Shannon capacity and the categorical product

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Abstract

Shannon OR-capacity $C_{\rm OR}(G)$ of a graph G, that is the traditionally more often used Shannon AND-capacity of the complementary graph, is a homomorphism monotone graph parameter satisfying $C_{\rm OR}(F \times G) \leq \min\{C_{\rm OR}(F), C_{\rm OR}(G)\}$ for every pair of graphs, where $F \times G$ is the categorical product of graphs F and G. Here we initiate the study of the question when could we expect equality in this inequality. Using a strong recent result of Zuiddam, we show that if this "Hedetniemi-type" equality is not satisfied for some pair of graphs then the analogous equality is also not satisfied for this graph pair by some other graph invariant that has a much "nicer" behavior concerning some different graph operations. In particular, unlike Shannon capacity or the chromatic number, this other invariant is both multiplicative under the OR-product and additive under the join operation, while it is also nondecreasing along graph homomorphisms. We also present a natural lower bound on $C_{\rm OR}(F \times G)$ and elaborate on the question of how to find graph pairs for which it is known to be strictly less, than the upper bound $\min\{C_{\rm OR}(F), C_{\rm OR}(G)\}$. We present such graph pairs using the properties of Paley graphs.

1 Introduction

For two graphs F and G, their categorical (also called tensor or weak direct) product $F \times G$ is defined by

$$V(F \times G) = V(F) \times V(G),$$

$$E(F \times G) = \{\{(x,u),(y,v)\} : x,y \in V(F), u,v \in V(G), \{x,y\} \in E(F), \{u,v\} \in E(G)\}.$$

Hedetniemi's more than half a century old conjecture that was refuted recently by Yaroslav Shitov [30] (cf. also [34] and [19] for further developments) stated that the chromatic number of $F \times G$ would be equal to the smaller of the chromatic numbers of F and G, i.e., that

$$\chi(F \times G) = \min\{\chi(F), \chi(G)\}.$$

It is easy to see that the right hand side above is an upper bound on the left hand side. If $c : V(F) \to \{1, 2, \dots, \chi(F)\}$ is a proper coloring of F then $c' : (x,u) \mapsto c(x)$ is a proper coloring of $F \times G$ proving $\chi(F \times G) \leq \chi(F)$. As the same argument works if we start with a proper coloring of G, this proves the claimed inequality. Thus the real content of the conjecture was that the right hand side is also a lower bound on $\chi(F \times G)$. Though not true in general, this holds in several special cases. In particular, it is easy to prove when $\min\{\chi(F), \chi(G)\} \leq 3$ and it is also known to hold when this value is 4. The latter, however, is a highly nontrivial result of El Zahar and Sauer [11] and the general case was wide open until the already mentioned recent breakthrough by Shitov [30]. For several related results, see the survey papers [28, 33, 35].

A map $f : V(T) \to V(H)$ between the vertex sets of graphs T and H is called a graph homomorphism if it preserves edges, that is, if f satisfies $\{a,b\} \in E(T) \Rightarrow \{f(a), f(b)\} \in E(H)$. The existence of a graph homomorphism from T to H is denoted by $T \to H$.

Behind the validity of the inequality $\chi(F \times G) \leq \min\{\chi(F), \chi(G)\}$ is the fact that both $F \times G \to F$ and $F \times G \to G$ holds (just take the projection maps), while it is generally true, that $T \to H$ implies $\chi(T) \leq \chi(H)$.

This suggests that if $p(G)$ is any graph parameter which is monotone nondecreasing under graph homomorphism, that is for which $T \to H$ implies $p(T) \leq p(H)$, then an analogous question to Hedetniemi's conjecture is meaningful for it: we automatically have $p(F \times G) \leq \min\{p(F), p(G)\}$ and one may ask whether equality holds. (If it does, we will say that p satisfies the Hedetniemi-type equality.)

