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On the derivatives of the powers of trigonometric and hyperbolic sine and cosine

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Abstract

This work contains different expressions for the kth derivative of the nth power of the trigonometric and hyperbolic sine and cosine.

The first set of expressions follow from the complex definitions of the trigonometric and hyperbolic sine and cosine, and the binomial theorem.

The other expressions are polynomial-based. They are perhaps less obvious, and use only polynomials in $\sin x$ and $\cos x$, or in $\sinh x$ and $\cosh x$. No sines or cosines of arguments other than x appear in these polynomial-based expressions. The final expressions are dependent only on $\sin x$, $\cos x$, $\sinh x$, or $\cosh x$ respectively when k is even; and they only have a single additional factor $\cos x$, $\sin x$, $\cosh x$, or $\sinh x$ respectively when k is odd.

The derivatives of the powers of the natural exponential function

This works starts with expressions for the derivatives of the powers of the natural exponential function.

For $k \in \mathbb{N}$, and $n \in \mathbb{Z}$, the kth derivative of the function applying the nth power to the natural exponential is:

$$\frac{\partial^k}{\partial x^k}\left[\exp^n x\right] = n^k \exp^n x = n^k \exp(nx) = n^k \mathrm{e}^{nx}.$$

The natural exponential function, its powers, and their derivatives are all both $\mathbb{C} \to \mathbb{C}$ and $\mathbb{R} \to \mathbb{R}$ functions: they are defined for the whole domain of the complex numbers, returning complex values; yet the image of all real arguments are themselves real values.

The definedness of exponentation

For the purposed of this work, 0^0 is considered defined as $0^0 = 1$. For a short discussion and historical overview of this controversy, see Knuth (1992). So the following identity holds over the entire domain $\mathbb{C}$ (including $\mathbb{R}$, including $\{0\}$):

$$\exp^0 x = \frac{\partial^0}{\partial x^0}\left[\exp^0 x\right] = 0^0 \exp^0 x = 0^0 \exp 0 = 1.$$

For bases $z \in \mathbb{C}$, the power z^n is defined for all exponents $n \in \mathbb{Z}$. For bases $z \in \mathbb{R}_{\geqslant 0}$, the power z^n is also defined for all exponents $n \in \mathbb{R}$. As the power rule also applies for those non-integer exponents, the polynomial-based identities obtained in this work also apply for real powers, in the intervals for x in which $\sin x$, $\cos x$, $\sinh x$, or $\cosh x$ is real and nonnegative. Note that $\sinh x \in \mathbb{R}_{\geqslant 0}$ when $x \in \mathbb{R}_{\geqslant 0}$, and that $\cosh x \in \mathbb{R}_{\geqslant 0}$ when $x \in \mathbb{R}$. However, the 4 functions are not all nonnegative-real-valued (or even real-valued) for every $z \in \mathbb{C}$. In light of this note, only the (strictly too strict) condition $n \in \mathbb{Z}$ will be considered and mentioned in this work.

Using the complex definitions and the binomial theorem

The first set of expressions for the derivatives of the powers of trigonometric and hyperbolic sine and cosine follow from the complex definitions of the trigonometric and hyperbolic sine and cosine, which are:

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\,\mathrm{i}}, \qquad \cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}, \qquad \sinh x = \frac{\mathrm{e}^{x} - \mathrm{e}^{-x}}{2}, \qquad \cosh x = \frac{\mathrm{e}^{x} + \mathrm{e}^{-x}}{2}.$$

These functions are all both $\mathbb{C} \to \mathbb{C}$ and $\mathbb{R} \to \mathbb{R}$ functions.

It follows from these complex definitions that powers of the trigonometric and hyperbolic sine and cosine are:

$$\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n}(x) = \frac{1}{\left\{\begin{matrix}2\,\mathrm{i}\\ 2\\ 2\\ 2\end{matrix}\right\}^{n}} \cdot \left\{\begin{matrix}\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}\\ \mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}\\ \mathrm{e}^{x} - \mathrm{e}^{-x}\\ \mathrm{e}^{x} + \mathrm{e}^{-x}\end{matrix}\right\}^{n}.$$

