $s_i$:

**Proposition 3.36.** *If $r \in R_n$ then $1_n \cdot \pi_{\operatorname{code}(r)} = 1_n \cdot s_{\operatorname{code}(r)} = r$.*

*Proof.* We will prove it by induction on $n$. It is evident for $n = 0$. Assume that we have proved the result up to step $n-1$, and let $r \in R_n$.

If $n \in r$ then $r$ writes $r = \underline{b}n\underline{e}$, $r' = \underline{b}\underline{e}$ and $\operatorname{code}(r) = \operatorname{code}(r') \cdot (\ell(\underline{b}) + 1)$. By definition we have $\pi_{\operatorname{code}(r)} = \pi_{\operatorname{code}(r')} \boxed{\begin{smallmatrix} n \\ \vdots \\ \ell(\underline{b})+1 \end{smallmatrix}}$. By induction $1_{n-1} \cdot \pi_{\operatorname{code}(r')} = r'$. So $1_n \cdot \pi_{\operatorname{code}(r')} = r'n = \underline{b}\underline{e}n$, since $\pi_{\operatorname{code}(r')}$ only acts on the first $n-1$ coordinates. Since $0 < \ell(\underline{b}) + 1 \le n$, a direct calculation gives us $\underline{b}\underline{e}n \cdot \boxed{\begin{smallmatrix} n \\ \vdots \\ \ell(\underline{b})+1 \end{smallmatrix}} = \underline{b}n\underline{e} = r$. So $1_n \cdot \pi_{\operatorname{code}(r)} = r$.

Otherwise $n \notin r$. Then $r$ writes $r = \underline{b}0\underline{e}$ with $0 \notin \underline{b}$, $r' = \underline{b}\underline{e}$ and $\mathrm{code}(r) = \mathrm{code}(r') \cdot -\ell(\underline{b})$. We get in the exact same way $1_n \cdot \pi_{\mathrm{code}(r')} = r'n = \underline{b}\underline{e}n$. Since $-\ell(\underline{b}) \leq 0$, a simple calculation gives us $\underline{b}\underline{e}n \cdot \boxed{\begin{smallmatrix} n \\ \vdots \\ -\ell(\underline{b}) \end{smallmatrix}} = \underline{b}0\underline{e} = r$. So $1_n \cdot \pi_{\mathrm{code}(r)} = r$.

The same proof works *mutatis mutandis* for $s$. ☐

**Corollary 3.37.** *For all $n$, $|G_n^0| \geq |F_n^0| \geq |R_n| = |C_n|$ et $|G_n^1| \geq |F_n^1| \geq |R_n| = |C_n|$.*

*Proof.* All the functions $\pi_{\mathrm{code}(r)}$ and $s_{\mathrm{code}(r)}$ for $r \in R_n$ are distinct since they have a distinct action on identity $1_n$. We conclude with Corollary 3.28 and Remark 3.9. ☐

The next step is to transfer on $R$-codes the action on rooks:

**Definition 3.38.** *For $\mathsf{c} = \mathsf{c}_1 \ldots \mathsf{c}_n \in C_n$ and $t \in \{\pi_0, \pi_1, \ldots, \pi_{n-1}\} \subset G_n^0$ we define $\mathsf{c} \cdot t$ recursively the following way:*

- *If $n = 1$ and $t = \pi_0$ then $\mathsf{c} \cdot t := \mathsf{0}$.*

*Otherwise we proceed by induction depending on the sign of $c_n$ and the value of $t$:*

*Pos. If $\mathsf{c}_n = i \geq 1$:*

*a. If $t = \pi_i$ then $\mathsf{c} \cdot t := \mathsf{c}$.*

*b. If $t = \pi_{i-1}$ then $\mathsf{c} \cdot t := \mathsf{c}_1 \ldots \mathsf{c}_{n-1}(\mathsf{c}_n - 1)$.*

*c. If $t = \pi_j$ with $j < i - 1$ then $\mathsf{c} \cdot t := [(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_j]\, \mathsf{c}_n$.*

*d. If $t = \pi_j$ with $j > i$ then $\mathsf{c} \cdot t := [(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_{j-1}]\, \mathsf{c}_n$.*

*Neg. If $\mathsf{c}_n = -i \leq 0$:*

*a. If $t = \pi_i$ then $\mathsf{c} \cdot t := \mathsf{c}$.*

*b. If $t = \pi_j$ with $0 < j < i$ then $\mathsf{c} \cdot t := [(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_j]\, \mathsf{c}_n$.*

*c. If $t = \pi_j$ with $j > i + 1$ then $\mathsf{c} \cdot t := [(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_{j-1}]\, \mathsf{c}_n$.*

*d. If $t = \pi_0$ then $\mathsf{c} \cdot t := [(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_0 \ldots \pi_{i-1}]\, \mathsf{0}$. (In particular $\mathsf{c} \cdot t = \mathsf{c}$ if $i = 0$.)*

*e. If $t = \pi_{i+1}$ (thus $i \neq n$) we have two possibilities:*

*$\alpha$. If $m(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) = i$ then $\mathsf{c} \cdot t := \mathsf{c}$.*

*$\beta$. Otherwise $\mathsf{c} \cdot t := \mathsf{c}_1 \ldots \mathsf{c}_{n-1}\ \overline{i+1}$.*

**Lemma 3.39.** *For any code $\mathsf{c} = \mathsf{c}_1 \ldots \mathsf{c}_n \in C_n$ and generator $t \in \{\pi_0, \pi_1, \ldots, \pi_{n-1}\} \subset G_n^0$, then $\mathsf{c} \cdot t$ is a code of size $n$.*

*Proof.* We will prove the result by induction on $n$, and we will prove along the way that $m(\mathsf{c} \cdot t) \geq m(\mathsf{c})$ if $t \neq \pi_0$. It is evident if $n = 1$.

For all subcases of case Pos. of Definition 3.38 it is evident that we get a code by induction since the last value is positive which do not lead to difficulties (we add to $\mathsf{c}_1 \ldots \mathsf{c}_{n-1}$ either $c_n$ or $c_n - 1$). The property of function $m$ is clear for subcase a. In b. if $i - 1 \neq 0$ then $c_n - 1 > 0$ so $m(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}(\mathsf{c}_n - 1)) \geq m(\mathsf{c})$. In c. the induction gives us $m((\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_j) \geq m(\mathsf{c}_1 \ldots \mathsf{c}_{n-1})$ and we conclude with the definition of $m$ to get $m([(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}) \cdot \pi_j]\, \mathsf{c}_n) \geq m(\mathsf{c}_1 \ldots \mathsf{c}_{n-1}\mathsf{c}_n)$ (we do the same for d.).

The subcase Neg.a. is clear. We prove subcases Neg.b. and Neg.c. using the induction on the condition of $m$ and the fact that in these two subcases $m(\mathsf{c}\cdot t) = \mathsf{c}_n = m(\mathsf{c})$. The subcase Neg.d. is clear by induction (we do not have to prove the condition of $m$ here), as subcase Neg.e.$\alpha$. The subcase Neg.e.$\beta$ remains, whose condition gives us $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) > i$ (since $\mathsf{c} \in C_n$) so $\mathsf{c}\cdot t \in C_n$ and $m(\mathsf{c}\cdot t) = i+1 > m(\mathsf{c}) = i$. $\square$

It therefore makes sense to apply the decode algorithm to $\mathsf{c}\cdot t$. The crucial fact that motivated the definition of the action on a code is that, forall $R$-code $\mathsf{c}$

$$\operatorname{decode}(\mathsf{c}\cdot t) = \operatorname{decode}(\mathsf{c})\cdot t\,. \tag{3.14}$$

We could prove this fact right away, by a tedious explicit calculation, distinguishing all cases. We urge the reader who want to understand the motivation of Definition 3.38 to do so. For example, in case Neg.e.$\alpha$, the assumption that $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) = i = -\mathsf{c}_n$ shows that, using Proposition 3.26, $\mathrm{FZ}(\operatorname{decode}(\mathsf{c}_1 \dots \mathsf{c}_{n-1})) = i$. Therefore $\operatorname{decode}(\mathsf{c}_1 \dots \mathsf{c}_{n-1})$ is of the form

$$\operatorname{decode}(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) = r_1 \dots r_i 0 r_{i+2} \dots r_{n-1} n\,,$$

where none of the $r_j$ for $j \le i$ vanish. Decoding further, since $\mathsf{c}_n = -i$, on finds that

$$\operatorname{decode}(\mathsf{c}_1 \dots \mathsf{c}_n) = r_1 \dots r_i 00 r_{i+2} \dots r_{n-1}\,.$$

So that, $\operatorname{decode}(\mathsf{c})\cdot \pi_{i+1} = \operatorname{decode}(\mathsf{c})$. That's why, in case Neg.e.$\alpha$, we defined $\mathsf{c}\cdot\pi_{i+1} \coloneqq \mathsf{c}$. Instead of doing the proof in all other cases, we will get the properties as a corollary of the much stronger fact that $\pi_{\mathsf{c}\cdot t} \equiv_0 \pi_{\mathsf{c}} t$ using the morphism $\Phi_0 : G_n^0 \twoheadrightarrow F_n^0$.

We turn now to the proof of that later statement. It will use intensively the following technical lemma:

**Lemma 3.40.** *If $i > 0$, $k < 0$ and $j < i-1$ we have the following identities:*

$$\pi_j \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}} = \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}}\,\pi_j \quad \textit{if} \quad 0 < j < |k| \qquad \textit{and} \qquad \pi_j \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}} = \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}}\,\pi_{j+1} \quad \textit{if} \quad j > |k|. \tag{3.15}$$

*In particular, by immediate induction:*

$$\boxed{\begin{matrix} j \\ \vdots \\ l \end{matrix}} \cdot \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}} = \boxed{\begin{matrix} i \\ \vdots \\ k \end{matrix}} \cdot \boxed{\begin{matrix} j \\ \vdots \\ l \end{matrix}} \qquad \textit{if } 0 < l \le j < \min(i, |k|). \tag{3.16}$$

*Proof.* We will only use relations (RB1 to RB4) of Remark 3.9 written according to Corollary 3.6. For the first equality we just apply successively in this order RB4, RB2, RB4, RB2 and RB4. For the second we only apply RB4, RB2 and RB4. $\square$

We may now proceed to the main theorem of this section:

**Theorem 3.41.** *For a code $\mathsf{c} = \mathsf{c}_1 \dots \mathsf{c}_n \in C_n$ and a generator $t \in \{\pi_0, \pi_1, \dots, \pi_{n-1}\} \subset G_n^0$, the congruence $\pi_{\mathsf{c}\cdot t} \equiv_0 \pi_{\mathsf{c}} t$ holds. Furthermore $\ell(\pi_{\mathsf{c}\cdot t}) \le \ell(\pi_{\mathsf{c}}) + 1$.*

*Proof.* We will only use the relations of the proof of Lemma 3.40. We then prove the theorem by induction on $n$ depending on $c_n$ and $t$. The remark on the length can be checked systematically in all the cases, we left it to the reader.

If $n = 1$ and $t = \pi_0$ then $\mathsf{c}\cdot t = 0$. Then $\pi_{\mathsf{c}\cdot t} = \pi_0 = \pi_c t$ by RB1.

Otherwise we write $\mathsf{c}' \coloneqq \mathsf{c}_1 \dots \mathsf{c}_{n-1}$ and we recall that $\pi_{\mathsf{c}} = \pi_{\mathsf{c}'} \boxed{\begin{matrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{matrix}}$.

Pos. $\mathsf{c}_n = i \geq 1$

a. If $t = \pi_i$ then $\mathsf{c} \cdot t = \mathsf{c}$. Then $\pi_{\mathsf{c}'}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}t = \pi_{\mathsf{c}'}\pi_{n-1}\dots\pi_i\pi_i \equiv_0 \pi_{\mathsf{c}'}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}$ by RB1.

b. If $t = \pi_{i-1}$ then $\mathsf{c} \cdot t = \mathsf{c}'(\mathsf{c}_n - 1)$. The relation is just $\pi_{\mathsf{c}'}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}\pi_{i-1} = \pi_{\mathsf{c}'}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n-1 \end{smallmatrix}}$.

c. If $t = \pi_j$ with $j < i-1$ then $\mathsf{c} \cdot t = (\mathsf{c}' \cdot \pi_j)\mathsf{c}_n$. Then

$$\pi_{\mathsf{c}}\,t = \pi_{\mathsf{c}'}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}\pi_j \equiv_0 \pi_{\mathsf{c}'}\pi_j\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}} \equiv_0 \pi_{\mathsf{c}'\cdot\pi_j}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}} = \pi_{(\mathsf{c}'\cdot\pi_j)\mathsf{c}_n} = \pi_{\mathsf{c}\cdot t}.$$

Indeed, the first congruency is Lemma 3.40, and the second holds by induction.

d. If $t = \pi_j$ with $j > i$ then $\mathsf{c} \cdot t = (\mathsf{c}' \cdot \pi_{j-1})\mathsf{c}_n$. We do the same than in Pos.c. using this time Relation RB2 and Relation RB4.

Neg. $\mathsf{c}_n = -i \leq 0$

a. If $t = \pi_i$ we do the same than in Pos.a. with RB1.

b. If $t = \pi_j$ with $0 < j < i$ we do the same than in Pos.c. with RB4.

c. If $t = \pi_j$ with $j > i+1$ we do the same than in Pos.d. with RB2 and RB4.

d. If $t = \pi_0$ $(i \neq 0)$ then $\mathsf{c} \cdot t = [(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) \cdot \pi_0 \dots \pi_{i-1}]\,0$. Furthermore

$$\begin{aligned}
\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}}\pi_0 = {} & \pi_{n-1}\dots\pi_2\pi_1\pi_0\pi_1\pi_2\dots\pi_i\pi_0 & \\
\equiv_0 {} & \pi_{n-1}\dots\pi_2\pi_1\pi_0\pi_1\pi_0\pi_2\dots\pi_i & \text{by RB4} \\
\equiv_0 {} & \pi_{n-1}\dots\pi_2\pi_0\pi_1\pi_0\pi_2\dots\pi_i & \text{by RB3} \\
\equiv_0 {} & \pi_0\pi_{n-1}\dots\pi_2\pi_1\pi_0\pi_2\dots\pi_i = \pi_0\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ 0 \end{smallmatrix}}\pi_2\dots\pi_i & \text{by RB4.}
\end{aligned}$$

Now using iteratively Lemma 3.40, one gets

$$\pi_0\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ 0 \end{smallmatrix}}\pi_2\dots\pi_i \equiv_0 \pi_0\pi_1\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ 0 \end{smallmatrix}}\pi_3\dots\pi_i \equiv_0 \dots \equiv_0 \pi_0\dots\pi_{i-1}\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ 0 \end{smallmatrix}}. \tag{3.17}$$

Thus $\pi_{\mathsf{c}}\pi_0 \;\equiv_0\; \pi_{\mathsf{c}'}\,(\pi_0\dots\pi_{i-1})\boxed{\begin{smallmatrix} n-1 \\ \vdots \\ 0 \end{smallmatrix}} \;\equiv_0\; \pi_{\mathsf{c}'\cdot(\pi_0\dots\pi_{i-1})0} \;=\; \pi_{\mathsf{c}\cdot\pi_0}$.

e. If $t = \pi_{i+1}$ (so $i \neq n$) we have two possibilities:

$\alpha$. Either $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) = i$;

$\beta$. Or $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) \neq i$. In this second case $\mathsf{c} \cdot t = \mathsf{c}'\,\overline{i+1}$, and we proceeds as in case Pos.b.

The last remaining case is then $\mathsf{c}_n = -i \leq 0$ with $t = \pi_{i+1}$ and $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) = i$. In this case we have $\mathsf{c} \cdot t = \mathsf{c}$.

Let $k$ be the index of the last non-positive $c_k \leq 0$. Since, by hypothesis, $m(\mathsf{c}_1 \dots \mathsf{c}_{n-1}) = i$, there are $i - |\mathsf{c}_k| = i + \mathsf{c}_k$ further indexes where the value of $m$ increase, we write them as $k < j_1 < \dots < j_{i+\mathsf{c}_k} < n$. In other words, these are the steps of the inductive construction of decode($\mathsf{c}$) where the value of FZ change. For each such index $j_u$, we split the columns of the corresponding decoded word into two parts as

$$\boxed{\begin{matrix} j_u-1 \\ \vdots \\ \mathsf{c}_{j_u} \end{matrix}} = \boxed{\begin{matrix} j_u-1 \\ \vdots \\ |\mathsf{c}_k|+u+1 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+u \\ \vdots \\ \mathsf{c}_{j_u} \end{matrix}}. \tag{3.18}$$

For the other indexes not belonging to the $j_u$, we consider them as first parts, leaving their second parts empty. Thanks to Lemma 3.40, all the second parts commute with the first parts on their right so that:

$$\begin{aligned}
\pi_{\mathsf{c}}t = \; & \pi_{\mathsf{c}_1\ldots\mathsf{c}_{k-1}} \boxed{\begin{matrix} k-1 \\ \vdots \\ \mathsf{c}_k \end{matrix}} \ldots \boxed{\begin{matrix} j_1-1 \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} j_2-1 \\ \vdots \\ \mathsf{c}_{j_2} \end{matrix}} \ldots \boxed{\begin{matrix} j_{i+\mathsf{c}_k}-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \ldots \boxed{\begin{matrix} n-1 \\ \vdots \\ -i \end{matrix}} \pi_{i+1} \\
= \; & \pi_{\mathsf{c}_1\ldots\mathsf{c}_{k-1}} \boxed{\begin{matrix} k-1 \\ \vdots \\ \mathsf{c}_k \end{matrix}} \ldots \boxed{\begin{matrix} j_1-1 \\ \vdots \\ |\mathsf{c}_k|+2 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+1 \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} j_2-1 \\ \vdots \\ |\mathsf{c}_k|+3 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+2 \\ \vdots \\ \mathsf{c}_{j_2} \end{matrix}} \ldots \boxed{\begin{matrix} j_{i+\mathsf{c}_k}-1 \\ \vdots \\ |\mathsf{c}_k|+i+\mathsf{c}_k+1 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+i+\mathsf{c}_k \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \ldots \boxed{\begin{matrix} n-1 \\ \vdots \\ -i-1 \end{matrix}} \\
\equiv_0 \; & \pi_{\mathsf{c}_1\ldots\mathsf{c}_{k-1}} \boxed{\begin{matrix} k-1 \\ \vdots \\ \mathsf{c}_k \end{matrix}} \ldots \boxed{\begin{matrix} j_1-1 \\ \vdots \\ |\mathsf{c}_k|+2 \end{matrix}} \ldots \boxed{\begin{matrix} j_2-1 \\ \vdots \\ |\mathsf{c}_k|+3 \end{matrix}} \ldots \boxed{\begin{matrix} j_{i+\mathsf{c}_k}-1 \\ \vdots \\ i+1 \end{matrix}} \ldots \boxed{\begin{matrix} n-1 \\ \vdots \\ -i-1 \end{matrix}} \cdot \boxed{\begin{matrix} |\mathsf{c}_k|+1 \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+2 \\ \vdots \\ \mathsf{c}_{j_2} \end{matrix}} \ldots \boxed{\begin{matrix} i \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}}.
\end{aligned}$$

We similarly further split the column $\boxed{\begin{matrix} k-1 \\ \vdots \\ \mathsf{c}_k \end{matrix}}$ into its negative and positive part, and commute the negative part as

$$\equiv_0 \pi_{\mathsf{c}_1\ldots\mathsf{c}_{k-1}} \boxed{\begin{matrix} k-1 \\ \vdots \\ 1 \end{matrix}} \ldots \boxed{\begin{matrix} j_1-1 \\ \vdots \\ |\mathsf{c}_k|+2 \end{matrix}} \ldots \boxed{\begin{matrix} j_{i+\mathsf{c}_k}-1 \\ \vdots \\ i+1 \end{matrix}} \ldots \pi_0\pi_1\ldots\pi_{|\mathsf{c}_k|} \boxed{\begin{matrix} n-1 \\ \vdots \\ -i-1 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+1 \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}}.$$

We now focus on the product of the the second parts which we call $S$. Using RB4, and striping the second parts from their topmost element, we get:

$$\begin{aligned}
S := \; & \pi_0\pi_1\ldots\pi_{|\mathsf{c}_k|} \boxed{\begin{matrix} n-1 \\ \vdots \\ -i-1 \end{matrix}}\boxed{\begin{matrix} |\mathsf{c}_k|+1 \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \\
\equiv_0 \; & \pi_0 \boxed{\begin{matrix} n-1 \\ \vdots \\ -i-1 \end{matrix}} \pi_1\ldots\pi_{|\mathsf{c}_k|}\pi_{|\mathsf{c}_k|+1}\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \\
\equiv_0 \; & \pi_0 \boxed{\begin{matrix} n-1 \\ \vdots \\ 2 \end{matrix}} \pi_1\pi_0\pi_1\ldots\pi_i\pi_{i+1}\pi_1\ldots\pi_{|\mathsf{c}_k|}\pi_{|\mathsf{c}_k|+1}\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \\
\equiv_0 \; & \boxed{\begin{matrix} n-1 \\ \vdots \\ 2 \end{matrix}} \pi_0\pi_1\pi_0\pi_1\ldots\pi_i\pi_{i+1}\pi_1\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}}.
\end{aligned}$$

We can now use RB3 and redistribute the colors:

$$\equiv_0 \boxed{\begin{matrix} n-1 \\ \vdots \\ 2 \end{matrix}} \pi_0\pi_1\pi_0\pi_2\ldots\pi_{|\mathsf{c}_k|+1}\pi_{|\mathsf{c}_k|+2}\ldots\pi_{i+1}\pi_1\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}}.$$

Now thanks to Lemma 3.40:

$$\begin{aligned}
\equiv_0 \; & \pi_0 \boxed{\begin{matrix} n-1 \\ \vdots \\ 1 \end{matrix}} \pi_2\ldots\pi_{|\mathsf{c}_k|+1}\pi_{|\mathsf{c}_k|+2}\ldots\pi_{i+1}\pi_0\pi_1\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}} \\
\equiv_0 \; & \pi_0\pi_1\ldots\pi_{|\mathsf{c}_k|}\pi_{|\mathsf{c}_k|+1}\ldots\pi_i \boxed{\begin{matrix} n-1 \\ \vdots \\ 1 \end{matrix}} \pi_0\pi_1\ldots\pi_i \boxed{\begin{matrix} |\mathsf{c}_k| \\ \vdots \\ \mathsf{c}_{j_1} \end{matrix}} \ldots \boxed{\begin{matrix} i-1 \\ \vdots \\ \mathsf{c}_{j_{i+\mathsf{c}_k}} \end{matrix}}
\end{aligned}$$

$$= \pi_0\pi_1\dots\pi_{|\mathsf{c}_k|}\pi_{|\mathsf{c}_k|+1}\dots\pi_i \boxed{\substack{n-1\\ \vdots\\ -i-1}}\ \boxed{\substack{|\mathsf{c}_k|\\ \vdots\\ \mathsf{c}_{j_1}}} \cdots \boxed{\substack{i-1\\ \vdots\\ \mathsf{c}_{j_{i+\mathsf{c}_k}}}}.$$

Going back to the main computation we can undo the splitting of Equation 3.18:

$$\begin{aligned}
\pi_{\mathsf{c}}t &\equiv_0 \pi_{\mathsf{c}_1\dots\mathsf{c}_{k-1}} \boxed{\substack{k-1\\ \vdots\\ 1}} \cdots \boxed{\substack{j_1-1\\ \vdots\\ |\mathsf{c}_k|+2}} \cdots \boxed{\substack{j_{i+\mathsf{c}_k}-1\\ \vdots\\ i+1}}\ \pi_0\pi_1\dots\pi_{|\mathsf{c}_k|}\pi_{|\mathsf{c}_k|+1}\dots\pi_i \boxed{\substack{n-1\\ \vdots\\ -i-1}}\ \boxed{\substack{|\mathsf{c}_k|\\ \vdots\\ \mathsf{c}_{j_1}}} \cdots \boxed{\substack{i-1\\ \vdots\\ \mathsf{c}_{j_{i+\mathsf{c}_k}}}}\\
&\equiv_0 \pi_{\mathsf{c}_1\dots\mathsf{c}_{k-1}} \boxed{\substack{k-1\\ \vdots\\ \mathsf{c}_k}} \cdots \boxed{\substack{j_1-1\\ \vdots\\ |\mathsf{c}_k|+1}} \cdots \boxed{\substack{j_{i+\mathsf{c}_k}-1\\ \vdots\\ i}} \cdots \boxed{\substack{n-1\\ \vdots\\ -i}} \cdot \boxed{\substack{|\mathsf{c}_k|\\ \vdots\\ \mathsf{c}_{j_1}}} \cdots \boxed{\substack{i-1\\ \vdots\\ \mathsf{c}_{j_{i+\mathsf{c}_k}}}} \quad \text{by RB4}\\
&\equiv_0 \pi_{\mathsf{c}_1\dots\mathsf{c}_{k-1}} \boxed{\substack{k-1\\ \vdots\\ \mathsf{c}_k}} \cdots \boxed{\substack{j_1-1\\ \vdots\\ \mathsf{c}_{j_1}}} \cdots \boxed{\substack{j_{i+\mathsf{c}_k}-1\\ \vdots\\ \mathsf{c}_{j_{i+\mathsf{c}_k}}}} \cdots \boxed{\substack{n-1\\ \vdots\\ -i}} \quad \text{by Lemma 3.40.}
\end{aligned}$$

So that we have proved that $\pi_{\mathsf{c}}t = \pi_{\mathsf{c}}$ in the last remaining case.

As told at the beginning of the proof, the remark on the length has been checked through all cases. ☐

**Example 3.42.** Since this last calculation is huge using specific notations, we now give an explicit example of calculation in case Neg.e.$\alpha$. We take $\mathsf{c} = 1234\bar{2}126\bar{4}$. Then, with $t = \pi_5$:

$$\begin{aligned}
\pi_{\mathsf{c}}t &= \boxed{\substack{4\\ \vdots\\ -2}} \cdot \boxed{\substack{5\\ \vdots\\ 1}} \cdot \boxed{\substack{6\\ \vdots\\ 2}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ -4}}\ \pi_5\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ -2}} \cdot \boxed{\substack{5\\ \vdots\\ 4}} \cdot \boxed{\substack{6\\ \vdots\\ 5}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ -5}} \cdot \boxed{\substack{3\\ \vdots\\ 1}} \cdot \boxed{\substack{4\\ \vdots\\ 2}} \quad \text{by RB4 and Lemma 3.40}\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ 0}} \cdot \boxed{\substack{5\\ \vdots\\ 4}} \cdot \boxed{\substack{6\\ \vdots\\ 5}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ -5}}\ \pi_1\pi_2\pi_3\pi_4 \cdot \boxed{\substack{2\\ \vdots\\ 1}} \cdot \boxed{\substack{3\\ \vdots\\ 2}} \quad \text{by RB4}\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ 1}} \cdot \boxed{\substack{5\\ \vdots\\ 4}} \cdot \boxed{\substack{6\\ \vdots\\ 5}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ 2}}\ \pi_0\pi_1\pi_0\pi_1\pi_2\pi_3\pi_4\pi_5\pi_1\pi_2\pi_3\pi_4 \cdot \boxed{\substack{2\\ \vdots\\ 1}} \cdot \boxed{\substack{3\\ \vdots\\ 2}} \quad \text{by RB4}\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ 1}} \cdot \boxed{\substack{5\\ \vdots\\ 4}} \cdot \boxed{\substack{6\\ \vdots\\ 5}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ 2}}\ \pi_0\pi_1\pi_0\pi_2\pi_3\pi_4\pi_5\pi_1\pi_2\pi_3\pi_4 \cdot \boxed{\substack{2\\ \vdots\\ 1}} \cdot \boxed{\substack{3\\ \vdots\\ 2}} \quad \text{by RB3 and redistributing.}\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ 1}} \cdot \boxed{\substack{5\\ \vdots\\ 4}} \cdot \boxed{\substack{6\\ \vdots\\ 5}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \pi_0\pi_1\pi_2\pi_3\pi_4 \boxed{\substack{8\\ \vdots\\ 2}}\ \pi_1\pi_0\pi_1\pi_2\pi_3\pi_4 \cdot \boxed{\substack{2\\ \vdots\\ 1}} \cdot \boxed{\substack{3\\ \vdots\\ 2}} \quad \text{by RB2 and RB4}\\
&\equiv_0 \boxed{\substack{4\\ \vdots\\ -2}} \cdot \boxed{\substack{5\\ \vdots\\ 3}} \cdot \boxed{\substack{6\\ \vdots\\ 4}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ -4}} \cdot \boxed{\substack{2\\ \vdots\\ 1}} \cdot \boxed{\substack{3\\ \vdots\\ 2}} \equiv_0 \boxed{\substack{4\\ \vdots\\ -2}} \cdot \boxed{\substack{5\\ \vdots\\ 1}} \cdot \boxed{\substack{6\\ \vdots\\ 2}} \cdot \boxed{\substack{7\\ \vdots\\ 6}} \cdot \boxed{\substack{8\\ \vdots\\ -4}} = \pi_{\mathsf{c}} \text{ by Lemma 3.40.}
\end{aligned}$$

**Remark 3.43.** The Definition 3.38, the Lemma 3.39 and the Theorem 3.41 can be also adapted to the case of $G_n^1$, using the transformation $\pi_i \mapsto s_i$ for $i \neq 0$ and $\pi_0 \mapsto \pi_0$. There are only few cases which differ; they are precisely those where relation RB1 is used (with $i \neq 0$), that is case Pos.a. and Neg.a. The modifications in the definition are thus the followings:

Pos.a. $c_n = i > 0$ and $t = s_i$ then $\mathsf{c} \cdot s_i = \mathsf{c}_1 \dots \mathsf{c}_{n-1}(\mathsf{c}_n + 1)$.