Theorems of this kind already exist. A famous example is Zhu's celebrated result known as the fractional version of Hedetniemi's conjecture [37]. It shows equality above in case $p(G)$ is the fractional chromatic number. More recently Godsil, Roberson, Šamal, and Severini [14] proved a similar result for the Lovász theta number of the complementary graph and also investigated the analogous question for a closely related parameter called vector chromatic number introduced in [21]. (The vector chromatic number is different from the *strict vector chromatic number* which is known to be identical to the Lovász theta number of the complementary graph, cf. [14], [23].) They conjectured that the Hedetniemi-type equality also holds for this parameter and proved it in special cases. In a follow-up paper by Godsil, Roberson, Roomey, Šamal, and Varvitsiotis [15] the latter conjecture is proved in general as well.

Both the fractional chromatic number and the Lovász theta number of the complementary graph are well-known upper bounds on the Shannon OR-capacity of the graph which is the usual Shannon capacity, or Shannon AND-capacity of the complementary graph. This is also a homomorphism-monotone parameter, so the Hedetniemi-type question is meaningful for it. In this paper we initiate the study of this question. Given the very complex behavior of Shannon capacity there seems to be little reason to believe that the Shannon OR-capacity would also satisfy the analogous equality. However, if one has to argue why it looks unlikely, then the first argument that comes to mind is that Shannon OR-capacity famously does not satisfy two other simple equalities that both the fractional chromatic number and the Lovász theta number of the complementary graph do. (For more details of this, see the next section.) Our main observation is that this argument is rather weak. This will be a consequence of a strong recent result by Jeroen Zuiddam [38].

We will also elaborate on the question of finding graph pairs that provide potential counterexamples to the mentioned equality. This turns out to be challenging as well, largely because of the lack of knowledge about the general behaviour of Shannon capacity. We will give a natural lower bound on $C_{\mathrm{OR}}(F \times G)$ in Section 4. If we want to "test" whether Shannon OR-capacity satisfies the Hedetniemi-type equality in nontrivial cases, we need some graph pairs (F, G), for which our lower bound is strictly smaller than the upper bound $\min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\}$. Since the Shannon capacity value is not known in too many nontrivial cases, finding such graph pairs is not entirely trivial. We will present some graph pairs with this property in the second subsection of Section 4.

2 Shannon OR-capacity

The Shannon capacity of a graph involves a graph product which is different of the categorical product that appears in Hedetniemi's conjecture. In fact, traditionally, that is, in Shannon's original and in some subsequent papers, see [29, 22], it is defined via a product that is often called the AND-product, cf. e.g. [3]. Sometimes it is more convenient, however, to define graph capacity in a complementary way, cf. e.g. [9] (see Definition 11.3). The graph product involved then is the OR-product and the resulting notion is equivalent to the previous one defined for the complementary graph. To avoid confusion, we will call these two notions Shannon AND-capacity and Shannon OR-capacity, the latter being the one we will mostly use.

Definition 1. *Let F and G be two graphs. Both their AND-product $F \boxtimes G$ and OR-product $F \cdot G$ is defined on the Cartesian product $V(F) \times V(G)$ as vertex set. The edge* *set of the OR-product is given by*

$$E(F \cdot G) = \{\{(f,g),(f',g')\} : f,f' \in V(F), g,g' \in V(G), \{f,f'\} \in E(F) \text{ or } \{g,g'\} \in E(G)\}.$$

On the other hand, the edge set of the AND-product is given by

$$E(F \boxtimes G) = \{\{(f,g),(f',g')\} : f,f' \in V(F), g,g' \in V(G),$$

$$\{f,f'\} \in E(F) \text{ and } \{g,g'\} \in E(G), \text{ or } f = f', \{g,g'\} \in E(G), \text{ or } \{f,f'\} \in E(F), g = g'\}.$$

We denote the t-fold OR-product of a graph G with itself by G^t, while the t-fold AND-product of G with itself will be denoted by $G^{\boxtimes t}$.

Denoting the complementary graph of a graph H by $\overline{H}$, note that the above definitions imply that $\overline{F \cdot G} = \overline{F} \boxtimes \overline{G}$. In particular, $\omega(G^t) = \alpha(\overline{G^t}) = \alpha(\overline{G}^{\boxtimes t})$, where $\omega(H)$ and $\alpha(H)$ denote the clique number and the independence number of graph H, respectively.

