Crossings over permutations avoiding some pairs of three length-patterns

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Abstract

In this paper, we compute the distributions of the statistic number of crossings over permutations avoiding one of the pairs $\{321, 231\}$, $\{123, 132\}$ and $\{123, 213\}$. The obtained results are new combinatorial interpretations of two known triangles in terms of restricted permutations statistic. For some pairs of three length-patterns, we find relationships between the polynomial distributions of the crossings over permutations that avoid the pairs containing the pattern 231 on the first hand and the pattern 312 on the other hand.

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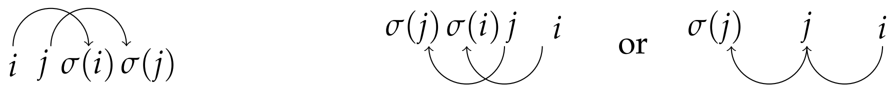
1 Introduction and main results

The statistic number of crossings is among the complicated statistics on permutations. Its survey arises from the works of A. de Médicis and X.G. Viennot [5], A. Randrianarivony[11, 12], S. Corteel [3], S. Burril et al. [2] to S. Corteel et al. [4]. Recently, the first author of this paper introduced the study of this statistic on permutations avoiding a single pattern of length three [10]. This one is devoted on the distribution of crossings on permutations avoiding a pair of patterns of length three. The technique we use in this paper differs from that of these known works who generally used other object like paths. Here, we simply manipulate the structure of our combinatorial objects and use some trivial bijections that we present in the next section.

A permutation σ of $[n] := \{1, 2, \ldots, n\}$ is a bijection from $[n]$ to itself which can be written linearly as $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. We shall refer n as the length of σ (i.e. $n = |\sigma|$) and we denote by S_n the set of all permutations of length n. A *crossing* of a given permutation σ is a pair of indexes (i, j) such that $i < j < \sigma(i) < \sigma(j)$ or $\sigma(i) < \sigma(j) \le i < j$. We denote by $\mathrm{cr}(\sigma)$ the number of crossings of σ. For graphical understanding, we usually draw arc diagrams, i.e. draw an upper (resp. a lower) arc from i to $\sigma(i)$ if $\sigma(i) > i$ (resp. $\sigma(i) < i$).



Upper crossing   Lower crossing

Figure 1: Arc diagrams of crossings.

Example: the crossings of the permutation $\pi = 4735126 \in S_7$ drawn in Figure 2 are $(1,2)$, $(5,6)$ and $(6,7)$. So we have $\mathrm{cr}(\pi) = 3$.

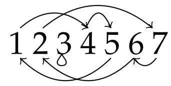


Figure 2: Arc diagrams of $\pi = 4735126 \in S_7$ with $\mathrm{cr}(\pi) = 3$.

Let $\sigma \in S_n$ and $\tau \in S_k$ with $1 \leq k \leq n$. For any given sequence of integers $i_1 < i_2 < \ldots < i_k$, we say that a subsequence $s = \sigma(i_1)\sigma(i_2)\ldots\sigma(i_k)$ of σ is an occurrence of τ if s and τ are in order isomorphic, i.e. $\sigma(i_x) < \sigma(i_y)$ if and only if $\tau(x) < \tau(y)$. If there is no occurrence of the pattern τ in σ, we say that σ is *τ-avoiding*. Example: the permutation $\pi = 4162375 \in S_7$ is 321-avoiding and it has five occurrences of the pattern 312 namely 312, 423, 623, 625 and 635. We will denote by $S_n(\tau)$ the set of all τ-avoiding permutations of $[n]$. For any subset of patterns $\mathrm{T} = \{\tau_1, \tau_2, \ldots\}$, we usually write $S_n(\tau_1, \tau_2, \ldots)$ for $S_n(T)$ and $S(\mathrm{T}) := \cup_{n \geq 0} S_n(\mathrm{T})$. There are three useful trivial involutions on S_n namely *reverse* r, *complement* c and *inverse* i defined as follows: for any permutation $\sigma \in S_n$,

- the reverse of σ is $\mathrm{r}(\sigma) = \sigma(n)\sigma(n-1)\ldots\sigma(1)$,
- the complement of σ is $\mathrm{c}(\sigma) = (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n))$,
- the inverse of σ is $\mathrm{i}(\sigma) = p(1)p(2)\ldots p(n)$ where $p(i)$ is the position of i in σ. We often write $\mathrm{i}(\sigma) = \sigma^{-1}$.

Example: for $\pi = 4135762 \in S_7$, we have $\mathrm{r}(\pi) = 2675314$, $\mathrm{c}(\pi) = 4753126$, $\pi^{-1} = 2731465$, $\mathrm{r} \circ \mathrm{c}(\pi) = 6213574$ and $\mathrm{r} \circ \mathrm{c} \circ \mathrm{i}(\pi) = 3247516$ where $\circ$ denotes the composition operation. Let us denote by $\mathrm{fg} := \mathrm{f} \circ \mathrm{g}$ for any involution f and g in $\{\mathrm{r}, \mathrm{c}, \mathrm{i}\}$. By composition $\circ$, these defined involutions generate the dihedral group $\mathcal{D} = \{\mathrm{id}, \mathrm{r}, \mathrm{c}, \mathrm{i}, \mathrm{rc}, \mathrm{ri}, \mathrm{ci}, \mathrm{rci}\}$ and they greatly simplify enumeration of pattern-avoiding permutations statistics through the fundamental property by Simion and Smith [14]

$$\varphi(S_n(\mathrm{T})) = S_n(\varphi(\mathrm{T})) \text{ for any } \varphi \in \mathcal{D} \text{ and any subset of patterns T.} \tag{1.1}$$

For any given statistic st, say that two subsets T_1 and T_2 are st-*Wilf-equivalent* if and only if the polynomial distributions of st over the sets $S_n(\mathrm{T}_1)$ and $S_n(\mathrm{T}_2)$ are the same for any integer n. In other word, we have

$$\sum_{\sigma \in S_n(\mathrm{T}_1)} x^{\mathrm{st}(\sigma)} = \sum_{\sigma \in S_n(\mathrm{T}_2)} x^{\mathrm{st}(\sigma)} \text{ for any integer } n.$$

