Reinterpreting Mütze's Theorem via Natural Enumeration of Ordered Rooted Trees

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Abstract

A reinterpretation of the proof of existence of Hamilton cycles in the middle-levels graphs is given via their dihedral quotients whose vertices are ordered rooted trees.

1 Introduction

Let $[2k+1] = \{0, \dots, 2k\}$, where $0 < k \in \mathbb{Z}$. The *middle-levels graph* M_k [4] is the subgraph induced by the k-th and $(k+1)$-th levels of the Hasse diagram [6] of the Boolean lattice $2^{[2k+1]}$. The dihedral group D_{4k+2} acts on M_k via translations mod $2k+1$ (Section 3) and complemented reversals (Section 4). We show that the sequence $\mathcal{S}$ [7] A239903 of *restricted-growth strings* (or *RGS's*) [1] page 325, revisited below, unifies the presentation of all M_k's. Recalling the *Catalan number* $C_k = \frac{(2k)!}{k!(k+1)!}$ [7] A000108, we prove that the first C_k terms of $\mathcal{S}$ stand for the orbits of $V(M_k)$ under D_{4k+2}-action. This is used to provide a reinterpretation of Mütze's theorem [5] on the existence of Hamilton cycles in M_k and its Gregor-Mütze-Nummenpalo proof [3] via *k-germs* (see Subsection 1.2) of RGS's. For history and motivation involved in the conjecture solved by Mütze's theorem, we refer to [5, 3].

1.1 Restricted-Growth Strings and Ordered Rooted Trees

Let $0 < k \in \mathbb{Z}$. The sequence of (pairwise different) RGS's $\mathcal{S} = (\beta(0), \dots, \beta(17), \dots) =$

$$(0, 1, 10, 11, 12, 100, 101, 110, 111, 112, 120, 121, 122, 123, 1000, 1001, 1010, 1011, \dots) \quad (1)$$

has the lengths of its contiguous pairs $(\beta(i-1), \beta(i))$ of RGS's constant unless $i = C_k$ for $0 < k \in \mathbb{Z}$, in which case $\beta(i-1) = \beta(C_k - 1) = 12\cdots k$ and $\beta(i) = \beta(C_k) = 10^k = 10\cdots0$.

In Section 5, the D_{4k+2}-action on M_k allows to project M_k onto a quotient graph R_k whose vertices are shown in Section 8 to stand for the first C_k terms of $\mathcal{S}$ via the lexical-matching colors $0, 1, \dots, k$ [4] on the $k+1$ edges incident to each vertex. In preparation for this, the RGS's β are converted in Section 2 into $(2k+1)$-strings $F(\beta)$ representing the *ordered rooted trees* on k edges (Remark 3) via a "castling" procedure that yields enumeration of all such trees. These ordered rooted trees (encoded as $F(\beta)$) are shown to represent the vertices of R_k via an "un-castling" procedure (Section 7). Adjacency in R_k is presented in Section 8.

In R_k, the first two lexical colors 0 and 1 yield a 2-factor W_{01}. In Section 9 an alternative way to R_k is presented. Employing it in Section 10, W_{01} is analyzed (Figure 5) from the

RGS-dihedral action viewpoint. From this, W_{01} is seen in Section 11 (as in [3]) morphing into Hamilton cycles of M_k via symmetric differences with 6-cycles as in Figure 6.

In particular, an integer sequence $\mathcal{S}_0$ is shown to exist so that for each integer $k > 0$ the neighbors via color k of the RGS's in R_k ordered as in $\mathcal{S}$ correspond to an idempotent permutation on the first C_k terms of $\mathcal{S}_0$. This and related properties hold for colors $1, \ldots, k+1$ (Theorems 9 and 16 and Remark 17) reflecting and extending properties of plane trees (i.e., classes of ordered rooted trees under root rotation) in Section 10, in particular Lemma 13. Moreover, Section 12 considers symmetry properties by reversing the designation of the roots in the ordered rooted trees so that each lexical color i ($0 \leq i \leq k$) adjacency can be seen from the color $(k-i)$ viewpoint. In addition, an all-RGS's binary tree is presented in Section 13 representing all the vertices of the R_k's.

1.2 Germs

To view the continuation of the sequence $\mathcal{S}$ in (1) for our purposes, each RGS $\beta = \beta(m)$, denoted simply by β, will be transformed for adequate $k > 1$ into a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ that we call a *k-germ* by prefixing to β enough zeros if necessary. Formally, a *k-germ* α with $1 < k \in \mathbb{Z}$ is defined to be a $(k-1)$-string $\alpha = a_{k-1}a_{k-2}\cdots a_2a_1$ such that:

(1) the leftmost position (called position $k-1$) of α contains entry $a_{k-1} \in \{0, 1\}$;

(2) given $1 < i < k$, the entry a_{i-1} (at position $i-1$) satisfies $0 \leq a_{i-1} \leq a_i + 1$.

Every k-germ $a_{k-1}a_{k-2}\cdots a_2a_1$ yields the $(k+1)$-germ $a_ka_{k-1}a_{k-2}\cdots a_2a_1 = 0a_{k-1}a_{k-2}\cdots a_2a_1$. A *non-null RGS*, again denoted α, is obtained by stripping a k-germ $\alpha = a_{k-1}a_{k-2}\cdots a_1 \neq 00\cdots 0$ off the null entries to the left of its leftmost position containing 1. We also say that the *null RGS* $\alpha = 0$ corresponds to every null k-germ α, for $0 < k \in \mathbb{Z}$. (We use the same notations $\alpha = \alpha(m)$ and $\beta = \beta(m)$ to denote both a k-germ and its associated RGS).

The k-germs are ordered as follows. Given two k-germs, say $\alpha = a_{k-1}\cdots a_2a_1$ and $\beta = b_{k-1}\cdots b_2b_1$, where $\alpha \neq \beta$, we say that α precedes β, written $\alpha < \beta$, whenever either:

(i) $a_{k-1} < b_{k-1}$ or

(ii) $a_j = b_j$, for $k-1 \leq j \leq i+1$, and $a_i < b_i$, for some $k-1 > i \geq 1$.

The resulting order on k-germs $\alpha(m)$ ($m \leq C_k$), corresponding biunivocally (via the assignment $m \to \alpha(m)$) with the natural order on m, yields a listing that we call the *natural (k-germ) enumeration*. Note that there are exactly C_k k-germs $\alpha = \alpha(m) < 10^k$, for every $k > 0$.

1.3 Nautral (Germ) Enumeration

To determine the RGS $\beta(m)$ of an integer $m \geq 0$, or associated k-germ $\alpha(m)$, we use *Catalan's triangle* Δ, namely a triangular arrangement of positive integers starting with the following successive rows Δ_j, for $j = 0, \ldots, 8$:

1								
1	1							
1	2	2						
1	3	5	5					
1	4	9	14	14				
1	5	14	28	42	42			
1	6	20	48	90	132	132		
1	7	27	75	165	297	429	429	
1	8	35	110	275	572	1001	1430	1430
..	..	..	..	..	..	..	..	..

where reading is linear, as in [7] A009766. The numbers τ_i^j in Δ_j ($0 \leq j \in \mathbb{Z}$), given by $\tau_i^j = (j+i)!(j-i+1)/(i!(j+1)!)$, are characterized by the following properties:

1. $\tau_0^j = 1$, for every $j \geq 0$;

2. $\tau_1^j = j$ and $\tau_j^j = \tau_{j-1}^j$, for every $j \geq 1$;

3. $\tau_i^j = \tau_i^{j-1} + \tau_{i-1}^j$, for every $j \geq 2$ and $i = 1, \ldots, j-2$;

4. $\sum_{i=0}^j \tau_i^j = \tau_j^{j+1} = \tau_{j+1}^{j+1} = C_j$, for every $j \geq 1$.

The determination of k-germ $\beta(m)$ proceeds as follows. Let $x_0 = m$ and let $y_0 = \tau_k^{k+1}$ be the largest member of the second diagonal of Δ with $y_0 \leq x_0$. Let $x_1 = x_0 - y_0$. If $x_1 > 0$, then let $Y_1 = \{\tau_{k-1}^j\}_{j=k}^{k+b_1}$ be the largest set of successive terms in the $(k-1)$-column of Δ with $y_1 = \sum Y_1 \leq x_1$. Either $Y_1 = \emptyset$, in which case we take $b_1 = -1$, or not, in which case we take $b_1 = |Y_1| - 1$. Let $x_2 = x_1 - y_1$. If $x_2 > 0$, then let $Y_2 = \{\tau_{k-2}^j\}_{j=k}^{k+b_2}$ be the largest set of successive terms in the $(k-2)$-column of Δ with $y_2 = \sum Y_2 \leq x_2$. Either $Y_2 = \emptyset$, in which case we take $b_2 = -1$, or not, in which case we take $b_2 = |Y_3| - 1$. Iteratively, we arrive at a null x_k. Then $\alpha(x_0) = a_{k-1}a_{k-2}\cdots a_1$, where $a_{k-1} = 1$, $a_{k-2} = 1 + b_1$, …, and $a_1 = 1 + b_k$.

We note that $\beta(m)$ is recovered from $\alpha(m) = \alpha(x_0)$ by removing the zeros to the left of the leftmost 1 in $\alpha(x_0)$. Given an RGS β or associated k-germ α, the considerations above can easily be played backwards to recover the corresponding integer x_0.

For example, if $x_0 = 38$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 38 - 14 = 24$, $y_1 = \tau_2^3 + \tau_2^4 = 5 + 9 = 14$, $x_2 = x_1 - y_1 = 24 - 14 = 10$, $y_2 = \tau_1^2 + \tau_1^3 + \tau_1^4 = 2 + 3 + 4 = 9$, $x_3 = x_2 - y_2 = 10 - 9 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - y_3 = 1 - 1 = 0$, so that $b_1 = 1$, $b_2 = 2$, and $b_3 = 0$, taking to $a_4 = 1$, $a_3 = 1 + b_1 = 2$, $a_2 = 1 + b_2 = 3$ and $a_1 = 1 + b_3 = 1$, determining the 5-germ $\alpha(38) = a_4a_3a_2a_1 = 1231$. If $x_0 = 20$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 20 - 14 = 6$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 1$, $y_2 = 0$ is an empty sum (since its possible summand $\tau_1^2 > 1 = x_2$), $x_3 = x_2 - y_2 = 1$, $y_3 = \tau_0^1 = 1$ and $x_4 = x_3 - x_3 = 1 - 1 = 0$, determining the 5-germ $\alpha(20) = a_4a_3a_2a_1 = 1101$. Moreover, if $x_0 = 19$, then $y_0 = \tau_3^4 = 14$, $x_1 = x_0 - y_0 = 19 - 14 = 5$, $y_1 = \tau_2^3 = 5$, $x_2 = x_1 - y_1 = 5 - 5 = 0$, determining the 5-germ $\beta(19) = a_4a_3a_2a_1 = 1100$.

2 Castling of Ordered Rooted Trees

Theorem 1. *To each k-germ $\alpha = a_{k-1}\cdots a_1 \neq 0^{k-1}$ with rightmost entry $a_i > 0$ ($k > i \geq 1$) corresponds a k-germ $\beta(\alpha) = b_{k-1}\cdots b_1 < \alpha$ with $b_i = a_i - 1$ and $a_j = b_j$, ($j \neq i$). Moreover, all k-germs form a tree $\mathcal{T}_k$ rooted at 0^{k-1}, each k-germ $\alpha \neq 0^{k-1}$ as a child of its $\beta(\alpha)$.*

Proof. The statement, illustrated by means of the first three columns of Table I, is straightforward. Table I also serves as illustration to the proof of Theorem 2. □

By representing $\mathcal{T}_k$ with each node β having its children α enclosed between parentheses following β and separating siblings with commas, we can write:

$$\mathcal{T}_4 = 000(001, 010(011(012)), 100(101, 110(111(121)), 120(121(122(123))))).$$

Theorem 2. *To each k-germ $\alpha = a_{k-1}\cdots a_1$ corresponds a $(2k+1)$-string $F(\alpha) = f_0f_1\cdots f_{2k}$ whose entries are the numbers $0, 1, \ldots, k$ (once each) and k asterisks ($*$) such that:*

(A) $F(0^{k-1}) = 012\cdots(k-2)(k-1)k*\cdots*$;

(B) *if* $\alpha \neq 0^{k-1}$, *then* $F(\alpha)$ *is obtained from* $F(\beta) = F(\beta(\alpha)) = h_0 h_1 \cdots h_{2k}$ *by means of the following "Castling Procedure" steps:*

1. *let* $W^i = h_0 h_1 \cdots h_{i-1} = f_0 f_1 \cdots f_{i-1}$ *and* $Z^i = h_{2k-i+1} \cdots h_{2k-1} h_{2k} = f_{2k-i+1} \cdots f_{2k-1} f_{2k}$ *be respectively the initial and terminal substrings of length* i *in* $F(\beta)$*;*

2. *let* $\Omega > 0$ *be the leftmost entry of the substring* $U = F(\beta) \setminus (W^i \cup Z^i)$ *and consider the concatenation* $U = X|Y$, *with* Y *starting at entry* $\Omega + 1$*; note that*$F(\beta) = W^i|X|Y|Z^i$*;*

3. *set* $F(\alpha) = W^i|Y|X|Z^i$.

