Number of Distinguishing Colorings and Partitions

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Abstract

A vertex coloring of a graph G is called distinguishing (or symmetry breaking) if no non-identity automorphism of G preserves it, and the distinguishing number, shown by $\mathrm{D}(G)$, is the smallest number of colors required for such a coloring. This paper is about counting non-equivalent distinguishing colorings of graphs with k colors. A parameter, namely $\Phi_k(G)$, which is the number of non-equivalent distinguishing colorings of a graph G with at most k colors, is shown here to have an application in calculating the distinguishing number of the lexicographic product and X-join of graphs. We study this index (and some other similar indices) which is generally difficult to calculate. Then, we show that if one knows the distinguishing threshold of a graph G, which is the smallest number of colors $\theta(G)$ so that, for $k \geq \theta(G)$, every k-coloring of G is distinguishing, then, in some special cases, counting the number of distinguishing colorings with k colors is vary easy. We calculate $\theta(G)$ for some classes of graphs including the Kneser graph $K(n, 2)$. We then turn to vertex partitioning by studying the distinguishing coloring partition of a graph G; a partition of vertices of G which induces a distinguishing coloring for G. There, we introduce $\Psi_k(G)$ as the number of non-equivalent distinguishing coloring partitions with at most k cells, which is a generalization to its distinguishing coloring counterpart.

Keywords: distinguishing coloring, distinguishing threshold, distinguishing partition, distinguishing coloring partition

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1 Introduction

Breaking symmetries of graphs via vertex coloring is a subject initiated by Babai's work [3] in 1977. There, he introduced the *asymmetric coloring of a graph*, and proved that a tree has an asymmetric coloring with two colors if all vertices have the same degree $\alpha \geq 2$, where α can be an arbitrary finite or infinite cardinal. The concept was later called *distinguishing coloring* in the literature, since the appearance of [1] by Albertson and Collins in 1996.

This paper is about counting *non-equivalent* distinguishing colorings and partitions of a given graph with k colors. Considering this number in case of 2 colors is as old as symmetry breaking in graphs; in 1977 Babai [3] tried to count distinguishing colorings of infinite trees, while in 1991 Polat and Sabidussi [19] tried to count *essentially different asymmetrising sets* in finite and infinite trees which are *distinguishing (coloring) partitions with 2 cells* in our terminology (see Section 4).

A vertex coloring of a graph G is called *distinguishing* if it is only preserved by the identity automorphism; in this case, we say that the coloring *breaks* all the symmetries of G. By a k*-distinguishing coloring* we mean a distinguishing coloring which uses exactly k colors. The *distinguishing number* of a graph G, denoted $\mathrm{D}(G)$, is the smallest number d such that there exists a distinguishing vertex coloring of G with d colors. The graph G is called d*-distinguishable* if there exists a distinguishing vertex coloring with d colors [1]. The distinguishing number of some important classes of graphs are as follows: $\mathrm{D}(K_n) = n$, $\mathrm{D}(K_{n,n}) = n + 1$, $\mathrm{D}(P_n) = 2$ for $n \geq 2$, $\mathrm{D}(C_3) = \mathrm{D}(C_4) = \mathrm{D}(C_5) = 3$ while $\mathrm{D}(C_n) = 2$ for $n \geq 6$ [1].

A whole wealth of results on the subject has already been generated. Among many, we can only mention a few, only those that have essentially important results, or those that have introduced new indices based on distinguishing colorings. For a connected finite graph G, it was independently proved by Collins and Trenk [5] and Klavžar et al. [18] that $\mathrm{D}(G) \leq \Delta + 1$, where Δ is the largest degree of G. Equality holds if and only if G is a complete graph $K_{\Delta+1}$, a balanced complete bipartite graph $K_{\Delta,\Delta}$, or C_5. Collins and Trenk [5] also mixed the concept of distinguishing colorings with proper vertex colorings to introduce the *distinguishing chromatic number* $\chi_D(G)$ of a graph G. It is defined as the minimum number of colors required to properly color the vertices of G such that this coloring is only preserved by the trivial automorphism. They also showed that, for a finite connected graph G, we have $\chi_D(G) \leq 2\Delta(G)$ and that equality holds only if G is isomorphic to $K_{\Delta,\Delta}$ or C_6.

Symmetry breaking can also happen via other kinds of graph colorings. An analogous index for an edge coloring, namely the *distinguishing index* $\mathrm{D}'(G)$, has been introduced by Kalinowski and Pilśniak in [16] as the minimum number of the required colors in an asymmetric edge-coloring of a connected graph $G \not\simeq K_2$. Moreover, they showed that $\mathrm{D}'(G) \leq \Delta(G)$ for a finite connected graph G, unless G is isomorphic to C_3, C_4 or C_5. Another analogous index is introduced by Kalinowski, Pilśniak and Woźniak in [17]; the *total distinguishing number* $\mathrm{D}''(G)$ is the minimum number of required colors in an asymmetric total coloring of G.

To generalize some results from the finite case to the infinite ones, Imrich, Klavžar and Trofimov [15] considered the distinguishing number for infinite graphs. They showed that for an infinite connected graph G we have $\mathrm{D}(G) \le \mathfrak{n}$, where $\mathfrak{n}$ is a cardinal number such that the degree of any vertex of G is not greater than $\mathfrak{n}$. Most symmetry breaking concepts have their relative counterparts in the infinite case, however, there are some (such as the *Infinite Motion Conjecture*) that arise only when we consider infinite graphs. As an instance, one can take a look at [13] by Imrich et al. which contains comparisons of some distinguishing indices of connected infinite graphs.

It was also interesting to know the distinguishing number for the product graphs. For example, Bogstad and Cowen [4] showed that for $k \ge 4$, every hypercube Q_k of dimension k, which is the Cartesian product of k copies of K_2, is 2-distinguishable. It has also been shown by Imrich and Klavžar in [14] that the distinguishing number of Cartesian powers of a connected graph G is equal to two except for K_2^2, K_3^2, K_2^3. Meanwhile, Imrich, Jerebic and Klavžar [12] showed that Cartesian products of relatively prime graphs whose sizes are close to each other can be distinguished with a small number of colors. Moreover, Estaji et al. in [8] proved that for every pair of connected graphs G and H with $|H| \le |G| < 2^{|H|} - |H|$, we have $\mathrm{D}(G \square H) \le 2$. Gorzkowska, Kalinowski and Pilśniak proved a similar result for the distinguishing index of the Cartesian product [10].