For $n \in \mathbb{N}$, the binomial theorem can be applied:

$$\begin{aligned}
\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n}(x) &= \frac{1}{\left\{\begin{matrix}2\,\mathrm{i}\\ 2\\ 2\\ 2\end{matrix}\right\}^{n}} \cdot \sum_{r=0}^{n} \left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{n-r} \cdot \binom{n}{r} \cdot \mathrm{e}^{\left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}\cdot r\cdot x} \cdot \mathrm{e}^{-\left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}\cdot (n-r)\cdot x}\\
&= \frac{\left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{n}}{\left\{\begin{matrix}2\,\mathrm{i}\\ 2\\ 2\\ 2\end{matrix}\right\}^{n}} \cdot \sum_{r=0}^{n} \left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{r} \cdot \binom{n}{r} \cdot \mathrm{e}^{\left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}\cdot (2r-n)\cdot x}\\
&= \frac{\left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{n}}{2^{n}} \cdot \sum_{r=0}^{n} \left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{r} \cdot \binom{n}{r} \cdot \left\{\begin{matrix}\mathrm{i}\\ 1\\ 1\\ 1\end{matrix}\right\}^{-n} \cdot \mathrm{e}^{\left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}\cdot (2r-n)\cdot x}.
\end{aligned}$$

And so the derivatives are:

$$\frac{\partial^{k}}{\partial x^{k}}\left[\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n}(x)\right] = \frac{\left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{n}}{2^{n}} \cdot \sum_{r=0}^{n} \left\{\begin{matrix}-1\\ 1\\ -1\\ 1\end{matrix}\right\}^{r} \cdot \binom{n}{r} \cdot \left\{\begin{matrix}\mathrm{i}\\ 1\\ 1\\ 1\end{matrix}\right\}^{-n} \cdot \left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}^{k} \cdot (2r-n)^{k} \cdot \mathrm{e}^{\left\{\begin{matrix}\mathrm{i}\\ \mathrm{i}\\ 1\\ 1\end{matrix}\right\}\cdot (2r-n)\cdot x}.$$

For trigonometric sine:

$$\begin{aligned}
\frac{\partial^{k}}{\partial x^{k}}\left[\sin^{n} x\right] &= \frac{(-1)^{n}}{2^{n}} \cdot \sum_{r=0}^{n} (-1)^{r} \cdot \binom{n}{r} \cdot (2r-n)^{k} \cdot \mathrm{i}^{k-n} \cdot \mathrm{e}^{\mathrm{i}(2r-n)x}\\
&= \frac{(-1)^{n}}{2^{n}} \cdot \sum_{r=0}^{n} (-1)^{r} \cdot \binom{n}{r} \cdot (2r-n)^{k} \cdot \mathrm{e}^{\mathrm{i}\left((k-n)\frac{\pi}{2} + (2r-n)x\right)}
\end{aligned}$$

$$= \frac{(-1)^{n}}{2^{n}} \cdot \sum_{r=0}^{n} (-1)^{r} \cdot \binom{n}{r} \cdot (2r-n)^{k} \cdot \left[\cos\left(\tfrac{(k-n)\pi}{2} + (2r-n)x\right) + \mathrm{i}\sin\left(\tfrac{(k-n)\pi}{2} + (2r-n)x\right)\right].$$

For trigonometric cosine:

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\big[\cos^n x\big] &= \frac{1}{2^n}\cdot\sum_{r=0}^{n}\binom{n}{r}\cdot(2r-n)^k\cdot \mathrm{i}^k\cdot \mathrm{e}^{\mathrm{i}(2r-n)x}\\
&= \frac{1}{2^n}\cdot\sum_{r=0}^{n}\binom{n}{r}\cdot(2r-n)^k\cdot \mathrm{e}^{\mathrm{i}\left(\frac{k\pi}{2}+(2r-n)x\right)}\\
&= \frac{1}{2^n}\cdot\sum_{r=0}^{n}\binom{n}{r}\cdot(2r-n)^k\cdot\Big[\cos\big(\tfrac{k\pi}{2}+(2r-n)x\big)+\mathrm{i}\sin\big(\tfrac{k\pi}{2}+(2r-n)x\big)\Big].
\end{aligned}$$

In the above results for trigonometric sine and cosine, the imaginary part evaluates to 0 for real arguments x.

The above results for trigonometric sine and cosine are not new, as they are already published by Qi (2015).