Various statistic-Wilf-equivalence classes for subset of patterns of length three are known in [1, 6, 8, 10, 13]. In [10], Rakotomamonjy particularly provided the Wilf-equivalence classes modulo cr for single pattern of length three. He proved bijectively that the only non singleton class is $\{132, 213, 321\}$, i.e.

$$\sum_{\sigma\in S_n(321)} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma\in S_n(132)} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma\in S_n(213)} q^{\mathrm{cr}(\sigma)}. \tag{1.2}$$

To prove the first identity of (1.2), he exploited the bijection $\Theta : S_n(321) \to S_n(132)$ exhibited by Elizalde and Pak in [7] and proved that Θ is cr-preserving. The second identity of (1.2) is simply obtained from the fact that the reverse-complement-inverse rci preserves the number of crossings (see Lemma 4.2 - [10]). Using the q,p-Catalan numbers defined by Randrianarivony [12], Rakotomamonjy also proved the following result.

Theorem 1.1. [10] *For any pattern $\tau \in \{321, 132, 213\}$, the polynomial $F_n(\tau; q) := \sum_{\sigma\in S_n(\tau)} q^{\mathrm{cr}(\sigma)}$ satisfies*

$$F_n(\tau; q) = F_{n-1}(\tau; q) + \sum_{k=0}^{n-2} q^k F_k(\tau; q) F_{n-1-k}(\tau; q).$$

Moreover, we have

$$\sum_{\sigma\in S(\tau)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z}{1 - \cfrac{qz}{1 - \cfrac{qz}{1 - \cfrac{q^2 z}{1 - \cfrac{q^2 z}{\ddots}}}}}}}.$$

Notice that finding $\sum_{\sigma\in S_n(\tau)} q^{\mathrm{cr}(\sigma)}$ or $\sum_{\sigma\in S(\tau)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|}$ are staying open for any $\tau \in \{123, 231, 312\}$. The first results of this paper is the following.

Theorem 1.2. *We have the following identities*

$$\sum_{\sigma\in S(231,321)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = \frac{1 - qz}{1 - (1+q)z - (1-q)z^2} \tag{1.3}$$

$$\sum_{\sigma\in S(123,\tau)} q^{\mathrm{cr}(\sigma)} z^{|\sigma|} = 1 + \frac{(1-qz)z}{(1-z)(1-(1+q)z)} \quad \textit{for any } \tau \in \{132, 213\}. \tag{1.4}$$

We observe throughout the paper of Bukata et al. [1] that identities (1.3) and (1.4) are respectively new combinatorial interpretations of the triangles A076791 and A299927 of the On-line Encyclopedia of Integer Sequences OEIS [15]. Bukata et al. interpreted these triangles in terms of number of double descents (ddes) and number of double ascents(dasc) over permutations avoiding some pairs of patterns of length 3 (see Proposition 7 and Proposition 11 in [1]). The statistics ddes and dasc are respectively defined by $\mathrm{ddes}(\sigma) := |\{i|\sigma(i) > \sigma(i+1) > \sigma(i+2)\}|$ and $\mathrm{dasc}(\sigma) := |\{i|\sigma(i) < \sigma(i+1) < \sigma(i+2)\}|$ for any permutation σ. Notice that the triangle A299927 is new in OEIS and it was first discovered by Bukata et al..

Let $\tau \in \{132, 213\}$. For any integer $n \geq 1$ and $k \geq 0$, as direct consequence of identity (1.4), we have

$$|\{\sigma \in S_n(123, \tau)|\mathrm{cr}(\sigma) = k\}| = \delta_{k,0} + \binom{n-1}{k+1}.$$

The next result of this paper concerns various relationships between the distributions of the number of crossing over permutations that avoid the pattern 231 on the first hand and permutations that avoid the pattern 312 on the second hand. For that, we denote by $F(\mathrm{T};q,z) := \sum_{\sigma\in S(\mathrm{T})} q^{\mathrm{cr}(\sigma)} z^{|\sigma|}$ for any subset of patterns T.

Theorem 1.3. *We have the following identities*

$$\begin{aligned} F(312;q,z) &= \frac{1}{1-zF(231;q,z)}, \\ F(312,123;q,z) &= 1+\left(\frac{z}{1-z}\right)^2 + zF(231,123;q,z), \\ \textit{and } F(312,\tau;q,z) &= 1+\left(\frac{z}{1-z}\right)F(231,\tau';q,z) \textit{ for any } (\tau,\tau')\in\{132,213\}^2. \end{aligned}$$

The aim of this paper is to find the polynomial distributions of the number of crossings over permutations avoiding any pair of patterns in S_3. The tool that we use is not sufficient to treat all cases. However, these relationships we found will obviously reduce the number of the remain cases to be processed.

We organize the rest of this paper in three sections. Section 2 is for notations and preliminaries in which we will prove one fundamental proposition that will play a central role in the proof of our results. In Section 3, we provide the proof of our main results. In Section 4, we end this paper with two additional results.

2 Notations and preliminaries

Let n be a positive integer. For any $k\in[n]$, we denote by $S_n^k := \{\sigma\in S_n|\sigma(k)=1\}$ and $S_{n,k} := \{\sigma\in S_n|\sigma(n)=k\}$. Let us also denote respectively by $F_n(\mathrm{T};q)$, $F_n^k(\mathrm{T};q)$ and $F_{n,k}(\mathrm{T};q)$ the polynomial distributions of cr over the set $S_n(\mathrm{T})$ and $S_n^k(\mathrm{T})$ and $S_{n,k}(\mathrm{T})$, for any subset of patterns T and any integer $k\in[n]$. We particularly denote by $F_n(q) := F_n(\varnothing;q)$, $F_n^k(q) := F_n^k(\varnothing;q)$ and $F_{n,k}(q) = F_{n,k}(\varnothing;q)$

Let m and n be two integers such that $m>1$. Let $\mathrm{T}\subset S_m$ and $k\in[n]$. We also denote by $\mathrm{T}^{-1} = \{\tau^{-1}|\tau\in\mathrm{T}\}$ and $\mathrm{T}(i) := \{\tau(i)|\tau\in\mathrm{T}\}$ for $i\in[m]$. In this section, we will prove the following fundamental proposition that will help us to solve our problems in the next sections.