Definition 2. *The Shannon OR-capacity of a graph G is defined as the always existing limit*

$$C_{\text{OR}}(G) := \lim_{t\to\infty} \sqrt[t]{\omega(G^t)}.$$

The Shannon AND-capacity is equal to $C_{\text{AND}}(G) := \lim_{t\to\infty} \sqrt[t]{\alpha(G^{\boxtimes t})} = C_{\text{OR}}(\overline{G})$.

We remark, that in information theory Shannon capacity is often defined as the logarithm of the above values (to emphasize its operational meaning), but we will omit logarithms as it is more customarily done in combinatorial treatments. We also note, that all graphs in our discussions are meant to be simple.

Proposition 1. *If G and H are two graphs such that $G \to H$, then $C_{\text{OR}}(G) \le C_{\text{OR}}(H)$.*

Proof. Let $f : V(G) \to V(H)$ be a graph homomorphism and $\boldsymbol{a} = a_1a_2\dots a_t, \boldsymbol{b} = b_1b_2\dots b_t$ be two adjacent vertices of G^t. Then for some i we have $\{a_i, b_i\} \in E(G)$ implying $\{f(a_i), f(b_i)\} \in E(H)$ and thus $\{f(a_1)f(a_2)\dots f(a_t), f(b_1)f(b_2)\dots f(b_t)\} \in E(H^t)$. Since our graphs are simple, this implies $\omega(G^t) \le \omega(H^t)$ for every t and thus the statement. □

Corollary 1.

$$C_{\text{OR}}(F \times G) \le \min\{C_{\text{OR}}(F), C_{\text{OR}}(G)\}.$$

Proof. The claimed inequality follows from our discussion in the Introduction: since the appropriate projection maps define graph homomorphisms from $F \times G$ to F and G, respectively, we have $F \times G \to F$ and $F \times G \to G$. By Proposition 1 this implies the statement. □

Thus the following question is indeed valid: For what graphs F and G do we have equality in Corollary 1? We elaborate on this question in the next two sections.

3 On the possibilities of equality

If one is asked whether believing in equality in Corollary 1 sounds plausible, then the most natural reaction seems to be to say "no" based mainly on the fact that the answer to two somewhat similar questions is negative, though neither is trivial. These two questions are the following.

Lovász asked in his celebrated paper [22], whether Shannon OR-capacity is multiplicative with respect to the OR-product. i.e., whether

$$C_{\mathrm{OR}}(F \cdot G) = C_{\mathrm{OR}}(F) C_{\mathrm{OR}}(G)$$

holds for all pairs of graphs F and G. (Formally the question was asked in the complementary language, but its mathematical content was equivalent to this.) This was answered in the negative by Haemers in [17].

The second question is from Shannon's paper [29] and to present it in our language we need the notion of *join* of two graphs.

Definition 3. *The join $F \oplus G$ of graphs F and G has the disjoint union of $V(F)$ and $V(G)$ as vertex set and its edge set is given by*

$$E(F \oplus G) = E(F) \cup E(G) \cup \{\{a, b\} : a \in V(F), b \in V(G)\},$$

that is, $F \oplus G$ is the disjoint union of graphs F and G with all edges added that has one endpoint in $V(F)$ and the other in $V(G)$.

Shannon [29] proved that $C_{\mathrm{OR}}(F \oplus G) \geq C_{\mathrm{OR}}(F) + C_{\mathrm{OR}}(G)$ for all pairs of graphs F and G and formulated the conjecture that equality always holds. This was refuted by Alon [1] only four decades after the question had been posed.

Two of the graph parameters, the fractional chromatic number $\chi_f(G)$ and the Lovász theta number of the complementary graph (or strict vector chromatic number) $\bar{\vartheta}(G) = \vartheta(\bar{G})$ that we mentioned in the Introduction as examples for graph parameters satisfying the Hedetniemi-type equality, i.e., for which we have

$$\chi_f(F \times G) = \min\{\chi_f(F), \chi_f(G)\}$$

and

$$\bar{\vartheta}(F \times G) = \min\{\bar{\vartheta}(F), \bar{\vartheta}(G)\},$$

respectively, also satisfy

$$\chi_f(F \cdot G) = \chi_f(F)\chi_f(G), \quad \bar{\vartheta}(F \cdot G) = \bar{\vartheta}(F)\bar{\vartheta}(G),$$

and

$$\chi_f(F \oplus G) = \chi_f(F) + \chi_f(G), \quad \bar{\vartheta}(F \oplus G) = \bar{\vartheta}(F) + \bar{\vartheta}(G).$$

(We remark that the chromatic number also trivially satisfies the second type of these equalities, i.e. $\chi(F \oplus G) = \chi(F) + \chi(G)$, but it does not satisfy the first one. At the same time it does satisfy the inequality $\chi(F \cdot G) \leq \chi(F)\chi(G)$.)