For hyperbolic sine:

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\big[\sinh^n x\big] &= \frac{(-1)^n}{2^n}\cdot\sum_{r=0}^{n}(-1)^r\cdot\binom{n}{r}\cdot(2r-n)^k\cdot \mathrm{e}^{(2r-n)x}\\
&= \frac{(-1)^n}{2^n}\cdot\sum_{r=0}^{n}(-1)^r\cdot\binom{n}{r}\cdot(2r-n)^k\cdot\Big[\cosh\big((2r-n)x\big)+\sinh\big((2r-n)x\big)\Big].
\end{aligned}$$

For hyperbolic cosine:

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}\big[\cosh^n x\big] &= \frac{1}{2^n}\cdot\sum_{r=0}^{n}\binom{n}{r}\cdot(2r-n)^k\cdot \mathrm{e}^{(2r-n)x}\\
&= \frac{1}{2^n}\cdot\sum_{r=0}^{n}\binom{n}{r}\cdot(2r-n)^k\cdot\Big[\cosh\big((2r-n)x\big)+\sinh\big((2r-n)x\big)\Big].
\end{aligned}$$

Using polynomials, intermediate step

Definitions of the intermediate polynomials

For $k\in\mathbb{N}$, $n\in\mathbb{Z}$, and $u\in\mathbb{C}$, let f_k, g_k, and h_k be sequences of curried polynomial functions, whose images $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ are polynomials in u, with coefficients that are, in turn, polynomials in n; and let those sequences be recursively defined (over k) as follows:

$$\begin{Bmatrix}f\\g\\h\end{Bmatrix}_0(n)(u)=1,$$

$$\begin{Bmatrix}f\\g\\h\end{Bmatrix}_{k+1}(n)(u)=\begin{Bmatrix}1\\-1\\1\end{Bmatrix}\cdot\left((n-k)\cdot u\cdot\begin{Bmatrix}f\\g\\h\end{Bmatrix}_k(n)(u)+\begin{Bmatrix}u^2-1\\u^2-1\\u^2+1\end{Bmatrix}\cdot\frac{\partial}{\partial u}\left[\begin{Bmatrix}f\\g\\h\end{Bmatrix}_k(n)(u)\right]\right).$$

Properties

The rank of the polynomials $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ is k.

The coefficient of the highest power with non-zero coefficient of $f_k(n)$ and of $h_k(n)$ is 1. The coefficient of the highest power with non-zero coefficient of $g_k(n)$ is 1 when k is even, or -1 when k is odd.

The polynomials $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ contain only non-zero coefficients for even powers of u when k is even, or only non-zero coefficients for odd powers of u when k is odd.

For $k \in \mathbb{N}$, $r \in \mathbb{N}$, when $r \leqslant k$, and when k and r have the same parity, the coefficients for u^r are polynomials in n with rank $\dfrac{k+r}{2}$.

$\forall\, k \in \mathbb{N}$, $n \in \mathbb{Z}$, and $u \in \mathbb{C}$: $g_k(n)(u) = (-1)^k f_k(n)(u)$.

The coefficients of the second-highest power with non-zero coefficient of $f_k(n)$ and of $g_k(n)$ are polynomials in n, of which the signs of the coefficients are alternating, and the absolute values of the coefficients appear in the lower-triangular matrix obtained through the multiplication of the following 3 lower-triangular matrices, followed by the removal of the left column of the lower-triangular product matrix, which is typeset in bold below:

$$\underbrace{\begin{bmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ & & \vdots \end{bmatrix}}_{\text{Pascal's triangle (1655)}} \times \begin{bmatrix} 1 \\ 1 & 1 \\ & 1 & 1 \\ & & 1 & 1 \\ & & & 1 & 1 \\ & & & & 1 & 1 \\ & & & & & \vdots \end{bmatrix} \times \underbrace{\begin{bmatrix} 1 \\ & 1 \\ & & 2 \\ & & & 4 \\ & & & & 8 \\ & & & & & 16 \\ & & & & & \vdots \end{bmatrix}}_{\text{Powers of 2}}$$

$$= \begin{bmatrix} \mathbf{1} \\ \mathbf{2} & 1 \\ \mathbf{3} & 3 & 2 \\ \mathbf{4} & 6 & 8 & 2 \\ \mathbf{5} & 10 & 20 & 20 & 8 \\ \mathbf{6} & 15 & 40 & 60 & 48 & 16 \\ & & \vdots \end{bmatrix}.$$

The sequence of numbers in this lower-triangular product matrix, including its left column, has been contributed by Adamson (2007) to the On-Line Encyclopedia of Integer Sequences (OEIS), where it is registered as sequence A133341.