In particular:

(a) *the leftmost entry of each* $F(\alpha)$ *is* 0*;* $k*$ *is a substring of* $F(\alpha)$, *but* $*k$ *is not;*

(b) *a number to the immediate right of any* $b \in [0, k)$ *in* $F(\alpha)$ *is larger than* b*;*

(c) W^i *is an (ascending) number* i*-substring and* Z^i *is formed by* i *of the* k *asterisks.*

Proof. Let $\alpha = a_{k-1} \cdots a_1 \neq 0^{k-1}$ be a k-germ. In the sequence of applications of items 1-3 along the path from root 0^{k-1} to α in $\mathcal{T}_k$, unit augmentation of a_i for larger values of i, $(0 < i < k)$, must occur earlier, and then in strictly descending order of the entries i of the intermediate k-germs. As a result, the length of the inner substring $X|Y$ is kept non-decreasing after each application. This is illustrated in Table I below, where the order of presentation of X and Y is reversed in successively decreasing steps. In the process, items (a)-(c) are seen to be fulfilled.

TABLE I

m	α	β	$F(\beta)$	i	$W^i \mid X \mid Y \mid Z^i$	$W^i \mid Y \mid X \mid Z^i$	$F(\alpha)$	α
0	0	–	–	–	–	–	$012**$	0
1	1	0	$012**$	1	$0 \mid 1 \mid 2* \mid *$	$0 \mid 2* \mid 1 \mid *$	$02*1*$	1
0	00	–	–	–	–	–	$0123***$	00
1	01	00	$0123***$	1	$0 \mid 1 \mid 23** \mid *$	$0 \mid 23** \mid 1 \mid *$	$023**1*$	01
2	10	00	$0123***$	2	$01 \mid 2 \mid 3* \mid **$	$01 \mid 3* \mid 2 \mid **$	$013*2**$	10
3	11	10	$013*2**$	1	$0 \mid 13* \mid 2* \mid *$	$0 \mid 2* \mid 13* \mid *$	$02*13**$	11
4	12	11	$02*13**$	1	$0 \mid 2*1 \mid 3* \mid *$	$0 \mid 3* \mid 2*3 \mid *$	$03*2*1*$	12
0	000	–	–	–	–	–	$01234****$	000
1	001	000	$01234****$	1	$0 \mid 1 \mid 234*** \mid *$	$0 \mid 234*** \mid 1 \mid *$	$0234***1*$	001
2	010	000	$01234****$	2	$01 \mid 2 \mid 34** \mid **$	$01 \mid 34** \mid 2 \mid **$	$0134**2**$	010
3	011	010	$0134**2**$	1	$0 \mid 134** \mid 2* \mid *$	$0 \mid 2* \mid 134** \mid *$	$02*134***$	011
4	012	011	$02*134***$	1	$0 \mid 2*1 \mid 34** \mid *$	$0 \mid 34** \mid 2*1 \mid *$	$034**2*1*$	012
5	100	000	$01234****$	3	$012 \mid 3 \mid 4* \mid ***$	$012 \mid 4* \mid 3 \mid ***$	$0124*3***$	100
6	101	100	$0124*3***$	1	$0 \mid 1 \mid 24*3** \mid *$	$0 \mid 24*3** \mid 1 \mid *$	$024*3**1*$	101
7	110	100	$0124*3***$	2	$01 \mid 24* \mid 3* \mid **$	$01 \mid 3* \mid 24* \mid **$	$013*24***$	110
8	111	110	$013*24***$	1	$0 \mid 13* \mid 24** \mid *$	$0 \mid 24** \mid 13* \mid *$	$024**13**$	111
9	112	111	$024**13**$	1	$0 \mid 24**1 \mid 3* \mid *$	$0 \mid 3* \mid 24**1 \mid *$	$03*24**1*$	112
10	120	110	$013*24***$	2	$01 \mid 3*2 \mid 4* \mid **$	$01 \mid 4* \mid 3*2 \mid **$	$014*3*2**$	120
11	121	120	$014*3*2**$	1	$0 \mid 14*3* \mid 2* \mid *$	$0 \mid 2* \mid 14*3* \mid *$	$02*14*3**$	121
12	122	121	$02*14*3**$	1	$0 \mid 2*34* \mid 3* \mid *$	$0 \mid 3* \mid 2*14* \mid *$	$03*2*14**$	122
13	123	122	$03*2*14**$	1	$0 \mid 3*2*1 \mid 4* \mid *$	$0 \mid 4* \mid 3*2*1 \mid *$	$04*3*2*1*$	123

The three successive subtables in Table I have C_k rows each, where $C_2 = 2$, $C_3 = 5$ and $C_4 = 14$; in the subtables, the k-germs α are shown both on the second and last columns via natural enumeration in the first column; the images $F(\alpha)$ of those α are on the penultimate column; the remaining columns in the table are filled, from the second row on, as follows: **(i)** $\beta = \beta(\alpha)$, arising in Theorem 1; **(ii)** $F(\beta)$, taken from the penultimate column in the previous row; **(iii)** the length i of W^i and Z^i ($k-1 \geq i \geq 1$); **(iv)** the decomposition $W^i|Y|X|Z^i$ of $F(\beta)$; **(v)** the decomposition $W^i|X|Y|Z^i$ of $F(\alpha)$, re-concatenated in the following, penultimate, column as $F(\alpha)$, with $\alpha = F^{-1}(F(\alpha))$ in the last column. □

To each $F(\alpha)$ corresponds a binary n-string $\theta(\alpha)$ of weight k obtained by replacing each number by 0 and each asterisk $*$ by 1. By attaching the entries of $F(\alpha)$ as subscripts to the corresponding entries of $\theta(\alpha)$, a subscripted binary n-string $\hat{\theta}(\alpha)$ is obtained. Let $\aleph(\theta(\alpha))$ be given by the *complemented reversal* of $\theta(\alpha)$, that is:

$$\text{if } \theta(\alpha) = a_0 a_1 \cdots a_{2k}, \text{ then } \aleph(\theta(\alpha)) = \bar{a}_{2k} \cdots \bar{a}_1 \bar{a}_0, \tag{2}$$

where $\bar{0} = 1$ and $\bar{1} = 0$. A subscripted version $\hat{\aleph}$ of $\aleph$ is obtained for $\hat{\theta}(\alpha)$, as shown subsequently in Table II, for $k = 2, 3$, with the subscripts of $\hat{\aleph}$ reversed with respect to $\aleph$. Each image under $\aleph$ is an n-string of weight $k+1$ and has the 1's indexed with number subscripts and the 0's indexed with asterisk subscripts. The number subscripts reappear in Sections 6-8 as *lexical colors* [4] for the graphs M_k, revisited below.

TABLE II

m	α	$\theta(\alpha)$	$\hat{\theta}(\alpha)$	$\hat{\aleph}(\theta(\alpha)) = \aleph(\hat{\theta}(\alpha))$	$\aleph(\theta(\alpha))$
0	0	00011	$0_0 0_1 0_2 1_* 1_*$	$0_* 0_* 1_2 1_1 1_0$	00111
1	1	00101	$0_0 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 1_0$	01011
0	00	0000111	$0_0 0_1 0_2 0_3 1_* 1_* 1_*$	$0_* 0_* 0_* 1_3 1_2 1_1 1_0$	0001111
1	01	0001101	$0_0 0_2 0_3 1_* 1_* 0_1 1_*$	$0_* 1_1 0_* 0_* 1_3 1_2 1_0$	0100111
2	10	0001011	$0_0 0_1 0_3 2_* 0_1 1_* 1_*$	$0_* 0_* 1_2 0_* 1_3 1_1 1_0$	0010111
3	11	0010011	$0_0 0_2 1_* 0_1 0_3 1_* 1_*$	$0_* 0_* 1_3 1_1 0_* 1_2 1_0$	0011011
4	12	0010101	$0_0 0_3 1_* 0_2 1_* 0_1 1_*$	$0_* 1_1 0_* 1_2 0_* 1_3 1_0$	0101011

Remark 3. We say that an *ordered rooted tree* [3] (or *ORT*) T is a tree with specified root vertex v_0, an embedding of T into the (coordinate) plane with v_0 on top, all edges descending from corresponding parent vertices to their children, and left-right ordering for these children. We use this ordering to perform a depth first search (DFS) on the ORT T with its vertices from v_0 downward denoted as v_i in a right-to-left breadth-first search (←BFS) fashion. Such DFS yields an n-string $F(\alpha)$ representing the ORT T by writing successively:

(i) the subindex i of each v_i in the DFS downward appearance and

(ii) an asterisk for the edge e_i ending at each child v_i in the DFS upward appearance.

(In Section 12, an opposite viewpoint on ORT's with a new root a_k is seen). Now, we write:

$$T = T_\alpha \text{ and } F(T_\alpha) = F(\alpha).$$

Then, Theorem 1 can be taken as a tree-surgery transformation from T_β onto T_α for each k-germ $\alpha \neq 0^{k-1}$ via the vertices v_i and edges e_i (whose parent vertices are generally reattached differently). This remark is used in Sections 10-12 in reinterpreting Mütze's theorem.

3 Translations mod $2k+1$

Let $n = 2k+1$. The *n-cube graph* H_n is the Hasse diagram of the Boolean lattice $2^{[n]}$ on the set $[n] = \{0, \ldots, n-1\}$. It is convenient to express each vertex v of H_n in either of three different equivalent ways, as:

(a) ordered set $A = \{a_0, a_1, \ldots, a_{j-1}\} = a_0a_1 \cdots a_{j-1} \subseteq [n]$ that v represents, $(0 < j \leq n)$;

(b) characteristic binary n-vector $B_A = (b_0, b_1, \ldots, b_{n-1})$ of ordered set A in item (a) above, where $b_i = 1$ if and only if $i \in A$, $(i \in [n])$;

(c) polynomial $\epsilon_A(x) = b_0 + b_1x + \cdots + b_{n-1}x^{n-1}$ associated to B_A in item (b) above.

Ordered set A and vector B_A in items (a) and (b) respectively are written for short as $a_0a_1 \cdots a_{j-1}$ and $b_0b_1 \cdots b_{n-1}$. A is said to be the *support* of B_A.

For each $j \in [n]$, let $L_j = \{A \subseteq [n] \text{ with } |A| = j\}$ be the *j-level* of H_n. Then, the middle-levels graph M_k is the subgraph of H_n induced by $L_k \cup L_{k+1}$, for $1 \leq k \in \mathbb{Z}$. By viewing the elements of $V(M_k) = L_k \cup L_{k+1}$ as polynomials, as in (c) above, a regular (i.e., free and transitive) *translation mod n* action Υ' of $\mathbb{Z}_n$ on $V(M_k)$ is seen to exist, given by:

$$\Upsilon' : \mathbb{Z}_n \times V(M_k) \to V(M_k), \text{ with } \Upsilon'(i, v) = v(x)x^i \pmod{1 + x^n}, \tag{3}$$

where $v \in V(M_k)$ and $i \in \mathbb{Z}_n$. Now, Υ' yields a quotient graph M_k/π of M_k, where π stands for the equivalence relation on $V(M_k)$ given by:

$$\epsilon_A(x)\pi\epsilon_{A'}(x) \Longleftrightarrow \exists\, i \in \mathbb{Z} \text{ with } \epsilon_{A'}(x) \equiv x^i\epsilon_A(x) \pmod{1 + x^n},$$

with $A, A' \in V(M_k)$. This is to be used in the proof of Theorem 5. Clearly, M_k/π is the graph whose vertices are the equivalence classes of $V(M_k)$ under π. Also, π induces a partition of $E(M_k)$ into equivalence classes, to be taken as the edges of M_k/π.