The following notion (adapted again for our complementary language) is from Zuiddam's recent paper [38] that borrows the terminology from Strassen's work [32] which it is based on.

Definition 4. *Let S be a collection of graphs closed under the join and the OR-product operations and containing the single vertex graph K_1. The* asymptotic spectrum $Y(S)$ *of S is the set of all maps $\varphi : S \to R_{\geq 0}$ which satisfy for all $G, H \in S$ the following four properties:*

- $\varphi(K_1) = 1$
- $\varphi(G \oplus H) = \varphi(G) + \varphi(H)$
- $\varphi(G \cdot H) = \varphi(G) \cdot \varphi(H)$
- *if $G \to H$, then $\varphi(G) \leq \varphi(H)$.*

Note that every $\varphi \in Y(S)$ provides an upper bound for the Shannon OR-capacity of graphs in S. Indeed, the first two properties imply $\varphi(K_n) = n$ for every n, which together with the fourth property imply $\omega(G) \leq \varphi(G)$ for every $G \in S$ and $\varphi \in Y(S)$. Using also the third property we obtain

$$C_{\mathrm{OR}}(G) = \lim_{t\to\infty} \sqrt[t]{\omega(G^t)} \leq \lim_{t\to\infty} \sqrt[t]{\varphi(G^t)} = \lim_{t\to\infty} \sqrt[t]{[\varphi(G)]^t} = \varphi(G).$$

Note also that $C_{\mathrm{OR}}(G)$ itself does not belong to $Y(S)$ by the above mentioned results of Haemers [17] and Alon [1].

Building on Strassen's theory of asymptotic spectra, Zuiddam proved the following surprising result (cf. also [13] for an independently found weaker version).

Theorem 1. *(Zuiddam's theorem [38]) Let S be a collection of graphs closed under the join and the OR-product operations and containing the single vertex graph K_1. Let $Y(S)$ be the asymptotic spectrum of S. Then for all graphs $G \in S$ we have*

$$C_{\mathrm{OR}}(G) = \min_{\varphi \in Y(S)} \varphi(G).$$

That is, Zuiddam's theorem states that the value of $C_{\mathrm{OR}}(G)$ is always equal to the value of one of its "nicely behaving" upper bound functions. Note that this would be trivial if $C_{\mathrm{OR}}(G)$ itself would be a member of $Y(S)$, but we have already seen that this is not the case. Zuiddam [38] gives a list of known elements of $Y(S)$. This list (translated to our complementary language) includes the fractional chromatic number, the Lovász theta number of the complementary graph, the so-called fractional Haemers bound (of the complementary graph) defined in [5] and further investigated in [8, 20], and another parameter called fractional orthogonal rank introduced in [10]. The fractional Haemers' bound also depends on a field and as Zuiddam also remarks, a separation result by Bukh and Cox [8] implies that this family of graph invariants has infinitely many different elements.

Now we are ready to prove our main result.

Theorem 2. *Either*

$$C_{\mathrm{OR}}(F \times G) = \min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\}$$

holds for graphs F and G or there exists some function φ satisfying the properties given in Definition 4 for which we have

$$\varphi(F \times G) < \min\{\varphi(F), \varphi(G)\}.$$

In short, Theorem 2 states that either Shannon OR-capacity satisfies the Hedetniemi-type equality, or if not, then there is some much "nicer behaving" graph invariant, too, which also violates it.