The derivatives of the powers of trigonometric and hyperbolic sine and cosine

With $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ defined above, the following identities will be proved:

$$\frac{\partial^k}{\partial x^k}\left[\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^n(x)\right] = \left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n-k}(x)\cdot\left\{\begin{matrix}f\\ g\\ f\\ h\end{matrix}\right\}_k(n)\left(\left\{\begin{matrix}\cos\\ \sin\\ \cosh\\ \sinh\end{matrix}\right\}(x)\right).$$

Proof

The proof is by induction on k.

The base case, for $k = 0$, is trivial:

$$\frac{\partial^0}{\partial x^0}\left[\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^n(x)\right] = \left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^n(x) = \left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n-0}(x)\cdot 1 = \left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n-0}(x)\cdot\left\{\begin{matrix}f\\ g\\ f\\ h\end{matrix}\right\}_0(n)\left(\left\{\begin{matrix}\cos\\ \sin\\ \cosh\\ \sinh\end{matrix}\right\}(x)\right).$$

For the inductive case, assume the induction hypothesis

$$\frac{\partial^k}{\partial x^k}\left[\left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^n(x)\right] = \left\{\begin{matrix}\sin\\ \cos\\ \sinh\\ \cosh\end{matrix}\right\}^{n-k}(x)\cdot\left\{\begin{matrix}f\\ g\\ f\\ h\end{matrix}\right\}_k(n)\left(\left\{\begin{matrix}\cos\\ \sin\\ \cosh\\ \sinh\end{matrix}\right\}(x)\right).$$

With $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ defined above, under this induction hypothesis, and using the product rule and the chain rule, the $(k+1)$th derivatives can intermediately be derived as follows:

$$\frac{\partial^{k+1}}{\partial x^{k+1}}\left[\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n}(x)\right]$$

$$=\frac{\partial}{\partial x}\left[\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k}(x)\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)\right]$$

$$=\frac{\partial}{\partial x}\left[\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k}(x)\right]\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)+\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k}(x)\cdot\frac{\partial}{\partial x}\left[\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)\right]$$

$$=(n-k)\cdot\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k-1}(x)\cdot\frac{\partial}{\partial x}\left[\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}(x)\right]\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)+\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k-1}(x)\cdot\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}(x)\cdot\frac{\partial}{\partial u}\left[\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)(u)\right]_{u=\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)}$$

$$\cdot\frac{\partial}{\partial x}\left[\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)\right]$$

$$=\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k-1}(x)\cdot(n-k)\cdot\begin{Bmatrix}1\\ -1\\ 1\\ 1\end{Bmatrix}\cdot\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)+\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k-1}(x)\cdot\begin{Bmatrix}-1\\ 1\\ 1\\ 1\end{Bmatrix}\cdot\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{2}(x)\cdot\frac{\partial}{\partial u}\left[\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)(u)\right]_{u=\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)}$$

$$=\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-k-1}(x)\cdot\begin{Bmatrix}1\\ -1\\ 1\\ 1\end{Bmatrix}\cdot\left((n-k)\cdot\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right)+\begin{Bmatrix}\cos^2 x-1\\ \sin^2 x-1\\ \cosh^2 x-1\\ \sinh^2 x+1\end{Bmatrix}\cdot\frac{\partial}{\partial u}\left[\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k}(n)(u)\right]_{u=\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)}\right)$$

$$=\begin{Bmatrix}\sin\\ \cos\\ \sinh\\ \cosh\end{Bmatrix}^{n-(k+1)}(x)\cdot\begin{Bmatrix}f\\ g\\ f\\ h\end{Bmatrix}_{k+1}(n)\left(\begin{Bmatrix}\cos\\ \sin\\ \cosh\\ \sinh\end{Bmatrix}(x)\right).$$