4 Complemented Reversals

Let $(b_0b_1 \cdots b_{n-1})$ denote the class of $b_0b_1 \cdots b_{n-1} \in L_i$ in L_i/π. Let $\rho_i : L_i \to L_i/\pi$ be the canonical projection given by assigning $b_0b_1 \cdots b_{n-1}$ to $(b_0b_1 \cdots b_{n-1})$, for $i \in \{k, k+1\}$. The definition of $\aleph$ in display (2) is easily extended to a bijection, again denoted $\aleph$, from L_k onto L_{k+1}. Let $\aleph_\pi : L_k/\pi \to L_{k+1}/\pi$ be given by $\aleph_\pi((b_0b_1 \cdots b_{n-1})) = (\bar{b}_{n-1} \cdots \bar{b}_1\bar{b}_0)$. Observe $\aleph_\pi$ is a bijection. Notice the commutative identities $\rho_{k+1}\aleph = \aleph_\pi\rho_k$ and $\rho_k\aleph^{-1} = \aleph_\pi^{-1}\rho_{k+1}$.

The following geometric representations will be handy. List vertically the vertex parts L_k and L_{k+1} of M_k (resp., L_k/π and L_{k+1}/π of M_k/π) so as to display a splitting of $V(M_k) = L_k \cup L_{k+1}$ (resp., $V(M_k)/\pi = L_k/\pi \cup L_{k+1}/\pi$) into pairs, each pair contained in a horizontal line, the two composing vertices of such pair equidistant from a vertical line ϕ (resp., ϕ/π, depicted through M_2/π on the left of Figure 1, Section 5 below). In addition, we impose that each resulting horizontal vertex pair in M_k (resp., M_k/π) be of the form $(B_A, \aleph(B_A))$ (resp., $((B_A), (\aleph(B_A)) = \aleph_\pi((B_A))))$, disposed from left to right at both sides of ϕ. A non-horizontal edge of M_k/π will be said to be a *skew edge*.

Theorem 4. *To each skew edge $e = (B_A)(B_{A'})$ of M_k/π corresponds another skew edge $\aleph_\pi((B_A))\aleph_\pi^{-1}((B_{A'}))$ obtained from e by reflection on the line ϕ/π. Moreover:*

(i) *the skew edges of M_k/π appear in pairs, with the endpoints in each pair forming two horizontal pairs of vertices equidistant from ϕ/π;*
(ii) *the horizontal edges of M_k/π have multiplicity ≤ 2.*

Proof. The skew edges $B_AB_{A'}$ and $\aleph^{-1}(B_{A'})\aleph(B_A)$ of M_k are reflection of each other about ϕ. Their endopoints form two horizontal pairs $(B_A, \aleph(B_{A'}))$ and $(\aleph^{-1}(B_A), B_{A'})$ of vertices. Now, ρ_k and ρ_{k+1} extend together to a covering graph map $\rho : M_k \to M_k/\pi$, since the edges accompany the projections correspondingly, exemplified for $k = 2$ as follows:

$$\aleph((B_A))=\ \aleph((00011))=\ \aleph(\{00011,10001,11000,01100,00110\})=\{00111,01110,11100,11001,10011\}=(00111),$$
$$\aleph^{-1}((B'_A))=\aleph^{-1}((01011))=\aleph^{-1}(\{01011,10110,10110,11010,10101\})=\{00101,10010,01001,10100,01010\}=(00101).$$

Here, the order of the elements in the image of class (00011) (resp., (01011)) mod π under $\aleph$ (resp., $\aleph^{-1}$) are shown reversed, from right to left (cyclically between braces, continuing on the right once one reaches the leftmost brace). Such reversal holds for every $k > 2$:

$$\aleph((B_A))=\ \aleph((b_0 \cdots b_{2k}))=\ \aleph(\{b_0 \cdots b_{2k},\ b_{2k} \ldots b_{2k-1},\ \ldots,\ b_1 \cdots b_0\})=\{\bar{b}_{2k} \cdots \bar{b}_0,\ \bar{b}_{2k-1} \cdots \bar{b}_{2k},\ \ldots,\ \bar{b}_1 \cdots \bar{b}_0\}=(\bar{b}_{2k} \cdots \bar{b}_0),$$
$$\aleph^{-1}((B'_A))=\aleph^{-1}((\bar{b}'_{2k} \cdots \bar{b}'_0))=\aleph^{-1}(\{\bar{b}'_{2k} \cdots \bar{b}'_0,\ \bar{b}'_{2k-1} \cdots \bar{b}'_{2k},\ \ldots,\ \bar{b}'_1 \cdots \bar{b}'_0\})=\{b'_0 \cdots b'_{2k},\ b'_{2k} \cdots b'_{2k-1},\ \ldots,\ b'_1 \cdots b'_0\}=(b'_0 \cdots b'_{2k}),$$

where $(b_0 \cdots b_{2k}) \in L_k/\pi$ and $(b'_0 \cdots b'_{2k}) \in L_{k+1}/\pi$. This establishes item (i) of the statement.

Every horizontal edge $v\aleph_\pi(v)$ of M_k/π has $v \in L_k/\pi$ represented by $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ in L_k, (so $v = (\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k)$). There are 2^k such vertices in L_k and at most 2^k corresponding vertices in L_k/π. For example, $(0^{k+1}1^k)$ and $(0(01)^k)$ are endpoints in L_k/π of two horizontal edges of M_k/π, each. To prove that this implies item (ii), we have to see that there cannot be more than two representatives $\bar{b}_k \cdots \bar{b}_1 b_0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 c_0 c_1 \cdots c_k$ of a vertex $v \in L_k/\pi$, with $b_0 = 0 = c_0$. Such a v is expressible as $v = (d_0 \cdots b_0 d_{i+1} \cdots d_{j-1} c_0 \cdots d_{2k})$, with $b_0 = d_i$, $c_0 = d_j$ and $0 < j - i \leq k$. Let the substring $\sigma = d_{i+1} \cdots d_{j-1}$ be said $(j - i)$-*feasible*. Let us see that every $(j - i)$-feasible substring σ forces in L_k/π only vertices ω leading to two different (parallel) horizontal edges in M_k/π incident to v. In fact, periodic continuation mod n of $d_0 \cdots d_{2k}$ both to the right of $d_j = c_0$ with minimal cyclic substring $\bar{d}_{j-1} \cdots \bar{d}_{i+1} 1 d_{i+1} \cdots d_{j-1} 0 = P_r$ and to the left of $d_i = b_0$ with minimal cyclic substring $0 d_{i+1} \cdots d_{j-1} 1 \bar{d}_{j-1} \cdots \bar{d}_{i+1} = P_\phi$ yields a two-way infinite string that winds up onto a class $(d_0 \cdots d_{2k})$ containing such an ω. For example, some pairs of feasible substrings σ and resulting vertices ω are:

(σ,ω) = ($\emptyset$,(oo1)), (0,(o0o11)), (1,(o1o)), (0^2,(o00o111)), (01,(o01o011)), (1^2,o11o0)),
(0^3,o000o1111)), (010,(o010o101101)), (01^2,(o011o)), (101,(o101o)), (1^3,(o111o00)),

with 'o' replacing $b_0 = 0$ and $c_0 = 0$, and where $k = \lfloor \frac{n}{2} \rfloor$ has successive values $k = 1, 2, 1, 3, 3, 2, 4, 5,\ 2, 2, 3$. If σ is a feasible substring and $\bar{\sigma}$ is its complemented reversal via $\aleph$, then the possible symmetric substrings $P_\phi \sigma P_r$ about oσo $= 0\sigma0$ in a vertex v of L_k/π are in order of ascending length:

$$\begin{array}{c} 0\sigma0, \\ \bar{\sigma}0\sigma0\bar{\sigma}, \\ 1\bar{\sigma}0\sigma0\bar{\sigma}1, \\ \sigma1\bar{\sigma}0\sigma0\bar{\sigma}1\sigma, \\ 0\sigma1\bar{\sigma}0\sigma0\bar{\sigma}1\sigma0, \\ \bar{\sigma}0\sigma1\bar{\sigma}0\sigma0\bar{\sigma}1\sigma0\bar{\sigma}, \\ 1\bar{\sigma}0\sigma1\bar{\sigma}0\sigma0\bar{\sigma}1\sigma0\bar{\sigma}1, \\ \cdots\cdots\cdots\cdots\cdots, \end{array}$$

where we use again '0' instead of 'o' for the entries immediately preceding and following the shown central copy of σ. The lateral periods of P_r and P_ϕ determine each one horizontal edge at v in M_k/π up to returning to b_0 or c_0, so no entry $e_0 = 0$ of $(d_0 \cdots d_{2k})$ other than b_0 or c_0 happens such that $(d_0 \cdots d_{2k})$ has a third representative $\bar{e}_k \cdots \bar{e}_1 0 e_1 \cdots e_k$ (besides $\bar{b}_k \cdots \bar{b}_1 0 b_1 \cdots b_k$ and $\bar{c}_k \cdots \bar{c}_1 0 c_1 \cdots c_k$). Thus, those two horizontal edges are produced solely from the feasible substrings $d_{i+1} \cdots d_{j-1}$ characterized above. □

To illustrate Theorem 4, let $1 < h < n$ in $\mathbb{Z}$ be such that $\gcd(h, n) = 1$ and let $\lambda_h : L_k/\pi \to L_k/\pi$ be given by $\lambda_h((a_0 a_1 \cdots a_n)) \to (a_0 a_h a_{2h} \cdots a_{n-2h} a_{n-h})$. For each such $h \leq k$, there is at least one h-feasible substring σ and a resulting associated vertex $v \in L_k/\pi$ as in the proof of Theorem 4. For example, starting at $v = (0^{k+1}1^k) \in L_k/\pi$ and applying λ_h repeatedly produces a number of such vertices $v \in L_k/\pi$. If we assume $h = 2h'$ with $h' \in \mathbb{Z}$, then an h-feasible substring σ has the form $\sigma = \bar{a}_1 \cdots \bar{a}_{h'} a_{h'} \cdots a_1$, so there are at least $2^{h'} = 2^{\frac{h}{2}}$ such h-feasible substrings.

5 The Dihedral Quotients

An *involution* of a graph G is a graph map $\aleph : G \to G$ such that $\aleph^2$ is the identity. If G has an involution, an $\aleph$-*folding* of G is a graph H, possibly with loops, whose vertices v' and edges or loops e' are respectively of the form $v' = \{v, \aleph(v)\}$ and $e' = \{e, \aleph(e)\}$, where $v \in V(G)$ and $e \in E(G)$; e has endvertices v and $\aleph(v)$ if and only if $\{e, \aleph(e)\}$ is a loop of G.

Note that both maps $\aleph : M_k \to M_k$ and $\aleph_\pi : M_k/\pi \to M_k/\pi$ in Section 4 are involutions. Let us denote each horizontal pair $\{(B_A), \aleph_\pi((B_A))\}$ of M_k/π by $\langle B_A \rangle$, where $|A| = k$. An $\aleph$-folding R_k of M_k/π is obtained whose vertices are the pairs $\langle B_A \rangle$ and having:

(1) an edge $\langle B_A \rangle \langle B_{A'} \rangle$ per skew-edge pair $\{(B_A)\aleph_\pi((B_{A'})), (B_{A'})\aleph_\pi((B_A))\}$;

(2) a loop at $\langle B_A \rangle$ per horizontal edge $(B_A)\aleph_\pi((B_A))$; because of Theorem 4, there may be up to two loops at each vertex of R_k.

Theorem 5. *R_k is a quotient graph of M_k under an action $\Upsilon : D_{2n} \times M_k \to M_k$.*

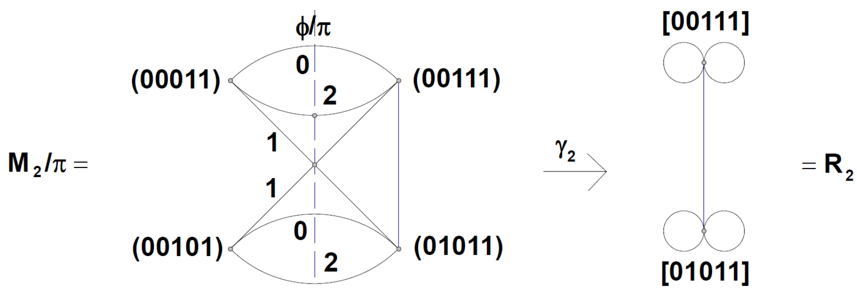

Figure 1: Reflection symmetry of M_2/π about a line ϕ/π and resulting graph map γ_2

Proof. D_{2n} is the semidirect product $\mathbb{Z}_n \rtimes_\varrho \mathbb{Z}_2$ via the group homomorphism $\varrho : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_n)$, where $\varrho(0)$ is the identity and $\varrho(1)$ is the automorphism $i \to (n-i)$, $\forall i \in \mathbb{Z}_n$. If $* : D_{2n} \times D_{2n} \to D_{2n}$ indicates group multiplication and $i_1, i_2 \in \mathbb{Z}_n$, then $(i_1, 0) * (i_2, j) = (i_1 + i_2, j)$ and $(i_1, 1) * (i_2, j) = (i_1 - i_2, \bar{j})$, for $j \in \mathbb{Z}_2$. Set $\Upsilon((i,j), v) = \Upsilon'(i, \aleph^j(v))$, $\forall i \in \mathbb{Z}_n, \forall j \in \mathbb{Z}_2$, where Υ' is as in display (3). Then, Υ is a well-defined D_{2n}-action on M_k. By writing

$(i,j)\cdot v = \Upsilon((i,j),v)$ and $v = a_0\cdots a_{2k}$, we have $(i,0)\cdot v = a_{n-i+1}\cdots a_{2k}a_0\cdots a_{n-i} = v'$ and $(0,1)\cdot v' = \bar{a}_{i-1}\cdots\bar{a}_0\bar{a}_{2k}\cdots\bar{a}_i = (n-i,1)\cdot v = ((0,1)*(i,0))\cdot v$, leading to the compatibility condition $((i,j)*(i',j'))\cdot v = (i,j)\cdot((i',j')\cdot v)$. □

Theorem 5 yields a graph projection $\gamma_k : M_k/\pi \to R_k$ for the action Υ, given for $k=2$ in Figure 1. In fact, γ_2 is associated with reflection of M_2/π about the dashed vertical symmetry axis ϕ/π so that R_2 (containing two vertices and one edge between them, with each vertex incident to two loops) is given as its image. Both the representations of M_2/π and R_2 in the figure have their edges indicated with colors 0,1,2, as arising inSection 6.