Neg.a. $c_n = -i \leq 0$ and $t = s_i$ then $\mathsf{c} \cdot s_i = \mathsf{c}_1 \dots \mathsf{c}_{n-1}(\mathsf{c}_n + 1)$.

The equivalent of Lemma 3.39 can be proved the same way. Finally the proof of Theorem 3.41 only use the relation $s_i^2 = 1$ in these two cases.

**Corollary 3.44.** *Let $1^{\mathsf{c}}_{\mathsf{n}}$ denote the code of the identity rook of size $n$. For any $\pi \in G^0_n$ and $s \in G^1_n$, the congruencies $\pi \equiv_0 \pi_{1^{\mathsf{c}}_{\mathsf{n}} \cdot \pi}$ et $s \equiv_1 s_{1^{\mathsf{c}}_{\mathsf{n}} \cdot s}$ hold.*

*Proof.* We use Theorem 3.41 and Remark 3.43 at $\mathsf{c} = 1^{\mathsf{c}}_{\mathsf{n}}$ and proceed by induction on the length of the words $\pi$ or $s$. □

We now have an easy proof of the identities that motivated Definition 3.38:

**Corollary 3.45.** *For any generator $t$ the following diagram is commutative:*

$$\begin{tikzcd} R_n \arrow[r, bend left, "\text{code}"] \arrow[d, "\cdot t"] & C_n \arrow[l, bend left, "\text{decode}"] \arrow[d, "\cdot t"] \\ R_n \arrow[r, bend left, "\text{code}"] & C_n \arrow[l, bend left, "\text{decode}"] \end{tikzcd}$$

*Proof.* We start by Theorem 3.41, $\pi_{\mathsf{c}\cdot t} \equiv_0 \pi_{\mathsf{c}} t$. Now since $\Phi_0 : G^0_n \to F^0_n$ is a morphism, we can apply this relation to the rook $1_n$. We obtain: $1_n \cdot \pi_{\mathsf{c}\cdot t} = 1_n \cdot (\pi_{\mathsf{c}}\, t) = (1_n \cdot \pi_{\mathsf{c}})\, t$. We conclude thanks to Proposition 3.36 and Theorem 3.27. □

**Corollary 3.46.** *The maps $\begin{cases} C_n \twoheadrightarrow G^0_n \\ c \;\mapsto \pi_c \end{cases}$ and $\begin{cases} C_n \twoheadrightarrow G^1_n \\ c \;\mapsto s_c \end{cases}$ are surjective; the following cardinalities coincide:*

$$|C_n| = |R_n| = |F^0_n| = |G^0_n| = |F^1_n| = |G^1_n|\,.$$

*Moreover, $F^0_n \simeq G^0_n$, $F^1_n \simeq G^1_n$ as monoids.*

*Proof.* Using both Remark 3.9 and Corollary 3.45, we get the following sequence of surjective maps: $C_n \twoheadrightarrow G^0_n \twoheadrightarrow F^0_n$. Furthermore $|F^0_n| \geq |C_n|$ by Corollary 3.37. Consequently $|C_n| = |F^0_n| = |G^0_n|$ and $F^0_n \simeq G^0_n$ as monoids. □

**Example 3.47.** Let $r = 240503$ and $t = \pi_0$. Then $r \cdot t = 040503$. Let us check our algorithm.

Firstly $\text{code}(r) = \mathsf{01323\overline{2}}$. Our algorithm gives us the following serie of operations:

$$\begin{aligned} \mathsf{01323\overline{2}} \cdot \pi_0 &= [(\mathsf{01323}) \cdot \pi_0\pi_1]\,\mathsf{0} \\ &= [((\mathsf{0132}) \cdot \pi_0)\,\mathsf{3} \cdot \pi_1]\,\mathsf{0} = [((\mathsf{013}) \cdot \pi_0)\,\mathsf{23} \cdot \pi_1]\,\mathsf{0} = [((\mathsf{01}) \cdot \pi_0)\,\mathsf{323} \cdot \pi_1]\,\mathsf{0} \\ &= [\mathsf{00323} \cdot \pi_1]\,\mathsf{0} = [\mathsf{0032} \cdot \pi_1]\,\mathsf{30} \\ &= \mathsf{003130} \end{aligned}$$

Finally we really have $\text{decode}(\mathsf{003130}) = 040503$.

Now, there is no need to distinguish between the monoids of functions from the presented monoids, since we have the proof that they are isomorphic.

**Notation 3.48.** *We denote $R^0_n := F^0_n \simeq G^0_n$ the 0-rook monoid.*

*For any rook $r$ we also denote $\pi_r := \pi_{\text{code}(r)}$.*

**Corollary 3.49.** *$\pi_r$ is the unique element of $R^0_n$ such that $1_n \cdot \pi_r = r$. With the identification $r \leftrightarrow \pi_r$, the action of $R^0_n$ on $R_n$ is nothing but the right multiplication in $R^0_n$: $\pi_r\pi_s = \pi_{r\cdot\pi_s}$.*

*Proof.* The identity $1_n \cdot \pi_r = r$ is Proposition 3.36, and $\pi_r$ is unique thanks to cardinalities. Finally, $1_n \cdot \pi_r\pi_s = (1_n \cdot \pi_r) \cdot \pi_s = r \cdot \pi_s$ and we conclude by unicity. □

We have, by the way, re-proven the presentation for the classical rook monoid:

**Corollary 3.50.** *For all $n$, We have the following isomorphisms of monoids: $F_n^1 \simeq R_n \simeq G_n^1$.*

*Proof.* The monoid morphism $\begin{cases} \langle s_1, \dots, s_{n-1}, \pi_0 \rangle \subseteq R_n & \longrightarrow & F_n^1 \subseteq \mathcal{F}(R_n, R_n) \\ r & \longmapsto & (r' \mapsto r' \cdot r) \end{cases}$ is well-defined, and surjective. By Corollary 3.46 we can deduce that $\langle s_1, \dots, s_{n-1}, \pi_0 \rangle \simeq R_n \simeq F_n^1$. □

Here is a further immediate consequence of the presentation:

**Corollary 3.51.** *The monoid $R_n^0$ is isomorphic to its opposite.*

*Proof.* It comes from the fact that the relations of the presentation of $R_n^0$ are symmetrical. □

## 3.4 A Matsumoto theorem for rook monoids

We now turn to the specific study of reduced words.

**Proposition 3.52.** *The words $s_{\mathrm{code}(r)}$ and $\pi_{\mathrm{code}(r)}$ are reduced expressions (i.e. of minimal length) respectively for $r \in R_n$ and $\pi_r \in R_n^0$.*

*Proof.* Corollary 3.44 tells us that every element of $R_n$ and $R_n^0$ can be written as $\pi_{\mathsf{c}}$ and $s_{\mathsf{c}}$ for some code $\mathsf{c}$. Moreover, according to Theorem 3.41 the rewriting of any word to $\pi_{\mathsf{c}}$ and $s_{\mathsf{c}}$ only decrease the length. To conclude, we still have to argue that $\pi_{\mathsf{c}}$ and $s_{\mathsf{c}}$ cannot be obtained with a different shorter code, which is clear from Proposition 3.36. □

**Remark 3.53.** The Corollary 3.44 gives us a standard expression for every element of $R_n^0$. We can now look back at Lemma 3.4 and realize that $P_n$ corresponds to the $R$-code $00\dots0$ ($n$ times), and thus to the action of replacing all the entries by 0.

A final important consequence of our construction is a proof of the analogue of Matsumoto's theorem, answering a question of Solomon [Solomon(2004), p. 209, bottom of the middle paragraph]:

**Theorem 3.54** (Matsumoto theorem for Rook monoids)**.** *If $\underline{u}$ and $\underline{v}$ are two reduced words over $\{\pi_0, s_1 \dots, s_{n-1}\}$ (resp. $\{\pi_0, \pi_1, \dots, \pi_{n-1}\}$) for the same element $r$ of $R_n^1$ (resp. $R_n^0$), then they are congruent using only the two Relations RB2 and RB4, namely the braid relations:*

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad 1 \le i \le n-2, \tag{Rs2}$$
$$s_i s_j = s_j s_i \qquad |i-j| \ge 2. \tag{Rs3}$$
$$\pi_0 s_j = s_j \pi_0 \qquad j \ne 1. \tag{Rs5.1}$$

*Respectively:*

$$\pi_i \pi_{i+1} \pi_i = \pi_{i+1} \pi_i \pi_{i+1} \qquad 1 \le i \le n-2, \tag{RB2}$$
$$\pi_i \pi_j = \pi_j \pi_i \qquad 0 \le i, j \le n-1, \qquad |i-j| \ge 2. \tag{RB4}$$

*Proof.* First of all, we only do the proof at $q = 0$, the $q = 1$ case is done similarly. Moreover, by transitivity, it is sufficient to work in the case where $\underline{v} = \pi_{\mathsf{c}}$ whith $\mathsf{c} = \mathrm{code}(1_n \cdot r)$. We proceed by induction on the common length $\ell$ of $\underline{u}$ and $\underline{v}$. It is obvious when $\ell = 0$. We now consider a reduced word $\underline{v} = \underline{v'}t$ for an element $r$. Then $\underline{v'}$ is also reduced for an element $r'$, so that $r't = r$. We assume by induction that $\underline{v'}$ is congruent to $\pi_{\mathsf{c}'}$ where $\mathsf{c}' := \mathrm{code}(1_n \cdot r')$ using only Relations RB2 and RB4. Therefore $\underline{v'}t$ and $\pi_{\mathsf{c}'}t$ are congruent too. In the proof of Theorem 3.41, we explicitely gave how to go from $\pi_{\mathsf{c}'}t$ to $\pi_{\mathsf{c}'\cdot t}$. Hence we only need to check that Relations RB1 and RB3 are only used in the case where $\underline{v'}t$ is not reduced that is when the length of $\underline{v'}t$ is larger that the length of $\pi_{\mathsf{c}'\cdot t}$. This indeed holds, namely, in cases Pos.a., Neg.a which use RB1 on one hand, and cases Neg.d, Neg.e.$\alpha$ which use RB3 on the other hand. □

As a consequence reduced words for $R_n^1$ and $R_n^0$ are the same:

**Corollary 3.55.** *Let $\underline{w}^1 \in G_n^1$ a word for a rook $r$ and $\underline{w}^0$ its corresponding word in $G_n^0$ obtained by replacing $s_i$ by $\pi_i$ and leaving $P_1$. Then $\underline{w}^1$ is reduced if and only if $\underline{w}^0$ is reduced. Moreover, when they are, for any $k = 0, \dots, |w|$, one has $1_n \cdot w_1^1 \cdots w_k^1 = 1_n \cdot w_1^0 \cdots w_k^0$ and the elements $(1_n \cdot w_1^0 \cdots w_k^0)_{k=0\dots|w|}$ are all distinct.*

*Proof.* Any reduced word is congruent by braid relations to a canonical one: $s_{\mathsf{c}}$ and $\pi_c$. Moreover, the canonical words corresponds by the exchange $s \leftrightarrow \pi$ and the braid relations keep this correspondence, so that the first statement holds. Now assume that a word $\underline{w}^i$ is reduced. Thanks to Corollary 3.49, we know that the sequence of elements are distinct, otherwise it would imply that some products $w_1^i \cdots w_k^i$ are equal for two different values of $k$ leading to a shorter word. Now Equation 3.5, prove the equality. □

As explained by Solomon [Solomon(2004)], this is sufficient to give a presentation of the $q$-rook algebra. Here is a quick sketch on how to do that: fix a parameter $q$ in a ring $\mathbf{R}$ and define an endomorphism $T_i$ of $\mathbf{R}R_n$ interpolating between $q = 1$ and $q = 0$ by

$$r \cdot T_i := q(r \cdot s_i) + (1 - q)(r \cdot (\pi_i - 1)), \tag{3.19}$$

for $i = 1, \dots, n - 1$ (where $s_i$ and $\pi_i$ acts according to Equations 3.3 and 3.5). It is well known [Lascoux(2003), Lascoux and Schützenberger(1987)] that these operators generate the Hecke algebra. We now consider the algebra generated by those generators plus $P_1$ defined as in Equation 3.5. Since $P_1$ commutes with $s_i$ and $\pi_i$ for $i \geq 2$, it commutes with $T_i$. Therefore for any rook $r$, it makes sense to define $T_r := T_{i_1}T_{i_2}\dots P_1 \dots T_{i_k}$ for any reduced word $s_{i_1}Ps_{i_2}\dots P_1 \dots s_{i_k}$. Due to the braid relations the result is independent from the chosen reduced word. Moreover for each of those words

$$1 \cdot T_r = r + \text{shorter terms}, \tag{3.20}$$

so that these $(T_r)_{r\in R_n}$ are linearly independent. It finally suffices to add four more relations which explain how to simplify non reduced words. Namely:

$$(T_i + 1)(T_i - q) = 0, \tag{3.21}$$
$$P_1^2 = P_i, \tag{3.22}$$
$$(P_1 - 1)T_1(P_1 - 1)T_1 = T_1(P_1 - 1)T_1(P_1 - 1), \tag{3.23}$$
$$P_1(T_1 - q)P_1(T_1(1 - P_1)T_1 - q) = 0. \tag{3.24}$$

We remark that this presentation is true over $\mathbb{Z}$ and therefore over any ring, and not only on fields. As far as we know, this was unknown before.

### 3.5 More actions of $R_n^0$

In Definition 3.8, we have given a right action of $R_n^0$ on $R_n$. It is now clear from Corollary 3.49 that this action is nothing but the right multiplication in $R_n^0$. Under this action, $P_j$ acts by killing the first $j$ entries:

$$(r_1 \dots r_n) \cdot P_j = 0 \dots 0\, r_{j+1} \dots r_n\,. \tag{3.25}$$

The inverse of a permutation matrix is its transpose. Transposing a rook matrix still gives a rook matrix, so that one can transfer the notion to rook vectors. It is computed as follows: for a rook $r$, the $i$-th coordinate of $r^t$ is the position of $i$ in $r$ if $i \in r$, and 0 otherwise. For instance $(105203)^t = 146030$.

Transposing the natural right action, we naturally get a left action of the opposite monoid on rooks. However $R_n^0$ is isomorphic to its oppose. It is therefore possible to define a left natural action:

**Definition 3.56.** *For $0 \le i \le n$ and $r = r_1 \dots r_n \in R_n$, define*

$$\pi_i \cdot r := (r^t \cdot \pi_i)^t \qquad \textit{so that} \qquad r \cdot \pi_i = (\pi_i \cdot r^t)^t\,. \tag{3.26}$$

*More explicitely, for $0 \le j \le n$, we write $j \in r$ if $j \in \{r_1, \dots, r_n\}$. Then for any rook $r$:*

- *$\pi_0$ replaces 1 by 0 in $r$ if $1 \in r$, and fixes $r$ otherwise.*
- *For $i > 0$, the action of $\pi_i$ on $r$ is*
  - *if $i, i+1 \in r$, call $k$ and $l$ their respective positions. Then $\pi_i$ fixes $r$ if $l < k$, otherwise it exchanges $i$ and $i+1$.*
  - *if $i \notin r$ and $i+1 \in r$, then $\pi_i$ replaces $i+1$ by $i$.*
  - *if $i+1 \notin r$ then $\pi_i$ fixes $r$.*

**Lemma 3.57.** *The previous definition is a left monoid action of $R_n^0$ on $R_n$ called the* left natural action. *Under this action, $P_j$ acts by replacing the entries smaller than $j$ by 0.*

**Example 3.58.** $\pi_0 \cdot 0342 = 0342$, $\quad \pi_1 \cdot 0342 = 0341$, $\quad \pi_2 \cdot 0342 = 0342$, $\quad \pi_3 \cdot 0342 = 0432$, $\pi_0 \cdot 132 = 032$.

This sheds some light on the link with the type $B_n$: it is well known that type $B_n$ can be realized using signed permutations. The quotient giving the 0-rook monoid can be realized by replacing the negative numbers by zeros.

**Proposition 3.59.** *$\pi_r$ is the unique element of $R_n^0$ such that $\pi_r \cdot 1_n = r$. With the identification $r \leftrightarrow \pi_r$, the left action of $R_n^0$ on $R_n$ is nothing but the left multiplication in $R_n^0$: $\pi_r \pi_s = \pi_{\pi_r \cdot s}$.*

*Proof.* For a rook $r$, let us call temporarily ${}_r\pi$ the reverse of the word $\pi_{r^t}$. Transposing Corollary 3.49 we get that ${}_r\pi$ is characterized by ${}_r\pi \cdot 1_n = r$ and ${}_r\pi_s\pi = {}_{\pi_r \cdot s}\pi$. However, at this stage it's not clear that ${}_r\pi = \pi_r$ (as element of $R_n^0$). Nevertheless, for generators that is words of length 1, the equality ${}_r\pi = \pi_r$ holds. Now given any reduced word $\underline{w} = w_1 \dots w_l$ for an element $x \in R_n^0$, set $r := 1_n \cdot \underline{w} = 1_n \cdot w_1 \cdot w_2 \cdots w_l$ so that $x = \pi_r$ in $R_n^0$. Since $\underline{w}$ is reduced, using Corollary 3.55, one gets that $r = \underline{w}^1$ (the product of the corresponding word in $R_n^1$ which is nothing but a matrix product). But this gives that $r = \underline{w}^1 \cdot 1_n$ so that using the transpose of Corollary 3.55, $r = \underline{w} \cdot 1_n$. By unicity, one concludes that ${}_r\pi = \pi_r$. $\square$

**Corollary 3.60.** *The natural left and right actions of $R_n^0$ on $R_n$ commute.*

*Proof.* Thanks to 3.49 and 3.59, this is just associativity in $R_n^0$. □

One can also extend the action of $H_n^0$ by isobaric divided differences on polynomials: the monoid $R_n^0$ acts also on the polynomials in $n$ indeterminates over any ring $k$, $k[X_1, \dots, X_n]$ in the following way.

**Lemma 3.61.** *Let $f \in k[X_1, \dots, X_n]$. Define*

$$f \cdot \pi_0 := f_{|X_1=0} = f(0, X_2, \dots X_n), \qquad \text{and} \qquad f \cdot \pi_i := \frac{X_i f - (X_i f) \cdot s_i}{X_i - X_{i+1}}. \tag{3.27}$$

*This definition is a right monoid action of $R_n^0$ over $k[X_1, \dots, X_n]$. Under this action,*

$$f \cdot P_j = f(0, \dots 0, X_j, \dots X_n). \tag{3.28}$$

*Proof.* It is a well-known fact [Lascoux and Schützenberger(1987)] that isobaric divided differences give an action of the Hecke algebra at $q = 0$. It remains only to show the relation $\pi_1 P_1 \pi_1 P_1 = P_1 \pi_1 P_1 = P_1 \pi_1 P_1 \pi_1$. We easily check by an explicit computation that the three members are equals to the operator $P_2$ defined by $f \cdot P_2 = f(0, 0, X_2, \dots, X_n)$. The action of $P_n$ can be easily obtained by induction with $P_{i+1} = P_i \pi_i P_i$. □

Actually, there is an extra relation, which can be checked by a explicit computation:

$$f \cdot \pi_1 \pi_0 \pi_1 = f \cdot \pi_0 \pi_1 \pi_0. \tag{3.29}$$

This shows that the monoid which is actually acting is $H^0(A_{n+1})$ (Cartan type $A_{n+1}$) thanks to the following sequence of surjective morphisms:

$$H^0(B_n) \twoheadrightarrow R_n^0 \twoheadrightarrow H^0(A_{n+1}). \tag{3.30}$$

Finally, we note that it is actually possible to get an action of the full generic $q$-rook algebra by taking the same definition as Relation 3.19.

## 4 The $\mathcal{R}$-order on rooks

In this section, we seek for combinatorial, order theoretic and geometric analogs of the permutohedron for rooks. Recall that the right Cayley graph of the symmetric group $\mathfrak{S}_n$ has several interpretations, namely:

- the Hasse diagram of the right weak order of $\mathfrak{S}_n$ seen as a Coxeter group, which is naturally a lattice [Guilbaud and Rosenstiehl(1963)];
- the Hasse diagram of Green's $\mathcal{R}$-order of the 0-Hecke monoid $H_n^0$ [Denton et al.(2010/11)Denton, Hivert,
- the skeleton of the polytope obtained as the convex hull of the set of points whose coordinates are permutations [Ziegler(1995), Example 0.10].

As we will see, some of these properties have an analog for rooks.

We first notice an important difference: on the contrary to $\mathfrak{S}_n$ the right order is not graded. This has been already noted for $R_2^0$. Indeed in the left part of Figure 4.1 we see two paths from 12 to 00 namely $\pi_0 \pi_1 \pi_0$ on the left and $\pi_1 \pi_0 \pi_1 \pi_0$ on the right. Starting with $n = 3$ the right order is moreover not isomorphic to its dual order.

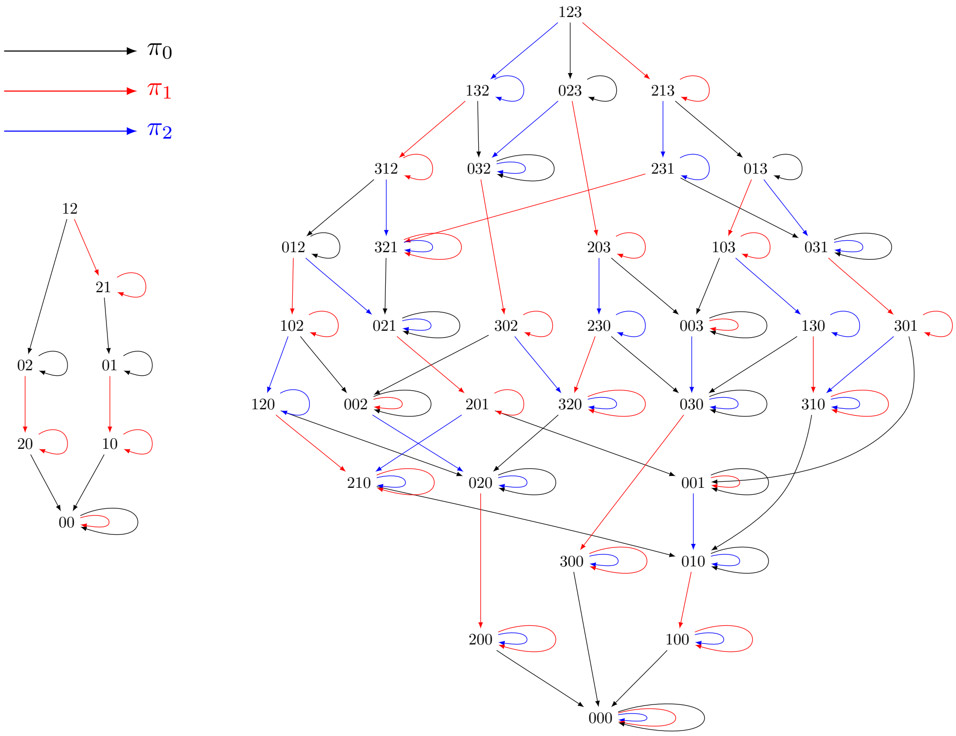



Figure 4.1: The right Cayley graph of $R_2^0$ and $R_3^0$.

## 4.1 $\mathcal{R}$-triviality of $R_n^0$

In this section we study the right Cayley graph of $R_n^0$ showing that except for loops (edge from a vertex to itself) it is acyclic. In monoid theoretic terminology, one says that $R_n^0$ is $\mathcal{R}$-trivial. From Coxeter group point of view, this is the analogue on rook of the (dual) right weak order. Note that the order considered here is different to the (strong) Bruhat order. Its analogue for rook is the subject of [Can and Renner(2012)].

Having shown this acyclicity, we will deduce from the symmetry of the relations of $R_n^0$ that the left sided Cayley graph is also acyclic. By a standard semigroup theory argument, this will imply that the two-sided Cayley graph is acyclic too, that is that $R_n^0$ is actually $\mathcal{J}$-trivial.

We first recall a combinatorial description of the $\mathcal{R}$-order of the 0-Hecke monoid (or equivalently the dual right-weak order of the symmetric group seen as a Coxeter group) [Björner and Brenti(2005)]. Recall that for two permutations $\sigma$ and $\tau$ one has $\sigma \leq_{\mathcal{R}} \tau$ if there exists a sequence $(i_1, \dots, i_k)$ with $0 < i_j < n$ such that $\sigma = \tau \cdot \pi_{i_1} \dots \pi_{i_k}$. Note that, in accord with the monoid convention and contrary to the Coxeter group convention, the identity is the largest element for this order. An algorithmic way to compare two permutations is to use values inversions (sometimes called co-inversions). We give here a definition which is also valid for rooks:

**Definition 4.1.** *For a rook $r$, the* set of inversions *of $r$ is defined by*

$$\mathrm{Inv}(r) := \{(r_i, r_j) \mid i < j \text{ and } r_i > r_j > 0\}. \tag{4.1}$$

It is a subset of $\Delta := \{(b, a) \mid n \geq b > a > 0\}$, but not all subsets are inversions sets of permutations and of rooks as we will see.

**Definition 4.2.** *A subset $I \subseteq \Delta$ is* transitive *if $(c,b) \in I$ and $(b,a) \in I$ implies $(c,a) \in I$.*

Here is a characterization of inversions sets:

**Lemma 4.3.** *Given a set $I \subseteq \Delta$, there exists a permutation $\sigma$ such that $\mathrm{Inv}(\sigma) = I$ if and only if $I$ and $\Delta \setminus I$ are both transitive. When this holds the permutation $\sigma$ is unique.*

*Proof.* This is a folklore result. To reconstruct $\sigma$ from its inversion set, one shows that the relation $I \cup \{(i,j) \mid (j,i) \in \Delta \setminus I\}$ is a total order, that is a permutation. □

Inversion sets allow to characterize the right order:

**Lemma 4.4** ([Björner and Brenti(2005)])**.** *Let $\sigma, \tau \in \mathfrak{S}_n$, then $\sigma \leq_{\mathcal{R}} \tau$ if and only if $\mathrm{Inv}(\tau) \subseteq \mathrm{Inv}(\sigma)$.*

**Proposition 4.5** ([Björner and Brenti(2005)])**.** *The right $\mathcal{R}$-order on permutations is a lattice. The meet $\sigma \wedge_{\mathcal{R}} \mu$ of $\sigma$ and $\tau$ is characterized by: $\mathrm{Inv}(\sigma \wedge_{\mathcal{R}} \mu)$ is the transitive closure of $\mathrm{Inv}(\sigma) \cup \mathrm{Inv}(\mu)$. The join of $\sigma$ and $\tau$ is characterized by: $\Delta \setminus \mathrm{Inv}(\sigma \vee_{\mathcal{R}} \mu)$ is the transitive closure of $(\Delta \setminus \mathrm{Inv}(\sigma)) \cup (\Delta \setminus \mathrm{Inv}(\mu))$.*

We now present how to adapt inversion sets to rooks. The idea is to record usual inversions as well as inversion with a 0 letter. Here is a way to do it:

**Definition 4.6.** *We call the* support *of a rook $r$ denoted $\mathrm{supp}(r)$ the set of non-zero letters appearing in its rook vector. For each letter $\ell \in \mathrm{supp}(r)$, we denote $Z_r(\ell)$ the number of $0$ which appear after $\ell$ in the rook vector of $r$.*

*We finally say that $(\mathrm{supp}(r), \mathrm{Inv}(r), Z_r)$ is the* rook triple associated to $r$.