$\square$

Using polynomials, final step

Definitions of the final polynomials

For $k \in \mathbb{N}$, $n \in \mathbb{Z}$, and $u \in \mathbb{C}$, let s_k, c_k, t_k, and d_k be sequences of curried polynomial functions, whose images $f_k(n)(u)$, $g_k(n)(u)$, and $h_k(n)(u)$ are polynomials in u, with coefficients that are, in turn, polynomials in n; and let those sequences be obtained as follows:

$$\begin{Bmatrix} s \\ c \\ t \\ d \end{Bmatrix}_k (n)(v) = \text{substitute (for all } m \in \mathbb{N}) \; u^{2m} \text{ with } \begin{Bmatrix} 1 - v^{2m} \\ 1 - v^{2m} \\ v^{2m} + 1 \\ v^{2m} - 1 \end{Bmatrix}$$

$$\text{in } \begin{Bmatrix} f \\ g \\ f \\ h \end{Bmatrix}_k (n)(u) \text{ when } k \text{ is even, or in } \frac{1}{u} \begin{Bmatrix} f \\ g \\ f \\ h \end{Bmatrix}_k (n)(u) \text{ when } k \text{ is odd.}$$

Properties

The rank of the polynomials $s_k(n)(v)$, $c_k(n)(v)$, $t_k(n)(v)$, and $d_k(n)(v)$ is always even: it k when k is even, or $k - 1$ when k is odd.

The coefficient of the highest power with non-zero coefficient of $t_k(n)(v)$ and of $d_k(n)(v)$ is 1. The coefficient of the highest power with non-zero coefficient of $s_k(n)(v)$ is 1 when $k \bmod 4 \in \{0,\, 1\}$, or -1 when $k \bmod 4 \in \{2,\, 3\}$. The coefficient of the highest power with non-zero coefficient of $c_k(n)(v)$ is 1 when $k \bmod 4 \in \{0,\, 3\}$, or -1 when $k \bmod 4 \in \{1,\, 2\}$.

The polynomials $s_k(n)(v)$, $c_k(n)(v)$, $t_k(n)(v)$ and $d_k(n)(v)$ contain only non-zero coefficients for even powers of v.

$\forall\, k \in \mathbb{N}$, $n \in \mathbb{Z}$, and $v \in \mathbb{C}$: $c_k(n)(v) = (-1)^k s_k(n)(v)$.

The derivatives of the powers of trigonometric and hyperbolic sine and cosine

With $s_k(n)(u)$, $c_k(n)(u)$, $t_k(n)(u)$ and $d_k(n)(u)$ defined above, and using the identities $\forall\, x \in \mathbb{C}$: $\sin^2(x) + \cos^2(x) = 1$ and $\cosh^2(x) - \sinh^2(x) = 1$, the kth derivatives can finally be derived as follows:

$$\frac{\partial^k}{\partial x^k}\left[\begin{Bmatrix} \sin \\ \cos \\ \sinh \\ \cosh \end{Bmatrix}^n (x)\right] = \begin{cases} \begin{Bmatrix} \sin \\ \cos \\ \sinh \\ \cosh \end{Bmatrix}^{n-k} (x) \cdot \begin{Bmatrix} s \\ c \\ t \\ d \end{Bmatrix}_k (n) \left(\begin{Bmatrix} \sin \\ \cos \\ \sinh \\ \cosh \end{Bmatrix}(x)\right) & \text{when } k \text{ is even,} \\ \begin{Bmatrix} \sin \\ \cos \\ \sinh \\ \cosh \end{Bmatrix}^{n-k} (x) \cdot \begin{Bmatrix} \cos \\ \sin \\ \cosh \\ \sinh \end{Bmatrix}(x) \cdot \begin{Bmatrix} s \\ c \\ t \\ d \end{Bmatrix}_k (n) \left(\begin{Bmatrix} \sin \\ \cos \\ \sinh \\ \cosh \end{Bmatrix}(x)\right) & \text{when } k \text{ is odd.} \end{cases}$$

Using polynomials, (partially) applied results

Both the intermediate and the final expressions are listed here. The expression in brackets is the intermediate or final polynomial, except for when the coefficient of the highest power with non-zero coefficient of $g_k(n)$, of $s_k(n)$, or of $c_k(n)$ is -1, in which case -1 is factored out.