6 Lexical Procedure

Let P_{k+1} be the subgraph of the unit-distance graph of $\mathbb{R}$ (the real line) induced by the set $[k+1] = \{0,\ldots,k\}$. We draw the grid $\Gamma = P_{k+1}\Box P_{k+1}$ in the plane $\mathbb{R}^2$ with a diagonal Δ traced from the lower-left vertex $(0,0)$ to the upper-right vertex (k,k). For each $v \in L_k/\pi$, there are $k+1$ n-tuples of the form $b_0b_1\cdots b_{n-1} = 0b_1\cdots b_{n-1}$ that represent v with $b_0 = 0$. For each such n-tuple, we construct a $2k$-path D in Γ from $(0,0)$ to (k,k) in $2k$ steps indexed from $i=0$ to $i=2k-1$. This leads to a lexical edge-coloring implicit in [4]; see the next statement and Figure 2, containing examples of such a $2k$-path D in thick trace.

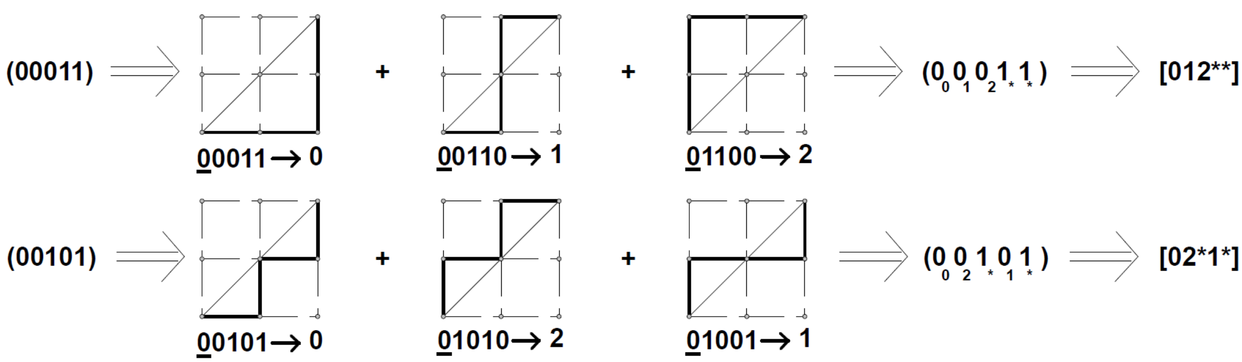

Figure 2: Representing lexical color assignment for $k=2$

Theorem 6. *[4] Each $v \in L_k/\pi$ has its $k+1$ incident edges assigned colors $0,1,\ldots,k$ by means of the following "Lexical Procedure", where $0 \le i \in \mathbb{Z}$, $w \in V(\Gamma)$ and D is a path in Γ. Initially, let $i=0$, $w=(0,0)$ and D contain solely the vertex w. Repeat $2k$ times the following sequence of steps (1)-(3), and then perform once the final steps (4)-(5):*
(1) *If $b_i = 0$, then set $w' := w+(1,0)$; otherwise, set $w' := w+(0,1)$.*
(2) *Reset $V(D) := v(D)\cup\{w'\}$, $E(D) := E(D)\cup\{ww'\}$, $i := i+1$ and $w := w'$.*
(3) *If $w \ne (k,k)$, or equivalently, if $i < 2k$, then go back to step (1).*
(4) *Set $\check{v} \in L_{k+1}/\pi$ to be the vertex of M_k/π adjacent to v and obtained from its representative n-tuple $b_0b_1\cdots\ b_{n-1} = 0b_1\cdots b_{n-1}$ by replacing the entry b_0 by $\bar{b}_0 = 1$ in $\check{v}$, keeping the entries b_i of v unchanged in $\check{v}$ for $i>0$.*
(5) *Set the color of the edge $v\check{v}$ to be the number c of horizontal (alternatively, vertical) arcs of D above Δ.*

Proof. If addition and subtraction in $[n]$ are taken modulo n and we write $[y,x)=\{y,y+1,y+2,\dots,x-1\}$, for $x,y\in[n]$, and $S^c=[n]\setminus S$, for $S=\{i\in[n]:b_i=1\}\subseteq[n]$, then the cardinalities of the sets $\{y\in S^c\setminus x:|[y,x)\cap S|<|[y,x)\cap S^c|\}$ yield all the edge colors, where $x\in S^c$ varies. □

The Lexical Procedure of Theorem 6, referred from now on as the *LP*, yields a 1-factorization not only for M_k/π but also for R_k and M_k. This is clarified by the end of Section 7.

7 Un-castling

A notation $\delta(v)$ is assigned to each pair $\{v,\aleph_\pi(v)\}\in R_k$, where $v\in L_k/\pi$, so that there is a unique k-germ $\alpha=\alpha(v)$ with $\langle F(\alpha)\rangle=\delta(v)$, (with notation $\langle\cdot\rangle$, as in $\langle B_A\rangle$ in Section 5). We exemplify this for $k=2$ in Figure 2 of Section 6, with the LP (indicated by arrows "$\Rightarrow$") departing from $v=(00011)$ (top) and $v=(00101)$ (bottom), passing to rightward sketches of Γ (separated by symbols "+"), one sketch (in which to trace the edges of $D\subset\Gamma$) per representative $b_0b_1\cdots b_{n-1}=0b_1\cdots b_{n-1}$ of v shown under the sketch (with $b_0=0$ underscored) and pointing via an arrow "$\to$" to the corresponding color $c\in[k+1]$, which is the number of horizontal arcs of D below Δ (as in step (5) of the LP).

In each of the two cases in Figure 2 (top, bottom), an arrow "$\Rightarrow$" to the right of the sketch triple points to a modification $\hat{v}$ of $b_0b_1\cdots b_{n-1}=0b_1\cdots b_{n-1}$ obtained by setting as a subindex of each 0 (resp., 1) its associated c (resp., an asterisk "$*$"). Further to the right, a third arrow "$\Rightarrow$" points to the n-tuple $\delta(v)$ formed by the string of subindexes of entries of $\hat{v}$ in the order they appear from left to right.

Theorem 7. *Let $\alpha(v^0)=a_{k-1}\cdots\alpha_1=00\cdots0$. To each $\delta(v)$ corresponds a sole k-germ $\alpha=\alpha(v)$ with $\langle F(\alpha)\rangle=\delta(v)$ by means of the following "Un-Castling Procedure": Given $v\in L_k/\pi$, let $W^i=01\cdots i$ be the maximal initial numeric (i.e., colored) substring in $\delta(v)$, where the length of W^i is $i+1$ $(0\le i\le k)$. If $i=k$, let $\alpha(v)=\alpha(v^0)$; else, set $m=0$ and:*

1. *set $\delta(v^m)=\langle W^i|X|Y|Z^i\rangle$, where Z^i is the terminal j_m-substring of $\delta(v^m)$, with $j_m=i+1$, and let X,Y (in that order) start at contiguous numbers Ω and $\Omega-1\ge i$;*
2. *set $\delta(v^{m+1})=\langle W^i|Y|X|Z^i\rangle$;*
3. *obtain $\alpha(v^{m+1})$ from $\alpha(v^m)$ by increasing its entry a_{j_m} by 1;*
4. *if $\delta(v^{m+1})=[01\cdots k*\cdots*]$, then stop; else, increase m by 1 and go to step 1.*

Proof. This is a procedure inverse to that of castling (Section 2), and items 1-4 follow. □

TABLE III

$j_0=0$	$\delta(v^1)$	$=$	$\langle 0\vert 3{\ast}2{\ast}1\vert 4{\ast}\vert{\ast}\rangle$	$=$	$\langle 03{\ast}2{\ast}14{\ast}{\ast}\rangle$	$=$	$\langle 0\vert 3{\ast}\vert 2{\ast}14{\ast}\vert{\ast}\rangle$	$\alpha(v^1)=001$	$\langle F(001)\rangle=\delta(v^1)$
$j_1=0$	$\delta(v^2)$	$=$	$\langle 0\vert 2{\ast}14{\ast}\vert 3{\ast}\vert{\ast}\rangle$	$=$	$\langle 02{\ast}14{\ast}3{\ast}{\ast}\rangle$	$=$	$\langle 0\vert 2{\ast}\vert 14{\ast}3{\ast}\vert{\ast}\rangle$	$\alpha(v^2)=011$	$\langle F(011)\rangle=\delta(v^2)$
$j_2=0$	$\delta(v^3)$	$=$	$\langle 0\vert 14{\ast}3{\ast}\vert 2{\ast}\vert{\ast}\rangle$	$=$	$\langle 014{\ast}3{\ast}2{\ast}{\ast}\rangle$	$=$	$\langle 01\vert 4{\ast}\vert 3{\ast}2\vert{\ast}{\ast}\rangle$	$\alpha(v^3)=012$	$\langle F(012)\rangle=\delta(v^3)$
$j_3=1$	$\delta(v^4)$	$=$	$\langle 01\vert 3{\ast}2\vert 4{\ast}\vert{\ast}{\ast}\rangle$	$=$	$\langle 013{\ast}24{\ast}{\ast}{\ast}\rangle$	$=$	$\langle 01\vert 3{\ast}\vert 24{\ast}\vert{\ast}{\ast}\rangle$	$\alpha(v^4)=112$	$\langle F(112)\rangle=\delta(v^4)$
$j_4=1$	$\delta(v^5)$	$=$	$\langle 01\vert 24{\ast}\vert 3{\ast}\vert{\ast}{\ast}\rangle$	$=$	$\langle 0124{\ast}3{\ast}{\ast}{\ast}\rangle$	$=$	$\langle 012\mid 4{\ast}\vert 3\vert{\ast}{\ast}\rangle$	$\alpha(v^5)=122$	$\langle F(122)\rangle=\delta(v^5)$
$j_5=2$	$\delta(v^6)$	$=$	$\langle 012\vert 3\vert 4{\ast}\vert{\ast}{\ast}{\ast}\rangle$	$=$	$\langle 01234{\ast}{\ast}{\ast}{\ast}\rangle$			$\alpha(v^6)=123$	$\langle F(123)\rangle=\delta(v^6)$

This un-castling leads to a finite sequence $\delta(v^0),\delta(v^1),\dots,\delta(v^s)$ of n-strings in L_k/π with parameters $j_0\ge j_1\ge\cdots\ge j_s$, and k-germs $\alpha(v^0),\alpha(v^1),\dots,\alpha(v^s)$. It also leads from $\alpha(v^0)$ to $\alpha(v)=\alpha(v^s)$ via unit incrementation of a_{j_i}, for $i=0,\dots,s$, with each incrementation yielding a corresponding $\alpha(v^i)$. Recall F is a bijection from the set $V(\mathcal{T}_k)$ of k-germs onto the set L_k/π, both sets being of cardinality C_k. Thus, to deal with $V(R_k)$ it is enough to deal with $V(\mathcal{T}_k)$, a fact useful in interpreting Theorem 8 below. For example $\delta(v) = \delta(v^0) = \langle 04*3*2*1*\rangle = \langle 0|4*|3*2*1|*\rangle = \langle W^0|X|Y|Z^0\rangle$ with $i = 0$ and $\alpha(v^0) = 000$, continued in Table III with $\delta(v^1) = \langle W^0|Y|X|Z^0\rangle$, finally arriving to $\alpha(v) = \alpha(v^s) = \alpha(v^6) = 123$.