Proposition 2.1. *For all integer $n\geq 1$, the following properties hold*

$$\textit{If } \min\mathrm{T}^{-1}(1)>1, \textit{ we have } F_n^1(\mathrm{T};q) = F_{n-1}(\mathrm{T};q). \tag{2.1}$$

$$\textit{If } \min\mathrm{T}^{-1}(1)>2, \textit{ we have } F_n^2(\mathrm{T};q) = qF_{n-1}(\mathrm{T};q)+(1-q)F_{n-2}(\mathrm{T};q). \tag{2.2}$$

$$\textit{If } \max\mathrm{T}^{-1}(1)<m-1, \textit{ we have } F_n^{n-1}(\mathrm{T};q) = qF_{n-1}(\mathrm{T}^{-1};q)+(1-q)F_{n-1,n-1}(\mathrm{T}^{-1};q). \tag{2.3}$$

$$\textit{If } \max\mathrm{T}^{-1}(1)<m, \textit{ we have } F_n^n(\mathrm{T};q) = F_{n-1}(\mathrm{T}^{-1};q). \tag{2.4}$$

For that, we need some notations to be defined and some lemmas to be proved. So we let $\sigma\in S_n$. We say that i is an upper transient (resp. lower transient) of σ if and only if $\sigma^{-1}(i)<i<\sigma(i)$ (resp. $\sigma(i)<i<\sigma^{-1}(i)$). We denote respectively by $\mathrm{ut}(\sigma)$ and $\mathrm{lt}(\sigma)$ the numbers of upper and lower transients of a given permutation σ. By this definition, we have the following remark.

Remark 2.2. *For any permutation σ, the index i is a lower transient of σ if and only if $(i, \sigma^{-1}(i))$ is a lower crossing of σ.*

For any given integer k, we also denote respectively by $Ut_k(\sigma) := \{i < k / \sigma^{-1}(i) < i < \sigma(i)\}$ and $Lt_k(\sigma) := \{i < k / \sigma(i) < i < \sigma^{-1}(i)\}$ the sets of upper and lower transients of σ less than k. Define also

$$\begin{aligned} \mathrm{ut}_k^-(\sigma) &:= |Ut_k(\sigma)| \text{ and } \mathrm{ut}_k^+(\sigma) := \mathrm{ut}(\sigma) - \mathrm{ut}_k^-(\sigma), \\ \mathrm{lt}_k^-(\sigma) &:= |Lt_k(\sigma)| \text{ and } \mathrm{lt}_k^+(\sigma) := \mathrm{lt}(\sigma) - \mathrm{lt}_k^-(\sigma), \\ \alpha_k(\sigma) &:= |\{i \geq k / \sigma(i) < k\}|. \end{aligned}$$

Observe that we particularly have $\mathrm{ut}_n^-(\sigma) = \mathrm{ut}_{n+1}^-(\sigma) = \mathrm{ut}(\sigma)$ and $\mathrm{lt}_n^-(\sigma) = \mathrm{lt}_{n+1}^-(\sigma) = \mathrm{lt}(\sigma)$, $\alpha_n(\sigma) = 1 - \delta_{n,\sigma(n)}$ and $\alpha_{n+1}(\sigma) = 0$ where δ is the usual Kronecker symbol. Now, let us recall some needed notations introduced in [10]. Given a permutation σ and two integers a and b, we denote by $\sigma^{(a,b)}$ the obtained permutation from σ by the following way:

- add by 1 each number in σ which is greater or equal to b,
- then, insert b at the a-th position of the modified σ.

We can simply write $\sigma^{-(a,b)}$ for $(\sigma^{-1})^{(a,b)}$. Example: we have $3142^{(2,3)} = 43152$ and $3142^{-(2,3)} = 23514$. Now, we prove here a fundamental lemma which is a particular case of Lemma 3.7 in [10].

Lemma 2.3. *Let $\sigma \in S_n$ and $k \in [n+1]$. We have*

$$\mathrm{cr}(\sigma^{(k,1)}) = \mathrm{cr}(\sigma) + \mathrm{ut}_k^-(\sigma) - \mathrm{lt}_k^-(\sigma) + \alpha_k(\sigma).$$

Proof. Let $\sigma \in S_n$ and $k \in [n+1]$. Firstly, we denote by $A_k(\sigma)$ (resp. $B_k(\sigma)$, $C_k(\sigma)$) the set of all crossings (i,j) of σ such that $j < k$ (resp. $i < k \leq j$, $k \leq i$). We obviously have $\mathrm{cr}(\sigma) = |A_k(\sigma)| + |B_k(\sigma)| + |C_k(\sigma)|$. Let us assume that $\pi = \sigma^{(k,1)}$. By definition, we have

$$\pi(i) = \sigma(i) + 1 \text{ if } i < k, \pi(k) = 1 \quad \text{and } \pi(i+1) = \sigma(i) + 1 \text{ if } i \geq k.$$

Let (i,j) be a pair of integer such that $i < j$. Based on this definition of π, we will examine the following three cases:

Case 1: Suppose that $j < k$. So we have $\pi(i) = \sigma(i) + 1$ and $\pi(j) = \sigma(j) + 1$.

- Assume that $(i,j) \in A_k(\sigma)$.
  - If $i < j < \sigma(i) < \sigma(j)$, then $i < j < \pi(i) < \pi(j)$ and $(i,j) \in A_k(\pi)$,
  - If $\sigma(i) < \sigma(j) \leq i < j$, then $\begin{cases} \pi(i) < \pi(j) \leq i < j & \text{if } \sigma(j) < i; \\ \pi(i) \leq i < \pi(j) = i + 1 \leq j & \text{if } \sigma(j) = i. \end{cases}$

  Thus, we have $\begin{cases} (i,j) \in A_k(\pi) & \text{if } \sigma(j) < i; \\ (i,j) \notin A_k(\pi) & \text{if } \sigma(j) = i. \end{cases}$
- Inversely, if $(i,j) \in A_k(\pi)$, the following properties hold:
  - if $i < j < \pi(i) < \pi(j)$, then $i < j \leq \sigma(i) < \sigma(j)$. So, we have $\begin{cases} (i,j) \in A_k(\sigma) & \text{if } \pi(i) > j + 1 \text{ (i.e. } \sigma(i) > j); \\ (i,j) \notin A_k(\sigma) & \text{if } \pi(i) = j + 1 \text{ (i.e. } \sigma(i) = j). \end{cases}$

- if $\pi(i) < \pi(j) \leq i < j$ then $\sigma(i) < \sigma(j) < i < j$, i.e. $(i,j) \in A_k(\sigma)$.