For $k \in \mathbb{N}$, $n \in \mathbb{Z}$, and $x \in \mathbb{C} \setminus \left\{ \begin{Bmatrix} m \\ (m + \frac{1}{2}) \\ \mathrm{i}\, m \\ \mathrm{i}\,(m + \frac{1}{2}) \end{Bmatrix} \pi \,\middle|\, n < k \wedge m \in \mathbb{Z} \right\}$:

The derivatives of the powers of trigonometric sine

$$\begin{aligned}
\frac{\partial^0}{\partial x^0}\big[\sin^n(x)\big] &= \sin^{n-0}(x)\big[1\big] \\
&= \sin^{n-0}(x)\big[1\big] = \sin^n(x), \\
\frac{\partial^1}{\partial x^1}\big[\sin^n(x)\big] &= \sin^{n-1}(x)\big[n\cos(x)\big] \\
&= \sin^{n-1}(x)\cos(x)\big[n\big], \\
\frac{\partial^2}{\partial x^2}\big[\sin^n(x)\big] &= \sin^{n-2}(x)\big[n^2\cos^2(x) - n\big] \\
&= -\sin^{n-2}(x)\big[n^2\sin^2(x) + (-n^2+n)\big], \\
\frac{\partial^3}{\partial x^3}\big[\sin^n(x)\big] &= \sin^{n-3}(x)\big[n^3\cos^3(x) + (-3n^2+2n)\cos(x)\big] \\
&= -\sin^{n-3}(x)\cos(x)\big[n^3\sin^2(x) + (-n^3+3n^2-2n)\big], \\
\frac{\partial^4}{\partial x^4}\big[\sin^n(x)\big] &= \sin^{n-4}(x)\big[n^4\cos^4(x) + (-6n^3+8n^2-4n)\cos^2(x) + (3n^2-2n)\big] \\
&= \sin^{n-4}(x)\big[n^4\sin^4(x) + (-2n^4+6n^3-8n^2+4n)\sin^2(x) + (n^4-6n^3+11n^2-6n)\big], \\
\frac{\partial^5}{\partial x^5}\big[\sin^n(x)\big] &= \sin^{n-5}(x)\big[n^5\cos^5(x) + (-10n^4+20n^3-20n^2+8n)\cos^3(x) \\
&\qquad + (15n^3-30n^2+16n)\cos(x)\big] \\
&= \sin^{n-5}(x)\cos(x)\big[n^5\sin^4(x) + (-2n^5+10n^4-20n^3+20n^2-8n)\sin^2(x) \\
&\qquad + (n^5-10n^4+35n^3-50n^2+24n)\big], \\
\frac{\partial^6}{\partial x^6}\big[\sin^n(x)\big] &= \sin^{n-6}(x)\big[n^6\cos^6(x) + (-15n^5+40n^4-60n^3+48n^2-16n)\cos^4(x) \\
&\qquad + (45n^4-150n^3+196n^2-88n)\cos^2(x) + (-15n^3+30n^2-16n)\big] \\
&= -\sin^{n-6}(x)\big[n^6\sin^6(x) + (-3n^6+15n^5-40n^4+60n^3-48n^2+16n)\sin^4(x) \\
&\qquad + (3n^6-30n^5+125n^4-270n^3+292n^2-120n)\sin^2(x) \\
&\qquad + (-n^6+15n^5-85n^4+225n^3-274n^2+120n)\big], \ \ldots
\end{aligned}$$