A pair of skew edges $(B_A)\aleph_\pi((B_{A'}))$ and $(B_{A'})\aleph((B_A))$ in M_k/π, to be called a *skew reflection edge pair* (*SREP*), provides a color notation for any $v \in L_{k+1}/\pi$ such that in each particular edge class mod π:

(I) all edges receive a common color in $[k+1]$ regardless of the endpoint on which the LP (or its modification immediately below) for $v \in L_{k+1}/\pi$ is applied;

(II) the two edges in each SREP in M_k/π are assigned a common color in $[k+1]$.

The modification in step (I) consists in replacing in Figure 2 each v by $\aleph_\pi(v)$ so that on the left we have now instead (00111) (top) and (01011) (bottom) with respective sketch subtitles

$$\begin{array}{lll} 0011\underline{1}\to 0, & 1001\underline{1}\to 1, & 1100\underline{1}\to 2,\\ 0101\underline{1}\to 0, & 1010\underline{1}\to 2, & 0110\underline{1}\to 1, \end{array}$$

resulting in similar sketches when the steps (1)-(5) of the LP are taken with right-to-left reading and processing of the entries on the left side of the subtitles (before the arrows "$\to$"), where now the values of each b_i must be taken complemented.

Since an SREP in M_k determines a unique edge ϵ of R_k (and vice versa), the color received by the SREP can be attributed to ϵ, too. Clearly, each vertex of either M_k or M_k/π or R_k defines a bijection from its incident edges onto the color set $[k+1]$. The edges obtained via $\aleph$ or $\aleph_\pi$ from these edges have the same corresponding colors.

Theorem 8. *A 1-factorization of M_k/π by the colors $0, 1, \ldots, k$ is obtained via the LP that can be lifted to a covering 1-factorization of M_k and subsequently collapsed onto a folding s-factorization of R_k. This insures the notation $\delta(v)$ for each $v \in V(R_k)$ so that there is a unique k-germ $\alpha = \alpha(v)$ with $\langle F(\alpha)\rangle = \delta(v)$.*

Proof. As pointed out in item (II) above, each SREP in M_k/π has its edges with a common color of $[k+1]$. Thus, the $[k+1]$-coloring of M_k/π induces a well-defined $[k+1]$-coloring of R_k. This yields the claimed collapsing to a folding 1-factorization of R_k. The lifting to a covering 1-factorization in M_k is immediate. The arguments above determine that the collapsing 1-factorization in R_k induces the claimed k-germs $\alpha(v)$. □

8 Germ Structure of the Lexical 1-Factorizations

From now on, each $v \in V(R_k)$ (omitting now its delimiting parentheses) is presented via $\delta(v)$ and also via the k-germ α for which $\delta(v) = \langle F(\alpha)\rangle$. Further, we view R_k as the graph whose vertices are the k-germs α, with adjacency inherited from that of their δ-notation via F^{-1} (i.e, un-castling). Moreover, $V(R_k)$ is presented as in the natural (germ) enumeration.

Examples of such presentation are shown in Table IV for $k = 2$ and 3, where m, $\alpha = \alpha(m)$ and $F(\alpha)$ are shown in the first three columns, for $0 \le m < C_k$. The neighbors of $F(\alpha)$ are presented in the central columns of the table as $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ respectively for the edge colors $k, k-1, \ldots, 0$, with notation given via the effect of function $\aleph$. The last four columns yield the k-germs α^k, α^{k-1}, …, α^0 associated via F^{-1} respectively to the listed neighbors $F^k(\alpha)$, $F^{k-1}(\alpha)$, …, $F^0(\alpha)$ of $F(\alpha)$ in R_k.

TABLE IV

m	α	$F(\alpha)$	$F^3(\alpha)$	$F^2(\alpha)$	$F^1(\alpha)$	$F^0(\alpha)$	α^3	α^2	α^1	α^0
0	0	$012**$	–	$012**$	$02*1*$	$12**0$	–	0	1	0
1	1	$02*1*$	–	$1*02*$	$012**$	$2*1*0$	–	1	0	1
0	00	$0123***$	$0123***$	$013*2**$	$023**1*$	$123***0$	00	10	01	00
1	01	$023**1*$	$1*023**$	$1*03*2*$	$0123***$	$2*13**0$	01	12	00	11
2	10	$013*2**$	$02*20**$	$0123***$	$03*2*1*$	$13*2**0$	11	00	12	10
3	11	$02*13**$	$013*2**$	$13**02*$	$02*13**$	$10**2*3$	10	11	11	01
4	12	$03*2*1*$	$2*1*03*$	$1*023**$	$013*2**$	$3*2*1*0$	12	01	10	12

For $k = 4, 5$, Tables V,VI, respectively, have a similar natural enumeration adjacency disposition. We can generalize these tables directly to *Colored Adjacency Tables* denoted *CAT*(k), for $k > 1$. This way, item (A) in Theorem 9 below is obtained as indicated in the aggregated row upending each of the Tables V,VI that cites the only non-asterisk entry, for each of $i = k, k-2, \ldots, 0$, as a number $j = (k-1), \ldots, 1$ that leads to entry equality in both columns $\alpha = a_{k-1} \cdots a_j \cdots a_1$ and $\alpha^i = a^i_{k-1} \cdots a^i_j \cdots a^i_1$, that is $a_j = a^i_j$. Other important properties are contained in Theorem 9, including item (B), that the columns α^0 in all CAT(k), ($k > 1$), yield an (infinte) integer sequence. Additional complementary properties are contained in Theorem 16 and Remark 17.

TABLE V

m	α	α^4	α^3	α^2	α^1	α^0	m	α	α^4	α^3	α^2	α^1	α^0
0	000	000	100	010	001	000	7	110	100	111	110	012	010
1	001	001	101	012	000	011	8	111	111	110	122	011	111
2	010	011	121	000	112	110	9	112	101	122	112	010	112
3	011	010	120	011	111	001	10	120	122	011	100	123	120
4	012	012	123	001	110	122	11	121	121	010	121	122	101
5	100	110	000	120	101	100	12	122	120	112	111	121	012
6	101	112	001	123	100	121	13	123	123	012	101	120	123
–	––	3∗∗	∗∗∗	3∗∗	∗2∗	∗∗1	–	––	3∗∗	∗∗∗	3∗∗	∗2∗	∗∗1

Theorem 9. *Let: $k > 1$, $j(\alpha^k) = k - 1$ and $j(\alpha^{i-1}) = i$, ($i = k-1, \ldots, 1$). Then:* **(A)** *each column α^{i-1} in* CAT(k), *for $i \in [k] \cup \{k+1\}$, preserves the respective $j(\alpha^{i-1})$-th entry of α;* **(B)** *the entries in column α^k (e.g. the leftmost column α^i in Tables IV-VI, for $0 \leq i \leq k > 1$) of all* CAT(k)*'s conform an RGS sequence; this determines a corresponding integer sequence $\mathcal{S}_0$, idempotent on its first C_k terms for each k;* **(C)** *the integer sequence $\mathcal{S}_1$ given by concatenating the m-indexed intervals $[0, 2), [2, 5), \ldots, [C_{k-1}, C_k)$, etc. in column α^{k-1} of the corresponding tables* CAT(2), CAT(3), … CAT(k), *etc. allows to encode all columns α^{k-1}'s;* **(D)** *for each $k > 1$, there i okay set eBay saless an idempotent permutation given in the m-indexed interval $[0, C_k)$ of the column α^{k-1} of* CAT(k)*; such permutation equals another one given in the interval $[0, C_k)$ in the column α^{k-2} of* CAT($k+1$).

Proof. Item (A) holds by Remarks 10-11, below. Let α be a k-germ. Then α shares with α^k all entries to the left of the leftmost entry 1, showing (B) via Remark 10. For (C), if $k = 3$: $m = 2, 3, 4$ yield for α^{k-1} the idempotent $(2, 0)(4, 1)$; etc. Item (D) is observable, too. □

TABLE VI

m	α	α^5	α^4	α^3	α^2	α^1	α^0	m	α	α^5	α^4	α^3	α^2	α^1	α^0
–	– –	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –	– –
0	0000	0000	1000	0100	0010	0001	0000	21	1110	1111	1100	1221	0110	1112	1110
1	0001	0001	1001	0101	0012	0000	0011	22	1111	1110	1111	1220	0122	1111	0111
2	0010	0011	1011	0121	0000	0112	0110	23	1112	1122	1101	1233	0112	1110	1222
3	0011	0010	1010	0120	0011	0111	0001	24	1120	1011	1222	1121	0100	1123	1120
4	0012	0012	1012	0123	0001	0110	0122	25	1121	1010	1221	1120	0121	1122	0101
5	0100	0110	1210	0000	1120	1101	1100	26	1122	1112	1220	1223	0111	1121	1122
6	0101	0112	1212	0001	1123	1100	1121	27	1123	1012	1233	1123	0101	1120	1223
7	0110	0100	1200	0111	1110	0012	0010	28	1200	1220	0110	1000	1230	1201	1200
8	0111	0111	1211	0110	1122	0011	1111	29	1201	1223	0112	1001	1234	1200	1231
9	0112	0101	1201	0122	1112	0010	0112	30	1210	1210	0100	1211	1220	1012	1011
10	0120	0122	1232	0011	1100	1223	1220	31	1211	1222	0111	1210	1233	1011	1221
11	0121	0121	1231	0010	1121	1222	1101	32	1212	1212	0101	1232	1223	1010	1212
12	0122	0120	1230	0112	1111	1221	0012	33	1220	1200	1122	1111	1210	0123	0120
13	0123	0123	1234	0012	1101	1220	1233	34	1221	1221	1121	1110	1232	0122	1211
14	1000	1100	0000	1200	1010	1001	1000	35	1222	1211	1120	1222	1222	0121	1112
15	1001	1101	0001	1201	1012	1000	1011	36	1223	1201	1223	1122	1212	0120	1123
16	1010	1121	0011	1231	1000	1212	1210	37	1230	1233	0122	1011	1200	1234	1230
17	1011	1120	0010	1230	1011	1211	1001	38	1231	1232	0121	1010	1231	1233	1201
18	1012	1123	0012	1234	1001	1210	1232	39	1232	1231	0120	1212	1221	1232	1012
19	1100	1000	1110	1100	0120	0101	0100	40	1233	1230	1123	1112	1211	1231	0123
20	1101	1001	1112	1101	0123	0100	0121	41	1234	1234	0123	1012	1201	1230	1234
–	– –	– –	– –	– –	– –	– –	– –	–	– –	– –	– –	– –	– –	– –	– –
		4***	****	4***	*3**	**2*	***1			4***	****	4***	*3**	**2*	***1

The sequences in (B)-(C) of Theorem 9 start as follows, with intervals ended in ";":

$\{0\}\cup\mathbb{Z}^+=$	0,	1;	2,	3,	4;	5,	6,	7,	8,	9,	10,	11,	12,	13;	14	15,	16,...
$(B)=$	0,	1;	3,	2,	4;	7,	9,	5,	8,	6,	12,	11,	10,	13;	19,	20,	25,...
$(C)=$	1,	0;	0,	3,	1;	0,	1,	8,	7,	12,	3,	2,	9,	4;	0,	1,	3,...

For the neighbors $\alpha^k, \alpha^{k-1}, \ldots, \alpha^0$ of a k-germ $\alpha = a_{k-1} \cdots a_1$, we need two substrings of α:

- **(a)** the *straight ascent* $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ of α is its maximal ascending substring;
- **(b)** the *landing ascent* $\alpha'_1 = a_{k-1} \cdots a_{k-i_1}$ of α is its maximal non-descending substring having at most two equal nonzero terms.

Note that $0 < i_1 < k$, in both (a) and (b). The following remarks yield steps to view directly the k-germs $\alpha^p = \beta = b_{k-1} \cdots b_1$ adjacent to α via colors $p = k, k-1, \ldots, 0$, independently of F^{-1} and F. Given a substring $\alpha' = a_{k-j} \cdots a_{k-i}$ of α, where $0 < j \leq i < k$, let: **(i)** $\psi(\alpha') = a_{k-i} \cdots a_{k-j}$ be the reverse string of α'; **(ii)** the *proper ascent* of α' be the straight ascent of α', if $a_{k-j} = 0$, and the landing ascent of α', if $a_{k-j} > 0$.