The lexicographic product was a subject of symmetry breaking via vertex and edge coloring by Alikhani and Soltani in [2], where they showed that under some conditions on the automorphism group of a graph G, we have $D(G) \le D(G^k) \le D(G) + k - 1$, where G^k is the kth lexicographic power of G, for any natural number k. As well, they showed that if G and H are connected graphs, then $D(H) \le D(G \circ H) \le |V(G)| \cdot D(H)$.

Coloring is not the only mean of symmetry breaking in graphs. For example, one might break the symmetries of a graph via a more general tool such as vertex partitioning. Ellingham and Schroeder introduced *distinguishing partition* of a graph as a partition of the vertex set that is preserved by no nontrivial automorphism [7]. Here, unlike coloring, some graphs have no distinguishing partition. Anyhow, for a graph G that admits a distinguishing partition, one may think of the minimum number of required cells in a distinguishing partition of the vertex set. Here, we show this index by $\mathrm{DP}(G)$.

In this paper, we introduce some further indices related to symmetry breaking of graphs by studying the number of non-equivalent distinguishing colorings of a graph with k colors and some other similar quantities. This is motivated by the problem of evaluating the distinguishing number of a lexicographic product or an X-join of some graphs, which we consider in Section 5.

The paper is organized as follows. In Section 2, we consider the number of non-equivalent distinguishing colorings of a graph G with (exactly) k colors, namely $\Phi_k(G)$ (and $\varphi_k(G)$) and, we calculate these indices for some simple types of graphs. Afterwards in Section 3, we introduce the *distinguishing threshold* as a dual index to the distinguishing number. It is shown that calculations of some indices introduced in Sections 2 and 4 are easier, in some cases, when we know the distinguishing threshold. Moreover, in Section 4, we consider the number of non-equivalent *distinguishing coloring partitions* of a graph G with (exactly) k cells, namely $\Psi_k(G)$ (and $\psi_k(G)$) and, we calculate them in some special cases. Additionally in Section 4, some other auxiliary indices are also introduced. Then, we present an application of one of the indices introduced here, namely $\Phi_k(G)$, in Section 5. We, finally, conclude the paper by shedding some lights on the future investigations in Section 6.

Here, we use the standard notation and terminology of graph theory, which can be found in [6]. We only remind that the set of neighbors of a vertex v in G is denoted by $N(v)$, while $N[v]$ stands for the set $N(v) \cup \{v\}$.

2 Non-equivalent Distinguishing Colorings

Two colorings c_1 and c_2 of a graph G are called *equivalent* if there is an automorphism α of G such that $c_1(v) = c_2(\alpha(v))$ for all $v \in V(G)$.

The number of non-equivalent distinguishing colorings of a graph G with $\{1, \ldots, k\}$ as the set of admissible colors is shown by $\Phi_k(G)$, while the number of non-equivalent k-distinguishing colorings of a graph G with $\{1, \ldots, k\}$ as the set of colors is shown by $\varphi_k(G)$. When G has no distinguishing colorings with exactly k colors, we put $\varphi_k(G) = 0$. It is also clear that $\Phi_{\mathrm{D}(G)}(G) = \varphi_{\mathrm{D}(G)}(G)$. Moreover, it is straightforward to show that

$$\Phi_k(G) = \sum_{i=\mathrm{D}(G)}^{k} \binom{k}{i} \varphi_i(G).$$

Note,aslo, that $\varphi_k(K_n)$ is nonzero only when $k = n$, for which we know $\varphi_n(K_n) = 1$. It is easy to prove that for $n \geq 2$ and $k \geq n$,

$$\Phi_k(K_n) = \binom{k}{n}.$$

In the following two theorems, we give some recursive formulas for $\Phi_k(P_n)$ and $\varphi_k(P_n)$. An interested reader can find some explicit formulas for these indices (and $\Phi_k(C_n)$ and $\varphi_k(C_n)$ as well), in the Online Encyclopedia of Integer Sequences under the relevant sequence number (see Appendix A).

Theorem 2.1. *For $n = 4, 5, \ldots$, we have*

$$\Phi_k(P_n) = \binom{k}{2} k^{n-2} + k\Phi_k(P_{n-2}),$$

while $\Phi_k(P_2) = \binom{k}{2}$ and $\Phi_k(P_3) = k\binom{k}{2}$.

Proof. The proof is clear for $n = 2, 3$. For $n \geq 4$, we know that the two end-vertices of P_n either have different colors or the same color. If they are different in colors, we can pick the two colors in $\binom{k}{2}$ different ways, and, the internal vertices can have any possible k^{n-2} combinations, because the only non-trivial automorphism of P_n has to map its end vertices onto one another. When the two end-vertices have the same color, we can choose their color in k different ways, but to be sure that the coloring is distinguishing, the remaining path of length $n - 2$ must be distinguishingly colored in $\Phi_k(P_{n-2})$ ways. □

In the next result, we make use of the well-known fact that the number of surjective functions from a set of n elements to a set of k elements is $k!\left\{ {n \atop k} \right\}$ where $\left\{ {n \atop k} \right\}$ is the Stirling number of the second type.

Theorem 2.2. *For $n = 4, 5, \ldots,$ we have*

$$\varphi_k(P_n) = k\big(\varphi_k(P_{n-2}) + \varphi_{k-1}(P_{n-2})\big) + \binom{k}{2}\Big((k-2)!\left\{ {n-2 \atop k-2} \right\} + 2(k-1)!\left\{ {n-2 \atop k-1} \right\} + k!\left\{ {n-2 \atop k} \right\}\Big).$$

Proof. With the same method of counting to the proof of Theorem 2.1, we know that either the colors of the two ends of P_n are the same or they are different. If they are the same, we can pick this color in k different ways. Since the coloring has to be distinguishing, the internal path of length $n-2$ has to be colored distinguishingly with either all k colors or the remaining $k-1$ colors. Therefore in this case we have $k\big(\varphi_k(P_{n-2}) + \varphi_{k-1}(P_{n-2})\big)$ non-equivalent distinguishing colorings for P_n.