The derivatives of the powers of trigonometric cosine

$$\begin{aligned}
\frac{\partial^0}{\partial x^0}\big[\cos^n(x)\big] &= \cos^{n-0}(x)\big[1\big] \\
&= \cos^{n-0}(x)\big[1\big] = \cos^n(x), \\
\frac{\partial^1}{\partial x^1}\big[\cos^n(x)\big] &= -\cos^{n-1}(x)\big[n\sin(x)\big] \\
&= -\cos^{n-1}(x)\sin(x)\big[n\big], \\
\frac{\partial^2}{\partial x^2}\big[\cos^n(x)\big] &= \cos^{n-2}(x)\big[n^2\sin^2(x) - n\big] \\
&= -\cos^{n-2}(x)\big[n^2\cos^2(x) + (-n^2+n)\big], \\
\frac{\partial^3}{\partial x^3}\big[\cos^n(x)\big] &= -\cos^{n-3}(x)\big[n^3\sin^3(x) + (-3n^2+2n)\sin(x)\big] \\
&= \cos^{n-3}(x)\sin(x)\big[n^3\cos^2(x) + (-n^3+3n^2-2n)\big], \\
\frac{\partial^4}{\partial x^4}\big[\cos^n(x)\big] &= \cos^{n-4}(x)\big[n^4\sin^4(x) + (-6n^3+8n^2-4n)\sin^2(x) + (3n^2-2n)\big] \\
&= \cos^{n-4}(x)\big[n^4\cos^4(x) + (-2n^4+6n^3-8n^2+4n)\cos^2(x) + (n^4-6n^3+11n^2-6n)\big], \\
\frac{\partial^5}{\partial x^5}\big[\cos^n(x)\big] &= -\cos^{n-5}(x)\big[n^5\sin^5(x) + (-10n^4+20n^3-20n^2+8n)\sin^3(x) \\
&\qquad + (15n^3-30n^2+16n)\sin(x)\big]
\end{aligned}$$

$$
\begin{aligned}
&= -\cos^{n-5}(x)\sin(x)\big[n^5\cos^4(x) + (-2n^5+10n^4-20n^3+20n^2-8n)\cos^2(x) \\
&\qquad + (n^5-10n^4+35n^3-50n^2+24n)\big], \\
\frac{\partial^6}{\partial x^6}\big[\cos^n(x)\big] &= \cos^{n-6}(x)\big[n^6\sin^6(x) + (-15n^5+40n^4-60n^3+48n^2-16n)\sin^4(x) \\
&\qquad + (45n^4-150n^3+196n^2-88n)\sin^2(x) + (-15n^3+30n^2-16n)\big] \\
&= -\cos^{n-6}(x)\big[n^6\cos^6(x) + (-3n^6+15n^5-40n^4+60n^3-48n^2+16n)\cos^4(x) \\
&\qquad + (3n^6-30n^5+125n^4-270n^3+292n^2-120n)\cos^2(x) \\
&\qquad + (-n^6+15n^5-85n^4+225n^3-274n^2+120n)\big],\ \dots
\end{aligned}
$$

The derivatives of the powers of hyperbolic sine

$$
\begin{aligned}
\frac{\partial^0}{\partial x^0}\big[\sinh^n(x)\big] &= \sinh^{n-0}(x)\big[1\big] \\
&= \sinh^{n-0}(x)\big[1\big] = \sinh^n(x), \\
\frac{\partial^1}{\partial x^1}\big[\sinh^n(x)\big] &= \sinh^{n-1}(x)\big[n\cosh(x)\big] \\
&= \sinh^{n-1}(x)\cosh(x)\big[n\big], \\
\frac{\partial^2}{\partial x^2}\big[\sinh^n(x)\big] &= \sinh^{n-2}(x)\big[n^2\cosh^2(x) - n\big] \\
&= \sinh^{n-2}(x)\big[n^2\sinh^2(x) + (n^2-n)\big], \\
\frac{\partial^3}{\partial x^3}\big[\sinh^n(x)\big] &= \sinh^{n-3}(x)\big[n^3\cosh^3(x) + (-3n^2+2n)\cosh(x)\big] \\
&= \sinh^{n-3}(x)\cosh(x)\big[n^3\sinh^2(x) + (n^3-3n^2+2n)\big], \\
\frac{\partial^4}{\partial x^4}\big[\sinh^n(x)\big] &= \sinh^{n-4}(x)\big[n^4\cosh^4(x) + (-6n^3+8n^2-4n)\cosh^2(x) + (3n^2-2n)\big] \\
&= \sinh^{n-4}(x)\big[n^4\sinh^4(x) + (2n^4-6n^3+8n^2-4n)\sinh^2(x) + (n^4-6n^3+11n^2-6n)\big], \\
\frac{\partial^5}{\partial x^5}\big[\sinh^n(x)\big] &= \sinh^{n-5}(x)\big[n^5\cosh^5(x) + (-10n^4+20n^3-20n^2+8n)\cosh^3(x) \\
&\qquad + (15n^3-30n^2+16n)\cosh(x)\big] \\
&= \sinh^{n-5}(x)\cosh(x)\big[n^5\sinh^4(x) + (2n^5-10n^4+20n^3-20n^2+8n)\sinh^2(x) \\
&\qquad + (n^5-10n^4+35n^3-50n^2+24n)\big], \\
\frac{\partial^6}{\partial x^6}\big[\sinh^n(x)\big] &= \sinh^{n-6}(x)\big[n^6\cosh^6(x) + (-15n^5+40n^4-60n^3+48n^2-16n)\cosh^4(x) \\
&\qquad + (45n^4-150n^3+196n^2-88n)\cosh^2(x) + (-15n^3+30n^2-16n)\big] \\
&= \sinh^{n-6}(x)\big[n^6\sinh^6(x) + (3n^6-15n^5+40n^4-60n^3+48n^2-16n)\sinh^4(x) \\
&\qquad + (3n^6-30n^5+125n^4-270n^3+292n^2-120n)\sinh^2(x) \\
&\qquad + (n^6-15n^5+85n^4-225n^3+274n^2-120n)\big],\ \dots
\end{aligned}
$$