Proof. Consider two graphs F and G and let S be a class of graphs satisfying the conditions in Zuiddam's theorem and containing all of F, G and $F \times G$. Then by Zuiddam's theorem there exists some $\varphi_0 \in Y(S)$ for which

$$C_{\mathrm{OR}}(F \times G) = \varphi_0(F \times G).$$

Assume that

$$\varphi_0(F \times G) = \min\{\varphi_0(F), \varphi_0(G)\}$$

holds. Then w.l.o.g. we may assume $\varphi_0(F) \leq \varphi_0(G)$ and thus $\varphi_0(F \times G) = \varphi_0(F)$. Since all elements in $Y(S)$ are upper bounds on Shannon OR-capacity, we also have that

$$\min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\} \leq \min\{\varphi_0(F), \varphi_0(G)\} = \varphi_0(F).$$

But

$$\varphi_0(F) = \varphi_0(F \times G) = C_{\mathrm{OR}}(F \times G),$$

so we have obtained

$$\min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\} \leq C_{\mathrm{OR}}(F \times G).$$

Since the opposite inequality is always true (by Corollary 1) this implies

$$\min\{C_{\mathrm{OR}}(F), C_{\mathrm{OR}}(G)\} = C_{\mathrm{OR}}(F \times G).$$

Consequently, if the latter equality does not hold, then we must have $\varphi_0(F \times G) \neq \min\{\varphi_0(F), \varphi_0(G)\}$ implying

$$\varphi_0(F \times G) < \min\{\varphi_0(F), \varphi_0(G)\} = \varphi(F_0)$$

by φ_0 satisfying the fourth property in Definition 4 and the fact that $F \times G \to F$. $\square$

Remark 1. While the proof of Theorem 2 is rather simple it may be worth noting how strong Zuiddam's theorem is on which it is based. An illustration of this is given in the last fifteen minutes of Zuiddam's lecture [39], where he shows an equally simple proof of the statement (translated to the language and notation we use here), that

$$C_{\rm OR}(F \cdot G) = C_{\rm OR}(F) C_{\rm OR}(G) \Leftrightarrow C_{\rm OR}(F \oplus G) = C_{\rm OR}(F) + C_{\rm OR}(G)$$

using his theorem. In other words, in posession of Zuiddam's theorem Haemers' 1979 result [17] about the non-multiplicativity of $C_{\rm OR}(G)$ with respect to the OR-product already implies Alon's breakthrough result refuting Shannon's conjecture that appeared only two decades later. $\Diamond$

Remark 2. Graph parameters that satisfy the Hedetniemi-type equality, but violate the conditions in Definition 4 exist. A simple example is the clique number that is not multiplicative with respect to the OR-product. (If it was, then the Shannon-capacity problem would be trivial.) A perhaps more artificial example is the reciprocal of the odd girth (taken to be 0 when the graph is bipartite) which also satisfies the Hedetniemi-type equality but fails to do so with all but the last one of the four conditions in Definition 4. $\Diamond$

4 A lower bound and identifying test cases

Due to the lack of knowledge of the Shannon capacity value for many graphs (note that even that is not known whether the computational problem given by it is decidable, see [2]), it is not entirely trivial how to find a pair of graphs on which one could at least try checking whether there is equality in Corollary 1 in any nontrivial way. In this section we establish a general lower bound for $C_{\rm OR}(F \times G)$ and present some graph pairs for which this lower bound is strictly smaller than the upper bound $\min\{C_{\rm OR}(F), C_{\rm OR}(G)\}$. Whether either of the two bounds is sharp in these cases remains an open problem.

4.1 Lower bound

The following Proposition gives our lower bound.

Proposition 2.

$$C_{\rm OR}(F \times G) \geq \max\{C_{\rm OR}(F'), C_{\rm OR}(G') : F' \subseteq F, F' \to G, G' \subseteq G, G' \to F\}.$$

Proof. Let F_0 denote the subgraph F' of F that admits a homomorphism to G with largest value of $C_{\rm OR}(F')$. Let G_0 be the analogous subgraph of G obtained when we exchange the letters F and G in the previous sentence. The statement is equivalent to the inequality $C_{\rm OR}(F \times G) \geq \max\{C_{\rm OR}(F_0), C_{\rm OR}(G_0)\}$.