Consequently, we obtain the following identity

$$|A_k(\sigma)| - |\{i|\sigma(i) < i < \sigma^{-1}(i) < k\}| = |A_k(\pi)| - |\{(i,j) \in A_k(\pi)|i < j < \pi(i) = j+1\}|. \tag{2.5}$$

Case 2: Suppose that $i < k \leq j$. We have $\pi(i) = \sigma(i) + 1$ and $\pi(j+1) = \sigma(j) + 1$.

- Assume that $(i,j) \in B_k(\sigma)$.
  - If $i < j < \sigma(i) < \sigma(j)$ then $(i, j+1) \in B_k(\pi)$,
  - If $\sigma(i) < \sigma(j) \leq i < j$ then $\begin{cases} (i,j+1) \in B_k(\pi) & \text{if } \sigma(j) < i; \\ (i,j+1) \notin B_k(\pi) & \text{if } \sigma(j) = i. \end{cases}$
- Inversely, if $(i,j) \in B_k(\pi)$,
  - if $i < j < \pi(i) < \pi(j)$, then $j > k$ since $\pi(k) = 1$. Thus, we have $i < j-1 < \sigma(i) < \sigma(j-1)$, i.e. $(i, j-1) \in B_k(\sigma)$,
  - if $\pi(i) < \pi(j) \leq i < j$, then $\sigma(i) < \sigma(j-1) < i < j-1$, i.e. $(i,j-1) \in B_k(\sigma)$.

Consequently, we obtain the following identity

$$|B_k(\sigma)| - |\{i|\sigma(i) < i < k \leq \sigma^{-1}(i)\}| = |B_k(\pi)|. \tag{2.6}$$

Case 3: Suppose now that $k \leq i < j$. We have $\pi(i+1) = \sigma(i) + 1$ and $\pi(j+1) = \sigma(j) + 1$.

- If $(i,j) \in C_k(\sigma)$, then we have
  - if $i < j < \sigma(i) < \sigma(j)$, then $(i+1, j+1) \in C_k(\pi)$,
  - if $\sigma(i) < \sigma(j) \leq i < j$ then $(i+1, j+1) \in C_k(\pi)$.
- Inversely, if $(i,j) \in C_k(\pi)$, we have
  - if $i < j < \pi(i) < \pi(j)$, then $k > i$ since $\pi(k) = 1$. Thus, we have $k \leq i-1 < j-1 < \sigma(i-1) < \sigma(j-1)$, i.e. $(i-1, j-1) \in C_k(\sigma)$,
  - if $\pi(i) < \pi(j) \leq i < j$, then $\sigma(i-1) < \sigma(j-1) \leq i-1 < j-1$. So, we get $\begin{cases} (i-1,j-1) \in C_k(\sigma) & \text{if } i > k; \\ (i-1,j-1) \notin C_k(\sigma) & \text{if } i = k. \end{cases}$

Similarly to the previous cases, we obtain

$$|C_k(\sigma)| = |C_k(\pi)| - |\{j > k|\pi(j) \leq k\}|. \tag{2.7}$$

By summing equations (2.5), (2.6) and (2.7), using the facts that $|\{i|\sigma(i) < i < \sigma^{-1}(i) < k\}| + |\{i|\sigma(i) < i < k \leq \sigma^{-1}(i)\}| = \mathrm{lt}_k^-(\sigma)$, $|\{(i,j) \in A_k(\pi)|i < j < \pi(i) = j+1\}| = |\{j < k|\sigma^{-1}(j) < j < \sigma(j)\}| = \mathrm{ut}_k^-(\sigma)$ and $|\{j > k|\pi(j) \leq k\}| = |\{j \geq k|\sigma(j) < k\}| = \alpha_k(\sigma)$, we get

$$\mathrm{cr}(\sigma) - \mathrm{lt}_k^-(\sigma) = \mathrm{cr}(\pi) - \mathrm{ut}_k^-(\sigma) - \alpha_k(\sigma). \tag{2.8}$$

We deduce from (2.8) the desired identity of our lemma. □

Lemma 2.4. *Let σ be a given permutation. If $\pi = \sigma^{-1}$ or $\mathrm{rc}(\sigma)$ then we have*

$$\mathrm{cr}(\pi) = \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma).$$

Proof. Let $\sigma \in S_n$. We have the following equivalences

$$i < \sigma(i) \Leftrightarrow n+1-\sigma(i) < n+1-i \Leftrightarrow \mathrm{rc}(\sigma)(n+1-i) < n+1-i.$$

This implies that inverse and reverse-complement exchanges lower and upper arcs including of course transients, i.e. $\mathrm{ut}(\pi) = \mathrm{lt}(\sigma)$ and $\mathrm{lt}(\pi) = \mathrm{ut}(\sigma)$. By this fact, Remark 2.2 explains how we get $\mathrm{cr}(\pi) = \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma)$ and we complete the proof of our lemma. □

Let n be an integer and $k \in [n]$. Let us now define a bijection $\Phi_{n,k}$ as follows

$$\begin{array}{rcl} \Phi_{n,k} : S_{n-1} & \longrightarrow & S_n^k \\ \sigma & \longmapsto & \sigma^{-(k,1)}. \end{array}$$

The properties of this bijection allow us to get some relations between F_n^{n-1}, F_n^n and F_n (see Proposition 2.5) and use its restricted version to prove Proposition 2.1.