When the two end-vertices of P_n have different colors, any arbitrary coloring of internal vertices makes the resulting coloring a distinguishing one. We can pick the two colors for the end vertices in $\binom{k}{2}$ ways while the rest of colors must be presented on the internal vertices. Thus, there are four different possibilities; either only the rest of $k-2$ colors are used to color the internal vertices of P_n, or these $k-2$ colors are used along with one of the two colors of the end vertices (two different possibilities), or all the k colors are presented on the internal vertices of P_n. In each case we must count the number of surjective functions from the set of available colors to the set of internal vertices. Therefore in this case we have $\binom{k}{2}\Big((k-2)!\left\{ {n-2 \atop k-2} \right\} + 2(k-1)!\left\{ {n-2 \atop k-1} \right\} + k!\left\{ {n-2 \atop k} \right\}\Big)$ non-equivalent distinguishing colorings for P_n with exactly k colors. □

We, additionally, calculate these indices for the complete bipartite graph $K_{n,n}$. Note that $\mathrm{D}(K_{n,n}) = n + 1$.

Theorem 2.3. *For $n = 2, 3, \ldots$ we have*

$$\Phi_k(K_{n,n}) = \frac{1}{2}\binom{k}{n}\left(\binom{k}{n} - 1\right),$$

and for $n + 1 \le k \le 2n$ we have

$$\varphi_k(K_{n,n}) = \frac{1}{2}\binom{k}{n}\binom{n}{k-n}.$$

Proof. The proof is easy for the first assertion since to color $K_{n,n}$ distinguishingly, one should color the vertices of each part different from other vertices within that part and the two parts of $K_{n,n}$ have to be colored differently. Since we have chosen n colors out of k colors to color the vertices of a part of $K_{n,n}$, we cannot color the other part by the same pallet of colors. Moreover, transposing the first and the second part makes the two resulting colorings equivalent. Therefore, we arrive at a conclusion for the first assertion.

For the second assertion, like the first one, we color the vertices of one part by choosing n colors out of k ones, then we use the remaining $k-n$ colors for the vertices of the other part. For the rest of the vertices, i.e. the $2n-k$ vertices in the other part, we must choose colors from the first set of n colors in $\binom{n}{2n-k} = \binom{n}{n-2n+k} = \binom{n}{k-n}$ ways. Again, by transposing the first and the second part, the two resulting colorings are equivalent. This makes the second assertion evident. □

In Appendix A, we have presented the parameters Φ_k and φ_k for the graphs P_n and C_n, for $n, k = 2, \ldots, 10$.

It might seem easy to calculate $\Phi_k(G)$ and $\varphi_k(G)$ when G is a path or a cycle. However, the calculations are not easy in the general case. Even for a computer algebra system, it might take very long to count the number of non-equivalent distinguishing colorings of a symmetric graph G on n vertices when $n \geq 10$. Anyhow, even when n is large, for some k the calculations are much easier.

Let G be a graph on n vertices. Assume that we desire to distinguishingly color the vertices of G with exactly n colors. Then every vertex must receive a color different from the others which gives rise to $n!$ colorings. However, in order to count non-equivalent colorings, we should consider the colorings modulo the automorphism group of G. Therefore, we have the following result, which coincides with Theorem 3.5 in the next section.

Proposition 2.4. *For any graph G on n vertices, we have*

$$\varphi_n(G) = \frac{n!}{|\mathrm{Aut}(G)|}. \quad \square$$

This result motivates us to ask whether there are numbers $k \leq n$ such that any k-coloring of a graph on n vertices is distinguishing. We will consider this problem in Section 3.

3 Distinguishing Threshold

For any graph G, we define the *distinguishing threshold* $\theta(G)$ to be the minimum number t such that for any $k \geq t$, any arbitrary coloring of G with k colors is distinguishing. For example $\theta(K_n) = \theta(\overline{K_n}) = n$ and $\theta(K_{m,n}) = m + n$. Note, also, that for an asymmetric graph G, we have $\theta(G) = \mathrm{D}(G) = 1$.

Let G be a graph on n vertices. If any pair of vertices of G have different neighborhoods from each other, then any $(n-1)$-coloring of G has to be distinguishing because in this case, no two vertices with the same color can be mapped to each other by a non-trivial color-preserving automorphism. Conversely, if there are two vertices of G whose neighborhoods are the same, then any $(n-1)$-coloring of G which assigns the same color to these two vertices, is not distinguishing. From this, we observe the following.

Lemma 3.1. *For any graph G on n vertices, $\theta(G) \leq n-1$ if and only if $N(v) \neq N(u)$, for any pair $v, u \in V(G)$.* □

For the cases of paths and cycles we can calculate the distinguishing threshold.

Proposition 3.2. *For any $n \geq 2$ we have*

$$\theta(P_n) = \lceil \frac{n}{2} \rceil + 1.$$

Proof. The automorphism group of P_n induces $\lceil \frac{n}{2} \rceil$ orbits on its vertices, while each orbit contains at most two vertices. By the pigeonhole principle, any combination of $\lceil \frac{n}{2} \rceil + 1$ colors on n vertices of P_n breaks at least one orbit. Since with $\lceil \frac{n}{2} \rceil$ colors, there is a non-distinguishing coloring, we must have $\theta(P_n) = \lceil \frac{n}{2} \rceil + 1$. □

Proposition 3.3. *For any $n \geq 3$ we have*

$$\theta(C_n) = \lfloor \frac{n}{2} \rfloor + 2$$

Proof. To show that $\theta(C_n) \geq \lfloor \frac{n}{2} \rfloor + 2$, we present an $(\lfloor \frac{n}{2} \rfloor + 1)$-coloring of an n-cycle with the vertex set $\{v_1, \ldots, v_n\}$ which is not a distinguishing coloring. If n is even, then color the vertices $v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor + 1}$ by colors $1, \ldots, \lfloor \frac{n}{2} \rfloor + 1$, and color the vertices $v_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, n$ by $\lfloor \frac{n}{2} \rfloor, \ldots, 2$, respectively. Similarly, if n is odd, color the vertices $v_1, \ldots, v_{\lfloor \frac{n}{2} \rfloor + 1}$ by colors $1, \ldots, \lfloor \frac{n}{2} \rfloor + 1$, and color $v_{\lfloor \frac{n}{2} \rfloor + 2}, \ldots, n$ by $\lfloor \frac{n}{2} \rfloor + 1, \ldots, 2$, respectively. This coloring is not distinguishing as it cannot break the reflection symmetry that fixes v_1 and maps v_2 on v_n. Figure 1 illustrates this coloring for C_8 and C_9.