**Example 4.7.** For example for $r = 2054001$, one gets $\mathrm{supp}(r) = \{1,2,4,5\}$, together with $\mathrm{Inv}(r) = \{(2,1),(4,1),(5,4),(5,1)\}$, $Z_r(1) = 0$, $Z_r(2) = 3$ and $Z_r(4) = Z_r(5) = 2$.

Here is a characterization of the rook triples:

**Proposition 4.8.** *A triple $(S, I, Z)$ where $S \subseteq \{1, \dots n\}$, $I \subseteq \Delta$ and $Z : S \mapsto \mathbb{N}$ is the rook triple of a rook $r$ if and only if the three following properties hold:*

- *the sets $I \subset \Delta \cap S^2$ and $I$ and $(\Delta \cap S^2) \setminus I$ are both transitive.*
- *for $\ell \in S$, one has $0 \leq Z(\ell) \leq n - |S|$;*
- *if $(b,a) \in I$ then $Z(b) \geq Z(a)$ else $Z(b) \leq Z(a)$.*

*Moreover, when these properties hold the corresponding rook $r$ is unique.*

*Proof.* We first prove the direct implication. The first statement says that if one erases the zeros from a rook, one gets a permutation of its support. The second statement says that there are $n - |\mathrm{supp}(r)|$ zeros. The third statement says that if $a$ is after $b$ in $r$, then there are less 0 to the right of $a$ than to the right of $b$.

Conversely, given such a triple, we can reconstruct a rook $r$ in two steps: the first condition ensures that there is a unique permutation $\sigma$ of the support $S$ with inversions set $I$. The third statement says that the function $Z$ is decreasing along the word $\sigma$. As a consequence, writing $\sigma_i^Z$ the subword of $\sigma$ composed by the letters $\ell$ such that $Z(\ell) = i$, one has

$$\sigma = \sigma^Z_{n-|\mathrm{supp}(r)|} \cdots \sigma^Z_2 \, \sigma^Z_1 \, \sigma^Z_0 \,. \tag{4.2}$$

Note that some of the $\sigma_i^Z$ may be empty. Then the rook

$$r = \sigma^Z_{n-|\operatorname{supp}(r)|}\, 0 \,\ldots\, 0\, \sigma^Z_2\, 0\, \sigma^Z_1\, 0\, \sigma^Z_0\,. \tag{4.3}$$

is indeed associated with the triple $(S, I, Z)$ and is by construction unique. □

**Example 4.9.** Going back to Example 4.7, consider the following triple with $n = 7$:

$$(S, I, Z) = (\{1, 2, 4, 5\}, \{(2, 1), (4, 1), (5, 4), (5, 1)\}, (\begin{smallmatrix}1 & 2 & 4 & 5\\ 0 & 3 & 2 & 2\end{smallmatrix}))\,.$$

There is a unique permutation $\sigma$ of $S$ with inversion set $I$, namely 2541. Writing $Z(i)$ below $i$ for each letter of $\sigma$, we get $(\begin{smallmatrix}2 & 5 & 4 & 1\\ 3 & 2 & 2 & 0\end{smallmatrix})$ and see that $Z$ is indeed decreasing. We then get that $\sigma^Z_3 = (2)$, $\sigma^Z_2 = (54)$, $\sigma^Z_1 = ()$, $\sigma^Z_0 = (1)$, so that we recover $r = 2054001$.

Our aim is now to show that the $\mathcal{R}$-order is actually an order. To do so, we start by defining combinatorially an order $r \leq_I u$, and then show that $\leq_I$ and $\leq_{\mathcal{R}}$ are actually equivalent.

**Definition 4.10.** *Let $r$ and $u \in R_n$. We write $r \leq_I u$ if and only if the three following properties hold:*

- $\operatorname{supp}(r) \subseteq \operatorname{supp}(u)$,
- $\{(b, a) \in \operatorname{Inv}(u) \mid b \in \operatorname{supp}(r)\} \subseteq \operatorname{Inv}(r)$,
- $Z_u(\ell) \leq Z_r(\ell)$ *for* $\ell \in \operatorname{supp}(r)$.

**Remark 4.11.** If $r$ and $u$ are permutations, then $\operatorname{supp}(r) = \operatorname{supp}(u) = \{1, \ldots, n\}$, so that $r \leq_I u$ if and only if $\operatorname{Inv}(u) \subset \operatorname{Inv}(r)$.

Moreover, as a consequence of the second condition, if $(b, a) \in \operatorname{Inv}(u)$ and $b \in \operatorname{supp}(r)$ then $a \in \operatorname{supp}(r)$. We abstract this fact with the following definition and lemma:

**Definition 4.12.** *Let $I \subseteq \Delta$ and $S \subset [\![1, n]\!]$. We say that $S$ is $I$-*compatible *if $(b, a) \in I$ and $b \in S$ implies $a \in S$, for all $b, a$.*

The previous remark now rephrases as:

**Lemma 4.13.** *If $r \leq_{\mathcal{R}} u$ then* $\operatorname{supp}(r)$ *is* $\operatorname{Inv}(u)$*-compatible.*

We will further need the following basic facts about compatibility:

**Lemma 4.14.** *The union $S_1 \cup S_2$ of two $I$-compatibles sets $S_1$ and $S_2$ is $I$-compatible.*
*If $S$ is $I_1$ and $I_2$-compatible, then it is $I_1 \cup I_2$-compatible.*
*If $S$ is $I$-compatible then it is compatible with the transitive closure of $I$.*

We get back to the study of $\leq_I$.

**Proposition 4.15.** *The set $R_n$ endowed with the relation $\leq_I$ is a poset with maximal element $1_n$ and minimal element $0_n = 0 \ldots 0$.*

*Proof.* The relation $\leq_I$ is reflexive, by definition.

If $r, u \in R_n$ are such that $r \leq_I u$ and $u \leq_I r$ then $\operatorname{supp}(r) = \operatorname{supp}(u)$ and therefore $\operatorname{Inv}(r) = \operatorname{Inv}(u)$ and $Z_r = Z_u$. As a consequence, the non-zero letters appear in the same order in $r$ and $u$ and the zeros are in the same places. Thus $\leq_I$ is antisymmetric.

Let $r \leq_I u \leq_I v$. Then $\operatorname{supp}(r) \subseteq \operatorname{supp}(v)$. Let $(b, a) \in \operatorname{Inv}(v)$ with $b \in \operatorname{supp}(r)$. Necessarily $b \in \operatorname{supp}(u)$ so that $(b, a) \in \operatorname{Inv}(u)$ and consequently $(b, a) \in \operatorname{Inv}(r)$. Finally if $\ell \in \operatorname{supp}(r)$ then $Z_v(\ell) \leq Z_u(\ell) \leq Z_r(\ell)$. Thus $\leq_I$ is transitive. □

**Theorem 4.16.** *Let $r, u \in R_n$. Then $\pi_r \leq_{\mathcal{R}} \pi_u$ if and only if $r \leq_I u$.*

*Proof.* By definition, $\pi_r \leq_{\mathcal{R}} \pi_u$ if there exists $\pi \in R_n^0$ such that $\pi_r = \pi_u \pi$. Using the identification $r \leftrightarrow \pi_r$ of Corollary 3.49, this is equivalent to $r = u \cdot \pi$. By abuse of notation in this proof we will therefore write $r \leq_{\mathcal{R}} u$ if there exists $\pi \in R_n^0$ such that $r = u \cdot \pi$.

For the direct implication, by induction and transitivity, it is sufficient to assume that $r = u \cdot \pi_i$ with $r \neq u$ and show $r <_I u$.

- If $i \neq 0$. Then $\operatorname{supp}(u) = \operatorname{supp}(r)$. Since $r \neq u$ we must have $u_i < u_{i+1}$ and also $r = u_1 \dots u_{i+1} u_i \dots u_n$. If $u_i \neq 0$ then $\operatorname{Inv}(r) = \operatorname{Inv}(u) \sqcup \{(u_{i+1}, u_i)\}$ and $Z_r = Z_u$. On the contrary, if $r_i = 0$, then $\operatorname{Inv}(r) = \operatorname{Inv}(u)$ and $Z_r(\ell) = Z_u(\ell)$ for $\ell \neq u_{i+1}$ and $Z_r(u_{i+1}) = Z_u(u_{i+1}) + 1$.
- If $i = 0$. Since $r \neq u$ we have $r_1 \neq 0$ and $u = 0 r_2 \dots r_n$. We can deduce that $\operatorname{supp}(r) = \operatorname{supp}(u) \cup \{r_1\}$. Furthermore,

  $$\operatorname{Inv}(r) = \{(u_i, u_j) \in \operatorname{Inv}(u) \mid i \neq 1\} = \{(u_i, u_j) \in \operatorname{Inv}(u) \mid u_i \in r\}. \tag{4.4}$$

  Finally for $\ell \in \operatorname{supp}(r)$, $Z_u(\ell) = Z_r(\ell)$.

For the converse implication, assume that $r <_I u$. By induction and transitivity it is sufficient to show that there exists $i$ such that $r \leq_I u \cdot \pi_i$ and $u \cdot \pi_i \neq u$. We proceed by a case analysis. First since $\operatorname{supp}(u) \subseteq \operatorname{supp}(u)$, we can distinguish whether $\operatorname{supp}(u) = \operatorname{supp}(u)$ or $\operatorname{supp}(u) \subsetneq \operatorname{supp}(u)$. In the equality case, we further distinguish whether $Z_u = Z_r$ or not.

- If $\operatorname{supp}(u) = \operatorname{supp}(r)$, and $Z_u \neq Z_r$, then there must exist $\ell \in \operatorname{supp}(r)$ such that $Z_u(\ell) < Z_r(\ell)$. Pick the leftmost $\ell$ in $u$ which verifies this condition. First, there must be some 0 on the left of $\ell$ in $u$ because there are $Z_u(\ell)$ on the right and at least $Z_r(\ell)$ in the word. Thus $\ell$ is not the first letter of $u$.

  Let $k$ be the letter immediately preceding $\ell$ in $u$. We claim that either $k = 0$ or $k$ is after $\ell$ in $r$. Indeed if $k \neq 0$ and $k$ is before $\ell$ in $r$ then we have $Z_r(k) \geq Z_r(\ell)$. Moreover $Z_u(\ell) = Z_u(k)$ because there is no zero in $u$ between $\ell$ and $k$. Therefore $Z_r(k) \geq Z_r(\ell) > Z_u(\ell) = Z_u(k)$ which contradicts our choice of $\ell$ as being the leftmost.

  Now, call $i$ the position of this $k$ in $u$. If $k = 0$, the only difference between the rook triples of $u$ and $u \cdot \pi_i$ is that $Z_{u \cdot \pi_i}(\ell) = Z_u(\ell) + 1$ so that $r \leq_I u \cdot \pi_i$. On the contrary, if $k \neq 0$, then the only difference between the rook triples of $u$ and $u \cdot \pi_i$ is that $\operatorname{Inv}(u \cdot \pi_i) = \operatorname{Inv}(u) \sqcup \{(l, k)\}$ so that again $r \leq_I u \cdot \pi_i$.
- If $\operatorname{supp}(u) = \operatorname{supp}(r)$, and $Z_u = Z_r$, then necessarily $\operatorname{Inv}(u) \subsetneq \operatorname{Inv}(r)$. Write $\tilde{r}$ and $\tilde{u}$ the words obtained by removing the zeros in $r$ and $u$. The inclusion of inversions

shows that $\tilde{u} \leq_S \tilde{r}$ where $\leq_S$ is the right order for permutations of $S = \operatorname{supp}(u)$. As a consequence, we know that it is possible to exchange two consecutive letters $a < b$ in $\tilde{u}$ to get a permutation $\tilde{v}$ of $\operatorname{supp}(u)$ such that

$$\operatorname{Inv}(\tilde{v}) = \operatorname{Inv}(\tilde{u}) \sqcup \{(b, a)\} \subset \operatorname{Inv}(\tilde{r})\,. \tag{4.5}$$

From the equality of $Z$, there cannot be any 0 between $a$ and $b$ in $u$, thus $a$ and $b$ are consecutive in $u$ as well. Writing $i$ for the position of $a$ in $u$, we have $r \leq_I u \cdot \pi_i$.

- The remaining case is $\operatorname{supp}(r) \subsetneq \operatorname{supp}(u)$. Let $\ell := \max(\operatorname{supp}(u) \setminus \operatorname{supp}(r))$. If $\ell$ is in position 1 in $u$ then $r \leq_I u \cdot \pi_0$ and we are done in this case.

  Otherwise if $\ell$ is not in position 1, we claim that the letter $k$ immediately preceding $\ell$ in $u$ is smaller than $l$. If not, then there is an inversion $(k, \ell)$ in $u$. Since $\operatorname{supp}(r)$ is $\operatorname{Inv}(u)$-compatible, then $k \notin \operatorname{supp}(r)$. This contradicts our choice of $\ell$ as being the maximum.

  Writing $i$ for the position of $k$ in $u$, we proceed as in the end of the first case: the only difference between the rook triples of $u$ and $u \cdot \pi_i$ is that $\operatorname{Inv}(u \cdot \pi_i) = \operatorname{Inv}(u) \sqcup \{(\ell, k)\}$ so that again $r \leq_I u \cdot \pi_i$. $\square$

We are now in position to prove the main result of this section:

**Corollary 4.17.** *The monoid $R_n^0$ is $\mathcal{R}$-trivial, $\mathcal{L}$-trivial and thus $\mathcal{J}$-trivial.*

*Proof.* A consequence Theorem 4.16 is that the $\mathcal{R}$-preorder is an order so that $R_n^0$ is $\mathcal{R}$-trivial. Moreover, it is isomorphic to its opposite by Corollary 3.51 and thus it is $\mathcal{L}$-trivial. We conclude with Lemma 2.7. $\square$

## 4.2 The lattice of the $\mathcal{R}$-order

Our goal here is to show that, similarly to the weak order of permutations, the $\mathcal{R}$-order for the rooks is a lattice. We start with an algorithm which computes the meet.

**Theorem 4.18.** *Let $u$ and $v$ be two rooks of size $n$. Define a new rook $r$ by the following algorithm:*

- *Let $I_0$ be the transitive closure of* $\operatorname{Inv}(u) \cup \operatorname{Inv}(v)$.
- *Let $S$ be the largest (for inclusion) $I_0$-compatible set contained in* $\operatorname{supp}(u) \cap \operatorname{supp}(v)$.
- *Let $I := I_0 \cap S^2$.*
- *Finally, for $x \in s$ let $Z(x) := \max\{Z_u(i), Z_v(i) \mid i = x$ or $(x, i) \in I\}$ with the convention that $Z_s(i) = 0$ if $i \notin \operatorname{supp}(s)$.*

*Then $(S, I, Z)$ is a rook triple whose associated rook $r$ is the meet $u \wedge_{\mathcal{R}} v$ of $u$ and $v$ for the $\mathcal{R}$-order.*

*Proof.* We first prove that $(S, I, Z)$ is indeed a rook triple.

- By definition, $I \subset \Delta \cap S^2$, let us show that $I$ and $(\Delta \cap S^2) \setminus I$ are transitive. We claim that $I$ is the transitive closure of $(\mathrm{Inv}(u) \cap S^2) \cup (\mathrm{Inv}(v) \cap S^2)$. Indeed, for any $(b,a) \in I$, then $(b,a) \in I_0$. By definition of the transitive closure, there exists a decreasing sequence of integer $b = c_1 > c_2 > \cdots > c_k = a$ such that $(c_i, c_{i+1}) \in \mathrm{Inv}(u) \cup \mathrm{Inv}(v)$ for $i = 1, \dots, k-1$. By induction, since $b \in S$, compatibility ensures that all of the $c_i$ belong to $S$. Hence the claim.

  As a consequence, using Proposition 4.5, $I$ is the inversion set of the meet in the permutohedron of the restriction of $u$ and $v$ to $S$ so that $I$ and $(\Delta \cap S^2) \setminus I$ are transitive.

- On has $|S| \leq \max(|\operatorname{supp}(u)|, |\operatorname{supp}(v)|)$. So that the condition $0 \leq Z(x) \leq n - |S|$ holds.

- Write $\mathcal{Z}(x) := \{Z_u(i), Z_v(i) \mid i = x \text{ or } (x,i) \in I\}$ so that $Z(x) := \max \mathcal{Z}(x)$. If $(b,a) \in I$, the transitivity of $I$ ensures that as sets $\mathcal{Z}(b) \supseteq \mathcal{Z}(a)$ so that $Z(b) \geq Z(a)$. Conversely write $\overline{I} := (\Delta \cap S^2) \setminus I$. If $(b,a) \in \overline{I}$, the transitivity of $\overline{I}$ shows that $(a,i) \in \overline{I}$ implies $(b,i) \in \overline{I}$. By contraposition, $(b,i) \in I$ implies $(a,i) \in I$ so that $\mathcal{Z}(b) \subseteq \mathcal{Z}(a)$ and therefore $Z(b) \leq Z(a)$.

Hence, we have proved that $(S, I, Z)$ is a rook triple. It remains to prove that its associated rook is the meet $u \wedge_{\mathcal{R}} v$. By construction, $r \leq_I u$ and $r \leq_I v$. So that we only need to prove that for any rook $s$ such that $s \leq_I u$ and $s \leq_I v$ then $s \leq_I r$.

- Using the rephrasing of Remark 4.11 we know that then $\operatorname{supp}(s)$ is $\mathrm{Inv}(u)$ and $\mathrm{Inv}(v)$-compatible and therefore compatible with the transitive closure of their union $I_0$. Since $S = \operatorname{supp}(r)$ is defined as the largest such set, $\operatorname{supp}(s) \subseteq \operatorname{supp}(r)$.

- Suppose $(b,a) \in \mathrm{Inv}(r)$, with $b \in \operatorname{supp}(s)$. Then by construction of $r$, there is a decreasing sequence $b = c_1 > c_2 > \cdots > c_k = a$ such that $(c_i, c_{i+1}) \in \mathrm{Inv}(u) \cup \mathrm{Inv}(v)$ for $i = 1, \dots, k-1$. By induction, having $s \leq_I u$ and $s \leq_I v$, one prove $c_i \in \operatorname{supp}(s)$ and $(c_i, c_{i+1}) \in \mathrm{Inv}(s)$. One concludes by transitivity that $(b,a) = (c_1, c_k) \in \mathrm{Inv}(s)$.

- Finally, assume $x \in \operatorname{supp}(s)$. Then $Z_s(x) \geq Z_u(x)$ and $Z_s(x) \geq Z_v(x)$. Moreover for any $i$ such that $(x,i) \in \mathrm{Inv}(r)$, by the preceding item, $i \in \operatorname{supp}(s)$ and $(x,i) \in \mathrm{Inv}(s)$. One deduces that $Z_s(x) \geq Z_s(i) \geq Z_u(i)$ and $Z_s(x) \geq Z_s(i) \geq Z_v(i)$. We just showed that $Z_s(x) \geq \max \mathcal{Z}(x)$. □

**Corollary 4.19.** *The $\mathcal{R}$-order of $R_n^0$ is a lattice.*

*Proof.* From the previous theorem, we know that $R_n^0$ is a meet semi-lattice. Now it is well known that a meet semi-lattice with a maximum element is a lattice. □

From the proof, we have a more explicit algorithm to compute the meet:

- Start with $S := \operatorname{supp}(u) \cap \operatorname{supp}(v)$. Then while one can find a $(b,a) \in \mathrm{Inv}(u) \cup \mathrm{Inv}(v)$ with $b \in S$ and $a \notin S$, remove $b$ from $S$. When no more such $(b,a)$ can be found, $S$ is the support of $u \wedge_{\mathcal{R}} v$.

- Using the usual algorithm for permutations of the set $S$ (see the sketch of the proof of Lemma 4.3), compute the meet of the restriction $u_{|S}$ and $v_{|S}$.

- Compute the $Z$ function using max as in the statement of Theorem 4.18.

- Finish inserting the zeros using $Z(x)$ as in the proof of Proposition 4.8.

**Example 4.20.** Let $u = 25104$ and $v = 12453$. So $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \{1, 2, 4, 5\}$. But $(4, 3)$ and $(5, 3) \in \mathrm{Inv}(v)$ and $3 \notin S$. So $S = \{1, 2\}$. We then get $I = \{(2, 1)\}$, So that $(u \wedge_{\mathcal{R}} v)_{|S} = 21$. It remains to insert the zeros. One compute $Z(2) = 1$ and $Z(1) = 1$ so that $u \wedge_{\mathcal{R}} v = 00210$. Here is a bigger example: Let us compute $r = 31086502 \wedge_{\mathcal{R}} 02178534$. One finds that $S = \{1, 2, 3\}$, and $I = \{(3, 2), (3, 1), (2, 1)\}$ and $Z = \left(\begin{smallmatrix}1&2&3\\2&2&2\end{smallmatrix}\right)$, so that $r = 00032100$. Similarly

$$30175082 \wedge_{\mathcal{R}} 02154738 = 00308210 \qquad \text{and} \qquad 43017582 \wedge_{\mathcal{R}} 02154738 = 75430821.$$

In the case of permutations, the involution $\sigma \to \tilde{\sigma} = \sigma\omega$ where $\omega$ is the maximal permutation (otherwise said, $\tilde{\sigma}$ is the mirror image of $\sigma$) is an isomorphism from the $\mathcal{R}$-order to its dual. A a consequence, one can compute the join using the meet: $\sigma \vee_{\mathcal{R}} \mu = \widetilde{\tilde{\sigma} \wedge_{\mathcal{R}} \tilde{\mu}}$. However, as seen for example on Figure 4.1 this trick does not work anymore for rooks. This ask for an algorithm to compute the join of two rooks. To describe this algorithm, we need a notion of non-inversion and a dual notion of compatibility:

**Definition 4.21.** *For any rook $r$, call* set of version *of $r$ the set:*

$$\overline{\mathrm{Inv}}(r) := (\Delta \setminus \mathrm{Inv}(r)) \;\cup\; \{(b, a) \in \Delta \mid a \notin r \text{ and } b \in r\}\,. \tag{4.6}$$

*Let $I \subseteq \Delta$ and $S \subset [\![1, n]\!]$. We say that $S$ is* dual $I$-compatible *if $(b, a) \in \Delta \setminus I$ and $a \in S$ implies $b \in S$.*

**Theorem 4.22.** *Let $u$ and $v$ be two rooks of size $n$. Define a new rook $r$ by the following algorithm:*

- *Let $I_0 := \Delta \setminus T$ where $T$ is the transitive closure of $\overline{\mathrm{Inv}}(u) \cap \overline{\mathrm{Inv}}(v)$.*
- *Let $S$ be the smallest dual $I_0$-compatible set containing $\mathrm{supp}(u) \cup \mathrm{supp}(v)$.*
- *Let $I := I_0 \cap S^2$.*
- *Finally, for $x \in s$ let $Z(x) := \min\{Z_u(i), Z_v(i) \mid i = x \text{ or } (x, i) \in \Delta \setminus I\}$, with the convention that $Z_s(i) = +\infty$ if $i \notin \mathrm{supp}(s)$.*

*Then $(S, I, Z)$ is a rook triple whose associated rook $r$ is the join $u \vee_{\mathcal{R}} v$.*

The proof is very similar to the one we did for the meet and is left to the reader.

**Example 4.23.** Let us compute $r = 30175082 \vee_{\mathcal{R}} 72185043$. One finds $S = \{1, 2, 3, 4, 5, 7, 8\}$, $I = \{(7, 5), (4, 3)\}$ and $Z = \left(\begin{smallmatrix}1&2&3&4&5&7&8\\1&0&0&0&0&0&0\end{smallmatrix}\right)$, so that $r = 10243758$.

We want to enumerate the join-irreducible elements. As in the classical permutohedron, they are related to descents, however, it the case of rooks, they are two different notions of descents.

**Definition 4.24** (Weak and strict descents)**.** *Let $r \in R_n$ be a rook. For any $0 \le i < n$, we say that $i$ is a* weak (right) descent *of $r$ if $r \cdot \pi_i = r$. We say that $i$ is a* strict (right) descent *if there exists a rook $s \neq r$ such that $s \cdot \pi_i = r$. Moreover, in the particular case $i = 0$, we say that $0$ is a strict descent with multiplicity $k$, if there are exactly $k$ rooks $s \neq r$ such that $s \cdot \pi_0 = r$.*

Any strict descent is a weak descent. Indeed if $s \cdot \pi_i = r$ then $r \cdot \pi = s \cdot \pi_i^2 = s \cdot \pi_i = r$. Weak descent and strict descents are equivalent when restricted to permutations, but they differ on rooks. For example, the rook 04003, has 3 weak descent namely $0, 2, 3$, but only $0, 2$ are strict ($04003 = 24003 \cdot \pi_0$ and $04003 = 00403 \cdot \pi_2$) and 0 has multiplicity 3: $04003 = 14003 \cdot \pi_0 = 24003 \cdot \pi_0 = 54003 \cdot \pi_0$.

**Lemma 4.25.** *The multiplicity of* $0$ *as a strict descent in a rook* $r$ *is* $0$ *if* $r$ *does not start with* $0$ *and is the number of* $0$ *in* $r$ *otherwise.*

**Definition 4.26.** *An element* $z$ *of a lattice* $L$ *is called* meet irreducible *if it can not be obtained as a non trivial meet that is* $z = z_1 \wedge z_2$ *implies* $z_1 = z$ *or* $z_2 = z$.

An equivalent definition is that $z$ has only one successor in the Hasse diagram of $L$. By definition, in a finite lattice, any element can be written as the meet of some meet irreducible elements. As a consequence, they form the minimal generating set of the meet semi-lattice.

For permutations, the number of meet irreducible for the $\mathcal{R}$-order (that is permutation with only one descent) is $a(n) = 2^n - n - 1$. It is a particular case of Eulerian numbers and is recorded as OEIS A000295. Here are the first values

$$0, 0, 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, 8178, 16369, 32752\,. \tag{4.7}$$

For rooks, the number of meet irreducibles has a very simple expression too:

**Proposition 4.27.** *The number of meet irreducibles for* $\leq_{\mathcal{R}}$ *is* $3^n - 2^n$.

This sequence is recorded as OEIS A001047. Here are the first values

$$0, 1, 5, 19, 65, 211, 665, 2059, 6305, 19171, 58025, 175099, 527345, 1586131\,. \tag{4.8}$$

We will actually prove a stronger statement, the previous one will follow thanks to the identity:

$$3^n - 2^n = \sum_{i=1}^{n} 3^{n-i}\, 2^{i-1}\,. \tag{4.9}$$

**Proposition 4.28.** *For any rook vector* $r$ *denote* $p(r)$ *the first value* $r_0$ *if its non zero, and* $1$ *if its zero. The number of meet-irreducibles* $r$ *of* $R_n$ *such that* $p(r) = i$ *is* $3^{n-i}\, 2^{i-1}$.