Proposition 2.5. *The bijection $\Phi_{n,n}$ preserves the number of crossings and the bijection $\Phi_{n,n-1}$ satisfies*

$$\mathrm{cr}(\Phi_{n,n-1}(\sigma)) = \begin{cases} \mathrm{cr}(\sigma) & \text{if } \sigma(n-1) = n-1; \\ \mathrm{cr}(\sigma)+1 & \text{if } \sigma(n-1) < n-1. \end{cases} \text{ for any } \sigma \in S_{n-1}.$$

Proof. Combining Lemma 2.3 and Lemma 2.4, it is not difficult to see that, for any $\sigma \in S_{n-1}$, we have

$$\mathrm{cr}(\sigma^{-(n,1)}) = \mathrm{cr}(\sigma) \text{ and } \mathrm{cr}(\sigma^{-(n-1,1)}) = \mathrm{cr}(\sigma) + 1 - \delta_{n-1,\sigma(n-1)}. \tag{2.9}$$

The proposition comes from (2.9). □

Let us denote by $\alpha \oplus \beta$ the *direct sum* of the two given permutations α and β defined as follows

$$\alpha \oplus \beta(i) = \begin{cases} \alpha(i), & \text{if } i \le |\alpha|; \\ |\alpha| + \beta(i - |\alpha|) & \text{if } i > |\alpha|. \end{cases}$$

Example: $1432 \oplus 4231 = 14328675$. An obvious property of the direct sum that we need is $\mathrm{cr}(\alpha \oplus \beta) = \mathrm{cr}(\alpha) + \mathrm{cr}(\beta)$ for any permutations α and β.

Proposition 2.6. *Let n be a non-negative integer. The following recurrences hold*

$$\begin{array}{rcl} F_n^n(q) & = & F_{n-1}(q) \text{ for } n \ge 1 \\ \text{and } F_n^{n-1}(q) & = & qF_{n-1}(q) + (1-q)F_{n-2}(q) \text{ for } n \ge 2. \end{array}$$

Proof. Since the bijection $\Phi_{n,n}$ is cr-preserving, we have $F_n^n(q) = F_{n-1}(q)$. Now, using the property of the bijection $\Phi_{n,n-1}$, we get

$$\begin{array}{rcl} F_n^{n-1}(q) & = & q \times \displaystyle\sum_{\sigma \in S_{n-1}, \sigma(n-1) \neq n-1} q^{\mathrm{cr}(\sigma)} + \sum_{\sigma \in S_{n-1}, \sigma(n-1) = n-1} q^{\mathrm{cr}(\sigma)} \\ & = & q\left(F_{n-1}(q) - F_{n-1,n-1}(q)\right) + F_{n-1,n-1}(q) \end{array}$$

Since $F_{n,n}(q) = \sum_{\sigma\oplus 1\in S_n} q^{\mathrm{cr}(\sigma\oplus 1)} = \sum_{\sigma\in S_n} q^{\mathrm{cr}(\sigma)} = F_{n-1}(q)$ for all $n \geq 1$, we consequently obtain

$$F_n^{n-1}(q) = qF_{n-1}(q) + (1-q)F_{n-2}(q) \text{ for all } n \geq 1.$$

This ends the proof of the proposition. □

We may observe that combination of relations (2.3) and (2.4) of Proposition 2.1 is restricted version of Proposition 2.6. The effect of the restriction totally changes the obtained relations. For example, we have $F_n^n(321;q) = 1 \neq F_{n-1}(321;q)$. We are now able to provide the proof of Proposition 2.1.

Proof of Proposition 2.1. Our proof is simply based on the following obvious fact. Let T be a subset of S_m for any integer $m > 1$. For any integer $n \geq m$, we have

(i) If $k < \min \mathrm{T}^{-1}(1)$, we have $\sigma^{(k,1)} \in S_n^k(\mathrm{T})$ if and only if $\sigma \in S_{n-1}(\mathrm{T})$.
(ii) If $n - m + \max \mathrm{T}^{-1}(1) < k \leq n$, we have $\sigma^{-(k,1)} \in S_n^k(\mathrm{T})$ if and only if $\sigma \in S_{n-1}(\mathrm{T}^{-1})$.

The two first relations (2.1) and (2.2) of Proposition 2.1 use the (i) of the fact. If $\min \mathrm{T}^{-1}(1) \neq 1$, then we have $1 \oplus \sigma \in S_n^1(\mathrm{T})$ if and only if $\sigma \in S_{n-1}(\mathrm{T})$ for any $n \geq 1$. Thus we get relation (2.1) as follows

$$F_n^1(\mathrm{T};q) = \sum_{1\oplus\sigma\in S_n^1(\mathrm{T})} q^{\mathrm{cr}(1\oplus\sigma)} = \sum_{\sigma\in S_{n-1}(\mathrm{T})} q^{\mathrm{cr}(\sigma)} = F_{n-1}(\mathrm{T};q).$$

By the same way, if $\min \mathrm{T}^{-1}(1) > 2$, we have $\sigma^{(2,1)} \in S_n^2(\mathrm{T})$ if and only if $\sigma \in S_{n-1}(\mathrm{T})$ for any $n \geq 1$. Moreover, we have $\mathrm{cr}(\sigma^{(2,1)}) = \mathrm{cr}(\sigma) + 1 - \delta_{1,\sigma(1)}$ for any permutation σ (see Lemma 2.3). By applying (2.1), we also get (2.2) as follows

$$\begin{aligned}
F_n^2(\mathrm{T};q) &= q \times \sum_{\sigma\in S_{n-1}(\mathrm{T}),\sigma(1)\neq 1} q^{\mathrm{cr}(\sigma)} + \sum_{\sigma\in S_{n-1}(\mathrm{T}),\sigma(1)=1} \\
&= q\left(F_{n-1}(\mathrm{T};q) - F_{n-1}^1(\mathrm{T};q)\right) + F_{n-1}^1(\mathrm{T};q) \\
&= qF_{n-1}(\mathrm{T};q) + (1-q)F_{n-2}(\mathrm{T};q).
\end{aligned}$$