Thus it is enough to show $C_{\mathrm{OR}}(F \times G) \geq C_{\mathrm{OR}}(F_0)$, the same argument will prove $C_{\mathrm{OR}}(F \times G) \geq C_{\mathrm{OR}}(G_0)$ when exchanging the role of F and G.

This readily follows from Proposition 1. Indeed, since $F_0 \times G \subseteq F \times G$ we have $C_{\mathrm{OR}}(F \times G) \geq C_{\mathrm{OR}}(F_0 \times G) \geq C_{\mathrm{OR}}(F_0)$, where the last inequality is a consequence of Proposition 1 and the fact that $F_0 \to G$ implies $F_0 \to F_0 \times G$. The latter follows by observing that if f is a homomorphism from F_0 to G, then $\{u, v\} \in E(F_0)$ implies $\{(u, f(u)), (v, f(v))\} \in E(F_0 \times G)$, therefore $f' : u \mapsto (u, f(u))$ is a homomorphism from F_0 to $F_0 \times G$. This completes the proof. □

Corollary 2. *If $F \to G$ then $C_{\mathrm{OR}}(F \times G) = C_{\mathrm{OR}}(F)$.*

Proof. This is an immediate consequence of Proposition 2 and Corollary 1. □

For example, since a longer odd cycle always admits a homomorphism to a shorter one (but not vice cersa) for arbitrary integers $1 \leq k \leq \ell$ we have $C_{\mathrm{OR}}(C_{2k+1} \times C_{2\ell+1}) = C_{\mathrm{OR}}(C_{2\ell+1})$. If $k = 1, \ell = 2$, then the above value is equal to $C_{\mathrm{OR}}(C_5) = \sqrt{5}$ by the celebrated result of Lovász [22] on the Shannon capacity of the 5-cycle. We remark that the exact value of $C_{\mathrm{OR}}(C_{2\ell+1})$ is unknown for all $\ell \geq 3$. A nontrivial result concerning these values is proven by Bohman and Holzman [7] who showed that $C_{\mathrm{OR}}(C_{2\ell+1}) > 2$ for every positive integer ℓ.

Naturally, if we would like to "test" whether the inequality in Corollary 1 can be strict then we need a pair of graphs F and G for which the upper bound on $C_{\mathrm{OR}}(F \times G)$ provided by Corollary 1 is strictly larger than the lower bound given in Proposition 2. As the exact value of Shannon capacity is known only in a few nontrivial cases, finding such a pair is not a completely trivial matter. We discuss this problem in the following subsection.

4.2 Paley graphs and variants

Definition 5. *Let q be an odd prime power satisfying $q \equiv 1 (\mathrm{mod}\ 4)$. The Paley graph P_q is defined on the elements of the finite field F_q as vertices. Two vertices form an edge if and only if their difference in F_q is a square in F_q.*

Note that the condition on q ensures that -1 has a square root in F_q and thus $a - b$ is a square in F_q if and only if $b - a$ is. Thus the definition is indeed meaningful and results in a(n undirected) graph. In the special case when q itself is a prime number p, edges of P_p are between vertices whose difference is a quadratic residue modulo p.

We also remark that P_5 is just the five-cycle C_5 and the graph P_{17} is well-known to be the unique graph on 17 vertices not having either a clique or an independent set of size 4, thus establishing the sharp lower bound on the largest known diagonal Ramsey number $R(4, 4) = 18$ [16]. (In fact, $P_5 \cong C_5$ is the unique graph establishing $R(3, 3) \geq 6$. For more on the connection between Ramsey numbers and Shannon capacity, cf. [3, 12, 24].)

Paley graphs are well-known to be self-complementary, vertex-transitive, and edge-transitive, cf. e.g. [36, 26]. The first two of these properties make them particularly useful for us by the following theorem of Lovász.

Theorem 3. *(Lovász [22]) If G is a vertex-transitive self-complementary graph on n vertices, then*

$$C_{\rm OR}(G) = \sqrt{n}.$$

Thus we have $C_{\rm OR}(P_q) = \sqrt{q}$ for all prime powers $q \equiv 1 (\bmod 4)$.