Remark 10. Consider case $p = k$. If $a_{k-1} = 1$, we take $0|\alpha$ in place of $\alpha = a_{k-1} \cdots a_1$, with $k-1$ in place of k, removing afterwards the added leftmost 0, etc, from the resulting β. So, let $\alpha_1 = a_{k-1} \cdots a_{k-i_1}$ be the proper ascent of α. Let $B_1 = i_1 - 1$, where $i_1 = ||\alpha_1||$ is the length of α_1. It can be seen that β has proper ascent $\beta_1 = b_{k-1} \cdots b_{k-i_1}$ with $\alpha_1 + \psi(\beta_1) = B_1 \cdots B_1$. If $\alpha \neq \alpha_1$, let α_2 be the proper ascent of $\alpha \setminus \alpha_1$. Then there is a $||\alpha_2||$-germ β_2 with $\alpha_2 + \psi(\beta_2) = B_2 \cdots B_2$ and $B_2 = ||\alpha_1|| + ||\alpha_2|| - 2$. Inductively when feasible for $j > 2$, let α_j be the proper ascent of $\alpha \setminus (\alpha_1|\alpha_2|\cdots|\alpha_{j-1})$. Then there is a $||\alpha_j||$-germ β_j with $\alpha_j + \psi(\beta_j) = B_j \cdots B_j$ and $B_j = ||\alpha_{j-1}|| + ||\alpha_j|| - 2$. This way, $\beta = \beta_1|\beta_2|\cdots|\beta_j|\cdots$.

Remark 11. Now consider case $k > p > 0$. If $p < k-1$, Theorem 9 implies $b_{p+1} = a_{p+1}$; in this case, let $\alpha' = \alpha \setminus \{a_{k-1} \cdots a_q\}$ with $q = p+1$. If $p = k-1$, let $q = k$ and let $\alpha' = \alpha$. In both cases (either $p < k-1$ or $p = k-1$) let $\alpha'_1 = a_{q-1} \cdots a_{k-i_1}$ be the proper ascent of α'. It can be seen that $\beta' = \beta \setminus \{b_{k-1} \cdots b_q\}$ has proper ascent $\beta'_1 = b_{k-1} \cdots b_{k-i_1}$ where $\alpha'_1 + \psi(\beta'_1) = B'_1 \cdots B'_1$ with $B'_1 = i_1 + a_q$. If $\alpha' \neq \alpha'_1$ then let α'_2 be the proper ascent of $\alpha' \setminus \alpha'_1$. Then there is a $||\alpha'_2||$-germ β'_2 where $\alpha'_2 + \psi(\beta'_2) = B'_2 \cdots B'_2$ with $B'_2 = ||\alpha'_1|| + ||\alpha'_2|| - 2$.

Inductively when feasible for $j > 2$, let α_j be the proper ascent of $\alpha' \setminus (\alpha'_1|\alpha'_2|\cdots|\alpha'_{j-1})$. Then there is a $||\alpha'_j||$-germ β'_j where $\alpha'_j + \psi(\beta'_j) = B'_j \cdots B'_j$ with $B'_j = ||\alpha'_{j-1}|| + ||\alpha'_j|| - 2$. This way, $\beta' = \beta'_1|\beta'_2|\cdots|\beta'_j|\cdots$.

We process the left-hand side from position q. If $p > 1$, we set $a_{a_q+2}\cdots a_q + \psi(b_{b_q+2}\cdots b_q)$ to be equal to a constant string $B\cdots B$ with $a_{a_q+2}\cdots a_q$ being a proper ascent and $a_{a_q+2} = b_{b_q+2}$. Expressing all numbers a_i, b_i above as a_i^0, b_i^0, respectively, in order to keep an inductive approach, let $a_q^1 = a_{a_q+2}$. While feasible, let $a_{q+1}^1 = a_{a_q+1}$, $a_{q+2}^1 = a_{a_q}$, etc. In this case, let $b_q^1 = b_{b_q+2}$, $b_{q+1}^1 = b_{b_q+1}$, $b_{q+2}^1 = b_{b_q}$, etc. Now, $a^1_{a^1_q+2}\cdots a^1_q + \psi(b^1_{b^1_q+2}\cdots b^1_q)$ is equal to a constant string with $a^1_{a^1_q+2}\cdots a^1_q$ being a proper ascent and $a^1_{a^1_q+2} = b^1_{b^1_q+2}$. The continuation of this procedure produces a subsequent string $a_q^2\cdots$, etc., until what remains to reach the leftmost entry of α is smaller than the needed space for the procedure itself to continue, in which case, a remaining initial proper ascent is shared by both α and β. This allows to form the left-hand side of $\alpha^p = \beta$ by concatenation.

9 Non-Germ Way to the Dihedral Quotients

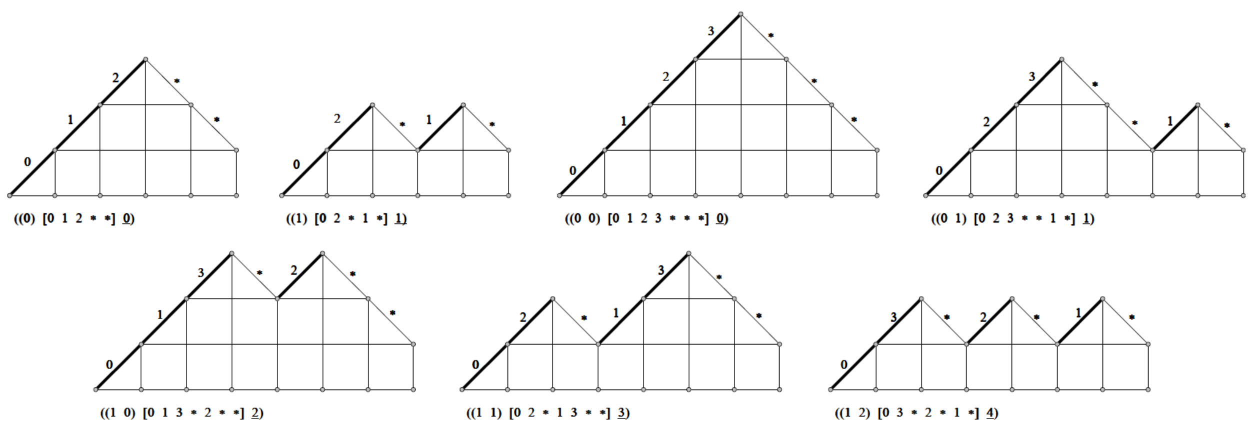

Figure 3: Alternative way to R_2 and R_3

An alternative way to get $\delta(v)$ (but not via its corresponding $\alpha = \alpha(v)$, which was the way we arrived to it) is to take the representation corresponding to the lexical color 0 in $\Gamma \subset \mathbb{R}^2$, rotating it around $(0,0)$ so that Δ is under the x-axis, then reflecting it on the x-axis and enlarging the grid with a scale factor of $\sqrt{2}$ (as in Figure 3 for the cases $k = 2, 3$). Then the lexical colors, labeling the diagonal edges of the form $(x,y)(x+1,y+1)$, are positioned in decreasing order from k to 0, from the top of the figure to its bottom and at each horizontal level from left to right. On the other hand, the diagonal edges of the form $(x,y)(x-1,y-1)$ carry an asterisk each. Each instance in Figure 3 is indicated by the corresponding k-germ α followed by its $F(\alpha)$ and then by its (underlined) order of presentation via the Castling Procedure. By recurring to all such piecewise linear (pl) representations for a fixed k, we have a way to obtain all elements of R_k alternative to that given via Section 2, but without recurring to α. This motivates the following definitions.

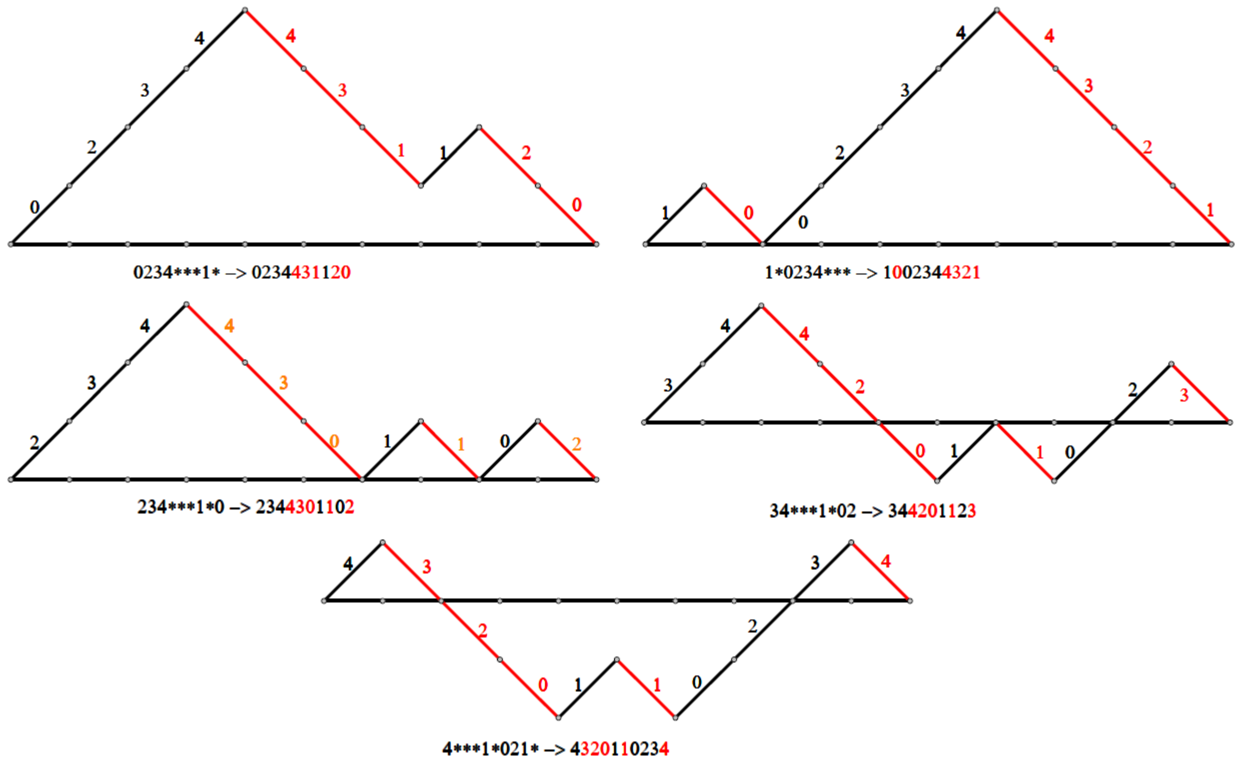

Figure 4: Examples of lexical-color adjacencies in R_4

To each $\mathbb{Z}_2$-edge-valued oriented n-path $P = v_0, v_1, \ldots, v_n$ we associate its *piecewise-linear* (pl) representation, i.e., the plane path obtained as the union of the successive segments $v_iv_{i+1} = (x_i, y_i), (x_i + d, y_i + d)$, where $d = 1$ (resp. $d = -1$) if the $\mathbb{Z}_2$-value of edge v_iv_{i+1} is 0 (resp. 1), starting at the origin $v_0 = (0, 0)$.

Theorem 12. *In each $\mathbb{Z}_2$-edge-valued oriented n-cycle graph C of weight k there is exactly one vertex v whose splitting into two vertices v' and v'' and removal of the resulting arc $v'v'$ transforms C into a $\mathbb{Z}_2$-edge-valued oriented n-path P_C of weight k and endvertices v', v'' whose associated pl representation touches the horizontal edge only at the origin, so that the lexical colors are assigned correctly to the 0-valued vertices, as in the Castling Procedure.*

Proof. Clearly, C can be interpreted as a vertex of R_k. By the discussion above and the method suggested via Figure 3, the vertex v must be selected as the starting vertex of the edge of C that is assigned the lexical color k. This leads to the statement. □

The color k adjacency can now be visualized by adding a rightmost edge extending each associated pl representation, as shown in the upper left case in Figure 4 for the vertex $F(\alpha) = F(001) = 0234***1*$ of R_4, where red color is used to indicate the $*$-edges, and using the same criterion used about Figure 3 to set "red" colors to $*$-edges from 0 to k always from top to bottom but from right to left (instead that from left to right, above). This yields a red-black path in the upper half of the cartesian plane with exactly two points touching the x-axis, namely the initial and terminal ones. Notice that this adjacency yields a notation for all the k colored edges or loops of R_k.

This procedural method yields a notation for the other edges or loops of all R_k; here, the corresponding pl representations may not only touch the x-axis more than twice but traverse to the lower half of the plane; the only extra provisions are that the first (black) edge is $(0,0)(1,1)$, the last (red) edge is $(n-1,1)(n,0)$ and that both these edges a fixed color in $[0,k)$. Figure 4 illustrates all these adjacencies and the resulting notation for the edges departing from the selected vertex of R_k.