It remains to show that for $k \geq \lfloor \frac{n}{2} \rfloor + 2$, every k-coloring of C_n is distinguishing. When n is odd, coloring the vertices of C_n with k colors results in at least three colors that are used only once. Similarly, when n is even, coloring the vertices of C_n with k colors results in at least four colors that are used only once. Hence, in any case, for any coloring of C_n, with k colors, there are at least three colors which are used only once. Now consider vertices v_1, v_2 and v_3 whose colors appeared only once in a k-coloring and suppose that P is the only path in C_n from v_1 to v_2 that does not contain v_3. Any color-preserving automorphism α of C_n must map P onto itself, which means that α is the identity. This completes the proof. □

Figure 1: Non-distinguishing colorings for C_8 and C_9 with 5 distinct colors.

It, therefore, seems an interesting problem on its own to calculate the distinguishing threshold of various families of graphs. As an example, one might ask this question for the Petersen graph P. In fact, this graph is a member of a well-known family of graphs, namely, the *Kneser* graphs. Let $0 \leq k \leq n/2$. Then, the Kneser graph $K(n, k)$ is a graph whose vertex set is the set of all k-subsets of $\{1,\ldots,n\}$ where two vertices are adjacent if and only if their corresponding sets do not intersect. It is easy to see that $K(n,k)$ is a vertex-transitive graph on $\binom{n}{k}$ vertices with valency $\binom{n-k}{k}$, and that $P = K(5,2)$. To study more on Kneser graphs see, for instance, [9].

We now evaluate the distinguishing threshold of the Kneser graphs $K(n,2)$.

Proposition 3.4. *For any $n \geq 5$, we have $\theta(K(n,2)) = \frac{1}{2}(n^2-3n+6)$. In particular, if P is the Petersen graph, then $\theta(P) = 8$.*

Proof. Assume that $k \geq \frac{1}{2}(n^2-3n+6)$ and, to get a contradiction, suppose that there is a k-coloring of $K(n,2)$ which is not distinguishing. This implies that there is a color-preserving automorphism α such that $\alpha(u) = v$, for some distinct vertices u and v. Let R be the set of common neighbors of u and v. Thus

$$r = |R| = \binom{n-4+i}{2},$$

where $i \in \{0,1\}$ is the possible size of the intersection of the corresponding 2-sets of u and v. Furthermore, let $S = N(u) \setminus N[v]$ and $S' = N(v) \setminus N[u]$. Therefore, we have

$$s = |S| = |S'| = \binom{n-2}{2} - r + i - 1.$$

Moreover, let

$$X = V(K(n,2)) \setminus (\{u,v\} \cup R \cup S \cup S').$$

Hence

$$x = |X| = \binom{n}{2} - (2 + r + 2s).$$

Note that, since we have $\alpha(N(u)) = N(v)$, the color-pallet of $N(u)$ must be the same as that of $N(v)$. Thus, the number of colors used for coloring the vertices in $\{u,v\} \cup R \cup S \cup S'$ is at most $1 + r + s$. Consequently, we have to color the vertices in X with $k - (1 + r + s)$ colors. But this is impossible because $k - (1 + r + s) > x$. The reason is as follows:

$$\begin{aligned}
k - (1 + r + s) - x &= k - 1 - r - s - \binom{n}{2} + 2 + r + 2s \\
&= k - \binom{n}{2} + 1 + s \\
&\geq \frac{1}{2}(n^2 - 3n + 6) - \binom{n}{2} + 1 + \binom{n-2}{2} - \binom{n-4+i}{2} + i - 1 \\
&\geq \frac{1}{2}(n^2 - 3n + 6) - \binom{n}{2} + 1 + \binom{n-2}{2} - \binom{n-3}{2} > 0
\end{aligned}$$

Now, suppose that $k = \frac{1}{2}(n^2-3n+4) = \frac{1}{2}(n^2-3n+6) - 1$. We show that there is a k-coloring of $K(n,2)$ which is not distinguishing. Color the vertices $\{3,1\}$ and $\{3,2\}$ by the color 1, the vertices $\{4,1\}$ and $\{4,2\}$ by the color 2, ..., and the vertices $\{n,1\}$ and $\{n,2\}$ by the color $n-2$, while the other $\binom{n}{2}-2n+4$ vertices receive the remaining $k-n+2$ colors. Note that

$$\binom{n}{2}-2n+4=k-n+2.$$

Now, consider the mapping $(\{3,1\},\{3,2\})(\{4,1\},\{4,2\})(\{5,1\},\{5,2\})\cdots(\{n,1\},\{n,2\})$ which is a nontrivial color-preserving automorphism of $K(n,2)$. ☐

The next result reveals the importance of the distinguishing threshold in counting the number of non-equivalent distinguishing colorings.

Theorem 3.5. *Let G be a graph on n vertices. For any $k \geq \theta(G)$ we have*

$$\varphi_k(G)=k!\left\{{n \atop k}\right\}/|\mathrm{Aut}(G)|.$$

Proof. When $k \geq \theta(G)$, any k-coloring of G is distinguishing. Therefore, to calculate $\varphi_k(G)$, we must only count the number of non-equivalent k-colorings. We know that the total number of k-colorings of G is equal to the number of surjective functions from the set of vertices to the set of colors. As it is noted just before Theorem 2.2, this number is $k!\left\{{n \atop k}\right\}$. Furthermore, for any automorphism $\alpha \in \mathrm{Aut}(G)$, the image of a k-coloring f under α is equivalent to f. Consequently, the result holds. ☐

Using Theorem 3.5 and Propositions 3.2 and 3.3 the following results follow immediately.

Corollary 3.6. *Let $n \geq 2$. For any $k \geq \lceil \frac{n}{2} \rceil + 1$ we have*

$$\varphi_k(P_n)=k!\left\{{n \atop k}\right\}/2. \quad \square$$

Corollary 3.7. *Let $n \geq 3$. For any $k \geq \lfloor \frac{n}{2} \rfloor + 2$ we have*

$$\varphi_k(C_n)=k!\left\{{n \atop k}\right\}/2n. \quad \square$$

Furthermore, it is immediate to observe that, for any $n \geq 4$, $\varphi_{n-1}(P_n)=\frac{1}{4}(n-1)n!$ and that for any $n \geq 5$, $\varphi_{n-1}(C_n)=\frac{1}{4}(n-1)(n-1)!$, which agree with the tables in Appendix A.

4 Non-equivalent Distinguishing Partitions

In this section, we turn our attention to the case of different distinguishing partitions of graphs. Let G be a graph and let P_1 and P_2 be two partitions of the vertices of G. We say P_1 and P_2 are *equivalent* if there is a non-trivial automorphism of G which maps P_1 onto P_2. The number of non-equivalent partitions of G, with at most k cells, is called the *partition number* of G and is denoted by $\Pi_k(G)$.