Let $p \equiv 1 (\bmod 4)$ be a prime number. The value of the clique number (or equivalently, the independence number) of P_p is not known and determining it is a well-known unsolved problem in number theory, the conjectured value being $O((\log p)^2)$, cf. [23]. It is not hard to see (and also follows from Theorem 3 above) that $\omega(P_p) \le \sqrt{p}$. Improving this bound by 1 for infinitely many primes p was already a nontrivial task that was achieved by Bachoc, Matolcsi, and Ruzsa [4] only a few years ago. Recently Hanson and Petridis [18] managed to improve this substantially by proving the general upper bound

$$\omega(P_p) \le \frac{\sqrt{2p-1}+1}{2} < \frac{\sqrt{2p}+1}{2}.$$

Notice that this upper bound immediately implies

$$\chi(P_p) > \frac{2p}{\sqrt{2p}+1}$$

by $\chi(P_p) \ge \frac{|V(P_p)|}{\alpha(P_p)} = \frac{|V(P_p)|}{\omega(P_p)}$. This in turn gives $\chi(P_p) > \sqrt{p}+1 \ge \lceil \sqrt{p} \rceil$, whenever $p > 20$ meaning that for primes at least 20 the largest subgraph of P_p that can be colored with $\lceil \sqrt{p} \rceil$ colors has strictly fewer vertices than P_p itself. There are only three primes of the form $4k+1$ below 20: $5, 13, 17$. As already mentioned above, P_{17} is well-known from [16] to be the graph establishing the largest known diagonal Ramsey number $R(4,4) = 18$, that is, it has no clique or independent set on more than 3 vertices. In fact, this also follows from the Hanson-Petridis bound as well as $\alpha(P_{13}) \le 3$. Therefore we have $\chi(P_{17}) \ge \lceil \frac{17}{3} \rceil = 6 > \lceil \sqrt{17} \rceil = 5$ as well as $\chi(P_{13}) \ge \lceil \frac{13}{3} \rceil = 5 > \lceil \sqrt{13} \rceil = 4$. (Obviously, the analogous inequality does not hold for $P_5 \cong C_5$.)

This suggests that if we knew that deleting a vertex of a Paley graph P_p its Shannon capacity becomes already smaller than $C_{\rm OR}(P_p) = \sqrt{p}$, then we could conclude that for $m = \lceil \sqrt{p} \rceil$ the graph pair (P_p, K_m) has the property, that the lower bound of Proposition 2 on $C_{\rm OR}(P_p \times K_m)$ is strictly smaller than its upper bound from Corollary 1, which is $\sqrt{p}$ in this case. (Here we use that $\max\{C_{OR}(H) : H \subseteq P_p, H \to K_m\} = \max\{C_{OR}(H) : H \subseteq P_p, \chi(H) \le m\}$, while $\max\{C_{OR}(T) : T \subseteq K_m, T \to P_p\} \le m-1$ if $\omega(P_p) < m$.) Let us denote the graph we obtain from P_p after deleting a vertex by Q_{p-1}. (Note that by the vertex-transitivity of P_p it does not matter which vertex is deleted.) Unfortunately, we do not have a proof that $C_{\rm OR}(Q_{p-1}) < C_{\rm OR}(P_p)$ always holds. Though we believe it is true for any prime p (of the form $4k+1$) it is clear that this will not follow simply from the symmetry properties of P_p (that one might believe at first sight), as the analogous statement is not true for all prime powers q. Indeed, if $q = p^k$ for k even, then it is known that $\omega(P_q) = \sqrt{q}$ [6], (cf. also [4]) and by vertex-transitivity this immediately implies $\omega(Q_{q-1}) = \sqrt{q}$ as well, that further implies $C_{\rm OR}(Q_{q-1}) = \sqrt{q}$ (where analogously to Q_{p-1}, Q_{q-1} denotes the graph obtained by deleting a vertex of P_q).