10 Union of 1-Factors 0 and 1

Given a k-germ α, let (α) represent the dihedral class $\delta(v) = \langle F(\alpha)\rangle$ with $v \in L_k/\pi$. Let W_{01} be the 2-factor given by the union of the 1-factors of colors 0 and 1 in M_k, namely those formed by lifting the edges $\alpha\alpha_0$ and $\alpha\alpha_1$ of R_k (instead of colors k and $k-1$, as in [3], but symmetrically equivalent). By an argument similar to that of Proposition 2 [3], the cycles of W_{01} can be obtained as in Figure 5 for $k = 2,3,4$ (denoted as cycle C_0 that starts with path $X(0)$ for $k = 2$; cycles C_0, C_1 that start with paths $X(0), X(1)$ for $k = 3$; and cycles C_0, C_1, C_2 that start with paths $X(0), X(1), X(2)$ for $k = 4$), where vertices of L_k (resp. L_{k+1}) have: **(a)** 0- (resp. 1-) entries replaced by their respective lexical colors $0,\dots,k$ (resp. asterisks); **(b)** 1- (resp. 0-) entries replaced by asterisks (resp. their respective lexical colors $0,\dots,k$); **(c)** delimiting chevron symbols ">" or "$\rangle$" (resp. "<" or "$\langle$"), instead of parentheses or brackets, meaning rightward or *forward* (resp. leftward or *reversed*) *reading*. Each such vertex v is shown to belong (via the set membership symbol expressed as ϵ in Figure 5) to (α_v), where α_v is the k-germ associated to v; such α_v is also expressed as the (underlined) decimal order of α_v. In each case, Figure 5 shows a vertically presented path of length $4k-1$ in the corresponding cycle C_i starting at the vertex $w = b_0b_1\cdots b_{2k} = 01\cdots *$ of smallest decimal order and proceeding by traversing the edges colored 1 and 0, alternatively. The terminal vertex of such subpath is $b_{2k}b_0b_1\cdots b_{2k-1} = *01\cdots b_{2k-1}$, obtained by translation mod n of w. Note that the initial entries of the succeeding vertices in C_i are presented first in the 0-column of $X(i)$, then in the $2k$-column of $X(i)$, $(2k-1)$-column of $X(i)$, etc., if necessary, as commented at the end of this section in relation to the symmetry properties of the plane trees of the next paragraph. Each cycle C_i in Figure 5 is encoded on its top right by a vertically presented sequence of expressions $(0\cdots 1)(1\cdots 0)$ that allow to obtain the actual sequence of initial entries of the succeeding vertices by interspersing asterisks between each two terms inside parentheses () and then removing those ().

In addition, an ORT $T = T_v$ (as in Remark 3 at the end of Section 2) for each v in question in each C as in Proposition 2(v) [3] is shown to the lower right of each exemplified $C = C_i$. Each of these T_v for a specific C_i corresponds to the k-germ α_v (so we write $F(\alpha_v) = F(T_{\alpha_v})$) and is headed in the figure by its (underlined) decimal order.

The trees corresponding to the k-germs in each case are obtained by applying a root rotation operation as in [3]. Such *root rotation operation* (*RRO*) consists in replacing the tree root by its leftmost child and redrawing the rooted planar tree accordingly. A *plane tree* is an equivalence relation of rooted trees under RROs. Applying the RRO has the same effect as traversing first an edge $\alpha\alpha_0$ in C_i and then the edge $\beta\beta_1$, also in C_i, where $\beta = \alpha_0$. For example, successive application of the RRO on the second cycle (C_1) for $k = 4$ in Figure 5 produces the cycle $(\underline{9}, \underline{2}, \underline{4}, \underline{11}, \underline{5}, \underline{6}, \underline{12}, \underline{7})$ in the corresponding equivalence class. This cycle

Figure 5: Cycles of W_{01} in M_k, $(k = 2, 3, 4)$

is observed starting downward from the second row, as well as (alternatively) upward from the third row, thus covering respectively the vertices or L_4 and L_5 in the class.

Based on [3], let D_i for $i \geq 0$ denote the set of all substrings of length $2i$ in any $F(\alpha)$ with exactly i color-entries and such that in every prefix, the number of asterisk-entries is at least as large as the number of color-entries. We define $D := \sum D_i$. We say that the elements of D are *Dyck words*. Note that each $F(\alpha)$ is of the form $0v1u*$, where u and v are Dyck words and 0, 1 are colors in $[k+1]$. Continuing our example above, the "forward words" $F(\alpha)$ in our considered cycle can be written with parentheses enclosing such Dick words v and u, namely $0()1(34**2*)*$, for $\underline{2}$; $0(3*24**)1()*$, for $\underline{9}$; $0()1(3*24**)*$, for $\underline{7}$; $0(3*2*)1(4*)*$, for $\underline{12}$; $0(24*3**)1()*$, for $\underline{6}$; $0()1(24*3**)*$, for $\underline{5}$; $0(2*)1(4*3*)*$, for $\underline{11}$; and $0(34**2*)1()*$, for $\underline{4}$. Similar treatment holds for the "backward words". This exemplifies and yields Corollary 2 [3]. Notice that as in Figure 5, each pair $(0\cdots1)(1\cdots0)$ represents two paths in the corresponding cycle, of lengths $2|(0\cdots1)|-1$ and $2|(1\cdots0)|$, adding up to $4k+2$. If the 0's delimiting these two paths are, say, $0v1u*$ and $0v'1u'*$ (this one read in reverse), then $|u|+2 = |(0\cdots1)|$ and $|u'|+2 = |(1\cdots0)|$. Recall that the first path here moves downward and the second path moves upward. Reading them starts at an entry 0 understood cyclically mod n and ends at an entry 1. However, notice that in reality the collections of paths obtained from the 1-factors under consideration are constant on the leftmost entry. So by combining those cycles with 6-cycles obtained below via symmetric difference, [3] yields their Hamilton cycles.

The 1-factor of color 0 makes the endvertices of each of its edges to have their representative plane trees obtained from each other by left-right reflection, denoted here by $\Phi = F\alpha^0F^{-1}$, on a vertical line (like the line ϕ in Section 4). For example, Figure 5 shows that for $k = 2$: both $\underline{0}, \underline{1}$ in $X(0)$ are fixed via Φ; for $k = 3$: $\underline{0}$ is fixed via Φ and $\underline{1}, \underline{3}$ correspond to each other via Φ, in $X(0)$; and $\underline{2}, \underline{4}$ are fixed via Φ, in $X(1)$; for $k = 4$: $\underline{0}, \underline{8}$ are fixed via Φ and $\underline{1}, \underline{3}$ correspond to each other via Φ, in $X(0)$; $\underline{5}, \underline{9}$ are fixed via Φ and the pairs $(\underline{5}, \underline{9})$, $(\underline{2}, \underline{7})$, $(\underline{4}, \underline{12})$ and $(\underline{6}, \underline{11})$ are pairs of correspondent plane trees via Φ, in $X(1)$; and $\underline{10}, \underline{13}$ are fixed via Φ, in $X(2)$. This left-right reflection symmetry arises from Theorem 4. It accounts for each pair of contiguous lines in any $X(i)$ corresponding to a 0-colored edge. For $k = 5$, this symmetry by Φ occurs in all cycles C_i $(i \in [6])$. But we also have $F(\underline{22}) = \rangle 024**135***\rangle$ for $\underline{22} = (1111)$ in C_0 and $F(\underline{39}) = \rangle 03*2*15*4**\rangle$ for $\underline{39} = (1232)$ in C_3, both having their 1-colored edges leading to their reversed reading between L_5 and L_6 again by Theorem 4. Moreover, $F((11\cdots1))$ has a similar property only if k is odd; if k is even, a 0-colored edge takes place instead of the 1-colored edge for k odd. These cases reflect the following lemma, which can alternatively be implied from Theorem 9 (B)-(C) via the correspondence $i \leftrightarrow k - i$, $(i \in [k + 1])$.

Lemma 13. **(A)** *Every 0-colored edge represents an adjacency via $\Phi = F\alpha^0F^{-1}$.* **(B)** *Every 1-colored edge represents an adjacency via the composition Ψ of Φ (first) and an RRO (second).*

By Theorem 4(ii), the number ξ of contiguous pairs of vertices of M_k with a common k-germ happens in pairs in each C_i. The first cases for which this ξ is null happens for $k = 6$, namely for $\rangle 012356**4****\rangle$ and $\rangle 01246*5**3***\rangle$, with respective reflections $\rangle 01235*46*****\rangle$ and $\rangle 0124*36*5****\rangle$ in different cycles C_i.

Observe that each two pairs of a forward and a reversed n-tuples have corresponding colored rooted planar trees related by mirror symmetry, so they are enantiomorphic. Thus, they yield two different pairs of *enantiomorphic* cycles C_i. For example, $k = 4$ offers $(\underline{1}, \underline{3})$ as the sole enantiomorphic pair in $X(0)$, and $(\underline{2}, \underline{7})$, $(\underline{4}, \underline{12})$, $(\underline{6}, \underline{11})$ as all the enantiomorphic pairs in $X(1)$.

Each enantiomorphic cycle C_i or each cycle C_i with $\xi = 2$ has $|C_i| = 2k(4k + 2)$, the maximal ξ provided by the corresponding $X(i)$. If $\xi = 2\zeta$ with $\zeta > 1$, then $|C_i| = \frac{2k(4k+1)}{\zeta}$. On account of these facts, we arrive to the following statement.

>2 * 1 3 * * 0> ε (11)= $\underline{3}$ ε X(0);	>0 1 3 * 2 * *> ε (10)= $\underline{2}$ ε X(1);	>0 1 3 4 * * 2 * *> ε (010)= $\underline{2}$ ε X(1);	>0 1 4 * 3 * 2 * *> ε (120)= $\underline{10}$ ε X(2);
<* 2 * 3 1 0 *< ε (10)= $\underline{2}$ ε X(1);	<* * * 3 2 1 0< ε (00)= $\underline{0}$ ε X(0);	<* * * * 4 3 2 1 0< ε (000)= $\underline{0}$ ε X(0);	<* * * 3 * 4 2 1 0< ε (100)= $\underline{5}$ ε X(1);
>3 * 2 * 1 * 0> ε (12)= $\underline{4}$ ε X(1);	>0 2 3 * * 1 *> ε (01)= $\underline{1}$ ε X(0);	>0 2 3 4 * * * 1 *> ε (001)= $\underline{1}$ ε X(0);	>0 2 4 * 3 * * 1 *> ε (101)= $\underline{6}$ ε X(1);
<* 3 2 0 * 1 *< ε (01)= $\underline{1}$ ε X(0);	<* 2 * 3 0 * 1< ε (12)= $\underline{4}$ ε X(1);	<* 2 * * 4 3 0 * 1< ε (012)= $\underline{4}$ ε X(1);	<* 2 * 3 * 4 0 * 1< ε (123)= $\underline{13}$ ε X(2);
>3 * * 0 2 * 1> ε (11)= $\underline{3}$ ε X(0);	>2 * 1 * 0 3 *> ε (12)= $\underline{4}$ ε X(1);	>2 * 1 4 * * 0 3 *> ε (122)= $\underline{12}$ ε X(1);	>3 * 2 * 1 * 0 4 *> ε (123)= $\underline{13}$ ε X(2);
<* 3 1 * 2 0 *< ε (11)= $\underline{3}$ ε X(0);	<* 1 * 2 * 3 0< ε (12)= $\underline{4}$ ε X(1);	<* 1 * * 4 2 * 3 0< ε (112)= $\underline{9}$ ε X(1);	<* 1 * 2 * 3 * 4 0< ε (123)= $\underline{13}$ ε X(2);
>2 * 1 3 * * 0> ε (11)= $\underline{3}$ ε X(0);	>0 1 3 * 2 * *> ε (10)= $\underline{2}$ ε X(1);	>0 1 3 4 * * 2 * *> ε (010)= $\underline{2}$ ε X(1);	>0 1 4 * 3 * 2 * *> ε (120)= $\underline{10}$ ε X(2);

Figure 6: Examples of 6-cycles toward Hamilton cycles for $k = 4, 5$

Theorem 14. *There is a bijection between the cycles of W_{01} and the plane trees whose vertices are labeled in $[k + 1]$. In the non-enantiomorphic cases, each such plane tree is represented by finitely many colored rooted planar trees corresponding each to the n-string $F(\alpha)$ of a k-germ α. On the other hand, each enantiomorphic pair of cycles disconnects* *the forward and reversed reading of each of its vertices, so that such n-strings $F(\alpha)$ are represented in both cycles, once forward in L_k and once reversed in L_{k+1}.*

