On sums of the small divisors of a natural number

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Abstract

We consider the positive divisors of a natural number that do not exceed its square root, to which we refer as the *small divisors* of the natural number. We determine the asymptotic behavior of the arithmetic function that adds the small divisors of a natural number, and we consider its Dirichlet generating series.

1 Introduction

By the *small divisors* of a natural number n, we mean the set of integers

$$\{d : d \mid n, 1 \leq d \leq \sqrt{n}\}.$$

The phrase "small divisors," as defined here, is not to be confused with classical small divisors problems of mathematical physics (see, e.g., Yoccoz [5]). Aside from an earlier paper by the author [3], our definition of this phrase seems absent from the literature. Define the arithmetic function a by

$$a(n) = \sum_{\substack{d \mid n \\ d \leq \sqrt{n}}} d,$$

the sum taken over natural numbers. Thus $a(n)$ adds the small divisors of n. The sequence $a(n)$ appears as sequence A066839 in the *OEIS* [4]. We have the trivial bound,

$$a(n) \leq \sum_{k=1}^{[\sqrt{n}]} k = \frac{1}{2}\left[\sqrt{n}\right]\left(\left[\sqrt{n}\right]+1\right) \leq \frac{1}{2}\left(n+\sqrt{n}\right) \leq n.$$

A. W. Walker has pointed out that

$$a(n) = \sum_{\substack{d|n \\ d\leq\sqrt{n}}} d \leq \sqrt{n}\sum_{d|n} 1 = \sqrt{n}\,\tau(n),$$

where $\tau(n)$ denotes the sum of *all* the positive divisors of n. As

$$\lim_{n\to\infty} \frac{\tau(n)}{n^{\delta}} = 0$$

for all $\delta > 0$ (see Apostol [1, Theorem 13.12]), it follows that

$$\lim_{n\to\infty} \frac{a(n)}{n^{\frac{1}{2}+\delta}} = 0$$

for all $\delta > 0$. Thus, it seems that $a(n)$ compares with $\sqrt{n}$. In § 2, we in fact prove that $a(n)$ has average order $\sqrt{n}$. In § 3 we obtain some properties of the Dirichlet generating series for $a(n)$.

We observe here that the function $a(n)$ is not multiplicative. It is, however, supermultiplicative:

Lemma 1. *If m and n are relatively prime natural numbers, then $a(mn) \geq a(m)a(n)$.*

Proof. Suppose d_1 and d_2 are small divisors of m, and d_1' and d_2' are small divisors of n. Since $\gcd(m,n) = 1$, we have $d_1d_1' = d_2d_2'$ if and only if $d_1 = d_2$ and $d_1' = d_2'$. Therefore the product

$$a(m)a(n) = \Big(\sum_{\substack{d|m \\ d\leq\sqrt{m}}} d\Big)\Big(\sum_{\substack{d'|n \\ d'\leq\sqrt{n}}} d'\Big) \tag{1}$$

gives a sum, all of whose addends are distinct small divisors of mn. Therefore $a(m)a(n) \leq a(mn)$. □

Note that $a(24) = 10$ and $a(36) = 16$. Yet, $26 \cdot 36 = 864$ and $a(864) = 130 < 160$. Hence $a(n)$ is not completely supermultiplicative.

2 Asymptotic behavior of $a(n)$

Two functions $f(x)$ and $g(x)$ are said to be *asymptotic* when

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

and we denote this by $f(x) \sim g(x)$. We shall use the notation of Bachmann and Landau, viz.,

$$f(x) = O\left(g(x)\right),$$

whenever $|f(x)| \leq C|g(x)|$ as $x \to \infty$ for some positive constant C independent of x.

If $f(n)$ and $g(n)$ are arithmetic functions, we say $f(n)$ is of *average order* $g(n)$ whenever

$$\sum_{k=1}^{n} f(k) \sim \sum_{k=1}^{n} g(k)$$

(e.g., see Hardy and Wright [2, § 18.2]).