*Proof.* A rook is meet irreducible if and only if it has a unique strict descent (counting multiplicities). Consider a meet irreducible rook $r$ with $p(r) = i$. There are two cases:

- if $i > 1$, then the rook is composed by two nondecreasing sequences, the first one starts with $i$. So each number smaller than $i$, either appears in the second subsequence or, do not appear at all so that the second sequence starts with some 0. Similarly each number larger than $i$, may appear in any of those two subsequences or not at all. So the number of choices is $2^{i-1}3^{n-i}$.

- if $i = 1$, then $r$ start either with 0 or 1. We want to show that the number of such rooks is $3^{n-1}$. We show that the set of those rooks is in bijection with the set of maps $f : [\![2, n]\!] \to \{0, 1, 2\}$.

  In the following, for any set $S$ of integers we write $W(S)$ the word obtained by writing the letter of $S$ in increasing order. Given a map $f : [\![2, n]\!] \to \{0, 1, 2\}$, one build a sequence starting with 1, then ordering the preimage of 0, putting as many zero as the preimage of 1, and then ordering the preimage of 2:

$$r(f) := 1 \cdot W(f^{-1}(0)) \cdot 0^{|f^{-1}(1)|} \cdot W(f^{-1}(2)) \,. \tag{4.10}$$

By definition, the result is a rook of size $n$ with at most one descent. Moreover, each rook with only one descent is obtained exactly once as the image of some $f$.

It remains to show that the maps which give rooks with no descent by the preceding construction are in bijection with rooks having 0 as unique descent with multiplicity 1. The point is the following: $r(f)$ has zero descents, that is $r(f)$ is nondecreasing, if and only if there exists a $1 \leq k \leq n$ such that

$$f(i) = \begin{cases} 0 & \text{if } i \leq k, \\ 2 & \text{otherwise.} \end{cases} \tag{4.11}$$

If it is the case, we redefine $r(f)$ as

$$r_1(f) := 0 \cdot W(\{i-1 \mid i \in f^{-1}(0)\}) \cdot W(\{f^{-1}(2)\}). \tag{4.12}$$

The set of the rooks obtained this way is the set of increasing rooks which start with a 0. According to Lemma 4.25, those are exactly the rooks having 0 as unique descent with multiplicity 1.

On conclude that there are exactly $3^{n-1}$ rooks starting either by 0 or 1. □

**Example 4.29.** Consider the function $f = \left(\begin{smallmatrix} 2&3&4&5&6&7&8&9 \\ 2&0&1&0&1&0&2&1 \end{smallmatrix}\right)$. Then $r(f) = 1 \cdot 357 \cdot 000 \cdot 28$ which has only one strict descent (the dots are only here to visualize the different part of the right hand side of Equation 4.10).

Now with $f = \left(\begin{smallmatrix} 2&3&4&5&6&7&8&9 \\ 0&0&0&0&0&2&2&2 \end{smallmatrix}\right)$, Equation 4.10 gives $r(f) = 1 \cdot 23456 \cdot \cdot 789$ which has no descent at all. So we take the second definition (Equation 4.12) and get the new value $r_1(f) = 0 \cdot 12345 \cdot 789$ which has 0 as unique strict descent.

As a concluding remark on irreducible elements, we note that, on the contrary to permutations, the poset is not self dual. So there is no reason why the number of meet irreducible elements should be equal to the number of join irreducible elements. They indeed differ and we do not have a formula for the number of join irreducibles. We give here the first values:

$$0, 1, 5, 16, 43, 106, 249 \,. \tag{4.13}$$

## 4.3 Chains in the rook lattice

We recall that a *chain* in a lattice $(L, \preceq)$ is a sequence of elements $(x_1, \ldots, x_r)$ such that $x_1 \preceq x_2 \preceq \cdots \preceq x_r$. A *maximal chain* is a chain which is not strictly included in another one. Denoting $m$ and $M$ the minimal and maximal elements of $L$, this is equivalent to $x_1 = m$, $x_r = M$ and for every $i < r$ there is no element between $x_i$ and $x_{i+1}$ for the order $\preceq$. For the weak order on permutations, maximal chains corresponds to reduced expression of the maximal permutation.

We now consider maximal chains of $R_n$ (thus also $R_n^0$ by Corollary 3.49). We see in Figure 4.1 that all the maximal chains are not of equal length. Experimental computation of the numbers of maximal chains give the following sequence: $1, 2, 23, 3625, 16489243$. We did not find any nice property: it is not refered in OEIS and the numbers contain big prime factor. A more interesting question is to only consider maximal chains of minimal length, that is reduced expressions of the maximal rook $P_n = 0 \dots 0 \in R_n$. Note by Lemma 3.4 that $\ell(P_n) = \binom{n+1}{2}$. We find the following numbers of such chains:

$$1, 2, 12, 286, 33592, 23178480\,. \tag{4.14}$$

This sequence is refered as OEIS A003121. It counts, among many other things, the number of maximal chains of length $\binom{n+1}{2}$ (hence maximal) in the *Tamari lattice* $\mathcal{T}_{n+1}$. This suggests that there is a bijection between the chains. It turns out that the coincidence is much stronger: the two posets restricted to the elements appearing in their respective chains of maximal length are isomorphic.

We first need to describe the elements appearing in a reduced expression of $P_n$. We need the following combinatorial definition for this:

**Definition 4.30.** *Let $\mathcal{A}$ be an alphabet and $a, b \in \mathcal{A}$, $\underline{u}, \underline{v}$ be two words over $\mathcal{A}$. The* shuffle product *is defined inductively by:*

$$a\underline{u} \sqcup\!\sqcup\, b\underline{v} = a(\underline{u} + b\underline{v}) + b(a\underline{u} + \underline{v})\,, \tag{4.15}$$

*the initial condition being that the empty word is the unit element.*

**Proposition 4.31.** *The rook vectors appearing as a left factor of a reduced expression of $P_n$ are the rooks:*

$$\mathcal{MCR}_n := \{0 \dots 0 \sqcup\!\sqcup\, (k+1) \dots n \mid 0 \leq k \leq n\}. \tag{4.16}$$

*Proof.* Let $r \in \mathcal{MCR}_n$ as defined by Equation 4.16. We assume that $r$ has $k$ zeros, so that the nonzero letters appearing in $r$ are $k+1, \dots, n$. Take the reduced expression for $r$ given by the $R$-code (Definition 3.34). Since the nonzero letters are in order, this expression if of length $\ell(r) = 1 + 2 + \dots + k + \sum_{i=k+1}^{n} Z_r(i)$. In order to bring $r$ to $P_n$ by right action we repeat the following steps until we reach $P_n$: let $i$ be the first nonzero letter and $p = i - Z_r(i)$ its position. Then multiplying $r$ on the right by $s_{p-1} \dots s_1 \pi_0$ brings $i$ to the front and kills it. The length of the word for $P_n$ obtained this way is equal to

$$\ell(r) + \sum_{i=k+1}^{n} (i - Z_r(i)) = \sum_{i=1}^{n} i = \binom{n+1}{2}. \tag{4.17}$$

This is the length of $P_n$, hence the expression is reduced, and $r$ appears in a maximal chain of minimal length.

Now we prove the converse inclusion by contradiction. Let $r \in R_n \setminus \mathcal{MCR}_n$, with $k$ zeros. We want to show that there is no reduced word for $P_n$ of the form $\underline{r}\,\underline{m}$ where $\underline{r}$ is a word for $r$. Assume that we have such a word. Since $r \notin \mathcal{MCR}_n$, then either there is a nonzero letter $k$ before a nonzero letter $k'$ with $k' < k$, or there is a nonzero letter $k'$ while a letter $k > k'$ is missing. The algorithm computing the canonical reduced word (Definition 3.34) shows that:

$$\ell(r) > 1 + 2 + \dots + k + \sum_{i \in r,\, i \neq 0} Z_r(i). \tag{4.18}$$

We call $\tilde{r} \in \mathcal{MCR}_n$ the rook vector obtained from $r$ by replacing the nonzero letters by $k+1, \dots, n$ in this order, so that $\sum_{i \in r,\, i \neq 0} Z_r(i) = \sum_{i=k+1}^{n} Z_{\tilde{r}}(i)$. Then $\tilde{r}\,\underline{m}$ gives $P_n$ as well. Thus $\ell(\underline{m}) \geq \sum_{i=k+1}^{n} (i - Z_{\tilde{r}}(i))$. So that $\ell(P_n) = |\underline{r}\,\underline{m}| > \binom{n+1}{2} = \ell(P_n)$, which is absurd. $\square$

In particular note that: $|\mathcal{MCR}_n| = \sum_{i=0}^{n} \binom{n}{i} = 2^n$.

**Example 4.32.** $\mathcal{MCR}_2 = \{12\} \cup \{0 \sqcup\!\sqcup 2\} \cup \{00\} = \{12\} \cup \{02, 20\} \cup \{00\}$

$$\begin{aligned}\mathcal{MCR}_3 &= \{123\} \cup \{0 \sqcup\!\sqcup 23\} \cup \{00 \sqcup\!\sqcup 3\} \cup \{000\} \\ &= \{123\} \cup \{023, 203, 230\} \cup \{003, 030, 300\} \cup \{000\},\end{aligned}$$

$$\begin{aligned}\mathcal{MCR}_4 &= \{1234\} \cup \{0 \sqcup\!\sqcup 234\} \cup \{00 \sqcup\!\sqcup 34\} \cup \{000 \sqcup\!\sqcup 4\} \cup \{0000\} \\ &= \{1234\} \cup \{0234, 2034, 2304, 2340\} \cup \{0034, 0304, 3004, 0340, 3040, 3400\} \\ &\quad \cup \{0004, 0040, 0400, 4000\} \cup \{0000\}.\end{aligned}$$

We now introduce a sequence of bijections from $\mathcal{MCR}_n$ to some special *Dyck paths*, that is vertices of the Tamari lattice. For now, recall that a *Dyck path* of length $n$ is a path in the plane starting from $(0, 0)$, ending in $(2n, 0)$ made with north-east (NE) $(1, 1)$ and south-east (SE) $(1, -1)$ such that the path is always above the line $y = 0$. We represent a Dyck path by a word of size $2n$ with $n$ letters 0 and $n$ letters 1, where 0 is a SE step, and 1 a NE step, and such that in every prefix of the word the number of 0 is less or equal to the number of 1. For instance 101100110 is a Dyck path. We will also represent it $1^1 0^1 1^2 0^2 1^2 0^1$.

The first bijection sends an element of $\mathcal{MCR}_n$ to a subset of $[n + 1]$ the following way:

$$\eta : \begin{cases} \mathcal{MCR}_n & \longrightarrow \quad [n+1] \\ r = r_1 \dots r_n & \longmapsto \quad \{i \mid r_i \neq 0\}. \end{cases} \tag{4.19}$$

This application is clearly a bijection since the nonzero letters of $r \in \mathcal{MCR}_n$ are $k + 1, \dots, n$ in this order, where $k$ is the number of zeros of $r$. Now that we have a subset of $[n]$ we can use the bijection C to compositions of $n+1$ introduced in Equation 2.5. If $I = (i_1, \dots, i_m) \vDash n+1$ the actions of the generators of $R_n^0$ through the bijection $\mathrm{C} \circ \eta$ are as follows:

$$I \cdot \pi_0 = (i_1 + i_2, i_3, \dots, i_m), \tag{4.20}$$
$$I \cdot \pi_j = (i_1, \dots, i_{j-1}, i_j - 1, i_{j+1} + 1, i_{j+2}, \dots, i_m) \text{ for } 0 < j < m. \tag{4.21}$$

We finally send a composition of $n + 1$ to a Dyck path as follows:

$$\delta : (i_1, \dots, i_m) \vDash n + 1 \longmapsto 1^{n-m}\, 0^{i_1}\, 1\, 0^{i_2}\, 1\, 0^{i_3} \dots 0^{i_{m-1}}\, 1\, 0^{i_m}. \tag{4.22}$$

It is easy to check that the Dyck paths we obtain this way are exactly those for whose the pattern 011 is forbidden. Note that the action of the generators of $R_n^0$ is thus to replace a 01 by 10 which pictorially inserts a diamond in a "valley". See Figure 4.2. We say that a Dyck path $D$ contains another Dyck path $D'$, and we denote it $D' \subseteq D$, if the path $D$ is above the path $D'$. Then the $\mathcal{R}$-order on $R_n^0$ is mapped to the order $\subseteq$ on Dyck paths avoiding the pattern 011 by the bijection $\delta \circ \mathrm{C} \circ \eta$. The maximal element of the poset on Dyck path is $1^{n+1}0^{n+1}$ and its minimal $(10)^{n+1}$. See the first line of Figure 4.5 to see all these isomorphisms. We finally remark that all these posets are actually lattices.

Now we briefly present the Tamari order, the reader should ref to [Tamari(1962)] for more details. A Dyck path is called *primitive* if it is not empty and has no other contact with the line $y = 0$ except at the starting and ending point. If $u$ is a Dyck path such that $u$ has a SE step $d$ followed by a primitive path $p$. Then the rotation on $u$ is to exchange the SE step

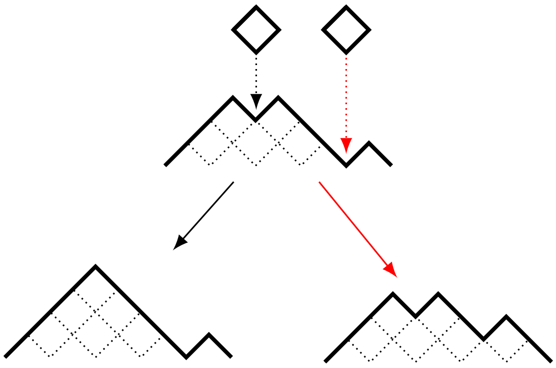


Figure 4.2: The flip of a valley in our special Dyck paths. The generator $\pi_i$ adds a diamond in the $i+1$th valley, counting from the left. Thus $\pi_0$ reduces the number of valley.

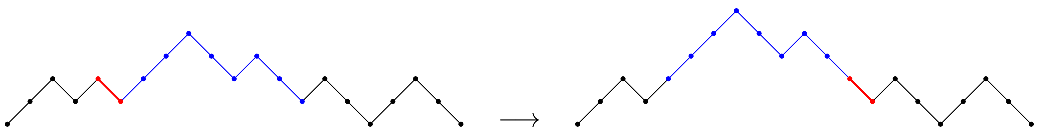


Figure 4.3: The rotation of Dyck words.

$d$ with the primitive path $p$. See Figure 4.3. These rotations are the cover relations of the Tamari order $\preceq_{\mathcal{T}}$.

We are interested in Dyck paths in a maximal chain of length $\binom{n}{2}$ in the Tamari lattice of size $n$. We denote by $\mathcal{MCT}_n$ their set.

**Proposition 4.33.** *The set $\mathcal{MCT}_n$ is exactly the set of Dyck paths avoiding* 011*. Furthermore the order $\preceq_{\mathcal{T}}$ restricted to $\mathcal{MCT}_n$ is equal to the order of inclusion $\subseteq$.*

*Proof.* The difference of diamonds between the minimal element $(10)^n$ and the maximal element $1^n0^n$ is exactly $\binom{n}{2}$, so that each rotation must add only one diamond. But a rotation on a SE step 0 followed by two NE steps 11 adds at least two diamonds, so that we can not rotate in such a SE step. Moreover the rotations on another licit SE step preserve the 011 pattern, so that an element with pattern 011 can not be in $\mathcal{MCT}_n$. On the contrary if $D$ is a Dyck path avoiding 011, then a rotation is exactly to add a diamond in a valley, and the resulting Dyck path also avoids 011.

Now that we have the description of elements of $\mathcal{MCT}_n$, doing a rotation corresponds to adding a diamond on a valley, so that the order $\preceq_{\mathcal{T}}$ implies the order $\subseteq$. Furthermore, by definition of the order $\preceq_{\mathcal{T}}$, the converse also holds. $\square$

As a consequence we have proven that the order on $\mathcal{MCT}_n$ obtained through the bijection $\delta \circ \mathrm{C} \circ \eta$ is exactly the Tamari order, so that the posets of $\mathcal{MCR}_n$ and $\mathcal{MCT}_{n+1}$ are isomorphic.

The elements appearing in $\mathcal{MCT}_n$ appears in many different contexts, see [Hohlweg and Lange(2007), Hohlweg et al.(2011)Hohlweg, Lange, and Thomas, Labbé and Lange(2018)] and the references in the latters. They correspond to binary trees which are chains, that is also binary trees with exactly one linear extension. For this reason they are called *singletons*. Equivalently they are permutations avoiding the patterns 132 and 312, or permutations with exactly one element in their sylvester class, that is common vertices between the associahedron and the permutahedron. Furthemore the historic definition of the associahedron is to keep only the faces of the permutahedron which contains such a singleton. See Figure 4.4, and [Hohlweg and Lange(2007)] for more details. See also Figure 4.5 for all the bijections seen in this section.

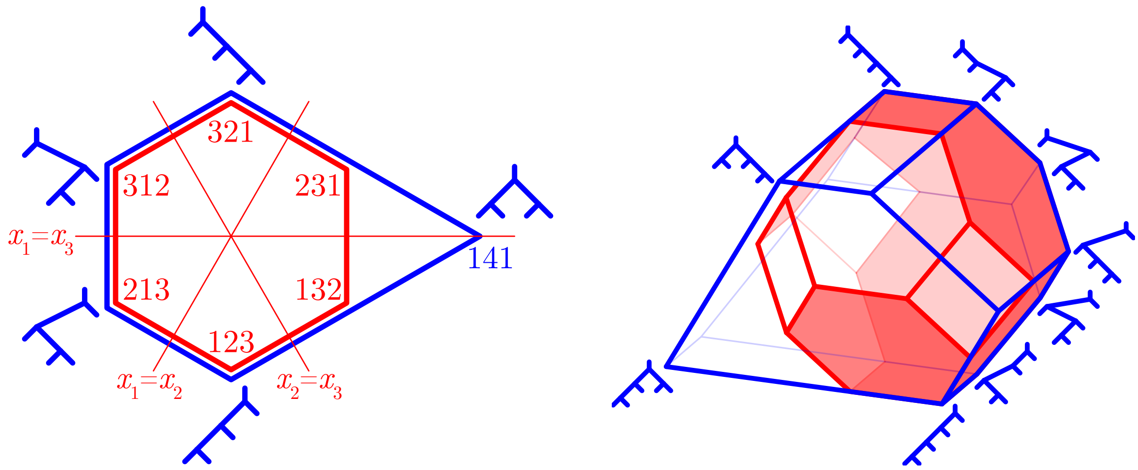



Figure 4.4: The Associahedron is obtained from the Permutahedron by keeping only faces containing a singleton.

## 4.4 Geometrical remarks

Recall that the right Cayley graph of the symmetric group $\mathfrak{S}_n$ is the 1-skeleton of a polytope namely the permutohedron [Ziegler(1995), Example 0.10]. It is defined as the convex hull of the set of points whose coordinates are permutations. It therefore lives in the hyperplane $\sum x_i = \frac{n(n+1)}{2}$, so that it is a polytope of dimension $n-1$.

Starting with $n = 3$, we can not hope that the right Cayley graph of $R_n$ could be the 1-skeleton of a polytope. Indeed in $R_n$ the element $1000\dots$ is always of degree 2, being linked only to $0000\dots$ and $0100\dots$, whereas the identity $123\dots$ is of degre $n$. Thus it is impossible to get a polytope.

Nevertheless, one can consider in a $n$-dimensional space the set of points whose coordinates are rook vectors (see Figure 4.6). The extremal points of its convex hull are the points in

$$\mathrm{Stell}_n := \{\mathfrak{S}_n(0\dots0k\dots n) \mid k \in [\![1, n]\!]\}\,. \tag{4.23}$$

This polyedron appeared under the name of stellohedron in [Manneville and Pilaud(2017), Figure 18] where it was defined as the graph associahedron of a star graph. It is also the secondary polytope of $\Delta_n \cup 2\Delta_n$ (see [Gelfand et al.(2008)Gelfand, Kapranov, and Zelevinsky]), two concentric copies of a $n$-dimensional simplex, which can also be defined as

$$\{e_i \mid i \in [n+1]\} \cup \{(n+2)e_i - \mathbf{1} \mid i \in [n+1]\}\,. \tag{4.24}$$

So we can see the Cayley graph of $R_n$ as being drawn on the face of the stellohedron. One can recover this graph from the permutohedron by taking all its projections on coordinate planes. Indeed, it is just saying that a rook can be obtained from a permutation replacing some entries by zeros and that edges are mapped to an identical edge or contracted.

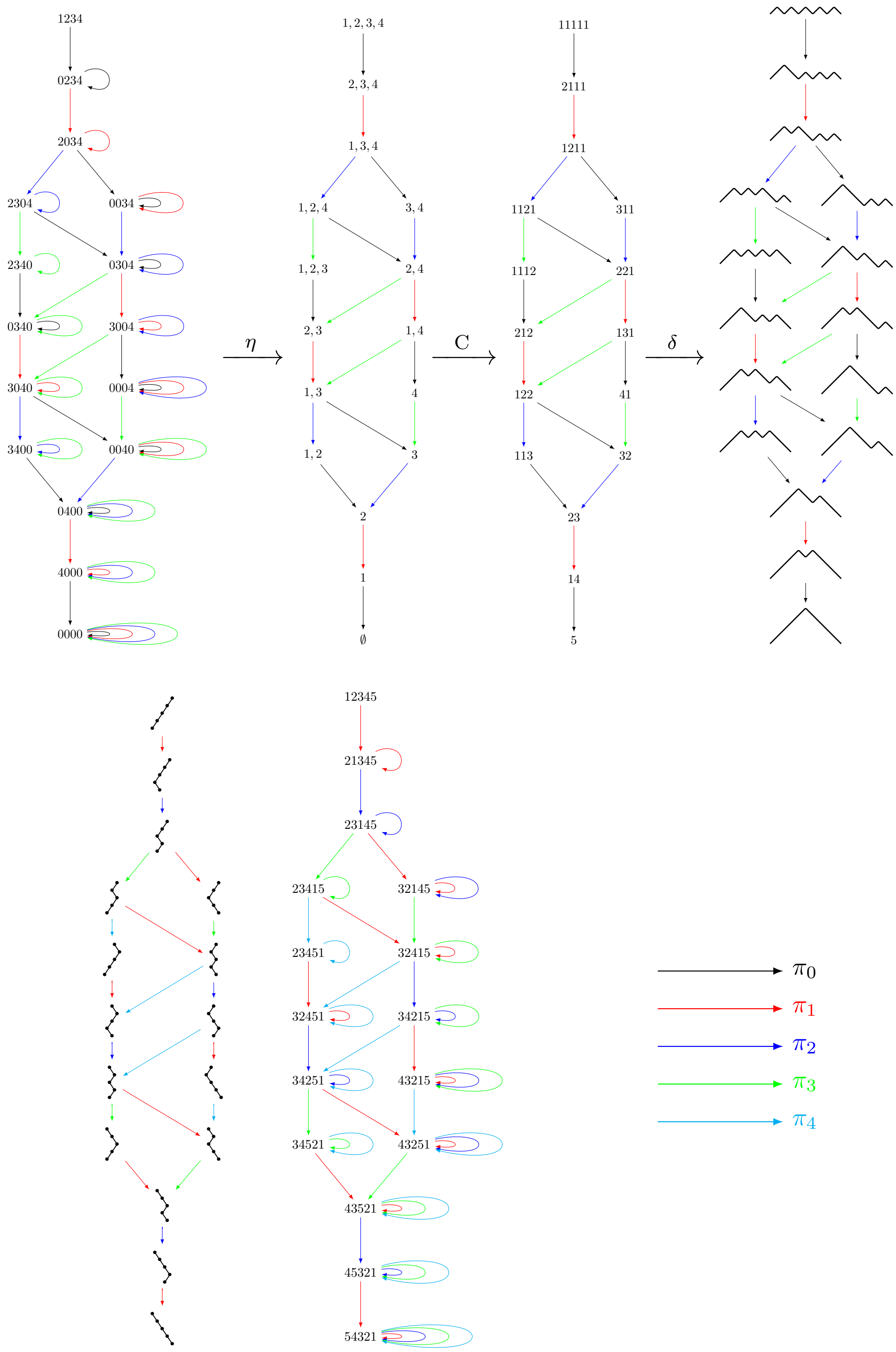



Figure 4.5: The lattice of $\mathcal{MCR}_4$, send to subsets of $[4]$, compositions of 5 and $\mathcal{MCT}_5$. On the second row we represent the poset $\mathcal{MCT}_5$ seen on binary trees which are chains, and permutations alone in their sylvester class or avoiding 132 and 312. We only represent loops on the rook vectors and the permutations, the other can be deduced by bijection. On the second line we apply generators of $H_5^0$ rather than $R_4^0$. Note that the bijection on the generators is only $\pi_i \mapsto \pi_{i+1}$.

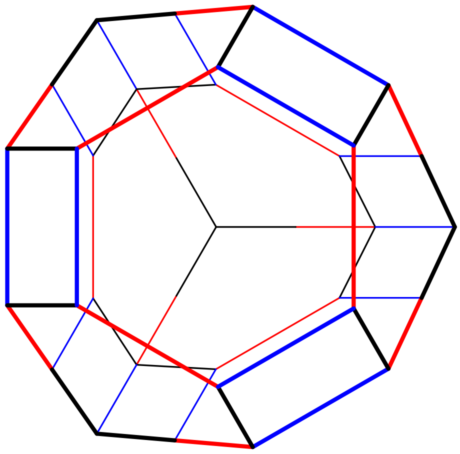


Figure 4.6: The Cayley graph of $R_3$ embedded in a 3-dimensional space.

### 4.5 A monoid associated to the stellohedron

The geometrical considerations raise the question whether there is a monoid structure giving the skeleton of the Stellohedron as Cayley graph. It turns out that the answer is true. Will we show moreover that there are monoids and lattice structures on graphs interpolating between the rook case and the stellar case.

**Definition 4.34.** *For any rook $r \in R_n$ denote $M(r) := \max\{i \in [\![1, n]\!] \mid i \notin r\}$ define $\mathrm{St}(r)$ to be the rook obtained by replacing by 0 all the letter smaller that $M(r)$ in $r$.*

**Example 4.35.** $M(104625) = 3$ and thus $\mathrm{St}(104625) = 004605$. Similarly $M(10806270) = 5$ and thus $\mathrm{St}(10806270) = 00806070$.

Clearly for any rook $r \in R_n$ then $\mathrm{St}(r) \in \mathrm{Stell}_n$, and for any $s \in \mathrm{Stell}_n$ one has $\mathrm{St}(s) = s$. We have then proved the following lemma:

**Lemma 4.36.** *The map $\mathrm{St}$ is a projection (i.e. $\mathrm{St} \circ \mathrm{St} = \mathrm{St}$) on $\mathrm{Stell}_n$.*

**Proposition 4.37.** *Denote $\mathrm{St}^0 : R_n^0 \to R_n^0$ the map corresponding to $\mathrm{St}$ in $R_n^0$, that is $\mathrm{St}^0(\pi_r) := \pi_{\mathrm{St}(r)}$. Then $\mathrm{St}^0$ is compatible with the product of $R_n^0$, namely for any $r, s \in R_n$*

$$\mathrm{St}^0(\mathrm{St}^0(\pi_r)\,\mathrm{St}^0(\pi_s)) = \mathrm{St}^0(\pi_r \pi_s). \tag{4.25}$$

*As a consequence, there is a unique monoid structure on $\mathrm{Stell}_n^0 := \mathrm{St}^0(R_n^0) = \{\pi_s \mid s \in \mathrm{Stell}_n\}$ such that such that $\mathrm{St}^0 : R_n^0 \to \mathrm{Stell}_n^0$ is a surjective monoid morphism.*

*Proof.* It is sufficient to show that for any $i \in [\![0, n-1]\!]$ and any $r \in R_n$ one has

$$\mathrm{St}^0(r \cdot \pi_i) = \mathrm{St}^0(\mathrm{St}^0(r) \cdot \pi_i) \qquad \text{and} \qquad \mathrm{St}^0(\pi_i \cdot r) = \mathrm{St}^0(\pi_i \cdot \mathrm{St}^0(r))\,. \tag{4.26}$$

Indeed these equalities means that the relation $\equiv$ defined by $r \equiv s$ if and only if $\mathrm{St}^0(r) = \mathrm{St}^0(s)$ is a monoid congruence. They are easily checked on the definition of the left and right action (Definitions 3.8 and 3.56). $\square$

We now explicit the left and right multiplication of the generator in $\mathrm{Stell}_n^0$:

**Proposition 4.38.** *We denote* $\overline{\pi}_i := \mathrm{St}^0(\pi_i)$*. Then* $\mathrm{Stell}_n^0$ *is generated by* $\{\overline{\pi}_i \mid 0 \leq i < n\}$*. And for* $i \in [\![1, n-1]\!]$ *and* $s = (s_1 \dots s_n) \in \mathrm{Stell}_n$*, one has* $\pi_s \overline{\pi}_i = \pi_s \pi_i$ *and* $\pi_{(s_1 \dots s_n)} \cdot \overline{\pi}_0 = \pi_u$ *where* $u$ *is the vector obtained by replacing all the element less or equal to* $s_1$ *by* $0$ *in* $s$*.*

*On the left, the product is given by* $\overline{\pi}_i \pi_{(s_1 \dots i+1 \dots s_n)} = \pi_{(s_1 \dots 0 \dots s_n)}$ *if* $i \notin s$ *and* $i+1 \in s$ *and* $\overline{\pi}_i \pi_r = \pi_i \pi_r$ *in all the other cases.*

We can moreover give a presentation for this new monoid:

**Theorem 4.39.** *The stellar monoid* $\mathrm{Stell}_n^0$ *is the quotient of the rook monoid by the relations*

$$\pi_i \pi_{i-1} \dots \pi_1 \pi_0 \pi_i \;\equiv\; \pi_i \pi_{i-1} \dots \pi_1 \pi_0 \tag{ST}$$

*for* $i < n-1$*.*

In order to prove the theorem, we need two preliminary lemmas.