Meanwhile, a *distinguishing coloring partition* of a graph G is a partition of the vertices of G such that it induces a distinguishing coloring for G. Note that the minimum number of cells required for such a partition equals the distinguishing number $\mathrm{D}(G)$. The number of non-equivalent distinguishing coloring partitions of a graph G with at most k cells is shown by $\Psi_k(G)$, while the number of non-equivalent distinguishing coloring partitions of a graph G with exactly k cells is shown by $\psi_k(G)$. It is not difficult to observe that

$$\Psi_k(G) = \sum_{j \leq k} \psi_j(G),$$

and that $\psi_n(G) = 1$, for any graph G on n vertices. In what follows, we deal with $\psi_k(G)$ and present some calculations where G is a path or a cycle. We start with the following observation which states how $\psi_2(P_n)$ is related to $\varphi_2(P_n)$.

Proposition 4.1. *For any $n \geq 1$, $\psi_2(P_{2n+1}) = \frac{1}{2}\varphi_2(P_{2n+1})$ and $\psi_2(P_{2n}) = \frac{1}{2}\varphi_2(P_{2n}) + 2^{n-2}$.*

Proof. First we consider the path P_{2n+1}. It is evident that any distinguishing coloring partition with two cells induces two distinguishing colorings. Furthermore, since only one of the two cells contains the middle vertex of P_{2n+1}, these two colorings are non-equivalent; this proves the first part.

To see the next part, we note that $\varphi_2(P_{2n})$ counts two different types of distinguishing coloring partitions: it counts type 1, which consists of the partitions in which swapping the two colors makes the resulting colorings equivalent, only once, while it counts type 2, which are the remaining partitions, twice. Hence, if we add the number of distinguishing coloring partitions of type 1 to $\varphi_2(P_{2n})$, then every distinguishing coloring partitions of P_{2n} is counted twice. On the other hand, it is not hard to see that the number of distinguishing coloring partitions of type 1 is equal to $2^n/2 = 2^{n-1}$. Therefore,

$$\psi_2(P_{2n}) = \frac{\varphi_2(P_{2n}) + 2^{n-1}}{2},$$

which completes the proof. □

An immediate consequence of Proposition 4.1 is that $\varphi_2(P_n)$ is always an even number. We can, furthermore, calculate $\psi_k(P_n)$ in the case $k = n - 1$.

Proposition 4.2. *Let $n \geq 2$. We have $\psi_{n-1}(P_n) = \lfloor \frac{n^2}{4} \rfloor$.*

Proof. As there are $n-1$ different cells, any partition of the vertices of $P_n = v_1 v_2 \cdots v_n$ will result in exactly one pair of vertices in the same part of the partition. So it suffices to count the number of different ways to choose two vertices such that including them in one cell and all the other vertices in singleton cells, results in a distinguishing coloring partition such that no two such partitions are mapped to each other using the non-trivial automorphism of P_n.

First, assume n is even. We split P_n to two halves: $A = \{v_1, \ldots, v_{\frac{n}{2}}\}$ and $B = \{v_{\frac{n}{2}+1}, \ldots, v_n\}$. There are two non-equivalent cases: (a) the two vertices are chosen from A, and (b) one is chosen from A and the other one is chosen from B. The case (a) contains $\binom{n/2}{2}$ ways. On the other hand, case (b), in turn, contains the following subcases: if one chooses v_1, then there are $|B| = \frac{n}{2}$ choices for the second vertex; if one chooses v_2, then there are $\frac{n}{2} - 1$ choices for the second vertex (note that the case (v_2, v_n) is equivalent to (v_1, v_{n-1}) which has already been counted). Continuing this argument, we will have

$$\frac{n}{2} + (\frac{n}{2} - 1) + \cdots + 1 = \frac{n^2}{8} + \frac{n}{4}$$

choices in case (b). Hence, the total number of ways is $n^2/4$ and the result follows.

In the case where n is odd, we set $A = \{v_1, \ldots, v_{\frac{n-1}{2}}\}$ and $B = \{v_{\frac{n+1}{2}+1}, \ldots, v_n\}$. Similar to the even case above, there are $\frac{(n-1)^2}{4}$ choices for the pairs to belong to the same cell of the partition, where either the pair is chosen from A or one vertex from A and the other one from B. In addition, there are $\frac{n-1}{2}$ further non-equivalent partitions in this case, in which the 2-vertex cell consists of a vertex of A along with the middle vertex $v_{\frac{n+1}{2}}$. We conclude that the total number of choices is $(n^2 - 1)/4$, which completes the proof. □

In the next proposition, we consider the same problem as in Proposition 4.2 for the case of cycles in which we make use of the distinguishing threshold of C_n.

Proposition 4.3. *Let $n \geq 3$. We have $\psi_{n-1}(C_n) = 0, 1$, if $n = 3$ and $n = 4$, respectively, and $\psi_{n-1}(C_n) = \lfloor \frac{n}{2} \rfloor$, if $n \geq 5$.*

Proof. It is easy to check the result in the small cases $n = 3, 4$ directly. We, therefore, consider the case $n \geq 5$. In this case, $n - 1 \geq \lfloor n/2 \rfloor + 2 = \theta(C_n)$ and, according to Proposition 3.2, any coloring of C_n with $n - 1$ colors is a distinguishing coloring. Thus, it is sufficient to count the number of non-equivalent partitions of the vertices $\{v_1, v_2, \cdots, v_n\}$ of C_n into $n - 1$ cells, such that no two such partitions are mapped to each other using an automorphism of C_n. It is not hard to see that there is a one-to-one correspondence between the family of such partitions and the set of all possible distances in C_n. In other words, the only such partitions are the ones including the cells $\{v_1, v_2\}$, $\{v_1, v_3\}$, ..., $\{v_1, v_{\lfloor \frac{n+1}{2} \rfloor}\}$. Therefore the result follows. □

We note that $\psi_k(G) = \Psi_k(G) - \Psi_{k-1}(G)$, for any graph G; in other words, in order to calculate $\psi_k(G)$, we can use $\Psi_k(G)$ if we already know the latter index of G. In the next theorem, we calculate $\Psi_k(P_n)$. To do so, we need the following notation. Recall that, for a given graph G, a distinguishing partition is a partition of the vertex set of G such that no nontrivial automorphism of G can preserve it (see [7]) and that $\mathrm{DP}(G)$ is the minimum number of cells in a distinguishing partition of G. It is evident that if G admits a distinguishing partition, then $\mathrm{DP}(G) \geq \mathrm{D}(G)$. For $n \geq 3$, the path P_n admits a distinguishing partition and $\mathrm{DP}(P_n) = 2$.