Theorem 2. *The function $a(n)$ is of average order $\sqrt{n}$. More precisely,*

$$\sum_{k=1}^{n} a(k) = \frac{2}{3} n\sqrt{n} + O(n \ln n). \tag{2}$$

Proof. Equation (2) proves the theorem, for, by elementary calculus,

$$\sum_{k=1}^{n} \sqrt{k} = \frac{2}{3} n\sqrt{n} + O(\sqrt{n}).$$

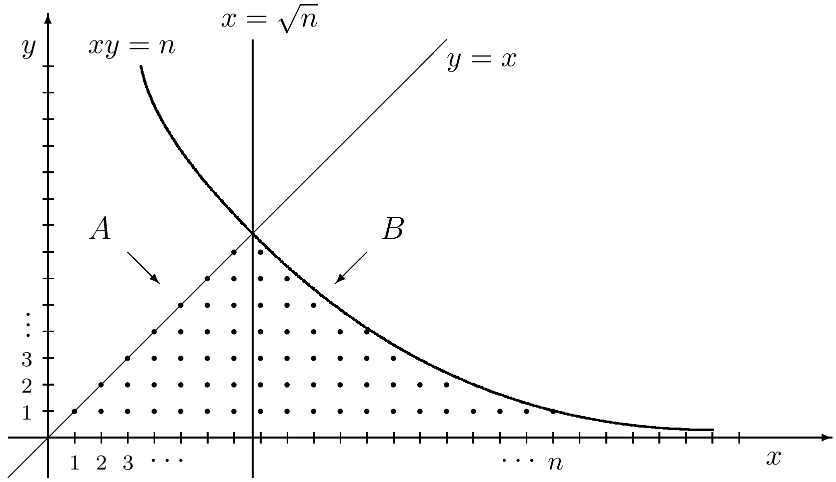
Note that

$$\sum_{k=1}^{n} a(k) = \sum_{k=1}^{n} \sum_{\substack{d|k \\ d\leq\sqrt{k}}} d = \sum_{(x,y)\in A} y + \sum_{(x,y)\in B} y, \tag{3}$$

where the ordered pairs (x, y) range over all lattice points (that is, where x, $y \in \mathbb{Z}$) of two regions, $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$, which are defined as follows,

$$A = \{(x, y) : 0 < y \leq x \leq \sqrt{n}\},$$
$$B = \{(x, y) : \sqrt{n} < x,\ 0 < y \leq n/x\},$$

Figure 1: Lattice points in regions A and B



and which are depicted in Figure 1.

Then,

$$\sum_{(x,y)\in A} y = \sum_{x=1}^{[\sqrt{n}]}\sum_{y=1}^{x} y = \frac{1}{2}\sum_{x=1}^{[\sqrt{n}]} x(x+1) = \frac{1}{2}\sum_{x=1}^{[\sqrt{n}]} x^2 + \frac{1}{2}\sum_{x=1}^{[\sqrt{n}]} x.$$

The first of these two sums yields

$$\frac{1}{2}\sum_{x=1}^{[\sqrt{n}]} x^2 = \frac{[\sqrt{n}]([\sqrt{n}]+1)(2[\sqrt{n}]+1)}{12} = \frac{(\sqrt{n}+O(1))^2(2\sqrt{n}+O(1))}{12},$$

while the second sum yields

$$\frac{1}{2}\sum_{x=1}^{[\sqrt{n}]} x = \frac{[\sqrt{n}]([\sqrt{n}]+1)}{4} = \frac{(\sqrt{n}+O(1))^2}{4},$$

where we have applied $[x] = x + O(1)$ for all real x. Therefore

$$\begin{aligned}\sum_{(x,y)\in A} y &= \frac{(\sqrt{n}+O(1))^2(2\sqrt{n}+O(1))}{12} + \frac{(\sqrt{n}+O(1))^2}{4} \\ &= \frac{1}{12}(\sqrt{n}+O(1))^2(2\sqrt{n}+O(1)),\end{aligned}$$

which yields

$$\sum_{(x,y)\in A} y = \frac{n\sqrt{n}}{6} + O(n). \tag{4}$$

Next, we have

$$\begin{aligned}\sum_{(x,y)\in B} y &= \sum_{x=[\sqrt{n}]+1}^{n} \sum_{y=1}^{[n/x]} y \\ &= \frac{1}{2} \sum_{x=[\sqrt{n}]+1}^{n} \left[\frac{n}{x}\right] \left(\left[\frac{n}{x}\right] + 1\right) \\ &= \frac{1}{2} \sum_{x=[\sqrt{n}]+1}^{n} \left(\frac{n}{x} + O(1)\right)^2,\end{aligned}$$

yielding

$$\sum_{(x,y)\in B} y = \frac{n^2}{2} \sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2} + O(n \ln n),$$

which follows because $\sum_{x=a}^{b} \frac{1}{x} = O(\ln b)$ for all a, $b \in \mathbb{N}$, $a < b$. By elementary calculus,

$$\sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2} = \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right),$$