**Lemma 4.40.** *Relation ST holds in* $\mathrm{Stell}_n^0$*.*

*Proof.* If we apply both side of Relation ST on the left on the identity rook, then $\pi_i$ exchange $i$ and $i+1$ and $\pi_i \pi_{i-1} \dots \pi_1 \pi_0$ kills all letters from $0$ to $i+1$. So both side are equal. □

**Lemma 4.41.** *In the rook monoid Relations ST implies the following relations:*

$$\pi_j \;\pi_i \pi_{i-1} \dots \pi_1 \pi_0 \;\equiv\; \pi_i \pi_{i-1} \dots \pi_1 \pi_0 \tag{ST'}$$

*for any* $0 \leq j \leq i < n$*.*

*Proof.* We distinguish three cases:

- $j = 0$. In this case, we have

$$\begin{aligned}
\pi_0 \pi_i \pi_{i-1} \dots \pi_1 \pi_0 &= \pi_i \pi_{i-1} \dots \pi_2 \pi_0 \pi_1 \pi_0 && \text{(by R4)}\\
&= \pi_i \pi_{i-1} \dots \pi_2 \pi_1 \pi_0 \pi_1 \pi_0 && \text{(by R3)}\\
&\equiv \pi_i \pi_{i-1} \dots \pi_2 \pi_1 \pi_0 \pi_0 && \text{(mod ST with } i = 1)\\
&= \pi_i \pi_{i-1} \dots \pi_2 \pi_1 \pi_0 && \text{(by R1).}
\end{aligned}$$

- $0 < j < i$. In this case, we have

$$\begin{aligned}
\pi_j \pi_i \pi_{i-1} \dots \pi_1 \pi_0 &= \pi_i \pi_{i-1} \dots \pi_j \pi_{j+1} \pi_j \dots \pi_1 \pi_0 && \text{(by R4)}\\
&= \pi_i \pi_{i-1} \dots \pi_{j+1} \pi_j \pi_{j+1} \dots \pi_1 \pi_0 && \text{(by R2)}\\
&= \pi_i \pi_{i-1} \dots \pi_{j+1} \pi_j \dots \pi_1 \pi_0 \pi_{j+1} && \text{(by R4)}\\
&\equiv \pi_i \pi_{i-1} \dots \pi_2 \pi_1 \pi_0 && \text{(mod ST with } i = j+1).
\end{aligned}$$

- $j = i$. In this case, we just have to apply R1. □

*Proof of Theorem 4.39.* From Corollary 3.49, for any rook $r \in R_n$, its $R$-code $\mathsf{c} = (\mathsf{c}_1, \dots, \mathsf{c}_n)$ verifies (with the notation of Definition 3.34):

$$\pi_r = \boxed{\begin{matrix} 0 \\ \vdots \\ \mathsf{c}_1 \end{matrix}} \cdot \boxed{\begin{matrix} 1 \\ \vdots \\ \mathsf{c}_2 \end{matrix}} \cdot \dots \cdot \boxed{\begin{matrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{matrix}}\,, \tag{4.27}$$

We denote $m = \max\{i \mid \mathsf{c}_i \leq 0\}$ with the convention that $m = 0$ if all the $\mathsf{c}_i$ are positive. Then thanks to relation ST' we know that

$$\pi_r \equiv \boxed{\begin{smallmatrix} m-1 \\ \vdots \\ 0 \end{smallmatrix}}\ \boxed{\begin{smallmatrix} m \\ \vdots \\ \mathsf{c}_{m+1} \end{smallmatrix}} \cdots\cdots \boxed{\begin{smallmatrix} n-1 \\ \vdots \\ \mathsf{c}_n \end{smallmatrix}} \qquad (\mathrm{mod}\ \mathrm{ST}), \tag{4.28}$$

where the first column is empty if $m = 0$. We call *stellar canonical word* any word appearing on the right hand side of this equation. In this case, the $(\mathsf{c}_i)_{i>m}$ verify $0 < \mathsf{c}_i \leq i$, so that there are

$$\sum_{m=0}^{n} (m+1)\dots(n-1)n = \sum_{m=0}^{n} \frac{n!}{m!} = \left|\mathrm{Stell}_n^0\right| \tag{4.29}$$

stellar canonical words. We have shown that each rook is congruent to a stellar canonical words modulo ST and that they are as many stellar canonical words as element of $\mathrm{Stell}_n^0$. As a consequence Relation ST is the only relations needed to get $\mathrm{Stell}_n^0$ from $R_n^0$. $\square$

### 4.5.1 The stelloid lattice

By analogy to the Rook lattice, we might wonder if the $\mathcal{R}$-order or $\mathcal{L}$-order of $\mathrm{Stell}_n^0$ are lattices (on the contrary to rooks, they are not isomorphic). It turns out that the $\mathcal{L}$-order is a lattice. See Figure 4.7 for a picture. We will show actually a stronger result:

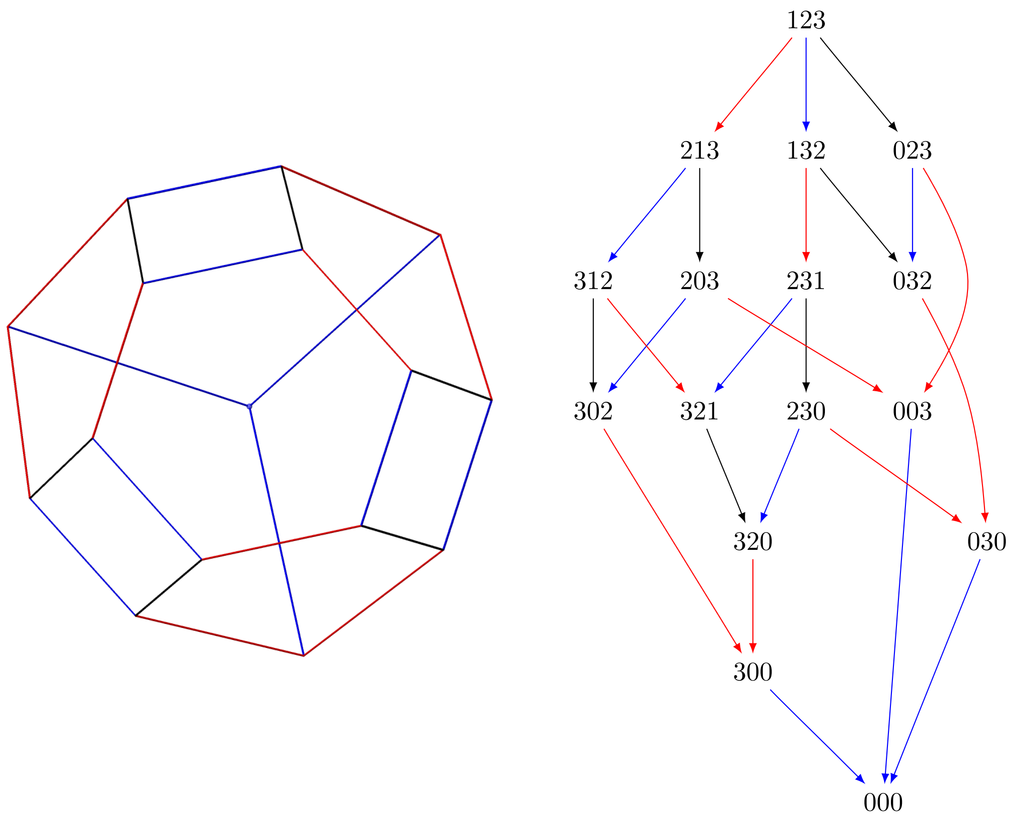



Figure 4.7: The left order of $\mathrm{Stell}_3^0$

**Theorem 4.42.** *The $\mathcal{L}$-order on* $\mathrm{Stell}_n^0$ *is a sublattice of the $\mathcal{L}$-order of* $R_n^0$.

*Proof.* We need to conjugate all the rooks to pass to the left order. The conjugate of a stellar rook $r$ is a rook such that all the zeroes are at the beginning. Equivalently this means that in its rook triple $(S_r, I_r, Z_r)$, the $Z_r$ function is the zero function. Now looking at the algorithm for computing the meet and join of two rooks, we have that

$$Z_{u\wedge_{\mathcal{R}}v}(x) := \max\{Z_u(i), Z_v(i) \mid i = x \text{ or } (x,i) \in I_{u\wedge_{\mathcal{R}}v}\}\,, \tag{4.30}$$

$$Z_{u\vee_{\mathcal{R}}v}(x) := \min\{Z_u(i), Z_v(i) \mid i = x \text{ or } (x,i) \in \Delta \setminus I_{u\wedge_{\mathcal{R}}v}\}\,. \tag{4.31}$$

As a consequence both $Z_{u\wedge_{\mathcal{R}}v}$ and $Z_{u\vee_{\mathcal{R}}v}$ are zero functions so that $u \wedge_{\mathcal{R}} v$ and $u \vee_{\mathcal{R}} v$ are conjugate stellar rooks too. □

**Remark 4.43.** The preimage of the stellar rook 300 is $\{300, 301, 310\}$ which is not a interval of the $\mathcal{L}$-order. As a consequence, $\mathrm{St}^0$ can't be a lattice morphism and the $\mathcal{L}$-order of $\mathrm{Stell}_n^0$ is a not a lattice quotient of the $\mathcal{L}$-order of $R_n^0$.

### 4.5.2 Higher order Stelloid monoid and lattices

The proofs of the two previous theorem makes it clear that the $\mathrm{Stell}_n^0$ monoid together with its $\mathcal{L}$-lattice is a particular case of a more general construction: for $k \geq 0$, define $\mathrm{St}_k$ the map from rooks to rooks which replace by 0 all the letter $i$ such that there is $k$ or more missing letter larger that $i$ (the usual St map is the case $k = 1$). For example $\mathrm{St}_2(3057016) = (3057006)$ and $\mathrm{St}_2(3407016) = (3407006)$. Also, $\mathrm{St}_i \circ \mathrm{St}_j = \mathrm{St}_{\max(i,j)}$. So that for all $n$, we have the inclusion of sets:

$$\{0^n\} = \mathrm{St}_0(R_n) \subset \mathrm{St}_1(R_n) \subset \mathrm{St}_2(R_n) \subset \cdots \subset \mathrm{St}_n(R_n) = R_n\,. \tag{4.32}$$

The following array give the cardinality of $\mathrm{St}_k(R_n)$.

| $k\backslash n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | | 2 | 5 | 16 | 65 | 326 | 1957 | 13700 |
| 2 | | | 7 | 31 | 165 | 1031 | 7423 | 60621 |
| 3 | | | | 34 | 205 | 1456 | 11839 | 108214 |
| 4 | | | | | 209 | 1541 | 13165 | 127289 |
| 5 | | | | | | 1546 | 13321 | 130656 |
| 6 | | | | | | | 13327 | 130915 |
| 7 | | | | | | | | 130922 |

Then the proof of Proposition 4.37 and Theorem 4.42 generalize to this new cases:

**Theorem 4.44.** *Denote* $\mathrm{St}_k^0 : R_n^0 \to R_n^0$ *the map corresponding to* $\mathrm{St}_k$ *in* $R_n^0$. *Then this map is a surjective monoid morphism to* $\mathrm{St}_k^0(R_n^0)$. *Moreover, the $\mathcal{L}$-order of* $\mathrm{St}_k^0(R_n^0)$ *is a sublattice of the $\mathcal{L}$-order of* $R_n^0$.

Hence Equation 4.32 is actually a sequence of inclusion of lattices. It also give rise to the following sequence of monoid morphisms:

$$\{0^n\} = \mathrm{St}_0(R_n^0) \leftarrow \mathrm{St}_1(R_n^0) \leftarrow \mathrm{St}_2(R_n^0) \leftarrow \cdots \leftarrow \mathrm{St}_n(R_n^0) = R_n^0\,. \tag{4.33}$$

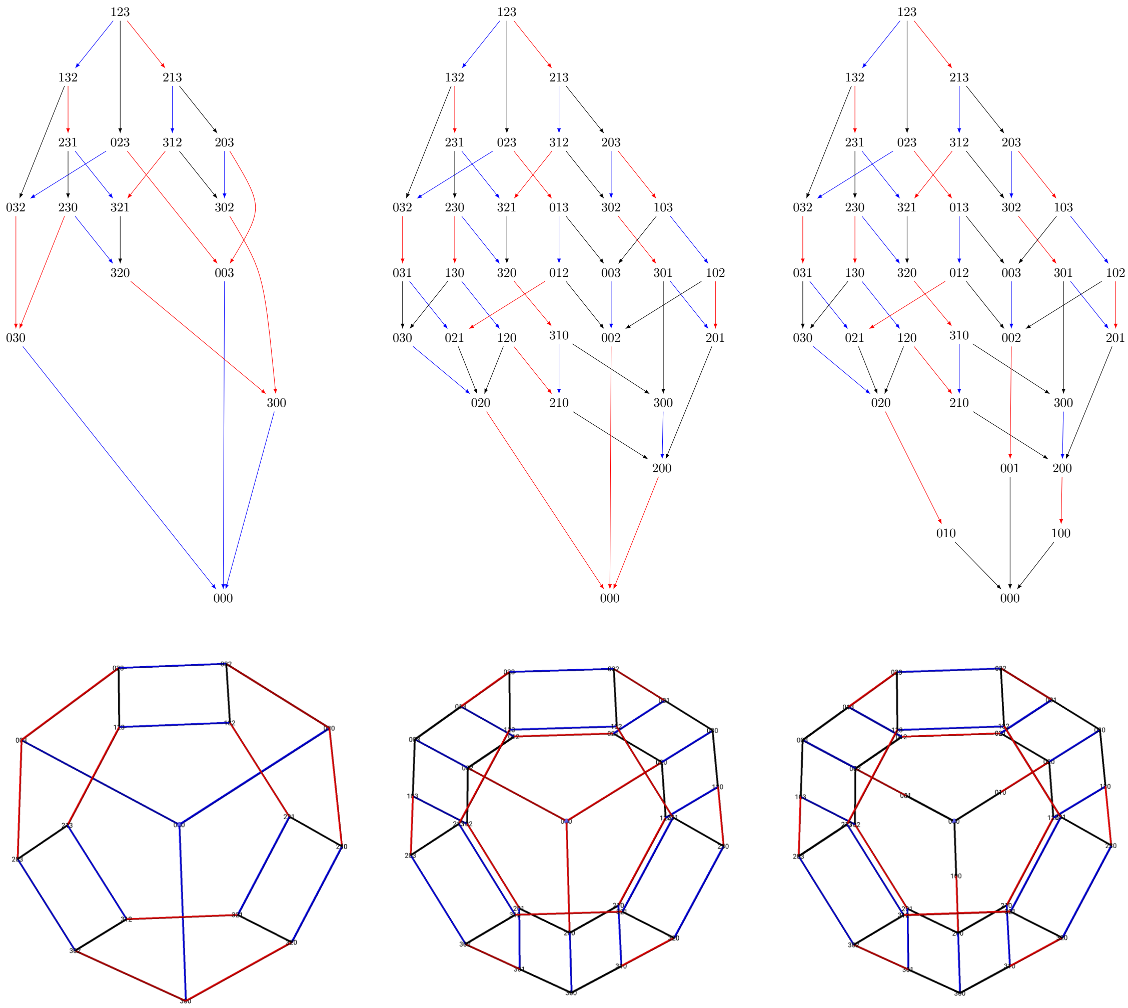


Figure 4.8: The left order of $\mathrm{St}_k^0(R_3)$ for $k = 1, 2, 3$

Figure 4.8 shows the three consecutive quotients of $\mathrm{St}_k^0(R_3)$ together with their geometric counterpart.

We finally remark that the quotient morphism are in the opposite direction of the inclusion of lattices of Equation 4.32. This suggest some kind of duality, but we haven't been able to give a formulation.

# 5 Representation theory of the 0-Rook monoid $R_n^0$

The goal of this section is to investigate the representation theory of $R_n^0$. We write $\mathbb{C}[R_n^0]$ the monoid algebra of $R_n^0$. In the sequel of the article $P_1$ will rather be denoted by $\pi_0$. Moreover, $r$ will usually denote an element of $R_n^0$. Also, we know from Corollary 3.49 and Proposition 3.59 that for any $r \in R_n^0$ there is a unique rook $\mathbf{r} := 1_n \cdot r = r \cdot 1_n$ such that $\pi_{\mathbf{r}} = r$. So when there is a need to distinguish, we will denote in normal letter $r$ the elements of the monoid and in boldface as $\mathbf{r}$ their associated rooks.

We start by summarizing the main results (in particular Corollary 3.49) of the Section 3 which concerns the representations:

**Proposition 5.1.** *The maps*

$$f_R : \left| \begin{array}{ccc} \mathbb{C}[R_n^0] & \longrightarrow & \mathbb{C}R_n \\ x & \longmapsto & 1_n \cdot x\,, \end{array} \right. \qquad \text{and} \qquad f_L : \left| \begin{array}{ccc} \mathbb{C}[R_n^0] & \longrightarrow & \mathbb{C}R_n \\ x & \longmapsto & x \cdot 1_n\,, \end{array} \right.$$

*extended by linearity, are two isomorphisms of representations of $R_n^0$ between the left and right regular representations and the natural one (acting on $R_n$).*

## 5.1 Idempotents and Simple modules

As for any algebra, the representation theory of $\mathbb{C}[R_n^0]$ (or equivalently $R_n^0$) is largely governed by its idempotents, however since $R_n^0$ is a $\mathcal{J}$-trivial monoid, as shown in [Denton et al.(2010/11)Denton, Hivert
it is sufficient to look for idempotents in the monoid $R_n^0$ itself.

**Proposition 5.2.** *For any $S \subset [\![0, n-1]\!]$, we denote $\pi_S$ the zero of the so-called parabolic submonoid generated by $\{\pi_i \mid i \in S\}$.*

*Proof.* This submonoid is finite since the ambient monoid $R_n^0$ is finite. By Proposition 4.17 it contains a unique minimal element for the $\mathcal{J}$-order, which is a zero. □

**Proposition 5.3.** *For any $S \subset [\![0, n-1]\!]$, write $S^c := [\![0, n-1]\!] \setminus S$ its complement and $I = C(S^c) = (i_1, \ldots, i_\ell)$ its associated extended composition. Then $\pi_S = \pi_{\mathbf{r}}$ where $\mathbf{r}$ is the block diagonal rook matrix of size $n$ whose block are anti diagonal matrices of $1$ of size $(i_1, \ldots, i_\ell)$, except the first block which is a zero matrix.*

Note that if $0 \notin S$ then the first part of $I$ is zero, so that the first zero block is of size 0 and therefore vanishes.

**Example 5.4.** If $n = 12$ and $S = \{0, 1, 2, 5, 7, 8, 11\}$. Then $S^c = \{3, 4, 6, 9, 10\}$ so that $I = C(S^c) = (3, 1, 2, 3, 1, 2)$. Similarly, if $T = \{2, 4, 5, 7, 8, 9, \}$, then $T^c = \{0, 1, 3, 6, 10, 11\}$ so that $J = C(T^c) = (0, 1, 2, 3, 4, 1, 1)$. Therefore the associated matrices are:

$$\pi_S = \left(\begin{array}{c|c|c|c|c|c} \begin{smallmatrix}0&0&0\\0&0&0\\0&0&0\end{smallmatrix} & & & & & \\ \hline & 1 & & & & \\ \hline & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & & & \\ \hline & & & \begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix} & & \\ \hline & & & & 1 & \\ \hline & & & & & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} \end{array}\right) \qquad \text{and} \qquad \pi_T = \left(\begin{array}{c|c|c|c|c|c} 1 & & & & & \\ \hline & \begin{smallmatrix}0&1\\1&0\end{smallmatrix} & & & & \\ \hline & & \begin{smallmatrix}0&0&1\\0&1&0\\1&0&0\end{smallmatrix} & & & \\ \hline & & & \begin{smallmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{smallmatrix} & & \\ \hline & & & & 1 & \\ \hline & & & & & 1 \end{array}\right)$$

*Proof.* We fix some $S$ and consider $\mathbf{r}$ the associated rook matrix. The block diagonal structure ensures that $\pi_{\mathbf{r}}$ belongs to the parabolic submonoid $\langle \pi_i \mid i \in S\rangle$. Indeed, suppose that there is a reduced word $\underline{w}$ for $\pi_{\mathbf{r}}$ with some $w_i \notin S$. Recall, that from Corollary 3.49, this means that $1_n \cdot \underline{w} = \mathbf{r}$. Choose the smallest such $i$. There are two cases whether $w_i = \pi_0$ or not.

- if $w_i = \pi_0$ with $0 \notin S$, then when computing $1_n \cdot w_1 \cdots w_{i-1} \cdot w_i$, the action of $\pi_0$ will be to kill a column. In this case, the killed column will never appear again so that there is no way to get the correct matrix.
- if $w_i = \pi_i$ with $i \neq 0$, when computing $1_n \cdot w_1 \cdots w_{i-1} \cdot w_i$, the action of $w_i$ is to exchange two columns from two different blocks. However, acting by any $\pi_j$ will never exchange those two columns again, so that it is not possible to get them back in the correct order.

Hence, we have proven that $\underline{w}$ only contains $\pi_i$ with $i \in S$ that is $r \in \langle \pi_i \mid i \in S\rangle$. Furthermore, using the action on matrices one sees that $r \cdot \pi_i = r$ or equivalently that $\pi_{\mathbf{r}}\pi_i = \pi_{\mathbf{r}}$ if and only if $i \in S$. This shows that $\pi_{\mathbf{r}}$ is the zero of $\langle \pi_i \mid i \in S\rangle$. □

**Remark 5.5.** If we decompose the set $S$ into its maximal components of consecutive letters $S_1 \cup S_2 \cup \cdots \cup S_r$, then $\pi_S = \prod_{1\leq i\leq r} \pi_{S_i}$ where the product commutes. Moreover, if $0 \in S$ then $\pi_{S_1} = P_m$ where $m$ is the size of the first block.

During the proof, we got the following Lemma:

**Lemma 5.6.** *Let $S \subset [\![0, n-1]\!]$. Then $\pi_S\pi_i = \pi_S = \pi_i\pi_S$ if $i \in S$, and $\pi_S\pi_i \neq \pi_S$ and $\pi_i\pi_S \neq \pi_S$ otherwise.*

**Proposition 5.7.** *The monoid $R_n^0$ has exactly $2^n$ idempotents: these are the zeros of every parabolic submonoid.*

*Proof.* We already know that $R_n^0$ has at least $2^n$ idempotents. We now have to prove this exhaust the idempotents of $R_n^0$.

Let $e$ an idempotent of $R_n^0$. Recall that $\mathrm{cont}(e)$ is the set of the $\pi_i$ with $i \in [\![0, n-1]\!]$ appearing in any reduced word of $e$. Let us show that $e = \pi_{\mathrm{cont}(e)}$, that is the zero of the parabolic submonoid $\langle \pi_i \mid i \in \mathrm{cont}(e)\rangle$. Indeed for $a \in \mathrm{cont}(e)$, one can write $e = \underline{uav}$ for some $\underline{u}$ and $\underline{v}$ in $R_n^0$. By definition of the $\mathcal{J}$-order, this means that $e \leq_{\mathcal{J}} a$. Using [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Lemma 3.6], this is equivalent to $ea = e$ and to $ae = e$, so that $e$ is stable under all its support. □

**Theorem 5.8.** *The monoid $R_n^0$ has $2^n$ left (and right) simple modules, all one-dimensional, indexed by the subsets of $[\![0, n-1]\!]$. Let $S \subset [\![0, n-1]\!]$. Its associated simple module $S_S$ is the one-dimensional module generated by $\varepsilon_S$ with the following action of generators:*

$$\pi_i \cdot \varepsilon_S = \begin{cases} \varepsilon_S & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

*Proof.* We apply [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Proposition 3.1] using Lemma 5.6. □

Recall that we write $x^\omega$ any sufficiently large power of $x$ which becomes idempotent, and that the star product of two idempotents is defined as $e * f = (ef)^\omega$. This endows the set of idempotents with a structure of a lattice where $*$ is the meet [Denton et al.(2010/11)Denton, Hivert, Schilling, Theorem 3.4]. We now explicitely describe this lattice:

**Proposition 5.9.** *Let $S, T \subset [\![0, n-1]\!]$. Then $\pi_S * \pi_T = \pi_{S \cup T}$.*

*Proof.* First we note that $\pi_S * \pi_T$ is inside the parabolic $S \cup T$. It is enough to show that it is a zero of this submonoid, and then conclude by unicity. The product formual is is a consequence of [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Lemma 3.6]. $\square$

**Corollary 5.10.** *The quotient $\mathbb{C}[R_n^0]/\operatorname{Rad}(\mathbb{C}[R_n^0])$ is isomorphic to the algebra of the lattice of the $n$-dimensionnal cube.*

### 5.2 Indecomposable projective modules

Recall that the indecomposable projective $H_n^0$-modules are spanned by descent classes (see Section 2.5 and reference therein for more details). Extending the definition from the Hecke monoid, we define left and right $R$-descents sets of a rook as:

$$D_R(r) = \{0 \le i \le n-1 \mid r\pi_i = r\} \quad (\text{resp. } D_L(r) = \{0 \le i \le n-1 \mid \pi_i r = r\})\,. \tag{5.2}$$

**Example 5.11.** Let $r = \mathbf{0423007} \in R_n^0$. We have $0 < 4 \ge 2 < 3 \ge 0 \ge 0 < 7$, and the first letter is 0. So $D_R(r) = \{0, 2, 4, 5\}$.

**Notation 5.12.** *We choose to represent an element $r \in R_n^0$ by a ribbon notation the usual way, with the difference that two zeros are vertical and not horizontal:* $\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline\end{array}$ *and not* $\begin{array}{|c|c|}\hline 0 & 0 \\ \hline\end{array}$.

This change of convention compared to *e.g.* [Krob and Thibon(1997)] is due to our choice of taking the $\pi$'s and not the $T_i$'s for generators of the Hecke algebra. As a consequence, the eigenvalues 0 and 1 are exchanged.