Although we do not have a general proof for $C_{\rm OR}(Q_{p-1}) < C_{\rm OR}(P_p)$, using the computability of the Lovász theta number we can decide in several cases that this indeed happens. With the help of the online available Python code [31] to compute the Lovász theta number for specific graphs[1], we can obtain for example the following values for the first five relevant numbers:

$$\bar{\vartheta}(Q_{12}) \approx 3.4927 < \sqrt{13}, \quad \bar{\vartheta}(Q_{16}) \approx 4.0035 < \sqrt{17},$$

$$\bar{\vartheta}(Q_{28}) \approx 5.3069 < \sqrt{29}, \quad \bar{\vartheta}(Q_{36}) \approx 6.0025 < \sqrt{37}, \quad \bar{\vartheta}(Q_{40}) \approx 6.3493 < \sqrt{41}.$$

It is worth calculating the Lovász theta number also for graphs we obtain when deleting two vertices of a Paley graph. By the edge-transitivity of P_p there are only two non-isomorphic such graphs (that are complementary to each other). We denote the one obtained when deleting two adjacent vertices by $Z^{(a)}_{p-2}$, and the one obtained when deleting two non-adjacent vertices by $Z^{(n)}_{p-2}$. Since p is odd and P_p is self-complementary, Q_{p-1} is self-complementary as well (cf. [27]). Since any self-complementary graph G on n vertices has a clique of size n in its second OR power (simply consider the vertices $(v, f(v))$, where f is a complementing permutation), we have

$$C_{\rm OR}(Q_{p-1}) \geq \sqrt{p-1}.$$

In fact, it is a natural question whether equality holds here. This is obviously so for $p = 5$, but seems to be open for all other relevant values of p. (Note that the above numerical values of $\bar{\vartheta}(Q_{p-1})$ are all strictly larger than $\sqrt{p-1}$.)

By the foregoing, in case $\max\{\bar{\vartheta}(Z^{(a)}_{p-2}), \bar{\vartheta}(Z^{(n)}_{p-2})\} < \sqrt{p-1}$, it implies

$$\max\{C_{\rm OR}(Z^{(a)}_{p-2}), C_{\rm OR}(Z^{(n)}_{p-2})\} < \sqrt{p-1},$$

and thus the graph pair (Q_{p-1}, K_ℓ) with $\ell = \lceil\sqrt{p-1}\rceil$ also has the property that our lower and upper bounds do not coincide for $C_{\rm OR}(Q_{p-1} \times K_\ell)$ provided that $K_\ell \not\to Q_{p-1}$ (that we are ensured of by the Hanson-Petridis upper bound on $\omega(P_p)$) and $Q_{p-1} \not\to K_\ell$, that is, $\chi(Q_{p-1}) > \ell$. The latter condition is not yet true for $p = 13$, but follows from the Hanson-Petridis bound [18] for all the larger relevant values of p. This is particularly appealing when $\sqrt{p-1}$ is an integer itself. This happens in several cases, starting (disregarding

[1] I am grateful to Anna Gujgiczer for showing me how this code can be used and also for providing several of the required calculations.

$p = 2, 5$ that are not relevant for us) with $p = 17, 37, 101$ (cf. sequence A002496 of The Online Encyclopedia of Integer Sequences [25]). Whether this sequence is infinite (as it is believed to be) is a famous open problem in number theory (one of the four problems called Landau's problems along with Goldbach's conjecture, the twin-prime conjecture, and Legendre's conjecture).

Using again the Python code [31], we obtain that

$$\max\{\bar{\vartheta}(Z_{15}^{(a)}), \bar{\vartheta}(Z_{15}^{(n)})\} = \min\{3.8726, 3.8849\} < 4,$$

$$\max\{\bar{\vartheta}(Z_{35}^{(a)}), \bar{\vartheta}(Z_{35}^{(n)})\} = \min\{5.9128, 5.9251\} < 6,$$

and

$$\max\{\bar{\vartheta}(Z_{99}^{(a)}), \bar{\vartheta}(Z_{99}^{(n)})\} = \min\{9.9496, 9.9574\} < 10.$$

Thus each of the graph pairs $(Q_{16}, K_4), (Q_{36}, K_6)$, and (Q_{100}, K_{10}) provide "test cases" for investigating the possibility of equality in Corollary 1. Let us stress again, that in the light of Theorem 2 any proof showing for example $C_{\mathrm{OR}}(Q_{16} \times K_4) < 4$ would imply the existence of a graph parameter that satisfies all the four conditions in Definition 4 and yet fails to satisfy the Hedetniemi-type equality.

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