hence

$$\sum_{(x,y)\in B} y = \frac{n\sqrt{n}}{2} + O(n \ln n). \tag{5}$$

As (2) follows immediately from (3), (4), and (5), the proof is complete. □

It is thus natural to consider the behavior of the sequence $a(n)/\sqrt{n}$. Perhaps unsurprisingly, this behavior is irregular. For instance, it is clear that

$$\liminf \frac{a(n)}{\sqrt{n}} = 0,$$

as $a(p) = 1$ for all primes p. On the other hand, it is easy to see that

$$\limsup \frac{a(n)}{\sqrt{n}} = \infty.$$

For, we need only consider the sequence $s_n = p_1^2 p_2^2 \cdots p_n^2$, where the primes are enumerated as $p_1 = 2$, $p_2 = 3$, and so on. Every number of the form $p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_n^{\epsilon_n}$, where $\epsilon_k = 0$ or 1 for $1 \leq k \leq n$, is a small divisor of s_n. Therefore

$$a(s_n) \geq \sum p_1^{\epsilon_1} p_2^{\epsilon_2} \cdots p_n^{\epsilon_n} = \prod_{k=1}^{n} (p_k + 1),$$

where the sum ranges over all n-tuples $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$ where $\epsilon_k = 0$ or 1 for $1 \leq k \leq n$. Hence

$$\lim_{n\to\infty} \frac{a(s_n)}{\sqrt{s_n}} \geq \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{1}{p_k}\right) = \infty.$$

The average order of $a(n)$ is interesting when compared to that of the function $\sigma(n)$, which adds *all* the positive divisors of n,

$$\sigma(n) = \sum_{d|n} d.$$

The sequence $\sigma(n)$ appears as A000203 in the *OEIS*. The average order of $\sigma(n)$ is $\frac{\pi^2}{6} n$ (see Hardy and Wright [2, § 18.3, Theorem 324]), i.e., we have a nonunit multiple of n ($\frac{\pi^2}{6} \approx 1.645$), as compared to Theorem 2 (merely $\sqrt{n}$ for the average order of $a(n)$).

3 The Dirichlet series of $a(n)$

An arithmetic function $f(n)$ is said to have a *Dirichlet generating series*, defined by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

Following Riemann, we let s be a complex variable and write

$$s = \sigma + it,$$

where σ and t are real; in particular $\sigma = \mathrm{Re}(s)$. Hence $|n^s| = n^\sigma$, therefore

$$\sum_{n=1}^{\infty} \left| \frac{a(n)}{n^s} \right| = \sum_{n=1}^{\infty} \frac{a(n)}{n^\sigma}.$$

Since $a(n) \geq 1$ for all $n \in \mathbb{N}$, it follows that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^{\sigma}} \tag{6}$$

diverges for all $\sigma \leq 1$; similarly, as $a(n) \leq n$ for all n, it follows that the series (6) converges for all $\sigma > 2$ (see Apostol [1, Theorem 11.8]). Therefore, there exists $\alpha \in \mathbb{R}$, $1 < \alpha \leq 2$, such that the Dirichlet series $L(s, a)$ converges on the half-plane $\sigma > \alpha$, but does not converge on the half-plane $\sigma < \alpha$. Here, α is called the *abscissa of convergence* of $L(s, a)$ (see Apostol [1, Theorem 11.9]).

Recall that the Dirichlet series $L(s, 1)$ is the Riemann zeta function when $\sigma > 1$, and that $L(s, 1)$ has $\alpha = 1$ as its abscissa of convergence. We write $\zeta(s) = L(s, 1)$.

Thus it follows that $L(s, \sqrt{n}) = \zeta\left(s - \frac{1}{2}\right)$, and has as its abscissa of convergence $\alpha = 3/2$. Therefore, in light of Theorem 2, we expect the same abscissa of convergence for $a(n)$.

Theorem 3. *The abscissa of convergence for the Dirichet series $L(s, a)$ is given by $\alpha = 3/2$.*

Proof. We need only show that the series (6) diverges at $\sigma = 3/2$, and converges for $3/2 < \sigma < 2$ (for, $L(\sigma, a)$ decreases as $\sigma \in \mathbb{R}$ increases).

First we consider

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a(k)}{k^{3/2}} &= \sum_{k=1}^{n} \frac{1}{k^{3/2}} \sum_{\substack{d \mid k \\ d \leq \sqrt{k}}} d \\
&= \sum_{k=1}^{n} \sum_{\substack{d \mid k \\ d \leq \sqrt{k}}} \frac{1}{(k/d)^{3/2}} \cdot \frac{1}{d^{1/2}} \\
&= \sum_{(x,y) \in A} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}} + \sum_{(x,y) \in B} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}},
\end{aligned}$$

where A and B are defined as in the proof of Theorem 2 (see Figure 1).