For the two last relations (2.3) and (2.4) of the proposition, we obviously use (ii) of the fact and we also exploit the bijections $\Phi_{n,n}$ and $\Phi_{n,n-1}$. If $\max \mathrm{T}^{-1}(1) < m - 1$ (i.e. $n - m + \max \mathrm{T}^{-1}(1) < n - 1$), we have $\sigma^{-(n-1,1)} \in S_n^{n-1}(\mathrm{T})$ if and only if $\sigma \in S_{n-1}(\mathrm{T}^{-1})$. This implies that we have $\Phi_{n,n-1}(S_{n-1}(\mathrm{T}^{-1})) = S_n^{n-1}(\mathrm{T})$. Using the property of the bijection $\Phi_{n,n-1}$ described in Theorem 2.5, we get (2.3) as follows

$$\begin{aligned}
F_n^{n-1}(\mathrm{T};q) &= q \times \sum_{\sigma\in S_{n-1}(\mathrm{T}^{-1}),\sigma(n-1)\neq n-1} q^{\mathrm{cr}(\sigma)} + \sum_{\sigma\in S_{n-1}(\mathrm{T}^{-1}),\sigma(n-1)=n-1} \\
&= q\left(F_{n-1}(\mathrm{T}^{-1};q) - F_{n-1,n-1}(\mathrm{T}^{-1};q)\right) + F_{n-1,n-1}(\mathrm{T}^{-1};q) \\
&= qF_{n-1}(\mathrm{T}^{-1};q) + (1-q)F_{n-1,n-1}(\mathrm{T}^{-1};q).
\end{aligned}$$

Notice that we generally have $F_{n,n}(\mathrm{T};q) \neq F_{n-1}(\mathrm{T};q)$ since the set $S_{n,n}(\mathrm{T})$ depends on T. By the same way we obtain the last relation (2.4) using the cr-preserving of the bijection $\Phi_{n,n}$. This ends the proof of Proposition 2.1.

Let us end this preliminaries section with illustration examples of Proposition 2.1. Since $\max\{321\}^{-1}(1) = 3 > 2$, we get by applying (2.2)

$$F_n^2(321;q) = qF_{n-1}(321;q) + (1-q)F_{n-2}(321;q) \text{ for } n \geq 2.$$

Since $123^{-1} = 123$ and $\max\{123\}^{-1}(1) = 1 < 2$, we can also apply (2.3) and get

$$F_n^{n-1}(123;q) = qF_{n-1}(123;q) + (1-q).F_{n-1,n-1}(321;q).$$

Since $S_{n,n}(123) = \{(n-1)\dots 21n\}$, then we have $F_{n,n}(123;q) = 1$ and we consequently obtain

$$F_n^{n-1}(123;q) = qF_{n-1}(123;q) + (1-q).$$

3 Proof of the main results

In this section, we will establish the proof of our results presented in Section 1 by using Proposition 2.1 as fundamental tool. For that, we denote by $F(\mathrm{T};q,z) := \sum_{\sigma\in S(\mathrm{T})} q^{\mathrm{cr}(\sigma)} z^{|\sigma|}$ for any pattern T.

3.1 Proof of Theorem 1.2

Proof. It is obvious to see that we have $S_n(321,231) = S_n^1(321,231) \cup S_n^2(321,231)$ for all n. So we get

$$F_n(321,231;q) = F_n^1(321,231;q) + F_n^2(321,231;q).$$

Since $\min\{321,231\}^{-1}(1) = 3 > 2$, we can apply the relations (2.1) and (2.2) of proposition 2.1 and we get

$$F_n(321,231;q) = (1+q)F_{n-1}(321,231;q) + (1-q)F_{n-2}(321,231;q), \text{ for } n \geq 2. \tag{3.1}$$

Recurrence (3.1) is associated with the following functional equation

$$F(321,231;q,z) = 1 + z + (1+q)z(F(321,231;q,z) - 1) + (1-q)z^2F(321,231;q,z).$$

Solving it by $F(321,231;q,z)$, we obtain the following identity equivalent to identity 1.3 of Proposition 2.1

$$F(321,231;q,z) = \frac{1-qz}{1-(1+q)z-(1-q)z^2}.$$

As structure, we have $S_n(123,132) = S_n^{n-1}(123,132) \cup S_n^n(123,132)$. Since $\max\{123,132\}^{-1}(1) = 1 < 2$, we can also apply the relations (2.3) and (2.4) of Proposition 2.1. Thus, since $\{123,132\}^{-1} = \{123,132\}$, we get

$$F_n^n(123,132;q) = F_{n-1}(123,132;q). \tag{3.2}$$

Moreover, since $F_{n,n}(123,132;q) = 1$, we get from (2.4)

$$F_n^{n-1}(123,132;q) = qF_{n-1}(123,132;q) + 1 - q. \tag{3.3}$$

Summing (3.2) and (3.3), we obtain the following recurrence

$$F_n(123,132;q) = (1+q)F_{n-1}(123,132;q) + 1 - q \text{ for } n \geq 2. \tag{3.4}$$

Recurrence (3.4) corresponds to the functional equation

$$F(123,132;q,z) = 1 + z + (1+q)z(F(123,132;q) - 1) + z\left(\frac{1}{1-z} - 1 - z\right).$$

When solving this functional equation by $F(123,132;q,z)$, we obtain

$$F(123,132;q,z) = 1 + \frac{z(1-qz)}{(1-z)(1-(1+q)z)}.$$

Finally, since $\{123,213\} = \mathrm{rci}(\{123,132\})$, we also have $F(123,132;q,z) = F(123,213;q,z)$. This completes the proof of identity (1.4) of Theorem 1.2 and Theorem 1.2 itself. ☐

Notice that when we solve the recurrence (3.4) with the initial condition $F_1(123,132;q) = 1$, we obtain the closed form

$$\sum_{\sigma \in S_n(123,\tau)} q^{\mathrm{cr}(\sigma)} = \frac{(1+q)^{n-1} - 1 + q}{q} \quad \text{for } n \geq 1 \text{ and } \tau \in \{132,213\}.$$

Furthermore, when we substitute $F_{n-1}(123,132;q)$ by $\frac{(1+q)^{n-2}-1+q}{q}$ for $n \geq 2$, we also get from (3.3)

$$\sum_{\sigma \in S_n^{n-1}(123,132)} q^{\mathrm{cr}(\sigma)} = (1+q)^{n-2} \quad \text{for } n \geq 2.$$

Since $\mathrm{rci}(\mathrm{S}_\mathrm{n}^{\mathrm{n}-1}(123,132)) = \mathrm{S}_{\mathrm{n},2}(123,213)$, we also have

$$\sum_{\sigma \in S_{n,2}(123,213)} q^{\mathrm{cr}(\sigma)} = (1+q)^{n-2} \quad \text{for } n \geq 2.$$

Corollary 3.1. *For $n \geq 2$ and $k \geq 0$, we have*

$$|\{\sigma \in S_n^{n-1}(123,132) | \mathrm{cr}(\sigma) = k\}| = |\{\sigma \in S_{n,2}(123,213) | \mathrm{cr}(\sigma) = k\}| = \binom{n-2}{k}.$$

We observe that Corollary 3.1 is a new combinatorial interpretation of the Pascal triangle A007318 in terms of crossings over restricted permutations.