For example, $r = \mathbf{0423007}$ is represented by the ribbon $\begin{array}{cccc} 0 & 4 & & \\ & 2 & 3 & \\ & & 0 & \\ & & 0 & 7 \end{array}$ Figure 5.1 shows the ribbon together with their associated descent sets. Figure 5.2 depicts the associated boolean lattice. With this notation we can easily find the idempotents of each $R$-descent set:

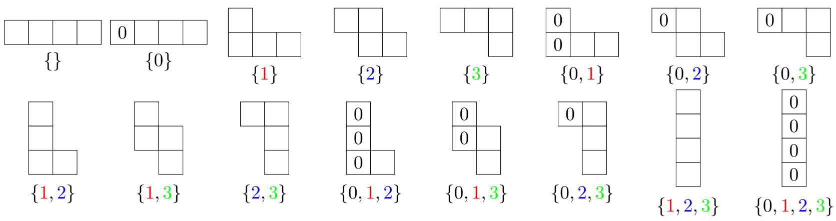



Figure 5.1: $R$-descent sets for $R_4^0$.

**Proposition 5.13.** *In each $R$-descent class there is a unique idempotent. It is obtained by filling ribbon shape by numbers 1 to $n$ in this order, going through the columns left to right and bottom to up. Then if 0 is in the descent class, fill the first column with zeros.*

*Proof.* The existence and the uniqueness come from Corollary 5.6. The way to fill in comes from Proposition 5.3. $\square$

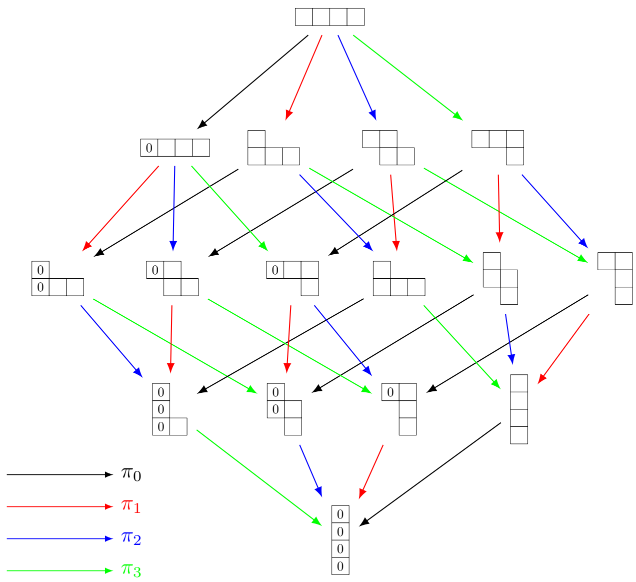
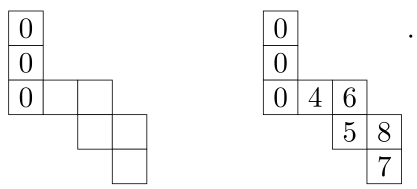



Figure 5.2: The lattice of $R$-descent sets for $R_4^0$.

**Example 5.14.** Consider the $R$-descent set $\{0, 1, 2, 5, 6, 7\}$ in size 8. We show below its associated ribbon shape and its idempotent



We now follow [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Section 3.4], specializing it to the combinatorics of $R_n^0$. Recall that the automorphism sub-monoid $\mathrm{rAut}(x)$ and $\mathrm{lAut}(x)$ are defined by

$$\mathrm{rAut}(x) := \{u \in M \mid xu = x\} \qquad \text{and} \qquad \mathrm{lAut}(x) := \{u \in M \mid ux = x\}\,. \tag{5.3}$$

**Proposition 5.15.** *Let $r \in R_n^0$.*

$$\mathrm{rAut}(r) = \langle \pi_i \mid i \in D_R(r) \rangle \qquad \textit{and} \qquad \mathrm{lAut}(r) = \langle \pi_i \mid i \in D_L(r) \rangle. \tag{5.4}$$

*Proof.* We do the proof for rAut. The first inclusion $\langle \pi_i \mid i \in D_R(r) \rangle \subseteq \mathrm{rAut}(r)$ is clear.

Let $u \in \mathrm{rAut}(r)$. So $ru = r$. Assume that $u \notin \langle \pi_i \mid i \in D_R(r) \rangle$. Let $\pi_{i_1} \dots \pi_{i_m}$ a reduced expression of $u$. Let $j$ be the smallest index such that $i_j \notin D_R(r)$. Then $ru = r\pi_{i_j} \dots \pi_{i_m}$ by minimality. Since $i_j \notin D_R(r)$, $r\pi_{i_j} <_{\mathcal{J}} r$ and by $\mathcal{J}$-triviality we get $ru <_{\mathcal{J}} r$. This contradict the minimality. □

From [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Proposition 3.16], we get the following corollary:

**Corollary 5.16.** *Let $r \in R_n^0$*

$$\mathrm{rfix}(r) = \pi_{D_R(r)} \qquad \textit{and} \qquad \mathrm{lfix}(r) = \pi_{D_L(r)}. \tag{5.5}$$

Then, applying Theorem 2.10, we get:

**Theorem 5.17.** *The indecomposable projective $R_n^0$-modules are indexed by the $R$-descents sets and isomorphic to the quotient of the associated $R$-descent class by the finer $R$-descent classes.*

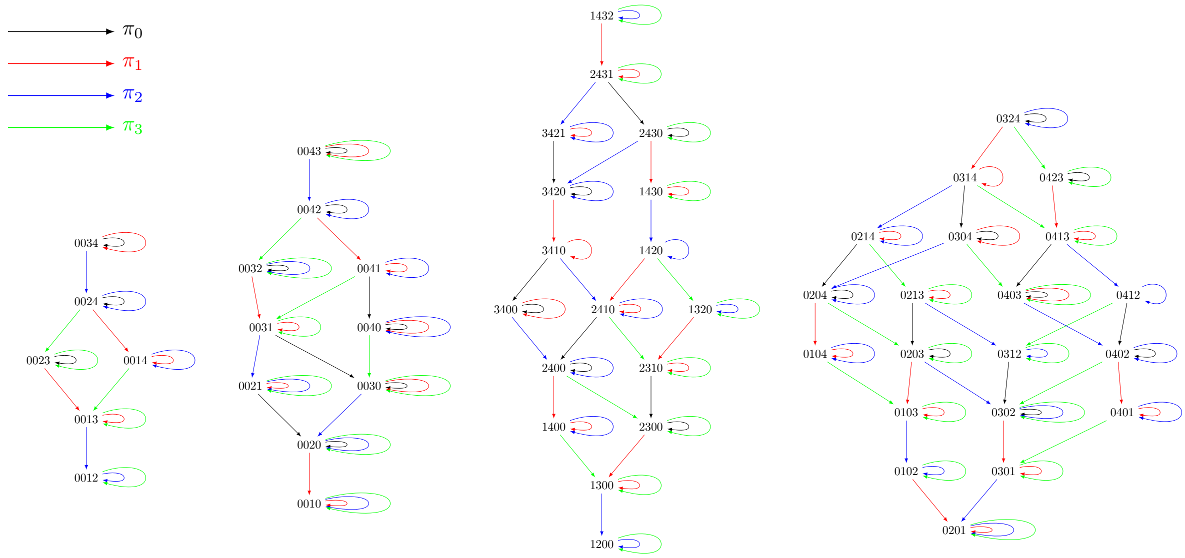



Figure 5.3: The $R$-descent classes $\{0,1\}$, $\{0,1,3\}$, $\{2,3\}$ and $\{0,2\}$.

Finally [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Theorem 3.20] gives the coefficients of the Cartan matrix of $R_n^0$ as the number of rooks with a given left and right descent set. We give it in Annex B, for $n = 1, 2, 3, 4$.

**Remark 5.18.** Contrary to the classical case [Krob and Thibon(1997)] these quotients are not intervals of the $\mathcal{R}$-order: the descent class depicted in Figure 5.4 has two bottom elements.

## 5.3 Ext-Quivers

The Ext-quiver of $H_n^0$ were first computed in [Duchamp et al.(2002)Duchamp, Hivert, and Thibon] in type $A$, and later in [Fayers(2005)] in the other types. Moreover, [Denton et al.(2010/11)Denton, Hivert, Sch describes an algorithm to compute the quiver of any $\mathcal{J}$-trivial monoid. This algorithm is implemented in `sage-semigroups` from the second author, Franco Saliola and Nicolas M. Thiéry [Hivert et al.(2012– It turns out that the quiver of rook monoids are not different from 0-Hecke monoids:

**Theorem 5.19.** *The kernel of the two following algebra morphisms*

$$\mathbb{C}[H^0(B_n)] \twoheadrightarrow \mathbb{C}[R_n^0] \qquad \textit{and} \qquad \mathbb{C}[R_n^0] \twoheadrightarrow \mathbb{C}[H^0(A_{n+1})] \tag{5.6}$$

*are included in the square radical of their respective domains. As a consequence, these three algebras share the same quiver.*

*Proof.* Recall that all of these algebras are monoid algebras of $\mathcal{J}$-trivial monoids. Thanks to [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Corollary 3.8], their radical is generated by commutators. Therefore, the following non zero elements: $\pi_0\pi_1\pi_0 - \pi_0\pi_1$ and

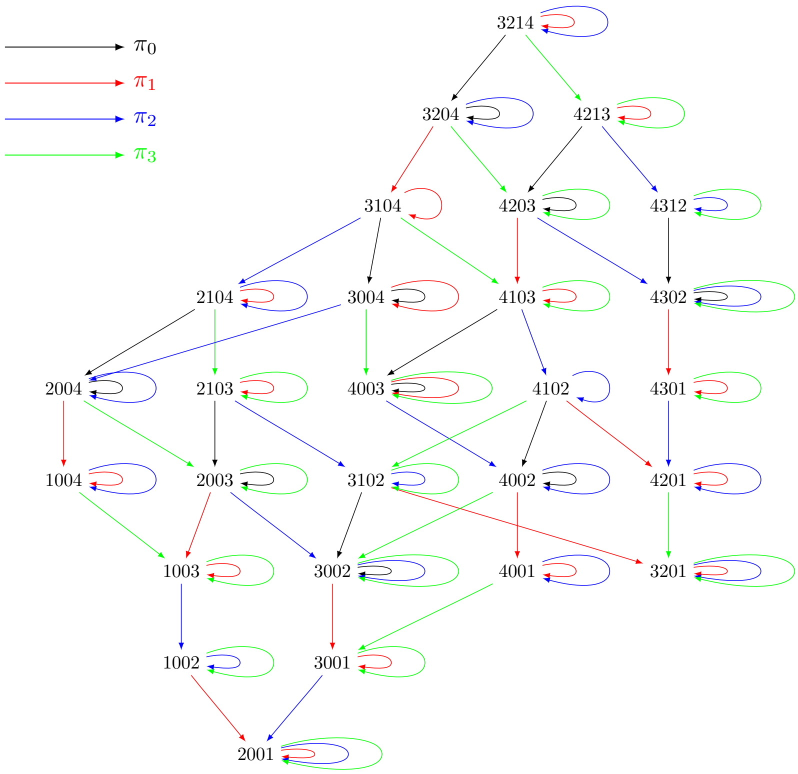



Figure 5.4: An example of a $R$-descent class which is not an interval of the $\mathcal{R}$-order.

$\pi_0\pi_1\pi_0 - \pi_1\pi_0$ lie in the radical of each of these three algebras. The first map has for kernel the ideal generated by the relation

$$\pi_0\pi_1\pi_0 - \pi_0\pi_1\pi_0\pi_1 = (\pi_0\pi_1\pi_0 - \pi_0\pi_1)(\pi_0\pi_1\pi_0 - \pi_1\pi_0)\,.$$

which thus lies in the square radical. Similarly the kernel of the second map is the ideal generated by

$$\pi_1\pi_0\pi_1 - \pi_1\pi_0\pi_1\pi_0 = (\pi_0\pi_1\pi_0 - \pi_1\pi_0)(\pi_0\pi_1\pi_0 - \pi_0\pi_1)\,.$$ □

We refer the reader who want to see actual picture of the quiver to [Duchamp et al.(2002)Duchamp, Hivert, Except for trivial cases, they are not of known type so that the representation theory of $R_n^0$ starting from $n = 3$ is wild.

## 5.4 Restriction functor to $H_n^0$

We now further examine the links between representations of $H_n^0$ and $R_n^0$. Indeed, since $H_n^0$ is a submonoid of $R_n^0$, it acts by multiplication on $R_n^0$. We can see $R_n^0$ as an $H_n^0$-module. Moreover, we can transport modules back and between $H_n^0$ and $R_n^0$ trough the induction and restriction functors.

We first look at simple modules whose restriction rule is immediate:

**Proposition 5.20.** *Let $J \subset [\![0, n-1]\!]$, with associated simple $R_n^0$-module $S_J$. Then:*

$$\operatorname{Res}_{H_n^0}^{R_n^0} S_J = S^H_{J\setminus\{0\}}\,, \tag{5.7}$$

*where $S_I^H$ is the simple $H_n^0$-module generated by the parabolic $I \subseteq [\![1, n-1]\!]$.*

The rule of induction for simple $H_n^0$-modules to $R_n^0$-modules is otherwise quite intricate and would be very technical. It would be very similar to what we will do in section 5.5.1 for the induction of simple modules of $R_n^0$ to another $R_m^0$, which is already very technical.

We now look at indecomposable $R_n^0$-projective modules.

**Proposition 5.21.** *Let $I \subset [\![1, n-1]\!]$ and $P_I^H$ the associated indecomposable $H_n^0$-projective module. Then:*

$$\operatorname{Ind}_{H_n^0}^{R_n^0} P_I^H = P_I \oplus P_{I\cup\{0\}}\,. \tag{5.8}$$

*Proof.* This is a consequence of Proposition 5.20, thanks to Frobenius reciprocity (see *e.g.* [Curtis and Reiner(1 Indeed, since the simple module $S_J^R$ is the quotient of the indecomposable projective $P_J^R$ by its radical, the multiplicity of $P_J^R$ in a projective module $P$ is equal to $\dim \operatorname{Hom}_R(P, S_J^R)$. By Frobenius reciprocity,

$$\operatorname{Mult}_{P_J^R}(\operatorname{Ind}_H^R P_I^H) = \dim \operatorname{Hom}_R(\operatorname{Ind}_H^R P_I^H, S_J^R) = \dim \operatorname{Hom}_H(P_I^H, \operatorname{Res} S_J^R)\,. \tag{5.9}$$

Now, Proposition 5.20 says that this dimension is 1 only if $I = J \setminus \{0\}$, otherwise it is 0. □

The restriction of projective modules from $R_n^0$ to $H_n^0$ is much more interesting. We will show that $R_n^0$-projective modules are still projective when restricted to $H_n^0$, and give a precise combinatorial rule.

**Definition 5.22.** *Let $I \subset \{1, \dots, n\}$ of size $k$, and $\sigma = i_1 \dots i_n \in \mathfrak{S}_n$. We define $\varphi_I(\sigma)$ to be the rook obtained by removing the first $k$ entries of $\sigma$ and inserting zeros in positions indexed by the elements of $I$.*

*We also denote $\psi : R_n \to \mathfrak{S}_n$ the map which takes a rook, put all zeros at the beginning of the word and replace them by the missing letters in decreasing order.*

**Example 5.23.** For instance $\varphi_{\{1,3\}}(14235) = 02035$ and $\psi(02410) = 53241$.

For the next results, we will consider $R_n^0$ to be a left $H_n^0$-module by left multiplication. Thus the action is on values as in Definition 3.56.

**Theorem 5.24.** *$\mathbb{C}R_n^0$ is projective over $H_n^0$. As a consequence any projective $R_n^0$-module remains projective when restricted to $H_n^0$.*

*Proof.* The main remark is that according to Definition 3.56, the left action of $\pi_i$ for $i > 0$ on any rook does not change the zeros: they remain at the same positions and no one are added.

For any $I \subset [\![0, n-1]\!]$, let $C_I$ the set of rooks with zeros in the positions indexed by $I$. Since the action of $H_n^0$ does not move zeros, we have the following decomposition in $H_n^0$-modules:

$$\mathbb{C}R_n^0 \simeq \bigoplus_{I\subset[\![0,n-1]\!]} \mathbb{C}C_I\,. \tag{5.10}$$

It is enough to prove that each summand $\mathbb{C}C_J$ are projective since direct sums of projective modules are projective.

For such a summand where zeros are in positions $i \in I$, the linearization of the map $\psi$ of Definition 5.22 is an injective $H_n^0$-module morphism. Its image is the set of permutations which start with $|I| - 1$ descents which is a well known projective $H_n^0$-module. Indeed, it is the $H_n^0$-module generated by the element $i, i-1, \dots, 2, 1, i+1, i+2, \dots, n$. This element is the zero of the parabolic submonoid generated by $\{\pi_1, \dots \pi_{i-1}\}$, hence idempotent. Consequently it generates a projective modules. This shows that $C_I$ is projective on $H_n^0$. □

We now describe explicitely the restriction functor. Recall from Equation 2.8 that the induction product of two indecomposable projective $H_m^0$-Module (resp. $H_n^0$-Module) $P_I$ and $P_J$ is given by $P_I \star P_J := \mathrm{Ind}_{m,n}(P_I \otimes P_J) \simeq P_{I \cdot J} \oplus P_{I \triangleright J}$.

**Definition 5.25.** *Let $I$ be an extended composition of $n$. A zero-filling of $I$ is a ribbon of shape $I$ with boxes either empty, either with $0$ inside according to the following rules:*

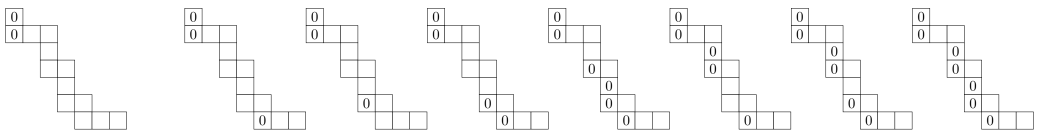
- *In the first column, either every box contains $0$ if $0 \in \mathrm{Des}(I)$, or none otherwise.*
- *Outside of the first column, if a box contains $0$ then there is no box on its left, and all the boxes below in the same column also contain zeros.*

*To each of these fillings $f$ we associate a tuple $\mathrm{Split}(f)$ of ribbon as follows*

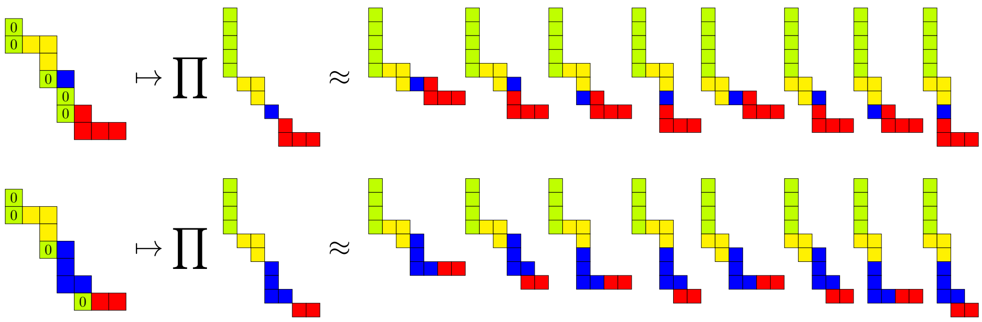
- *the first entry of the tuple is a column whose size is the total number of zeros in $f$*
- *the other entries of the tuple are the (down-right) connected components of $I$ where the boxes containing a $0$ in $f$ are removed.*

*To each splitting, it therefore makes sense to consider the $\star$-product $\prod_{r\in\mathrm{Split}(f)} P_r$.*

**Example 5.26.** The following picture shows an extended composition followed by some of its 0-fillings. There are $3 * 3 * 2$ of them.



We now consider two particular 0-fillings and show the ribbons appearing in the associated respective products (the colors are just to show what happens of each box):



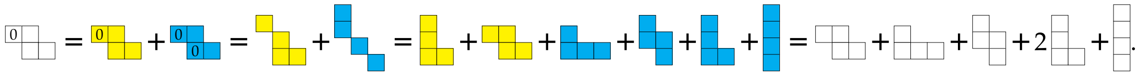
**Theorem 5.27.** *The indecomposable projective $R_n^0$-module $P_I^R$ associated to an extended composition $I$ splits as a $H_n^0$-module as*

$$P_I^R \simeq \bigoplus_f \prod_{r\in\mathrm{Split}(f)} P_r \,, \tag{5.11}$$

*where the direct sum spans along all the zero-fillings of $I$, and the product is for the induction product $\star$.*

Before giving a proof, we give a full example.

**Example 5.28.** We decompose restriction of the indecomposable projective $R_4^0$-module $P_{\{0,2\}}$ into indecomposable projective $H_4^0$-modules. The colors indicate the different products of zero-filling. Figure 5.5 depict the action of the generators.



*Proof.* Let $P_I$ be an indecomposable projective $R_n^0$-module and look at it inside the regular representation. We proceed as in the proof of Theorem 5.24: we cut $P_I$ according to the positions of zeros, which comes down to cutting along the zero-fillings. Indeed the conditions of zero-fillings give us only valid fillings, because they still have the good descent set. Moreover, we see all of them appearing in the descent class: for a given zero-filling $f$, we fill the diagram of $I$ column after column, left to right, down to up, by the entries starting from the number of zeros in the zero-filling plus 1 to $n$. We get a rook of descent set $I$ with zero in the positions given by $f$.

Let $F$ be a zero-filling of shape $I$ with $i$ zeros in positions indexed by elements of $D \subset [\![1,n]\!]$. Let $M_D \subset R_n^0$ be the associated $H_n^0$-projective indecomposable module. We consider the restriction $\psi_F := \psi_{|M_D}$. We need to describe the image of $\psi_F$. First they start with $i$ descents including zeros. We consider the connected components of $[\![1,n]\!] \setminus I$: the letters at these positions are moved to the right by $\psi_F$, but keep their relative order. It is only between the connected components that we can have either a rise or a descent. Then we are getting a subset from a product associated to $F$. And we get them all: take one of them, and fill it with the same rule as before; one gets a permutation and then apply $\varphi_I$ defined in 5.22 to get an element with the good descent set and positions of zeros which will be sent by $\psi_F$ to an element of the product. $\square$

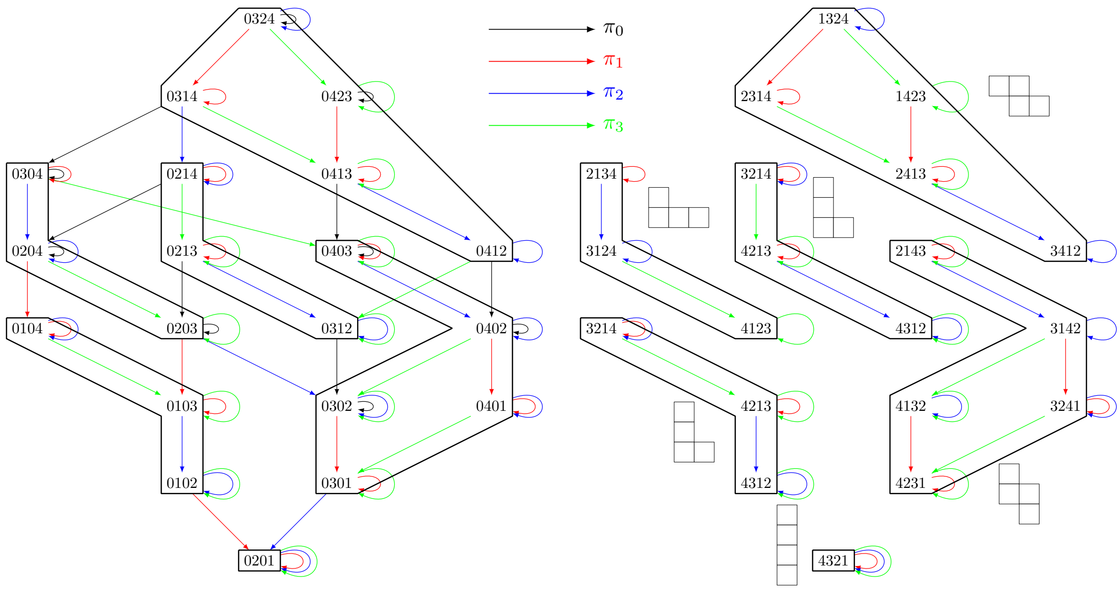


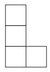
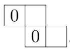

Figure 5.5: The decomposition of a $R_4^0$-projective module associated to $\{0,2\}$ into $H_4^0$-projective modules.

In Annex B, we give the decomposition functor from $R_n^0$ to $H_n^0$ for $n = 1, 2, 3, 4$.

We can be a little more precise:

**Proposition 5.29.** *Let $P^R$ be an indecomposable projective module of $R_n^0$. Write $P^R = \bigoplus P_I^H$ its decomposition into indecomposable $H_n^0$-projective modules. Then the isomorphism of $H_n^0$-module $\tilde{\varphi} : \bigoplus P_I^H \to P^R$ is triangular: $\tilde{\varphi}(e) = \varphi_I(e) + \sum_{e'<e} e'$, with $\varphi_I$ defined in 5.22 and $I$ the zero-set linked to $P_I^H$.*

*Proof.* We consider a $R_n^0$ indecomposable projective $P^R$, pick a $D \subset [\![1, n]\!]$ and denote as in the proof of Theorem 5.27 the $H_n^0$ submodule $M_D$ of rooks whose zeros are in positions indexed by the elements of $D$. The setwise map $\psi_{|M_D}$ extends linearly to an isomorphism to the projective but not necessarily indecomposable permutation module $\prod_{r \in \mathrm{Split}(f)} P_r$. Using [Denton et al.(2010/11)Denton, Hivert, Schilling, and Thiéry, Theorem 3.11 and Corollary 3.19], we know that the basis change decomposing this module to its indecomposable component is uni-triangular. The statement follows by inverting this map. □

**Example 5.30.** We know from Example 5.28 that there is a module  inside the Figure 5.5, coming from the zero-filling . This $H_n^0$-module is well-known to have the elements 3214, 4213 and 4312. So ours must contains $\varphi_{\{0,2\}}(3214) = 0104$, $\varphi_{\{0,2\}}(4213) = 0103$ and $\varphi_{\{0,2\}}(4312) = 0102$. See Figure 5.5.

### 5.5 Tower of monoids

The goal of this section is to investigate if the chain of submonoids

$$R_1^0 \subset R_2^0 \subset R_3^0 \subset \cdots \subset R_{n-1}^0 \subset R_n^0 \subset R_{n+1}^0 \subset \cdots \tag{5.12}$$

can be endowed with a structure of a tower of monoids [Bergeron and Li(2009)].

Recall that an associative tower of monoids is a sequence $(M_i)_{i \in \mathbb{N}}$ where $M_0 = \{1\}$ together with a collection of monoid morphisms $\rho_{n,m} : M_n \times M_m \to M_{n+m}$ such that the product $a \cdot b := \rho_{n,m}(a, b)$ defined on the disjoint union $\sqcup_{i \in \mathbb{N}} M_i$ is associative.

**Proposition 5.31.** *The maps*

$$\begin{array}{ccccccc} \rho_{n,m} : & R_n^0 & \times & R_m^0 & \longrightarrow & R_{n+m}^0 \\ & \pi_0, \dots \pi_{n-1} & & & \longmapsto & \pi_0, \dots \pi_{n-1} \\ & P_i & & & \longmapsto & P_i \\ & & & \pi_1, \dots \pi_{n-1} & \longmapsto & \pi_{n+1}, \dots \pi_{n+m-1} \\ & & & P_i & \longmapsto & P_{i+n} \end{array} \tag{5.13}$$

*defines an associative tower of monoids.*

**Notation 5.32.** *If $a \in R_n^0$ and $b \in R_m^0$ we denote $a \cdot b := \rho_{n,m}(a, b)$.*

*Furthermore, if $w$ is a word on nonnegative integers, $\overline{w}^n$ denotes the word $w$ where all nonzero entries have been increased by $n$.*

*Proof.* We first show that $\rho_{n,m}$ are morphisms. Let $a \in R_n^0$ et $b \in R_m^0$. Then, by relation of commutation and absorption we get $\rho(a, b) = \rho(a, 1) \cdot \rho(1, b) = \rho(1, b) \cdot \rho(a, 1)$.