By elementary calculus,

$$\sum_{y=1}^{x} \frac{1}{y^{1/2}} \geq \int_{1}^{x} \frac{dy}{y^{1/2}} = 2x^{1/2} - 2,$$

hence

$$\sum_{(x,y)\in A} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}} = \sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^{3/2}} \sum_{y=1}^{x} \frac{1}{y^{1/2}} \geq 2 \sum_{x=1}^{[\sqrt{n}]} \frac{1}{x} - 2 \sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^{3/2}}$$
$$\geq 2 \log \left[\sqrt{n}\right] - 2\zeta(3/2),$$

where we applied $\sum_{x=1}^{m} 1/x \geq \log m$ for all $m \in \mathbb{N}$. Clearly,

$$\sum_{(x,y)\in B} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}} \geq 0,$$

hence

$$\sum_{k=1}^{n} \frac{a(k)}{k^{3/2}} = \sum_{(x,y)\in A} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}} + \sum_{(x,y)\in B} \frac{1}{x^{3/2}} \cdot \frac{1}{y^{1/2}} \geq 2 \log \left[\sqrt{n}\right] - 2\zeta(3/2),$$

which diverges to infinity as $n \to \infty$; thus the series (6) diverges at $\sigma = 3/2$.

Next we consider $\frac{3}{2} < \sigma < 2$. We remark that for $M \in \mathbb{N}$ we have

$$\sum_{y=1}^{M} \frac{1}{y^{\sigma-1}} \leq 1 + \int_{1}^{M} \frac{dy}{y^{\sigma-1}} \leq \frac{M^{2-\sigma}}{2-\sigma}. \tag{7}$$

Here,

$$\sum_{k=1}^{n} \frac{a(k)}{k^{\sigma}} = \sum_{k=1}^{n} \frac{1}{k^{\sigma}} \sum_{\substack{d|k \\ d\leq\sqrt{k}}} d$$
$$= \sum_{k=1}^{n} \sum_{\substack{d|k \\ d\leq\sqrt{k}}} \frac{1}{(k/d)^{\sigma}} \cdot \frac{1}{d^{\sigma-1}}$$
$$= \sum_{(x,y)\in A} \frac{1}{x^{\sigma}} \cdot \frac{1}{y^{\sigma-1}} + \sum_{(x,y)\in B} \frac{1}{x^{\sigma}} \cdot \frac{1}{y^{\sigma-1}}.$$

Applying (7), we have both

$$\begin{aligned}\sum_{(x,y)\in A} \frac{1}{x^\sigma}\cdot\frac{1}{y^{\sigma-1}} &= \sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^\sigma} \sum_{y=1}^{x} \frac{1}{y^{\sigma-1}} \\ &\le \sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^\sigma}\cdot\frac{x^{2-\sigma}}{2-\sigma} \\ &= \frac{1}{2-\sigma}\sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^{2(\sigma-1)}},\end{aligned}$$

and

$$\begin{aligned}\sum_{(x,y)\in B} \frac{1}{x^\sigma}\cdot\frac{1}{y^{\sigma-1}} &= \sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^\sigma} \sum_{y=1}^{[n/x]} \frac{1}{y^{\sigma-1}} \\ &\le \sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^\sigma}\cdot\frac{[n/x]^{2-\sigma}}{2-\sigma} \\ &\le \frac{n^{2-\sigma}}{2-\sigma}\sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2},\end{aligned}$$

hence

$$\sum_{k=1}^{n} \frac{a(k)}{k^\sigma} \le \frac{1}{2-\sigma}\sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^{2(\sigma-1)}} + \frac{n^{2-\sigma}}{2-\sigma}\sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2}. \tag{8}$$

Clearly

$$\sum_{x=1}^{[\sqrt{n}]} \frac{1}{x^{2(\sigma-1)}} \le \zeta(2(\sigma-1)). \tag{9}$$

We remark that

$$\sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2} \le \frac{1}{\sqrt{n}}\,, \tag{10}$$

because

$$\sum_{x=[\sqrt{n}]+1}^{n} \frac{1}{x^2} \le \frac{1}{([\sqrt{n}]+1)^2} + \int_{[\sqrt{n}]+1}^{n} \frac{dx}{x^2} \le \frac{1}{n} + \int_{\sqrt{n}}^{n} \frac{dx}{x^2} = \frac{1}{\sqrt{n}}\,.$$