We define $\Xi_k(G)$ to be the number of non-equivalent distinguishing partitions of G with at most k cells. Correspondingly, $\xi_k(G)$ denotes the number of non-equivalent distinguishing partitions of G with exactly k cells. It is not hard to see that

$$\Xi_k(G) = \sum_{j\le k} \xi_j(G).$$

Note that, if G does not admit a distinguishing partition with exactly k cells, then $\xi_k(G) = 0$ and that $\xi_k(G) = \Xi_k(G) - \Xi_{k-1}(G)$.

Theorem 4.4. *Let $n \ge 2$. For any $k \ge 2$ we have*

$$\Psi_k(P_n) = \Pi_k(P_n) - \Pi_k(P_{\lceil \frac{n}{2}\rceil}) - \Xi_k(P_{\lceil \frac{n}{2}\rceil})$$

Proof. In order to count $\Psi_k(P_n)$, the number of non-equivalent distinguishing coloring partitions of P_n with at most k cells, we should subtract the number of non-distinguishing partitions from the total number of non-equivalent partitions of P_n with at most k cells, i.e. $\Pi_k(P_n)$. Note that if a partition of P_n is non-distinguishing, then its restriction to $P_{\lceil \frac{n}{2}\rceil}$, is either a non-distinguishing or a distinguishing partition. The number of the partitions of the former type is $\Pi_k(P_{\lceil \frac{n}{2}\rceil}) - \Xi_k(P_{\lceil \frac{n}{2}\rceil})$, while the number of the partitions of the latter type is $\Xi_k(P_{\lceil \frac{n}{2}\rceil})$

Let α and β be the non-trivial automorphisms of P_n and $P_{\lceil \frac{n}{2}\rceil}$, respectively. If a partition $\pi = \{\pi_1, \dots, \pi_r\}$ $(r \le k)$ of $P_{\lceil \frac{n}{2}\rceil}$ is non-distinguishing, i.e. $\beta(\pi) = \pi$, then the lifted partition

$$\pi' = \{\pi_1 \cup \alpha(\pi_1), \dots, \pi_r \cup \alpha(\pi_r)\}$$

of P_n is non-distinguishing. However, if a partition $\sigma = \{\sigma_1, \dots, \sigma_r\}$ $(r \le k)$ of $P_{\lceil \frac{n}{2}\rceil}$ is distinguishing, i.e. $\beta(\sigma) \ne \sigma$, then the two lifted partitions

$$\sigma' = \{\sigma_1 \cup \alpha(\sigma_1), \dots, \sigma_r \cup \alpha(\sigma_r)\} \quad \text{and} \quad \sigma'' = \{\beta(\sigma_1) \cup \alpha(\beta(\sigma_1)), \dots, \beta(\sigma_r) \cup \alpha(\beta(\sigma_r))\}$$

of P_n are non-equivalent non-distinguishing partitions. On the other hand, by the definition, the number of partitions σ is equal to $\Xi_k(P_{\lceil \frac{n}{2}\rceil})$. Therefore the total number of non-equivalent non-distinguishing partitions of P_n equals to

$$\Pi_k(P_{\lceil \frac{n}{2}\rceil}) - \Xi_k(P_{\lceil \frac{n}{2}\rceil}) + 2\Xi_k(P_{\lceil \frac{n}{2}\rceil}) = \Pi_k(P_{\lceil \frac{n}{2}\rceil}) + \Xi_k(P_{\lceil \frac{n}{2}\rceil}),$$

which completes the proof. □

Theorem 4.4 provides a nice connection among $\Psi_k(P_n)$, $\Pi_k(P_n)$ and $\Xi_k(P_n)$. See Appendix B for tables of these indices.

5 Distinguishing Lexicographic Products

In this section we provide an important application of one of the indices introduced in this paper, namely $\Phi_k(G)$. We start by recalling some preliminaries to the topic of lexicographic product of graphs.

Let X be a graph. The *X-join of* $\{Y_x | x \in V(X)\}$, is the graph Z with

$$V(Z) = \{(x, y) : x \in X, y \in Y_x\}$$

and

$$E(Z) = \{[(x, y), (x', y')] : [x, x'] \in E(X) \text{ or else } x = x' \text{ and } [y, y'] \in E(Y_x)\}.$$

Whenever, for all $x \in X$, we have $Y_x \simeq Y$, for a fixed graph Y, the graph Z is called the *lexicographic product of X and Y* and we write $Z = X \circ Y$.

We remind the reader that in [2], some bounds on the distinguishing number of the lexicographic product of graphs have been presented. In this section, we calculate the distinguishing number of a lexicographic product whenever this calculation is possible.

Automorphism groups of the X-join of $\{Y\}_x$ and lexicographic products were studied by Hemminger [11] and Sabidussi [20]. Hemminger defined natural isomorphisms of an X_1-join graph onto X_2-join graph as follows: let Z_i be an X_i-join of $\{Y_{ix}\}_{x \in X_i}$, $i = 1, 2$. Then a graph isomorphism μ of Z_1 onto Z_2 is called *natural* if for each $x_1 \in X_1$ there is an $x_2 \in X_2$ such that $\mu(Y_{1x_1}) = Y_{2x_2}$. Otherwise μ is called *unnatural*. He then characterized all the X-join graphs whose automorphism groups consists of all their natural automorphisms [11]. When the automorphisms of an X-join graph (or a lexicographic product) are all natural ones, it is easier to break them, as we do it here.