11 Reinterpretation of Mütze's Theorem

Based in Proposition 3 [3], we notice that in each cycle C_i of W_{01} there are two vertices u, v at distance 5 in C_i with $u \in L_{k+1}$ above $v \in L_k$ in $X(i)$ so that: **(i)** u and v are adjacent via a color $h \in \{2, \ldots, k\}$; **(ii)** u is of the (cyclic) reading $\langle \cdots * h0 * \cdots \langle$ and v is of the (cyclic) reading $\rangle \cdots * 0h * \cdots \rangle$. **(iii)** the column on which these two occurrences of h happen at distance 5 looks between u and v (included) as the transpose of $(h, *, 0, 0, *, h)$. In fact, note that each vertex u in a C_i of the (cyclic) reading $\langle \cdots * h0 * \langle$ with $h \in \{2, \ldots, k\}$ is at a forward distance 5 from a vertex v of C_i of the (cyclic) reading $\rangle \cdots * 0h*\rangle$, (and viceversa) and that the column containing both occurrences of the color h looks as in (ii) above. Then the vertices u' and v' in C_i that precede respectively u and v in X_i are the endvertices of a 3-path $u'u''v''v'$ in M_k with the edge $u''v'' \in C_j$ for some $j \neq i$. The 6-cycle $U = (uu'u''v''v'v)$ has a symmetric difference with $C_i \cup C_j$ resulting in a cycle covering all the vertices of C_i and of C_j. Repeated applications of this type of symmetric difference yields the claimed Hamilton cycles in [3]. Figure 6 illustrates such 6-cycles U for $k = 3, 4$, where the vertices $u', u'', v'', v', v, u, u'$ (again) are presented vertically in each of the four cases presented. The mentioned symmetric difference replaces the edges $u''v''$, $v'v$, uu' in $C_i \cup C_j$ by the the other three edges of U, namely $u'u''$, $v''v''$, vu. In the figure, vertically contiguous positions holding a common number g that indicates adjacency with an edge of that color number g are presented in red if $g \in \{0, 1\}$ (as for $u''v''$, where $g = 1$, in column say r_1, exactly at the position where u and v differ but have common color h) and in orange otherwise. The column, say r_2 (resp r_3) in each instance of Figure 5 containing color 1 in u' (resp. 0 in v') and a color $c \in \{2, \ldots, k\}$ in v' (resp. color $d \in \{2, \ldots, k\}$ in u') starts with $1, *, c, c,$ (resp. $d, d, *, 0$). Then, the only three columns having changes in U are r_1, r_2, r_3. All the other columns have their first four entries alternating two asterisks and two colors.

Theorem 15. *[5, 3] Let $0 < k \in \mathbb{Z}$. Then, Hamilton cycles in M_k are obtained by symmetric difference of W_{01} with 6-cycles of the form (u, u', u'', v'', v', v).*

12 Ordered Rooted Trees with Root k

How about changing 0 as a root of ORT's to k as a root? We start with some examples. Figure 7 shows on its left-hand side the 14 ORT's for $k = 4$ presented as in Table I. Each such tree T is headed by its k-germ, in which the entry i producing T via castling is in red. Such T has its vertices denoted on their left and edges denoted on their right, as established in Section 10. Castling as in Section 2 here is indicated in any particular child tree T by distinguishing in red the largest subtree common with that of the parent tree of T (in the castling process) whose castling reattachment produces T. This subtree corresponds with substring X in Theorem 2. In each case of such a parent tree, the vertex in which the corresponding tree surgery leads to such a child tree T is additionally labeled (on its right) with its k-germ, in which expression the entry to be modified in the case is set in red color.

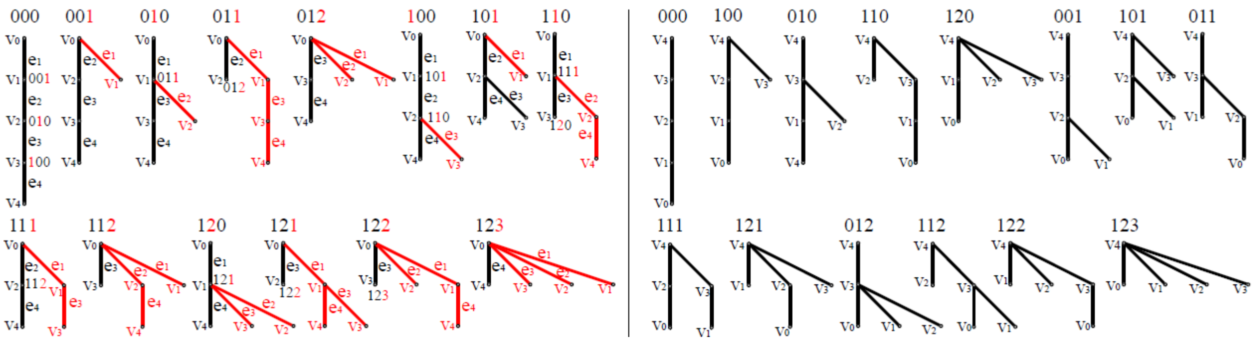

Figure 7: Generation of ORT's for $k = 4$

On the other hand, the 14 trees in the right-hand side of Figure 7 have their labels set by making the root to be k (instead of 0), then going down to 0 (instead of k) while gradually increasing (instead of decreasing) a unit sibling by sibling from left to right at each level. In this case, the header k-germs correspond to the new root viewpoint, which determines a bijection Θ established by correspondence of the old and the new header k-germs. This yields an involution formed by the pairs $(001, 100)$, $(011, 110)$, $(120, 012)$ and $(112, 121)$, with fixed 000, 010, 101, 111, 122 and 123.

The function Θ seen from the k-germ viewpoint, namely as the composition function $F^{-1}\Theta F$, behaves as follows. Let $A = a_{k-1}a_{k-2}\cdots a_2a_1$ be a k-germ and let a_i be the leftmost occurrence of its largest integer value. For example, consider the 17-germ $A' = 0123223423410121$. A substring B of A is said to be an *atom* if it is either (formed by) a sole 0 or a maximal strictly increasing substring of A not starting with 0. By enclosing successive atoms between parentheses, A' can be written as $A' = (0)123(2)(234)4(23)(1)(0)(12)(1)$, obtained from the *base string* $C' = 1\cdots a_i = 1234$ by inserting (at the start, in the middle or at the end of C') all the necessary (parenthesized) atoms. This atom-parenthesizing procedure works for every k-germ A and determines a corresponding base string C, like the C' in our example. Then, $F^{-1}\Theta F(A)$ is obtained by reversing the position of the parenthesized atoms, preceding each of them their corresponding appearance in C'. In the case of A', it is $F^{-1}\Theta F(A') = (1)(12)(0)(1)12(234)34(23)(2)(0)$.

Theorem 16. *Given a k-germ $A = a_{k-1}a_{k-2}\cdots a_2a_1$, let a_i be the leftmost occurrence of its largest integer value. Then, A is obtained from string $C = a_1\cdots a_i$ by inserting in C all atoms of $A \setminus C$ in their corresponding left-to-right order. Moreover, $F^{-1}\Theta F(A)$ is obtained by reversing the insertion of those atoms in C, in right-to-left fashion. Furthermore, for each of the involutions α^i $(0 < i < k)$, it holds that $\alpha^i\Theta = \Theta\alpha^{k-i}$. This implies that* **(A)** *every k-colored edge represents an adjacency via $\Phi' = F\alpha_k F^{-1}$ and* **(B)** *every $(k-1)$-colored edge represents an adjacency via $\Psi' = F\Psi F^{-1}$.*

Proof. It only rests to prove the last assertion in the statement. This arises by combining our discussion above with Lemma 13. □

Remark 17. The reflection symmetry of Φ' in Theorem 16 yields the sequence S_0 cited in Theorem 9(B). A similar observation yields the sequence S_1 cited in Theorem 9(C) from Ψ'.

13 An All-RGS's Binary Tree

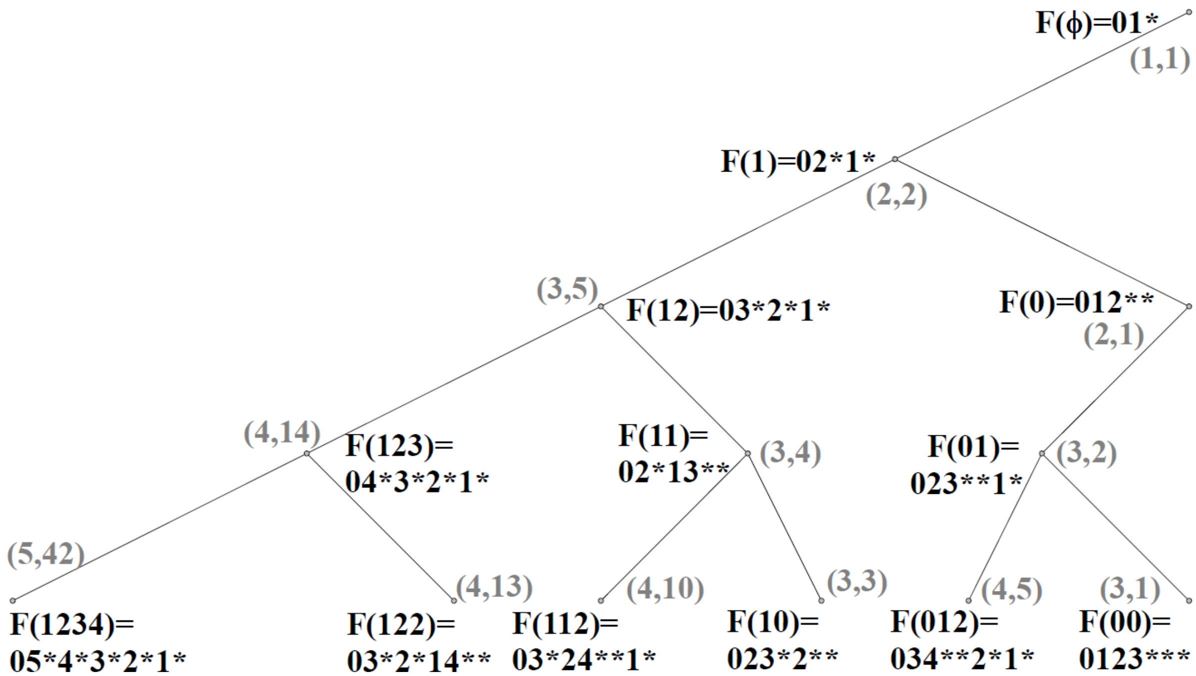

Figure 8: Restriction of T to its first five levels

Even though the graphs R_k were considered for $k > 1$, the graph R_1 is still defined, having just one vertex 001 with $\delta(001) = 01*$ (as in Section 7) and two loops. Thus, the only vertex of R_1 is denoted $01*$ and the correspondence F as in Section 2 can be extended by declaring $F(\emptyset) = 01*$. This is the root of a binary tree T that has $\cup_{k=1}^{\infty} V(R_k)$ as its node set and is defined as follows: **(A)** the root of T is $01*$; **(B)** the left child of a node $\delta(v) = 0|X$ in T with $||X|| = 2k$ exists and equals $0|X|1*$; **(C)** unless $\delta(v) = 012\cdots(k-1)k**\cdots*$, it is $\delta(v) = 0|X|Y|*$, where X and Y are strings starting at some $j > 1$ and at $j-1$, respectively, in which case there is a right child of $\delta(v)$, namely $0|Y|X|*$, by means of the Un-castling Procedure of Section 7.

With nodes expressed in terms of k-germs, T has each node $a_{k-1}a_{k-2}\cdots a_2a_1$ as a parent of a left child $b_kb_{k-1}\cdots b_1 = a_{k-1}a_{k-2}\cdots a_2a_1(a_1-1)$, and it has a right child ρ only if $a_1 > 0$, in which case $\rho = c_{k-1}c_{k-2}\cdots c_2c_1 = a_{k-1}a_{k-2}\cdots a_2(a_1-1)$. Figure 8 shows the first five levels of T with nodes expressed in terms of k-germs via F, in black color. The figure also assigns to each node a (dark-gray colored) ordered pair of positive integers (i, j), where $j \leq C_i$. The root, given by $F(\emptyset) = 01*$, is assigned $(i, j) = (1, 1)$. The left child of a node assigned (i, j) is assigned a pair $(k, j') = (i+1, j')$, where j' is the order of appearance of the k-germ α corresponding to (k, j') in its presentation via castling as in Table I; α becomes the RGS corresponding to j' in the sequence $\mathcal{S}$ (A239903), once the extra zeros to the left of its leftmost nonzero entry are removed; note $j' = j'(j)$ arises from the series associated to A076050, deducible from items 1-4 in Subsection 1.3. The right child of a node assigned (i, j) is defined only if $j > 1$ and in that case is assigned the pair $(i, j-1)$.

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