The derivatives of the powers of hyperbolic cosine

$$
\begin{aligned}
\frac{\partial^0}{\partial x^0}\big[\cosh^n(x)\big] &= \cosh^{n-0}(x)\big[1\big] \\
&= \cosh^{n-0}(x)\big[1\big] = \cosh^n(x), \\
\frac{\partial^1}{\partial x^1}\big[\cosh^n(x)\big] &= \cosh^{n-1}(x)\big[n\sinh(x)\big] \\
&= \cosh^{n-1}(x)\sinh(x)\big[n\big],
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2}\big[\cosh^n(x)\big] &= \cosh^{n-2}(x)\big[n^2\sinh^2(x)+n\big] \\
&= \cosh^{n-2}(x)\big[n^2\cosh^2(x)+(-n^2+n)\big], \\
\frac{\partial^3}{\partial x^3}\big[\cosh^n(x)\big] &= \cosh^{n-3}(x)\big[n^3\sinh^3(x)+(3n^2-2n)\sinh(x)\big] \\
&= \cosh^{n-3}(x)\sinh(x)\big[n^3\cosh^2(x)+(-n^3+3n^2-2n)\big], \\
\frac{\partial^4}{\partial x^4}\big[\cosh^n(x)\big] &= \cosh^{n-4}(x)\big[n^4\sinh^4(x)+(6n^3-8n^2+4n)\sinh^2(x)+(3n^2-2n)\big] \\
&= \cosh^{n-4}(x)\big[n^4\cosh^4(x)+(-2n^4+6n^3-8n^2+4n)\cosh^2(x) \\
&\qquad\qquad +(n^4-6n^3+11n^2-6n)\big], \\
\frac{\partial^5}{\partial x^5}\big[\cosh^n(x)\big] &= \cosh^{n-5}(x)\big[n^5\sinh^5(x)+(10n^4-20n^3+20n^2-8n)\sinh^3(x) \\
&\qquad\qquad +(15n^3-30n^2+16n)\sinh(x)\big] \\
&= \cosh^{n-5}(x)\sinh(x)\big[n^5\cosh^4(x)+(-2n^5+10n^4-20n^3+20n^2-8n)\cosh^2(x) \\
&\qquad\qquad\qquad +(n^5-10n^4+35n^3-50n^2+24n)\big], \\
\frac{\partial^6}{\partial x^6}\big[\cosh^n(x)\big] &= \cosh^{n-6}(x)\big[n^6\sinh^6(x)+(15n^5-40n^4+60n^3-48n^2+16n)\sinh^4(x) \\
&\qquad\qquad +(45n^4-150n^3+196n^2-88n)\sinh^2(x)+(15n^3-30n^2+16n)\big] \\
&= \cosh^{n-6}(x)\big[n^6\cosh^6(x)+(-3n^6+15n^5-40n^4+60n^3-48n^2+16n)\cosh^4(x) \\
&\qquad\qquad +(3n^6-30n^5+125n^4-270n^3+292n^2-120n)\cosh^2(x) \\
&\qquad\qquad +(-n^6+15n^5-85n^4+225n^3-274n^2+120n)\big],\ \ldots
\end{aligned}
$$

Source code

Source code for a .NET implementation, to list and calculate and evaluate this work's polynomials and expressions, written in F#, can be found at `http://fssnip.net/7Xf`.

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