Theorem 5.1. *Suppose that $X \circ Y$ represents the lexicographic product of the two graphs X and Y. Then $\mathrm{D}(X \circ Y) = k$ where k is the least integer that $\Phi_k(Y) \geq \mathrm{D}(X)$, provided that all the automorphisms of $X \circ Y$ be the natural ones.*

Proof. When all the automorphisms of $G = X \circ Y$ are natural ones, any distinguishing coloring of G has to break symmetries inside Y_x for all $x \in V(X)$ and for each automorphism α of X that maps u on v ($u \neq v$), colorings of Y_u and Y_v must be non-equivalent. Therefore, whenever G has a distinguishing coloring with k colors, we must have $\Phi_k(Y) \geq \mathrm{D}(X)$. □

By a similar argument, we find an upper bound for the distinguishing number of the X-join of a set of graphs $\{Y_x\}_{x \in X}$. To do so, we need some notation in our argument. Let f be a distinguishing coloring of X with $\mathrm{D}(X)$ colors. For $x \in X$ let

$$C(x) = \{w \in X : f(x) \neq f(w) \text{ and } Y_x \simeq Y_w\} \cup \{x\}$$

and

$$D_x = \min\{k : \Phi_k(Y_x) \geq |C(x)|\}.$$

For each $x \in X$, we obviously have $|C(x)| \geq 1$ and $D_x \geq \mathrm{D}(Y_x)$. Moreover, put

$$d_f = \max\{D_x : x \in X\}.$$

Theorem 5.2. *Let Z be the X-join of $\{Y_x\}_{x \in X}$ whose automorphism group is the set of natural automorphisms. Let S be the set of all (non-equivalent) distinguishing colorings of X with $\mathrm{D}(X)$ colors. Then $\mathrm{D}(Z) \leq \min\{d_f : f \in S\}$.*

Proof. Let $f \in S$ and suppose that for each $x \in X$, we colored Y_x distinguishingly. For each $w \in C(x) \setminus \{x\}$, if Y_x and Y_w are colored non-equivalently, then we can guarantee that the resulting coloring of Z is distinguishing. For such a coloring we do not need more than d_f colors. Consequently, $\mathrm{D}(Z) \leq d_f$. □

We must point this out to the reader that some times it is possible that $\mathrm{D}(Z)$ becomes strictly less than $\min\{d_f : f \in S\}$. For example, suppose that X is a large cycle on $\{v_1, v_2, \ldots, v_n\}$ as its set of vertices. Let Y_1 be an asymmetric tree on $m_1 \geq 7$ vertices and Y_3 be another asymmetric tree on $m_3 \geq 7$ vertices, and $Y_1 \not\simeq Y_3$. Suppose also that for $i = 2, 4, 5, \ldots, n$, all Y_is are isomorphic to K_1. Then, the X-join of $\{Y_i\}_{i=1}^n$ is an asymmetric graph (whose distinguishing number equals to 1), while $\min\{d_f : f \in S\} = 2$. However, the stated bound in Theorem 5.2 is the best that can be generally found about $\mathrm{D}(Z)$, because it is attainable by the lexicographic product of two graphs (e. g. $C_n \circ K_1$).

6 Conclusion

We have seen in Section 5 that counting the number of non-equivalent distinguishing colorings, Φ, has an application in finding the distinguishing number of lexicographic products. Moreover, other indices have shown to have deep interactions with each other and also with Φ. It should be noted that calculating these indices are not always easy. Even when the automorphism group is very small and simple, counting non-equivalent distinguishing colorings or distinguishing (coloring) partitions faces with several calculation obstacles.

In the appendices, there are tables of the indices introduced in this paper for small paths and cycles. Considering these tables suggests that most of these indices are new generators of integer sequences.

To make calculations more comfortable, in some special cases, we introduced the notion of distinguishing threshold in Section 3, which enables us to reduce the required calculations in a computer algebra system to an acceptable level. However, this index has the importance to be considered separately, as it is a dual to the distinguishing number; *there is a distinguishing coloring with a number of colors greater than or equal to the distinguishing number while there is a non-distinguishing coloring with a number of colors less than the distinguishing threshold.* Among many good questions, one might consider this index for some families of graphs, or, study the distinguishing threshold for the Cartesian product.

We, finally, point out that one might consider infinite graphs or some notions other than distinguishing coloring for defining the parameters that we have introduced in this paper; e. g. distinguishing edge coloring, distinguishing total coloring, etc.

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Appendix A

Tables of Φ_k and φ_k for small paths and cycles

n \ k	2	3	4	5	6	7	8	9	10
2	1	3	6	10	15	21	28	36	45
3	2	9	24	50	90	147	224	324	450
4	6	36	120	300	630	1176	2016	3240	4950
5	12	108	480	1500	3780	8232	16128	29160	49500
6	28	351	2016	7750	23220	58653	130816	265356	499500
7	56	1053	8064	38750	139320	410571	1046528	2388204	4995000
8	120	3240	32640	195000	839160	2881200	8386560	21520080	49995000
9	240	9720	130560	975000	5034960	20168400	67092480	193680720	499950000
10	496	29403	523776	4881250	30229200	141229221	536854528	1743362676	4999950000

Table 1: Some different values of $\Phi_k(P_n)$ (Sequence A293500)

n \ k	2	3	4	5	6	7	8	9	10
3	0	1	4	10	20	35	56	84	120
4	0	3	15	45	105	210	378	630	990
5	0	12	72	252	672	1512	3024	5544	9504
6	1	37	266	1120	3515	9121	20692	42456	80565
7	2	117	1044	5270	19350	57627	147752	338364	709290
8	6	333	3788	23475	102690	355446	1039248	2673810	6222150
9	14	975	14056	106950	555990	2233469	7440160	21493836	55505550
10	30	2712	51132	483504	3009426	14089488	53611992	174189024	499720518

Table 2: Some different values of $\Phi_k(C_n)$ (Sequence A309528)

n \ k	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0
3	2	3	0	0	0	0	0	0	0
4	6	18	12	0	0	0	0	0	0
5	12	72	120	60	0	0	0	0	0
6	28	267	780	900	360	0	0	0	0
7	56	885	4188	8400	7560	2520	0	0	0
8	120	2880	20400	63000	95760	70560	20160	0	0
9	240	9000	93120	417000	952560	1164240	725760	181440	0
10	496	27915	409140	2551440	8217720	14817600	15120000	8164800	1814400

Table 3: Some different values of $\varphi_k(P_n)$ (Sequence A309785)

n \ k	2	3	4	5	6	7	8	9	10
3	0	1	0	0	0	0	0	0	0
4	0	3	3	0	0	0	0	0	0
5	0	12	24	12	0	0	0	0	0
6	1	34	124	150	60	0	0	0	0
7	2	111	588	1200	1080	360	0	0	0
8	6	315	2484	7845	11970	8820	2520	0	0
9	14	933	10240	46280	105840	129360	80640	20160	0
10	30	2622	40464	254664	821592	1481760	1512000	816480	181440

Table 4: Some different values of $\varphi_k(C_n)$ (Sequence A309651)

Appendix B

Tables of Ψ_k, ψ_k, Π_k, π_k, Ξ_k and ξ_k for small paths and cycles

n \ k	2	3	4	5	6	7	8	9	10
2	1	1	1	1	1	1	1	1	1
3	1	2	2	2	2	2	2	2	2
4	4	8	9	9	9	9	9	9	9
5	6	20	26	27	27	27	27	27	27
6	16	65	102	111	112	112	112	112	112
7	28	182	364	440	452	453	453	453	453
8	64	560	1436	1978	2120	2136	2137	2137	2137
9	120	1640	5560	9082	10428	10670	10690	10691	10691
10	256	4961	22136	43528	55039	58019	58409	58434	58435