The proof of the associativity rely on the following lemma:

**Lemma 5.33.** *Let $a \in R_n^0$ and $b \in R_m^0$. Then*

$$a \cdot b = \begin{cases} \mathbf{a}\overline{\mathbf{b}}^n & \text{if } 0 \notin \mathbf{b}, \\ 0 \dots 0\overline{\mathbf{b}}^n & \text{otherwise.} \end{cases} \tag{5.14}$$

*Proof.* Indeed $\rho(a, b) = \rho(a, 1)\rho(1, b)$. If $0 \notin b$ then $\pi_0$ does not appear in any reduced expression of $b$, thus the reduced expressions of $a$ and $b$ contain generators which do not act on $1_{n+m}$ on the same positions. Otherwise $P_{n+1}$ appear in $\rho(1, b)$, and since all elements of $\rho(1, a)$ commute with those of $\rho(1, b)$, $P_{n+1}$ absorbs all the $\rho(a)$. □

We now can compute explicitely the products $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$, do the four cases whether $0 \in B$ or not and $0 \in C$ and check associativity. □

**Remark 5.34.** The embedding $\rho$ is not injective since $\forall a, a' \in R_n^0$, and $b \in R_m^0$ with $0 \in b$ : $a \cdot b = a' \cdot b$ by Lemma 5.33. So we do not have a tower of monoid in the sense of [Bergeron and Li(2009)].

**Remark 5.35.** To map $R_n^0 \times R_m^0$ to $R_{n+m}^0$, Remark 3.10 prevents us to use the trivial map $(a, b) \mapsto \mathbf{a}\overline{\mathbf{b}}^{\mathrm{n}}$: it is not a monoid morphism.

### 5.5.1 Restriction and induction of simple modules

The goal of this section is to describe the restriction and induction rule of the tower of the 0-rook monoids. Recall that for $H_n^0$, this gives the multiplication and comultiplication rule of the Hopf algebra of quasi-symmetric functions in the fundamental basis [Krob and Thibon(1997)].

#### Restriction of simples modules

**Theorem 5.36.** *Let $J \subset [\![0, n+m-1]\!]$ a parabolic of $R_{n+m}^0$. Then:*

$$\operatorname{Res}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_J = \begin{cases} S_{J \cap [\![0,n-1]\!]} \otimes S_{\overline{J \cap [\![n+1,n+m-1]\!]}} & \text{if } J \cap [\![0, n]\!] \neq [\![0, n]\!], \\ S_{[\![0,n-1]\!]} \otimes S_{\{0\} \cup \overline{J \setminus [\![0,n]\!]}} & \text{otherwise.} \end{cases} \tag{5.15}$$

*where $\overline{X} := \{x - n \mid x \in X\}$.*

*Proof.* We know that $S_J = \langle \varepsilon_J \rangle$, and that $\varepsilon_J \cdot \pi_i = \varepsilon_J$ if $i \in J$, and 0 otherwise. The action of $R_n^0 \otimes 1_m$ on $S_J$ gives us $S_{J \cap [\![0,n-1]\!]}$. The generators $1_n \otimes \pi_1, \dots, 1_n \otimes \pi_{m-1}$ of $1_n \otimes R_m^0$ act as $\pi_{n+1}, \dots, \pi_{n+m-1}$. It remains only to see how $1_n \otimes \pi_0 = P_{n+1}$ acts on $S_J$. By Lemma 3.4 we have that $P_{n+1} = \pi_0\pi_1\pi_0\pi_2\pi_1\pi_0 \dots \pi_n \dots \pi_2\pi_1\pi_0$. So if there is $0 \le i \le n$ with $i \notin J$, $\varepsilon_J \cdot \pi_i = 0$ thus $\varepsilon_J \cdot P_{n+1} = 0$. Otherwise, for all $i \in [\![0, n]\!]$, $\varepsilon_J \cdot \pi_i = \varepsilon_J$ and so $\varepsilon_J \cdot P_{n+1} = \varepsilon_J$. □

**Induction of simple modules** We can compute the induction of simple module thanks to Virmaux [Virmaux(2014), Theorem 4.3], which we reformulate in our context here. The comparisons are done with the $\mathcal{R}$-order in $R_n^0$, which we described in Theorem 4.16.

**Theorem 5.37** ([Virmaux(2014), Theorem 4.3])**.** *If $e \in E(R_n^0)$ and $f \in E(R_m^0)$, then*

$$\operatorname{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_e \otimes S_f = {}^{(e \cdot f) R_{n+m}^0} \big/ {}_{[(R_{<e} \cdot f) + (e \cdot R_{<f})]\, R_{n+m}^0}, \tag{5.16}$$

*where $R_{<e}$ is the set of elements of $R_n^0$ strictly smaller than $e$, and $R_{<f}$ those of $R_m^0$ strictly smaller than $f$.*

**Notation 5.38.** *In Equation 5.16, we will denote by $Q(e,f)$ the right hand side of the equality. It is a $R^0_{n+m}$-module. It is also a quotient which is compatible with the canonical basis. By abuse of language, we will say that an element $r \in R^0_{n+m}$ remains in $Q(e,f)$ and write $r \in Q(e,f)$ if $r$ is not mapped to zero in the quotient.*

Our first goal is to rephrase Theorem 5.37 in a more combinatorial way.

**Notation 5.39.** *Until now, we used the notation $\pi_I$ to design the idempotent of the parabolic submonoid associated to $I$ in $R^0_n$. In order to avoid confusion, we will now denote it by $\pi_{I,n}$. Note that as long as $n, m \geq \max I + 1$, then $\pi_{I,n}$ and $\pi_{I,m}$ have the same reduced expressions and thus the same action of the first $\min(n,m)$-letters on the identity of size $\max(n,m)$.*

In the sequel of this section, we fix $I \subseteq [\![0, n-1]\!]$ and $J \subseteq [\![0, m-1]\!]$. They encode the data of two simple modules of $R^0_n$ and $R^0_m$ respectively, or equivalently of two idempotents. We denote $e := \pi_{I,n}$ and $f := \pi_{J,m}$ these two idempotents.

Before giving the induction of the simple modules, we go for a serie of lemmas.

**Lemma 5.40.** *The image of $(e,f) \in R^0_n \times R^0_m$ in $R^0_{n+m}$ is the element of $R^0_{n+m}$ associated to $\mathbf{e}\overline{\mathbf{f}}^n$ if $0 \notin J$ and to $0 \dots 0\overline{\mathbf{f}}^n$ otherwise. In particular we have the following cases:*

- *If $J = \emptyset$ then $e \cdot f = \mathbf{e}\overline{12\dots m}^n = \pi_{I,n+m}$.*
- *If $I = \emptyset$ and $0 \notin J$ then $e \cdot f = 1 \dots n\overline{\mathbf{f}}^n = \pi_{\overline{J}^n, n+m}$.*
- *If $I = [\![0, n-1]\!]$ and $0 \in J$ then $e \cdot f = 0 \dots 0\overline{\mathbf{f}}^n = \pi_{\left([\![0,n]\!] \cup \overline{J \setminus \{0\}}^n\right), n+m}$.*

*Proof.* It is straightforward application of Lemma 5.33. □

**Remark 5.41.** Note that because of the form of idempotents, $0 \notin I \Leftrightarrow 0 \notin \mathbf{e}$.

**Lemma 5.42.** *Assume that $0 \in J$ and $I \neq [\![0, n-1]\!]$. Then $Q(e,f) = 0$.*

*Proof.* Since $0 \in J$ then $e \cdot f = 0 \dots 0\mathbf{f}$ according to Lemma 5.40. On the other hand, let $j \in [\![0, n-1]\!] \setminus I$. Then in $Q(e,f)$ we are doing a quotient by $(e \cdot \pi_j) \cdot 1_m$ which is above $e \cdot f$ by Theorem 4.16. Hence $Q(e,f) = 0$. □

We are now considering cases where $0 \notin J$, writing $\mathbf{f} = f_0 \dots f_m$.

**Lemma 5.43.** *Assume $0 \notin J$. Let $r$ be an element of $R^0_{n+m}$ which does not vanish in the quotient $Q(e,f)$. Let $a$ and $b$ be two letters of $\mathbf{r}$, not both zero. If $a$ and $b$ appear both in $\mathbf{e}$ (resp. $a-n$ and $b-n$ appear both in $\mathbf{f}$) then they appear in $\mathbf{r}$ in the same order as in $\mathbf{e}$ (resp. $\mathbf{f}$). Furthermore, all the nonzero letters of $\mathbf{e}$ appear in $\mathbf{r}$. Finally, if $f_i + n$ is not in $\mathbf{r}$ then $f_j + n$ is not in $\mathbf{r}$, for all $j < i$.*

**Example 5.44.** If $\mathbf{e} = \mathbf{023}$ and $\mathbf{f} = \mathbf{213}$ then neither $\mathbf{042356}$, or $\mathbf{005463}$, or $\mathbf{025306}$ remain in $Q(e,f)$, respectively because of the first, second and third rule.

*Proof.* For the first point, it is sufficient to do the proof when the two letters are consecutive in $\mathbf{e}$. Let $\mathbf{r} \in Q(e,f)$. So $r \leq \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$. Assume $\mathbf{e} = \underline{L}ab\underline{R}$ with $a$ and $b$ non both zero, and both present in $\mathbf{r}$.

Suppose first that $a > b$, so that $a \neq 0$ and $b \neq 0$ since $0 \notin J$. Since $r < \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ we deduce that $a$ is before $b$ in $\mathbf{r}$.

Otherwise, $a < b$. Let $i := \ell(\underline{L})$ be the position of $a$ in $\mathbf{e}$. So $i \notin I$. Then $e \cdot \pi_i < e$. Also $\mathrm{Inv}(e \cdot \pi_i) = \mathrm{Inv}(e) \cup \{(b, a)\}$. Thus, $\mathrm{Inv}((\mathbf{e} \cdot \pi_i)\overline{\mathbf{f}}^{\mathrm{n}}) = \mathrm{Inv}(\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}) \cup \{(b, a)\}$ while $(b, a) \notin \mathrm{Inv}(\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}})$. Since $r < \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ we get $\{(r_i, r_j) \in \mathrm{Inv}(\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}) \mid r_i \in \mathbf{r}\} \subseteq \mathrm{Inv}(r)$. Assume that $b$ is left to $a$ in $\mathbf{r}$. In this case we have $\{(r_i, r_j) \in \mathrm{Inv}(\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}) \cup \{(b, a)\} \mid r_i \in \mathbf{r}\} \subseteq \mathrm{Inv}(r)$, so $r < (\mathbf{e} \cdot \pi_i)\overline{\mathbf{f}}^{\mathrm{n}}$, the latter being an element by which we quotient in $Q(e, f)$. It is a contradiction.

The proof is the same when $a$ and $b$ both come from $\mathbf{f}$ once decreased. The only change are that both letter are nonzero, and that we have to decrease by $n$.

Let us prove the second point by contradiction, assuming that a nonzero letter $b$ in $\mathbf{e}$ is not in $\mathbf{r}$. We first show that the first nonzero letter of $\mathbf{e}$, say $a$, is not in $\mathbf{r}$ either. By contradiction, assume that $a \in \mathbf{r}$. If $a > b$ then $e$ has descent $(a, b)$. So $r$ must also have it since $r < \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ and $a \in \mathbf{r}$, but it is not the case since $b \notin \mathbf{r}$, which is a contradiction. Otherwise $a < b$. Since $a \in \mathbf{r}$ and $b \notin \mathbf{r}$, and that the generator $\pi_0$ can only delete the first letter, $r$ is in the $\mathcal{R}$-order between $\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ and a rook $r'$ in which $a$ is there and $b$ is in first position. Because of the first point, this element $r'$ has been sent to 0 in the quotient, and thus $r$ which is below as well. So $r = 0$, again this is a contradiction.

The same argument also apply to the third case, with some minor adaptation.

Thus if there is a nonzero letter of $\mathbf{e}$ lacking in $\mathbf{r}$, the first one at least is lacking. We now look at $\mathbf{e}$. If $0 \notin I$, $\mathbf{e}$ begins with $a$. Then $\mathbf{q} := (\mathbf{e} \cdot \pi_0)\overline{\mathbf{f}}^{\mathrm{n}}$ is an element by which we quotient. We have $r < \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ and $a \notin \mathbf{r}$ so $r < q$, thus $r = 0$, and we get a contradiction.

Otherwise $0 \in I$ so $\mathbf{e} = 0 \dots 0a \dots$. We denote by $i$ the position of the last 0 and $q := (\mathbf{e} \cdot \pi_i)\overline{\mathbf{f}}^{\mathrm{n}}$ is an element by which we quotient. Since $a \notin \mathbf{r}$, $r$ is in the $\mathcal{R}$-order between $\mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ and a rook $r'$ in which $a$ is there in first position. In particular in $\mathbf{r}$, we have a 0 right to $a$. So $r < r' < \mathbf{e}\overline{\mathbf{f}}^{\mathrm{n}}$ and $(a, 0) \in \mathrm{Inv}(r')$, so $r' < q$ and thus $r = 0$, for a final contradiction. $\square$

**Remark 5.45.** Let $K \subset [\![1, n-1]\!]$ and $g \in R_n^0$ the associated idempotent (hence $0 \notin g$). We write $\mathbf{g} = g_1 g_2 \dots g_n$. Because of Proposition 5.13 we have that if $g_1 = \ell$ then $g_2 = \ell - 1$, $g_3 = \ell - 2$, $\dots$, $g_{\ell-1} = 2$ and $g_\ell = 1$. Furthermore $\ell \notin K$ (since $g_{\ell+1} > g_\ell$) and $\ell = \min([\![1, n-1]\!] \setminus I)$.

We are now in position to state the formula giving the induction of simple modules. Recall that $\sqcup\!\sqcup$ denote the so-called shuffle product introduced in Definition 4.30. We also denote $0^i$ the word $00 \dots 0$ with $i$ letters 0.

**Theorem 5.46.** *For $n, m \in \mathbb{N}$, we fix $I \subseteq [\![0, n-1]\!]$ and $J \subseteq [\![0, m-1]\!]$. Denoting $e := \pi_{I,n}$ and $f := \pi_{J,m}$, the induction of simple modules $S_I = S_e$ and $S_J = S_f$ is given by*

1. *If $0 \in J$ and $I \neq [\![0, n-1]\!]$ then $\mathrm{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_I \otimes S_J = 0$.*

2. *If $0 \in J$ and $I = [\![0, n-1]\!]$ then $\mathrm{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_I \otimes S_J = \langle \mathbf{e}\overline{\mathbf{f}}^{n} \rangle \simeq S_{[\![0,n]\!] \cup \overline{J \setminus \{0\}}^{n}}$.*

3. *If $0 \notin J$ and $I = [\![0, n-1]\!]$ then $\mathrm{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_I \otimes S_J = \langle 0^n \sqcup\!\sqcup \overline{\mathbf{f}}^{n} \rangle$.*

4. *If $0 \notin J$ and $0 \in I$, $I \neq [\![0, n-1]\!]$, let $\ell := f_1$ be the first letter of $\mathbf{f} = f_1 \dots f_m$. Then:*

$$\mathrm{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_I \otimes S_J = \left\langle\ 0^i \mathbf{e} \sqcup\!\sqcup \overline{f_{i+1} \dots f_m}^{n} \mid i = 0, \dots, \ell\ \right\rangle . \tag{5.17}$$

5. *If $0 \notin J$ and $0 \notin I$, let $\ell := f_1$ be the first letter of $\mathbf{f} = f_1 \dots f_m$. Then*

$$\operatorname{Ind}_{R_n^0 \times R_m^0}^{R_{n+m}^0} S_I \otimes S_J = \left\langle\, 0^i \sqcup \mathbf{e} \sqcup \overline{f_{i+1} \dots f_m}^{n} \mid i = 0, \dots, \ell \,\right\rangle . \tag{5.18}$$

*Proof.* 1. This case follows directly from Lemma 5.42.

2. Let $K := [\![0, n]\!] \cup \overline{J \setminus \{0\}}^{\mathrm{n}}$. Then by Lemma 5.40,$e \cdot f = \pi_{K,n+m}$. Since $I = [\![0, n-1]\!]$ then

$$Q(e, f) = {}^{\pi_{K,n+m} R_{n+m}^0} \diagup_{[(0 \dots 0 \cdot R_{<f})]\, R_{n+m}^0} . \tag{5.19}$$

On the other hand, let $g := \pi_{K,n+m}$ be the idempotent associated to $K$ in $R_{n+m}^0$. By Theorem 5.37,

$$S_g = \operatorname{Ind}_{1 \times R_{n+m}^0}^{R_{n+m}^0} 1 \otimes S_g = {}^{g R_{n+m}^0} \diagup_{[R_{<g}]\, R_{n+m}^0} . \tag{5.20}$$

But since $I = [\![0, n-1]\!]$ on has $R_{<g} = 0 \dots 0 \cdot R_{<f}$, so that $Q(e, f) \simeq S_g$.

3. Since $I = [\![0, n-1]\!]$ then $\mathbf{e} = 0 \dots 0$ and $e \cdot f = 0 \dots 0\underline{\mathbf{f}}$. Let $\mathbf{r} \in 0 \dots 0 \sqcup \overline{\mathbf{f}}^{\mathrm{n}}$. Clearly $r < e \cdot f$. We know that $\mathbf{r}$ has the same number of zeros than $\mathbf{e} \cdot \mathbf{f}$ and also that its inversions are those of $f$ increased by $n$. We deduce that $r$ is not below $\mathbf{e}\overline{(\mathbf{f} \cdot \pi_j)}^{\mathrm{n}}$ in the $\mathcal{R}$-order for $j \in [\![0, m-1]\!] \setminus J$. Thus $r \in Q(e, f)$.

   Conversely let $r \in Q(e, f)$. Since $0 \notin J$ then $r \not< \mathbf{e}\overline{(\mathbf{f} \cdot \pi_0)}^{\mathrm{n}}$. So the first letter of $\mathbf{f}$ increased by $n$ is in $\mathbf{r}$. By Lemma 5.43 all the letters of $\mathbf{f}$ increased by $n$ are in $\mathbf{r}$. Again by Lemma 5.43 they are in the same order, and so $\mathbf{r} \in 0 \dots 0 \sqcup \overline{\mathbf{f}}^{\mathrm{n}}$.

4. Denote $S_{ef} := \mathbf{e} \sqcup \overline{\mathbf{f}}^{\mathrm{n}} + 0\mathbf{e} \sqcup \overline{f_2 \dots f_m}^{\mathrm{n}} + \dots + 0 \dots 0\mathbf{e} \sqcup \overline{f_{\ell+1} \dots f_m}^{\mathrm{n}}$ and let $\mathbf{r} \in S_{ef}$. The same argument than the third point shows that $r \in Q(e, f)$.

   Conversely, let $r \in Q(e, f)$. Since $0 \in I$ (or equivalently, $0 \in \mathbf{e}$) Lemma 5.43 tells us that the eventual new zeros of $\mathbf{r}$ are before the nonzero letters of $\mathbf{e}$. By the same lemma, the letters of $\mathbf{f}$ disappear in the same order than in $\mathbf{f}$. So that we have proven:

$$\mathbf{r} \in T_{ef} := \mathbf{e} \sqcup \overline{\mathbf{f}}^{\mathrm{n}} + 0\mathbf{e} \sqcup \overline{f_2 \dots f_m}^{\mathrm{n}} + \dots + 0 \dots 0\mathbf{e} \sqcup \overline{f_m}^{\mathrm{n}} + 0 \dots 0\mathbf{e}. \tag{5.21}$$

   We recall that $\ell = f_1$. We have to show that elements of $T_{ef} \setminus S_{ef}$ are not in $Q(e, f)$. A first immediate remark is that all these elements are below $t = 0 \dots 0\mathbf{e}\overline{f_{\ell+2} \dots f_m}^{\mathrm{n}}$. But $t < \mathbf{e} f_1 \dots f_{\ell-1} f_{\ell+1} f_\ell f_{\ell+2} \dots f_m = (\mathbf{e} \cdot (\mathbf{f} \cdot \pi_\ell))$. Thus, since $\ell \notin J$ (by Remark 5.45), $t = 0$ in $Q(e, f)$, and so all $T_{ef} \setminus S_{ef}$ also, hence the result.

5. Denote $S_{ef} := \mathbf{e} \sqcup \overline{\mathbf{f}}^{\mathrm{n}} + 0 \sqcup \mathbf{e} \sqcup \overline{f_2 \dots f_m}^{\mathrm{n}} + \dots + 0 \dots 0 \sqcup \mathbf{e} \sqcup \overline{f_{\ell+1} \dots f_m}^{\mathrm{n}}$. Let $\mathbf{r} \in S_{ef}$. The argument of the third point proves that $r \in Q(e, f)$.

   Conversely, for $r \in Q(e, f)$, the argument of the fourth point shows that $\mathbf{r} \in S_{ef}$. □

Recall that the corresponding rule for $H_n^0$ is the multiplication of the fundamental basis $(F_I)$ of quasi-symmetric function [Krob and Thibon(1997)]. This rule can be computed as follows [Gessel(1984), Duchamp et al.(2002)Duchamp, Hivert, and Thibon]. Let $I$ and $J$ be two compositions. Choose any permutation $\sigma \in \mathfrak{S}_n$ whose descent composition is $\mathrm{C}(\sigma) = I$, for example $\pi_I$ whose corresponding $H_n^0$ element is idempotent, and $\mu$ such that $\mathrm{C}(\mu) = J$. Then

$$F_I\, F_j = \sum_{\nu \in \sigma \sqcup \overline{\mu}^{\mathrm{n}}} F_{\mathrm{C}(\nu)} . \tag{5.22}$$

As explained by Virmaux [Virmaux(2014)] this is a direct consequence of Theorem 5.37.

To get the analogue of the product of quasi-symmetric functions, one has to use the Theorem 5.46 and then get the projection of the induced module in the Grothendieck ring. This amounts to compute the $R$-descent of every rook vector appearing in the sum $Q(e,f)$ according to Jordan-Hölder's theorem.

**Example 5.47.** If $n=2$, $m=3$, $I=\{0,1\}$ and $J=\{1\}$. Then $\mathbf{e}=00$ and $\mathbf{f}=213$. Theorem 5.46 says that

$$Q(e,f)=\langle 00 \sqcup\!\sqcup 435\rangle = \langle \mathbf{00435}, \mathbf{04035}, \mathbf{04305}, \mathbf{04350}, \mathbf{40035}, \mathbf{40305}, \mathbf{40350}, \mathbf{43005}, \mathbf{43050}, \mathbf{43500}\rangle.$$

This gives the following $R$-descent classes:

| Element | **00435** | **04035** | **04305** | **04350** | **40035** | **40305** | **40350** | **43005** | **43050** | **43500** |
|---|---|---|---|---|---|---|---|---|---|---|
| Descents | 0,1,3 | 0,2 | 0,2,3 | 0,2,4 | 1,2 | 1,3 | 1,4 | 1,2,3 | 1,2,4 | 1,3,4 |

Finally: $$\operatorname{Ind} S^2_{\{0,1\}} \times S^3_{\{1\}} = S^5_{\{0,1,3\}} + S^5_{\{0,2\}} + S^5_{\{0,2,3\}} + S^5_{\{0,2,4\}} + S^5_{\{1,2\}} + S^5_{\{1,3\}} + S^5_{\{1,4\}} + S^5_{\{1,2,3\}} + S^5_{\{1,2,4\}} + S^5_{\{1,3,4\}}$$

**Example 5.48.** If $n=3$, $m=2$, $I=\{0,1\}$ and $J=\{1\}$. Then $\mathbf{e}=003$ and $\mathbf{f}=21$. Theorem 5.46 says that

$$\begin{aligned} Q(e,f) &= \langle 003 \sqcup\!\sqcup 21 + 0003 \sqcup\!\sqcup 1 + 00003\rangle \\ &= \langle \{\mathbf{00321}, \mathbf{00231}, \mathbf{00213}, \mathbf{02031}, \mathbf{02013}, \mathbf{02103}, \mathbf{20031}, \mathbf{20013}, \mathbf{20103}, \mathbf{21003}\} \\ &\qquad \cup \{\mathbf{00031}, \mathbf{00013}, \mathbf{00103}, \mathbf{01003}, \mathbf{10003}\} \cup \{\mathbf{00003}\}\rangle \end{aligned}$$

Then:

| Element | **00321** | **00231** | **00213** | **02031** | **02013** | **02103** | **20031** | **20013** |
|---|---|---|---|---|---|---|---|---|
| Descents | 0,1,3,4 | 0,1,4 | 0,1,3 | 0,2,4 | 0,2 | 0,2,3 | 1,2,4 | 1,2 |
| Element | **20103** | **21003** | **00031** | **00013** | **00103** | **01003** | **10003** | **00003** |
| Descents | 1,3 | 1,2,3 | 0,1,2,4 | 0,1,2 | 0,1,3 | 0,2,3 | 1,2,3 | 0,1,2,3 |

$$\begin{aligned} \operatorname{Ind} S^3_{\{0,1\}} \times S^2_{\{1\}} = {} & S^5_{\{0,1,3,4\}} + S^5_{\{0,1,4\}} + 2S^5_{\{0,1,3\}} + S^5_{\{0,2,4\}} + S^5_{\{0,2\}} + 2S^5_{\{0,2,3\}} + S^5_{\{1,2,4\}} \\ & + S^5_{\{1,2\}} + S^5_{\{1,3\}} + S^5_{\{1,2\}} + S^5_{\{0,1,2,4\}} + S^5_{\{0,1,2\}} + 2S^5_{\{1,2,3\}} + S^5_{\{0,1,2,3\}} \end{aligned}$$

This defines the left (resp. right) dual branching graph, where the arrows $I \mapsto J$ are labelled by the multiplicity of $S_J$ in the induction of $S_I$ along the morphism $\rho_{1,n}$ (resp. $\rho_{n,1}$). The beginning of those two graphs are illustrated in Figures 5.6 and 5.7.

**Hopf algebra** On the contrary to $H^0_n$, we do not get a Hopf algebra. Indeed, the following diagram that express the compatibility of the product with the co-product does not commute:

$$\begin{tikzcd} R^0_{a+b} \times R^0_{c+d} \arrow[r, "\mathrm{Ind}"] \arrow[d, "\mathrm{Res}\times\mathrm{Res}"'] & R^0_{a+b+c+d} \arrow[d, "\mathrm{Res}"] \\ R^0_a \times R^0_b \times R^0_c \times R^0_d \arrow[r, "\mathrm{Ind}\times\mathrm{Ind}"] & R^0_{a+c} \times R^0_{b+d} \end{tikzcd}$$

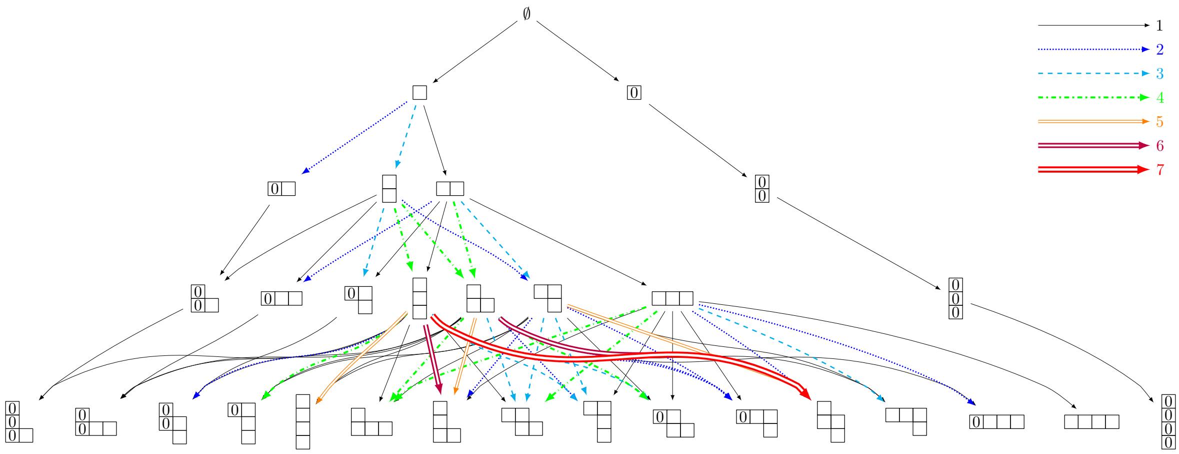



Figure 5.6: The left dual branching graph of $R_n^0$.