3.2 Proof of Theorem 1.3

In this subsection, we will establish the proof of the result concerning some relationships between the distributions of crossings over the sets $S_n(312,\mathrm{T})$ and $S_n(231,\mathrm{T})$, where T is empty or a singleton of $\{123,132,213\}$. As we did in the preceding subsection, we will first find recurrences and we then compute the corresponding generating functions to get the desired relations.

Proposition 3.2. *For all integer $n \geq 1$, we have*

$$F_n(312;q) = \sum_{j=0}^{n-1} F_j(231;q)F_{n-1-j}(312;q). \tag{3.5}$$

Proof. We have $S_n^j(312) = \{\sigma_1 \oplus \sigma_2 | \sigma_1 \in S_j^j(312), \sigma_2 \in S_{n-j}(312)\}$ for all $j \geq 1$. So, we get using (2.4) the following identities

$$F_n^j(312;q) \quad = \quad F_j^j(312;q)F_{n-j}(312;q) = F_{j-1}(231;q)F_{n-j}(312;q) \text{ for } 1 \leq j \leq n.$$

By summing $F_n^j(312;q)$ over $j \in [n]$, we obtain the desired relationship for $F_n(312;q)$. □

Proposition 3.3. *For all integer $n \geq 2$, we have*

$$F_n(123,312;q) = n - 1 + F_{n-1}(123,231;q). \tag{3.6}$$

Proof. We have $S_n(123,312) = \{\pi_1, \pi_2, \dots \pi_{n-1}\} \cup S_n^n(123,312)$ with $\pi_j = j \dots 21n(n-1)\dots(j+1)$ for all $j \in [n]$. So we get

$$F_n(123,312;q) = \sum_{j=1}^{n-1} q^{\mathrm{cr}(\pi_j)} + F_n^n(123,312;q).$$

It is not difficult to see that we have $\mathrm{cr}(\pi_j) = 0$ for all $j \in [n]$. Thus, we immediately obtain the proposition using again (2.4). □

Proposition 3.4. *For any τ_1, τ_2 and $\tau_3 \in \{132, 213\}$ and for all $n \geq 2$, we have*

$$F_n(312,\tau_1;q) = F_{n-1}(312,\tau_2;q) + F_{n-1}(231,\tau_3;q). \tag{3.7}$$

Proof. Since $S_n(312,213) = S_n^1(312,213) \cup S_n^n(312,213)$, we get

$$F_n(312,213;q) \quad = \quad F_n^1(312,213;q) + F_n^n(231,213;q).$$

So for all $n \geq 2$ we get from (2.1) and (2.2) the following identity

$$F_n(312,213;q) \quad = \quad F_{n-1}(312,213;q) + F_{n-1}(231,213;q).$$

To complete the proof of the proposition, we just use the facts that $\mathrm{rci}(\{312,132\}) = \{312,213\}$ and $\mathrm{rci}(\{231,132\}) = \{231,213\}$. □

Now, to prove Theorem 1.3, we just compute the corresponding generating functions of the three recurrences (3.5), (3.6) and (3.7) and deduce the desired relations.

From (3.5), we obtain the functional equation

$$F(312;q,z) = 1 + zF(312;q,z).F(231;q,z)$$

which leads to

$$F(312;q,z) = \frac{1}{1 - zF(231;q,z)}. \tag{3.8}$$

The associated generating function with (3.6) is

$$F(123,312;q,z) = 1 + z + \left(\frac{z}{1-z}\right)^2 + z(F(123,231;q,z) - 1).$$

This functional equation is equivalent to the following one

$$F(312,123;q,z)1 + \left(\frac{z}{1-z}\right)^2 + zF(231,123;q,z). \tag{3.9}$$

From (3.7), when we set $\tau = \tau_1 = \tau_2$ and $\tau' = \tau_3$ we get the functional equation

$$F(312,\tau;q,z) = 1 + z + z\left(F(312,\tau;q,z) + F(231,\tau';q,z) - 2\right).$$

Solving it for $F(312,\tau;q,z)$, we obtain

$$F(312,\tau;q,z) = 1 + \left(\frac{z}{1-z}\right) F(231,\tau';q,z) \text{ for any } (\tau,\tau') \in \{132,213\}^2. \tag{3.10}$$

This completes the proof of Theorem 1.3.

4 Additional results

We end this paper with two additional results. The first one is about $F_n(321,213;q)$ and $F_n(321,132;q)$. The second one is inspired from the first section and is about a cr-preserving bijection between S_n^k and S_n^{n+1-k}.

For the first result, we remark that the distribution of cr over the set of permutations avoiding one of the pairs $\{321,213\}$ and $\{321,132\}$ can be computed. One of the tools that we may use is an interesting relationship proved by Randrianarivony [12]. He showed how the statistic cr is related to other usual statistics through the following identity

$$\mathrm{cr}(\sigma) = \mathrm{inv}(\sigma) - \mathrm{exc}(\sigma) - 2\mathrm{nes}(\sigma), \tag{4.1}$$

where $\mathrm{inv}(\sigma) = \{(i,j) | i < j \text{ and } \sigma(i) > \sigma(j)\}$ is the number of inversions of σ, $\mathrm{exc}(\sigma) = \{i | \sigma(i) > i\}$ is the number of excedances of σ and $\mathrm{nes}(\sigma) = \{(i,j) | i < j < \sigma(j) < \sigma(i) \text{ or } \sigma(j) < \sigma(i) < i < j\}$ is the number of nestings of σ, for any permutation σ. Let us end this paper with the following result in which we try to use identity (4.1) to get the proof.