Table 5: Some different values of $\Psi_k(P_n)$ (Sequence A309635)

n \ k	2	3	4	5	6	7	8	9	10
3	0	1	1	1	1	1	1	1	1
4	0	1	2	2	2	2	2	2	2
5	0	4	6	7	7	7	7	7	7
6	1	9	19	22	23	23	23	23	23
7	1	26	58	74	77	78	78	78	78
8	4	66	195	279	306	310	311	311	311
9	7	183	651	1084	1255	1292	1296	1297	1297
10	18	488	2294	4554	5803	6141	6195	6200	6201

Table 6: Some different values of $\Psi_k(C_n)$ (Sequence A309785)

n \ k	2	3	4	5	6	7	8	9	10
2	1	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0
4	4	4	1	0	0	0	0	0	0
5	6	14	6	1	0	0	0	0	0
6	16	49	37	9	1	0	0	0	0
7	28	154	182	76	12	1	0	0	0
8	64	496	876	542	142	16	1	0	0
9	120	1520	3920	3522	1346	242	20	1	0
10	256	4705	17175	21392	11511	2980	390	25	1

Table 7: Some different values of $\psi_k(P_n)$ (Sequence A309748)

n \ k	2	3	4	5	6	7	8	9	10
3	0	1	0	0	0	0	0	0	0
4	0	1	1	0	0	0	0	0	0
5	0	4	2	1	0	0	0	0	0
6	1	8	10	3	1	0	0	0	0
7	1	25	32	16	3	1	0	0	0
8	4	62	129	84	27	4	1	0	0
9	7	176	468	433	171	37	4	1	0
10	18	470	1806	2260	1248	338	54	5	1

Table 8: Some different values of $\psi_k(C_n)$ (Sequence A309784)

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2	2	2	2
3	1	3	4	4	4	4	4	4	4	4
4	1	6	10	11	11	11	11	11	11	11
5	1	10	25	31	32	32	32	32	32	32
6	1	20	70	107	116	117	117	117	117	117
7	1	36	196	379	455	467	468	468	468	468
8	1	72	574	1451	1993	2135	2151	2152	2152	2152
9	1	136	1681	5611	9134	10480	10722	10742	10743	10743
10	1	272	5002	22187	43580	55091	58071	58461	58486	58487

Table 9: Some different values of $\Pi_k(P_n)$ (Sequence A320750)

n \ k	1	2	3	4	5	6	7	8	9	10
3	1	2	3	3	3	3	3	3	3	3
4	1	4	6	7	7	7	7	7	7	7
5	1	4	9	11	12	12	12	12	12	12
6	1	8	22	33	36	37	37	37	37	37
7	1	9	40	73	89	92	93	93	93	93
8	1	18	100	237	322	349	353	354	354	354
9	1	23	225	703	1137	1308	1345	1349	1350	1350
10	1	44	582	2433	4704	5953	6291	6345	6350	6351

Table 10: Some different values of $\Pi_k(C_n)$ (Sequence A320748)

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	0	0
2	1	1	0	0	0	0	0	0	0	0
3	1	2	1	0	0	0	0	0	0	0
4	1	5	4	1	0	0	0	0	0	0
5	1	9	15	6	1	0	0	0	0	0
6	1	19	50	37	9	1	0	0	0	0
7	1	35	160	183	76	12	1	0	0	0
8	1	71	502	877	542	142	16	1	0	0
9	1	135	1545	3930	3523	1346	242	20	1	0
10	1	271	4730	17185	21393	11511	2980	390	25	1

Table 11: Some different values of $\pi_k(P_n)$ (Sequence A284949)

n \ k	1	2	3	4	5	6	7	8	9	10
3	1	1	1	0	0	0	0	0	0	0
4	1	3	2	1	0	0	0	0	0	0
5	1	3	5	2	1	0	0	0	0	0
6	1	7	14	11	3	1	0	0	0	0
7	1	8	31	33	16	3	1	0	0	0
8	1	17	82	137	85	27	4	1	0	0
9	1	22	202	478	434	171	37	4	1	0
10	1	43	538	1851	2271	1249	338	54	5	1

Table 12: Some different values of $\pi_k(C_n)$ (Sequence A152176)

n \ k	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	0	0	0	0	0	0	0	0	0	0
3	0	1	1	1	1	1	1	1	1	1
4	0	2	4	4	4	4	4	4	4	4
5	0	6	16	20	20	20	20	20	20	20
6	0	12	52	80	86	86	86	86	86	86
7	0	28	169	336	400	409	409	409	409	409
8	0	56	520	1344	1852	1976	1988	1988	1988	1988
9	0	120	1600	5440	8868	10168	10388	10404	10404	10404
10	0	240	4840	21760	42892	54208	57108	57468	57488	57488

Table 13: Some different values of $\Xi_k(P_n)$ (Sequence A320751)

n \ k	2	3	4	5	6	7	8	9	10
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	4	6	6	6	6	6	6	6
7	1	13	30	34	34	34	34	34	34
8	2	45	127	176	185	185	185	185	185
9	7	144	532	871	996	1011	1011	1011	1011
10	12	416	1988	3982	5026	5280	5304	5304	5304

Table 14: Some different values of $\Xi_k(C_n)$ (Sequence A324803)

n \ k	2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0
3	1	0	0	0	0	0	0	0	0
4	2	2	0	0	0	0	0	0	0
5	6	10	4	0	0	0	0	0	0
6	12	40	28	6	0	0	0	0	0
7	28	141	167	64	9	0	0	0	0
8	56	464	824	508	124	12	0	0	0
9	120	1480	3840	3428	1300	220	16	0	0
10	240	4600	16920	21132	11316	2900	360	20	0

Table 15: Some different values of $\xi_k(P_n)$ (Sequence A320525)

n \ k	2	3	4	5	6	7	8	9	10
3	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0
6	0	4	2	0	0	0	0	0	0
7	1	12	17	4	0	0	0	0	0
8	2	43	82	49	9	0	0	0	0
9	7	137	388	339	125	15	0	0	0
10	12	404	1572	1994	1044	254	24	0	0

Table 16: Some different values of $\xi_k(C_n)$ (Sequence A324802)