Here is a counter example: Using Theorem 5.46, we get $\mathrm{Res}^{R_3^0}_{R_1^0\times R_2^0} S^3_{\{0,1\}} = S^1_{\{0\}} \otimes S^2_{\{0\}}$ and $\mathrm{Res}^{R_2^0}_{R_1^0\times R_1^0} S^2_{\{1\}} = S^1_{\{\}} \otimes S^1_{\{\}}$. Then

$$\begin{aligned}\mathrm{Ind}\times\mathrm{Ind}\left(\mathrm{Res}\times\mathrm{Res}\, S^3_{\{0,1\}} \otimes S^2_{\{1\}}\right) &= \mathrm{Ind}(S^1_{\{0\}} \otimes S^1_{\{\}}) \otimes \mathrm{Ind}(S^2_{\{0\}} \otimes S^1_{\{\}})\\ &= (S^2_{\{0\}} + S^2_{\{1\}}) \otimes (S^3_{\{0\}} + S^3_{\{0,1\}} + S^3_{\{0,2\}} + S^3_{\{1\}})\,.\end{aligned} \tag{5.23}$$

Hence this sum has 8 elements, with multiplicity. On the other hand, we saw in Example 5.48 that $\mathrm{Ind}\, S^3_{\{0,1\}} \times S^2_{\{1\}}$ is a sum of 16 elements (with multiplicity) and Theorem 5.36 shows that the multiplicity does not change by restriction. Hence the result is false.

**Induction with $H_n^0$** One can wonder what would happen if we rather consider the induction and restriction along the inclusion $R_n^0 \times H_m^0 \to R_{n+m}^m$. It is not a tower of monoids, but the morphisms $\tilde{\rho}_{n,m} := (\rho_{n,m})_{|R_n^0\times H_m^0}$ are injective. We just give the result of the induction of simple modules:

**Theorem 5.49.** *For $n, m \in \mathbb{N}$, let $I \subseteq [\![0, n-1]\!]$ and $J \subseteq [\![1, m-1]\!]$. Denoting $e := \pi_{I,n} \in R_n^0$ and $f := \pi_{J,m} \in H_m^0$, the induction of simple modules $S_I = S_e$ and $S_J = S_f$ is given by*

1. *If $0 \in I$, let $\ell$ be the first letter of $\mathbf{f} = f_1 \dots f_m$. Then:*

$$\begin{aligned}\mathrm{Ind}^{R^0_{n+m}}_{R_n^0\times H_m^0} S_I \otimes S_J = \big\langle \mathbf{e} \sqcup \overline{\mathbf{f}}^n + 0\mathbf{e} \sqcup \overline{f_2 \dots f_m}^n + 00\mathbf{e} \sqcup \overline{f_3 \dots f_m}^n +\\ \dots \qquad + 0\dots 0\mathbf{e} \sqcup \overline{f_{\ell+1} \dots f_m}^n \big\rangle\,,\end{aligned} \tag{5.24}$$

   *where the last term begins with $\ell$ letters 0.*

2. *If $0 \notin I$, let $\ell$ be the first letter of $\mathbf{f} = f_1 \dots f_m$. Then*

$$\mathrm{Ind}^{R^0_{n+m}}_{R_n^0\times H_m^0} S_I \otimes S_J = \big\langle \mathbf{e} \sqcup \overline{\mathbf{f}}^n + 0 \sqcup \mathbf{e} \sqcup \overline{f_2 \dots f_m}^n + 00 \sqcup \mathbf{e} \sqcup \overline{f_3 \dots f_m}^n +$$

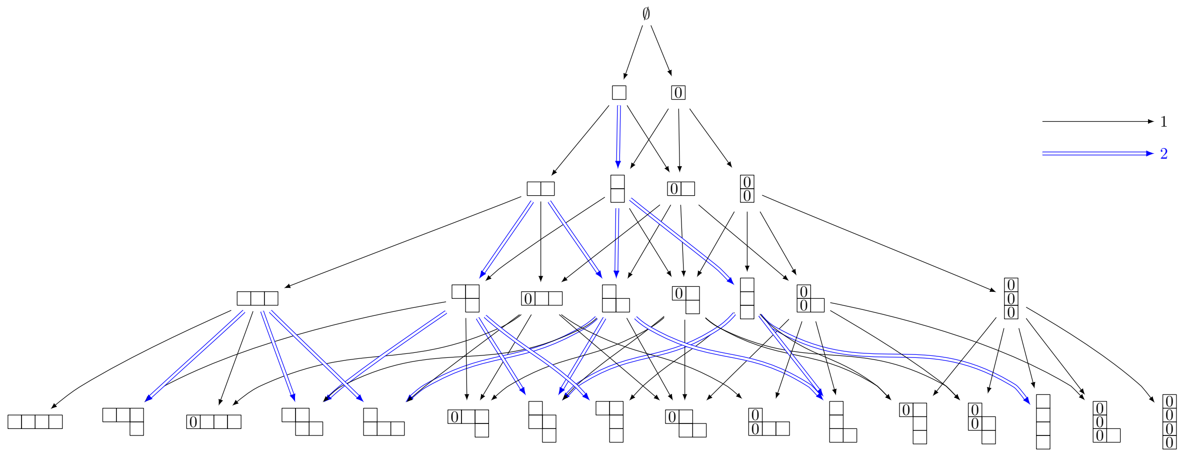



Figure 5.7: The right dual branching graph of $R_n^0$.

$$\dots \qquad + 0\dots 0 \sqcup \mathbf{e} \sqcup \overline{f_{\ell+1}\dots f_m}^n \rangle \,, \quad (5.25)$$

*where the last term begins with $\ell$ letters* 0.

*Proof.* This is a consequence of Theorem 5.46. □

### 5.5.2 Projective indecomposable modules

**Restriction of indecomposable projective modules** In order to get a co-product on the Grothendieck ring of projective modules, $\mathcal{K}_0$, we need that $R^0_{m+n}$ is projective over $R^0_m \times R^0_n$. Unfortunately, this is not the case. We will moreover give counterexamples to the fact that $R^0_n$ is projective over $R^0_{n-1}$ for both embedding $\rho_{n-1,1}$ and $\rho_{1,n-1}$. This forbids to have any analogues of Bratelli diagrams for projective modules.

Let us take $P_{\{0,2,3\}}$. We want to restrict this projective indecomposable module of $R^0_4$ to $R^0_2 \times R^0_2$. In Figure 5.8 we have on the left the module $P^4_{\{0,2,3\}}$ where we deleted the arrows of $\pi_2$ and showed the action of $P_3$. Here we see that $P_3$ has a stable subspace of dimension 1. On the right we represent what would be a necessary part of the decomposition of $P^4_{\{0,2,3\}}$, that is $P^2_{\{0\}} \otimes P^2_{\{1\}}$. Here we see that $P_3$ (that is the $\pi_0$ of the right $R^0_2$ according to the embedding 5.31) as a stable subspace of dimension 2. Hence it is impossible to cut the left one to get a sum of projective indecomposable modules since the right one must be there and can not be.

We give now two counterexamples which show that it does not work also for the restriction along both embeddings $\rho_{n-1,1}$ and $\rho_{1,n-1}$. On the left of Figure 5.9 we have the projective module $P^4_{\{2\}}$. We see that no element of this module has two zeros, hence $P_2$ send every element to zero. In the middle of the figure we have the same module where we forgot the action of $\pi_3$, that is we are looking at the restriction $R^0_4 \to R^0_3 \otimes R^0_1$. In the left one we forgot the action of both $\pi_0$ and $\pi_1$ but put the action of $P_2$ (none here): we are looking at the restriction along $R^0_4 \to R^0_1 \otimes R^0_3$. If the middle and right modules were projective, these figures could be cut as projective modules of $R^0_3$. We proceed step by step on the middle one. First we recognise the first chain of five elements which is $P^3_{\{2\}}$. Then the element 1423 is $P^3_{\{\}}$. All the cycles below with element on top 2413 is $P^3_{\{1\}}$. The element 1203 is again $P^3_{\{\}}$. But the

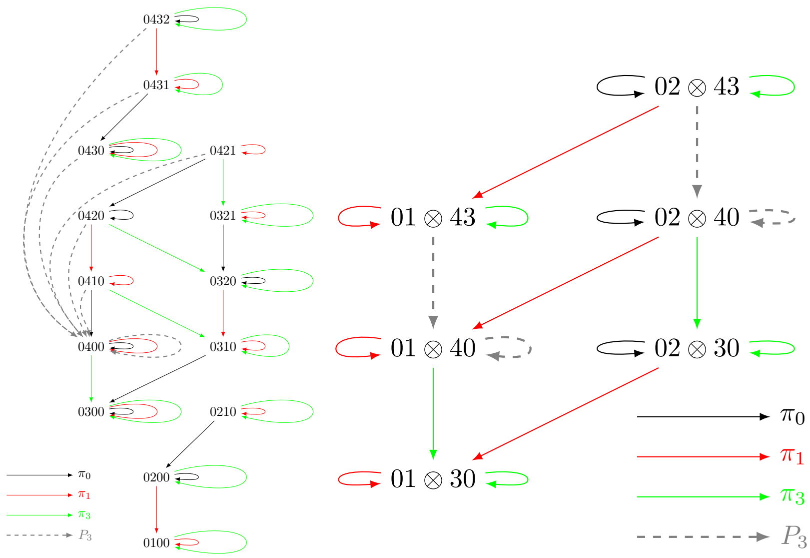



Figure 5.8: First counterexample for the restriction of projective modules.

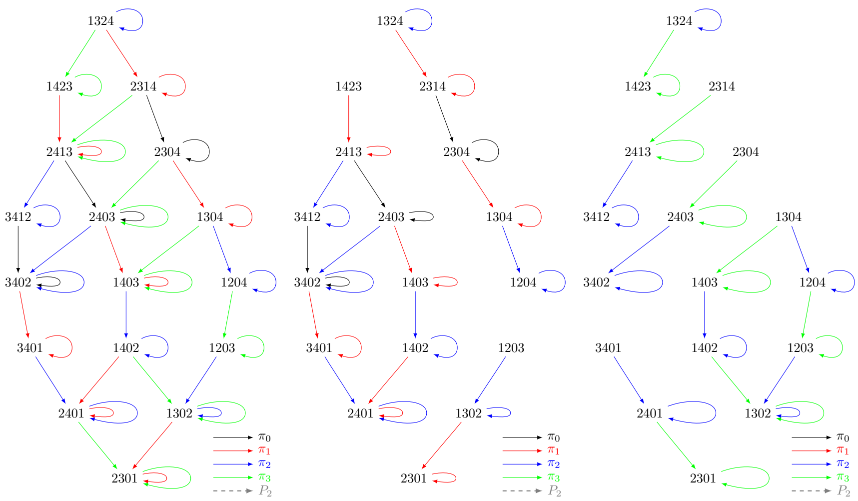



Figure 5.9: Second counterexample for the restriction of projective modules.

last two elements do not correspond to any projective modules of $R_3^0$ (it should correspond to $P^3_{\{2\}}$ since 1302 only has the loop of $\pi_2$, which is not the case).

We proceed the same way for the right module. We immediatly have a contradiction with the first element which should generate $P^3_{\{1\}}$ (be careful of the labels!) which is not the case.

As a conclusion of this paragraph, since we do not have the restriction of indecomposable projective modules, we will not be able to have a tower of monoids as for the case of $H_n^0$ to get **NCSF** and **QSym** [Krob and Thibon(1997)].

**Induction of indecomposable projective modules** For this one we can use Frobenius reciprocity as we did in Proposition 5.21, using Theorem 5.36:

**Theorem 5.50.** *Let $I \subset [\![0, n-1]\!]$ and $J \subset [\![0, m-1]\!]$. Then*

$$\operatorname{Ind}_{R^0_n \times R^0_m}^{R^0_{n+m}} P_I \otimes P_J = \begin{cases} P_{I \cup \overline{J}^n} \oplus P_{I \cup \{n\} \cup \overline{J}^n} & \text{if } 0 \notin J \\ P_{[\![0,n]\!] \cup \overline{J \setminus \{0\}}^n} & \text{if } 0 \in J \text{ and } I = [\![0, n-1]\!] \\ 0 & \text{otherwise.} \end{cases}.$$

*Proof.* We reason as in the proof of Proposition 5.21, using Frobenius reciprocity:

$$\operatorname{Hom}_{R^0_{n+m}}\left(\operatorname{Ind}_{R^0_n \times R^0_m}^{R^0_{n+m}} P_I \otimes P_J, S_K\right) = \operatorname{Hom}_{R^0_n \otimes R^0_m}\left(P_I \otimes P_J, \operatorname{Res}_{R^0_n \times R^0_m}^{R^0_{n+m}} S_K\right). \tag{5.26}$$

We are looking for sets $K \subset [\![0, n+m-1]\!]$ such that the simple $R^0_{n+m}$-module $S_K$ restricts to $S_I \otimes S_J$ over $R^0_n \times R^0_m$. If $0 \notin J$ then $K \cap [\![0, n-1]\!] = I$ and $K \cap [\![n+1, n+m-1]\!] = \overline{J}^n$. We conclude considering the two cases whether $n \in K$ or not. On the contrary, if $0 \in J$ then we are in the second case of Theorem 5.36. So either $K \cap [\![0, n]\!] = [\![0, n]\!]$ that is $I = [\![0, n-1]\!]$, and we have the second case, either it is wrong and in this case no restriction can be obtained. □

As we have seen, the natural tower of monoids structure of $(R^0_n)_{n\in\mathbb{N}}$ described here does not have a very nice representation theory. However, this is not the only tower structure, and they may be nice tower structure on their algebras involving linear combination.

# A Implementation

A large part of the algorithms here are implemented in `Sagemath` [Stein et al.(2018)]. The representation theory where computed using `sage_semigroups` [Hivert et al.(2012–2018)Hivert, Saliola, and Thié from the second author, F. Saliola and N. Thiéry. The code is freely accessible at

https://github.com/hivert/Jupyter-Notebooks

Thanks to the binder technology, one can experiment with it online at

https://mybinder.org/v2/gh/hivert/Jupyter-Notebooks/master?filepath=rook-0.ipynb

# B Tables

**Decomposition functor** We give the decomposition functor from projective $R^0_n$-modules into $H^0_n$-modules. They where computed according to Theorem 5.27.

$$\begin{aligned}
P^R_{(1)} &\simeq P_{(1)} \\
P^R_{(2)} &\simeq P_{(2)} \\
P^R_{(1,1)} &\simeq 2P_{(1,1)} + P_{(2)} \\
P^R_{(3)} &\simeq P_{(3)} \\
P^R_{(2,1)} &\simeq P_{(1,2)} + P_{(2,1)} + P_{(3)} \\
P^R_{(1,2)} &\simeq P_{(1,1,1)} + 2P_{(1,2)} + P_{(2,1)} + P_{(3)} \\
P^R_{(1,1,1)} &\simeq 3P_{(1,1,1)} + P_{(1,2)} + P_{(2,1)}
\end{aligned}$$

$$\begin{aligned}
P^R_{(0,1)} &\simeq P_{(1)} \\
P^R_{(0,2)} &\simeq P_{(1,1)} + P_{(2)} \\
P^R_{(0,1,1)} &\simeq P_{(1,1)} \\
P^R_{(0,3)} &\simeq P_{(1,2)} + P_{(3)} \\
P^R_{(0,2,1)} &\simeq 2P_{(1,1,1)} + P_{(1,2)} + P_{(2,1)} \\
P^R_{(0,1,2)} &\simeq P_{(1,1,1)} + P_{(1,2)} \\
P^R_{(0,1,1,1)} &\simeq P_{(1,1,1)}
\end{aligned}$$

$$
\begin{aligned}
P^R_{(4)} &\simeq P_{(4)}\\
P^R_{(0,4)} &\simeq P_{(1,3)} + P_{(4)}\\
P^R_{(3,1)} &\simeq P_{(1,3)} + P_{(3,1)} + P_{(4)}\\
P^R_{(0,3,1)} &\simeq P_{(1,1,2)} + P_{(1,2,1)} + P_{(1,3)} + P_{(3,1)}\\
P^R_{(2,2)} &\simeq P_{(1,2,1)} + P_{(1,3)} + P_{(2,2)} + P_{(3,1)} + P_{(4)}\\
P^R_{(0,2,2)} &\simeq P_{(1,1,1,1)} + 2P_{(1,1,2)} + P_{(1,2,1)} + P_{(1,3)} + P_{(2,2)}\\
P^R_{(2,1,1)} &\simeq P_{(1,1,2)} + P_{(1,2,1)} + P_{(1,3)} + P_{(2,1,1)} + P_{(3,1)}\\
P^R_{(0,2,1,1)} &\simeq 3P_{(1,1,1,1)} + P_{(1,1,2)} + P_{(1,2,1)} + P_{(2,1,1)}\\
P^R_{(1,3)} &\simeq P_{(1,1,2)} + 2P_{(1,3)} + P_{(2,2)} + P_{(4)}\\
P^R_{(0,1,3)} &\simeq P_{(1,1,2)} + P_{(1,3)}\\
P^R_{(1,2,1)} &\simeq 2P_{(1,1,1,1)} + 2P_{(1,1,2)} + 3P_{(1,2,1)} + P_{(1,3)} + P_{(2,1,1)} + P_{(2,2)} + P_{(3,1)}\\
P^R_{(0,1,2,1)} &\simeq 2P_{(1,1,1,1)} + P_{(1,1,2)} + P_{(1,2,1)}\\
P^R_{(1,1,2)} &\simeq 2P_{(1,1,1,1)} + 3P_{(1,1,2)} + P_{(1,2,1)} + P_{(1,3)} + P_{(2,1,1)} + P_{(2,2)}\\
P^R_{(0,1,1,2)} &\simeq P_{(1,1,1,1)} + P_{(1,1,2)}\\
P^R_{(1,1,1,1)} &\simeq 4P_{(1,1,1,1)} + P_{(1,1,2)} + P_{(1,2,1)} + P_{(2,1,1)}\\
P^R_{(0,1,1,1,1)} &\simeq P_{(1,1,1,1)}
\end{aligned}
$$

**Cartan matrices** We show below the first Cartan matrices of the 0-rook monoids $R_n$ for $n = 2, 3, 4, 5$. The column on the left shows the associated idempotents.

$$
\begin{array}{c} 12\\ 02\\ 21\\ 00 \end{array}
\begin{pmatrix}
1 & . & . & .\\
. & 1 & 1 & .\\
. & 1 & 2 & .\\
. & . & . & 1
\end{pmatrix}
\qquad
\begin{array}{c} 123\\ 023\\ 213\\ 003\\ 132\\ 032\\ 321\\ 000 \end{array}
\begin{pmatrix}
1 & . & . & . & . & . & . & .\\
. & 1 & 1 & . & 1 & . & . & .\\
. & 1 & 3 & . & 2 & 1 & 1 & .\\
. & . & . & 1 & . & 1 & 1 & .\\
. & 1 & 2 & . & 2 & . & . & .\\
. & . & 1 & 1 & . & 2 & 2 & .\\
. & . & 1 & 1 & . & 2 & 3 & .\\
. & . & . & . & . & . & . & 1
\end{pmatrix}
\qquad
\begin{array}{c} 1234\\ 0234\\ 2134\\ 0034\\ 1324\\ 0324\\ 3214\\ 0004\\ 1243\\ 0243\\ 2143\\ 0043\\ 1432\\ 0432\\ 4321\\ 0000 \end{array}
\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .\\
. & 1 & 1 & . & 1 & . & . & . & 1 & . & . & . & . & . & . & .\\
. & 1 & 3 & . & 3 & 1 & 1 & . & 2 & 1 & 2 & . & 1 & . & . & .\\
. & . & . & 1 & . & 1 & 1 & . & . & 1 & 1 & . & 1 & . & . & .\\
. & 1 & 3 & . & 4 & 1 & 1 & . & 2 & 1 & 3 & . & 1 & . & . & .\\
. & . & 1 & 1 & 1 & 3 & 3 & . & . & 2 & 4 & 1 & 2 & 1 & 1 & .\\
. & . & 1 & 1 & 1 & 3 & 5 & . & . & 2 & 6 & 1 & 3 & 2 & 2 & .\\
. & . & . & . & . & . & . & 1 & . & . & . & 1 & . & 1 & 1 & .\\
. & 1 & 2 & . & 2 & . & . & . & 2 & . & . & . & . & . & . & .\\
. & . & 1 & 1 & 1 & 2 & 2 & . & . & 2 & 3 & . & 2 & . & . & .\\
. & . & 2 & 1 & 3 & 4 & 6 & . & . & 3 & 9 & 1 & 4 & 2 & 2 & .\\
. & . & . & . & . & 1 & 1 & 1 & . & . & 1 & 2 & . & 2 & 2 & .\\
. & . & 1 & 1 & 1 & 2 & 3 & . & . & 2 & 4 & . & 3 & . & . & .\\
. & . & . & . & . & 1 & 2 & 1 & . & . & 2 & 2 & . & 3 & 3 & .\\
. & . & . & . & . & 1 & 2 & 1 & . & . & 2 & 2 & . & 3 & 4 & .\\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

$$
\begin{array}{r}
12345\\ 02345\\ 21345\\ 00345\\ 13245\\ 03245\\ 32145\\ 00045\\ 12435\\ 02435\\ 21435\\ 00435\\ 14325\\ 04325\\ 43215\\ 00005\\ 12354\\ 02354\\ 21354\\ 00354\\ 13254\\ 03254\\ 32154\\ 00054\\ 12543\\ 02543\\ 21543\\ 00543\\ 15432\\ 05432\\ 54321\\ 00000
\end{array}
\begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 1 & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & 3 & . & 3 & 1 & 1 & . & 3 & 1 & 2 & . & 1 & . & . & . & 2 & 1 & 2 & . & 2 & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & 1 & . & 1 & 1 & . & . & 1 & 1 & . & 1 & . & . & . & . & 1 & 1 & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . \\
. & 1 & 3 & . & 5 & 1 & 1 & . & 4 & 2 & 5 & . & 2 & . & . & . & 2 & 1 & 4 & . & 4 & 1 & 1 & . & 1 & . & 1 & . & . & . & . & . \\
. & . & 1 & 1 & 1 & 3 & 3 & . & 1 & 3 & 5 & 1 & 3 & 1 & 1 & . & . & 2 & 4 & 1 & 4 & 2 & 2 & . & 2 & 1 & 2 & . & 1 & . & . & . \\
. & . & 1 & 1 & 1 & 3 & 5 & . & 1 & 3 & 8 & 1 & 5 & 2 & 2 & . & . & 2 & 6 & 1 & 6 & 3 & 4 & . & 3 & 2 & 4 & . & 2 & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & 1 & . & 1 & 1 & . & . & . & . & 1 & . & 1 & 1 & . & . & 1 & 1 & . & 1 & . & . & . \\
. & 1 & 3 & . & 4 & 1 & 1 & . & 4 & 1 & 3 & . & 1 & . & . & . & 2 & 1 & 3 & . & 3 & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & 1 & 1 & 2 & 3 & 3 & . & 1 & 4 & 7 & 1 & 4 & 1 & 1 & . & . & 2 & 5 & 1 & 5 & 3 & 3 & . & 2 & 1 & 3 & . & 1 & . & . & . \\
. & . & 2 & 1 & 5 & 5 & 8 & . & 3 & 7 & 21 & 2 & 11 & 5 & 5 & . & . & 3 & 13 & 2 & 14 & 9 & 13 & . & 4 & 4 & 12 & 1 & 4 & 1 & 1 & . \\
. & . & . & . & . & 1 & 1 & 1 & . & 1 & 2 & 3 & 1 & 3 & 3 & . & . & . & 1 & 2 & 1 & 4 & 4 & 1 & . & 2 & 4 & 1 & 2 & 1 & 1 & . \\
. & . & 1 & 1 & 2 & 3 & 5 & . & 1 & 4 & 11 & 1 & 7 & 2 & 2 & . & . & 2 & 7 & 1 & 8 & 4 & 6 & . & 3 & 2 & 6 & . & 2 & . & . & . \\
. & . & . & . & . & 1 & 2 & 1 & . & 1 & 5 & 3 & 2 & 5 & 5 & . & . & . & 2 & 2 & 2 & 6 & 8 & 1 & . & 3 & 7 & 2 & 3 & 2 & 2 & . \\
. & . & . & . & . & 1 & 2 & 1 & . & 1 & 5 & 3 & 2 & 5 & 7 & . & . & . & 2 & 2 & 2 & 6 & 10 & 1 & . & 3 & 9 & 2 & 4 & 3 & 3 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . & 1 & . & 1 & 1 & . \\
. & 1 & 2 & . & 2 & . & . & . & 2 & . & . & . & . & . & . & . & 2 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & 1 & 1 & 1 & 2 & 2 & . & 1 & 2 & 3 & . & 2 & . & . & . & . & 2 & 3 & . & 3 & . & . & . & 2 & . & . & . & . & . & . & . \\
. & . & 2 & 1 & 4 & 4 & 6 & . & 3 & 5 & 13 & 1 & 7 & 2 & 2 & . & . & 3 & 10 & 1 & 10 & 4 & 5 & . & 4 & 2 & 5 & . & 2 & . & . & . \\
. & . & . & . & . & 1 & 1 & 1 & . & 1 & 2 & 2 & 1 & 2 & 2 & . & . & . & 1 & 2 & 1 & 3 & 3 & . & . & 2 & 3 & . & 2 & . & . & . \\
. & . & 2 & 1 & 4 & 4 & 6 & . & 3 & 5 & 14 & 1 & 8 & 2 & 2 & . & . & 3 & 10 & 1 & 11 & 4 & 6 & . & 4 & 2 & 6 & . & 2 & . & . & . \\
. & . & . & . & 1 & 2 & 3 & 1 & . & 3 & 9 & 4 & 4 & 6 & 6 & . & . & . & 4 & 3 & 4 & 9 & 11 & 1 & . & 4 & 10 & 2 & 4 & 2 & 2 & . \\
. & . & . & . & 1 & 2 & 4 & 1 & . & 3 & 13 & 4 & 6 & 8 & 10 & . & . & . & 5 & 3 & 6 & 11 & 18 & 1 & . & 5 & 16 & 3 & 6 & 4 & 4 & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & 1 & 1 & 1 & . & . & . & . & . & 1 & 1 & 2 & . & . & 1 & 2 & . & 2 & 2 & . \\
. & . & 1 & 1 & 1 & 2 & 3 & . & 1 & 2 & 4 & . & 3 & . & . & . & . & 2 & 4 & . & 4 & . & . & . & 3 & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & 2 & 1 & . & 1 & 4 & 2 & 2 & 3 & 3 & . & . & . & 2 & 2 & 2 & 4 & 5 & . & . & 3 & 5 & . & 3 & . & . & . \\
. & . & . & . & 1 & 2 & 4 & 1 & . & 3 & 12 & 4 & 6 & 7 & 9 & . & . & . & 5 & 3 & 6 & 10 & 16 & 1 & . & 5 & 15 & 2 & 6 & 3 & 3 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & 2 & 2 & 1 & . & . & . & . & . & 2 & 3 & 2 & . & . & 2 & 3 & . & 3 & 3 & . \\
. & . & . & . & . & 1 & 2 & 1 & . & 1 & 4 & 2 & 2 & 3 & 4 & . & . & . & 2 & 2 & 2 & 4 & 6 & . & . & 3 & 6 & . & 4 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & 2 & 3 & 1 & . & . & . & . & . & 2 & 4 & 2 & . & . & 3 & 3 & . & 4 & 4 & . \\
. & . & . & . & . & . & . & . & . & . & 1 & 1 & . & 2 & 3 & 1 & . & . & . & . & . & 2 & 4 & 2 & . & . & 3 & 3 & . & 4 & 5 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1
\end{pmatrix}
$$

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