Thus by (8), (9), and (10), we have

$$\begin{aligned}\sum_{k=1}^{n} \frac{a(k)}{k^\sigma} &\le \frac{\zeta(2(\sigma-1))}{2-\sigma} + \frac{n^{2-\sigma}}{2-\sigma} \cdot \frac{1}{\sqrt{n}} \\ &= \frac{1}{2-\sigma}\left(\zeta(2(\sigma-1)) + \frac{1}{n^{\sigma-\frac{3}{2}}}\right) \\ &\le \frac{1}{2-\sigma}\left(\zeta(2(\sigma-1)) + 1\right)\end{aligned}$$

for all $n \in \mathbb{N}$. Hence the series (6) converges for all σ such that $\frac{3}{2} < \sigma < 2$. $\square$

We may define the arithmetic function $b(n)$ by $b(1) = 1$, and, when n has unique prime factorization $n = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$,

$$b(n) = a(p_1^{\beta_1}) a(p_2^{\beta_2}) \cdots a(p_k^{\beta_k}).$$

Thus $b(n)$ is multiplicative, and $b(n) \le a(n)$ for all $n \in \mathbb{N}$ by Lemma 1. Note, then, that for all $\sigma > 3/2$ we have $L(\sigma, b) \le L(\sigma, a)$. Furthermore, as $b(n)$ is multiplicative, then $L(s, b)$ has an Euler product representation on its half-plane of convergence (see Apostol [1, Theorem 11.6]), given by

$$\begin{aligned}L(s,b) &= \prod_p \left(1 + \frac{b(p)}{p^s} + \frac{b(p^2)}{p^{2s}} + \frac{b(p^3)}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^s} + \frac{p+1}{p^{2s}} + \frac{p+1}{p^{3s}} + \frac{p^2+p+1}{p^{4s}} + \frac{p^2+p+1}{p^{5s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1}{p^{2s-1}} + \frac{1}{p^{4s-2}} + \cdots\right)\left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) \\ &= \prod_p \left(1 - \frac{1}{p^{2s-1}}\right)^{-1}\left(1 - \frac{1}{p^s}\right)^{-1} \\ &= \zeta(2s-1)\zeta(s),\end{aligned}$$

where the products are taken over all the primes p. Note that the second line follows because

$$b(p^n) = a(p^n) = 1 + p + \cdots + p^{[n/2]}$$

for all primes p and integers $n \ge 0$.

On the other hand, as $a(n) \leq n$ for all natural numbers n, we have for all $\sigma > 2$,

$$L(\sigma, a) \leq \sum_{n=1}^{\infty} \frac{n}{n^{\sigma}} = \zeta(\sigma - 1).$$

Hence for all $\sigma > 2$,

$$\zeta(2\sigma - 1)\zeta(\sigma) \leq L(\sigma, a) \leq \zeta(\sigma - 1). \tag{11}$$

In light of Theorem 2, this is unsurprising, as, recalling $L(\sigma, \sqrt{n}) = \zeta\left(\sigma - \frac{1}{2}\right)$, we see that the same bounds as in (11) hold for all $\sigma > 2$:

$$\zeta(2\sigma - 1)\zeta(\sigma) \leq L(\sigma, \sqrt{n}) \leq \zeta(\sigma - 1).$$

The latter inequality is immediate, while the former follows because

$$\left(1 - \frac{1}{p^{2\sigma-1}}\right)^{-1} \left(1 - \frac{1}{p^{\sigma}}\right)^{-1} \leq \left(1 - \frac{1}{p^{\sigma-\frac{1}{2}}}\right)^{-1}$$

for all primes p and all $\sigma > 2$.

Note that

$$\zeta(2s - 1) = \sum_{n=1}^{\infty} \frac{n}{n^{2s}},$$

hence $\zeta(2s - 1) = L(s, f)$, where

$$f(n) = \begin{cases} \sqrt{n}, & \text{if } n \text{ is a square;} \\ 0, & \text{otherwise.} \end{cases}$$

As $L(s, b) = \zeta(2s - 1)\zeta(s)$, then (see Apostol [1, Theorem 11.5])

$$b(n) = \sum_{d|n} f(d).$$

Thus $b(n)$ adds the square roots of the square divisors of n. For example, $b(72) = 1+2+3+6 = 12$; this compares to $a(72) = 1+2+3+4+6+8 = 24$.

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