Theorem 4.1. *Let us denote by* $[n]_q = 1 + q + \ldots + q^{n-1}$ *for any integer* $n \geq 1$. *For any* $\tau \in \{132,213\}$, *we have*

$$\sum_{\sigma \in S_n(321,\tau)} q^{\mathrm{cr}(\sigma)} = 1 + \sum_{k=1}^{n-1} [n-k]_{q^k}.$$

Proof. It is easy to see that we have $S_n(321,213) = S_n^1(321,213) \cup \{\alpha_2, \alpha_3, \ldots, \alpha_n\}$ where $\alpha_j = (n-j+2)\ldots(n-1)n12\ldots(n+1-j)$ for all $j \in [n]$. From this structure, we get

$$F_n(321,213;q) = F_n^1(321,213;q) + \sum_{j=2}^{n} q^{\mathrm{cr}(\alpha_j)}.$$

Since every 321-avoiding permutations are nonesting (see [10]), we have $\mathrm{cr}(\alpha_j) = \mathrm{inv}(\alpha_j) - \mathrm{exc}(\alpha_j) = (j-1)(n-j)$ for all j. Using the fact that $F_n^1(321, 213; q) = F_{n-1}(321, 213; q)$, we obtain

$$F_n(321, 213; q) \;=\; F_{n-1}(321, 213; q) + \sum_{j=2}^{n} q^{(j-1)(n-j)}.$$

When we solve this recurrence with the initial condition $F_1(321, 213; q) = 1$, we obtain

$$F_n(321, 213; q) = 1 + \sum_{k=1}^{n-1}\sum_{j=1}^{k} q^{j(k-j)} = 1 + \sum_{k=1}^{n-1} [n-k]_{q^k}.$$

From the fact that $F_n(321, 213; q) = F_n(321, 132; q)$ since $\{321, 132\} = \mathrm{rci}(\{321, 213\})$, we complete the proof of the theorem. □

For the last additional result, we notice first that we have $S_n^k = \{\sigma^{(k,1)} | \sigma \in S_{n-1}\}$. We will show that the following well defined and bijective map preserves the number of crossings:

$$\begin{array}{rccc} \Psi_{n,k}: & S_n^k & \longrightarrow & S_n^{n+1-k} \\ & \sigma^{(k,1)} & \longmapsto & \mathrm{rc}(\sigma)^{(n+1-k,1)}. \end{array}$$

Theorem 4.2. *The bijection $\Psi_{n,k}$ preserves the number of crossings for $1 \le k \le n$.*

Proof. Let $\sigma^{(k,1)} \in S_n^k$ and $\pi^{(n+1-k,1)} = \Psi_{n,k}(\sigma^{(k,1)})$ for $\sigma \in S_{n-1}$. Knowing that rc exchanges lower and upper arcs, it is not difficult to see that we have

$$\mathrm{ut}^-_{n+1-k}(\pi) = \mathrm{lt}^+_k(\sigma) \text{ and } \mathrm{lt}^-_{n+1-k}(\pi) = \mathrm{ut}^+_k(\sigma). \tag{4.2}$$

Moreover, since $|\{i < k/\sigma(i) \ge k\}| = |\{i \ge k/\sigma(i) < k\}|$, we get

$$\alpha_{n+1-k}(\pi) = \alpha_k(\sigma). \tag{4.3}$$

Indeed, we have

$$\begin{aligned} \alpha_{n+1-k}(\pi) &= |\{n-i \ge n+1-k/\pi(n-i) < n+1-k\}|, \\ &= |\{i \le k-1/n-\sigma(i) < n+1-k\}|, \\ &= |\{i < k/\sigma(i) > k-1\}|, \\ &= |\{i < k/\sigma(i) \ge k\}|, \\ &= \alpha_k(\sigma). \end{aligned}$$

Consequently, combining (4.2) and (4.3) with Lemma 2.3 and Lemma 2.4, we get

$$\begin{aligned} \mathrm{cr}(\pi^{(n+1-k,1)}) &= \mathrm{cr}(\pi) + \mathrm{ut}^-_{n+1-k}(\pi) - \mathrm{lt}^-_{n+1-k}(\pi) + \alpha_{n+1-k}(\pi), \\ &= \mathrm{cr}(\sigma) + \mathrm{ut}(\sigma) - \mathrm{lt}(\sigma) + \mathrm{lt}^+_k(\sigma) - \mathrm{ut}^+_k(\sigma) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma) + \big(\mathrm{ut}(\sigma) - \mathrm{ut}^+_k(\sigma)\big) - \big(\mathrm{lt}(\sigma) - \mathrm{lt}^+_k(\sigma)\big) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma) + \mathrm{ut}^-_k(\sigma) - \mathrm{lt}^-_k(\sigma) + \alpha_k(\sigma), \\ &= \mathrm{cr}(\sigma^{(k,1)}). \end{aligned}$$

This proves the cr-preserving of the bijection $\Psi_{n,k}$ and also ends the proof of Theorem 4.2. □

Corollary 4.3. *For any integers n and $k \in [n]$, we have the following equidistributions*

$$\sum_{\sigma \in S_{n,k}} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n^{n+1-k}} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_n^k} q^{\mathrm{cr}(\sigma)} = \sum_{\sigma \in S_{n,n+1-k}} q^{\mathrm{cr}(\sigma)}.$$

Proof. We have $S_n^{n+1-k} = \Psi_{n,k}(S_n^k)$ and $S_{n,n+1-k} = \mathrm{rci}(S_n^k)$ for any $k \in [n]$. So we get these identities from the facts that the bijections $\Psi_{n,k}$ and rci are cr-preserving. □

Corollary 4.4. *The number of permutations of $[2n]$ having r crossings is always even for all integers $n \geq 1$ and $r \geq 0$.*

Proof. The number of permutations of $[2n]$ having r crossings is $[q^r]F_{2n}(q)$ (the coefficient of the polynomial $F_{2n}(q)$) where $F_{2n}(q) = \sum_{k=1}^{2n} F_{2n}^k(q) = 2\sum_{k=1}^{n} F_{2n}^k(q)$. □

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