# Note on sequences A123192, A137396 and A300453

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**Abstract**

We give the connection between three polynomials that generate triangles in *The On-Line Encyclopedia of Integer Sequences* (A123192, A137396 and A300453). We show that they are related with the bracket polynomial for the $(2,n)$-torus knot.

Keywords: bracket polynomial, torus knot, cycle graph.

## 1 Introduction

Let $K_n$ denote the $(2,n)$-torus knot diagram (Figure 1 (a)). The corresponding bracket polynomial is given by the formula [2]

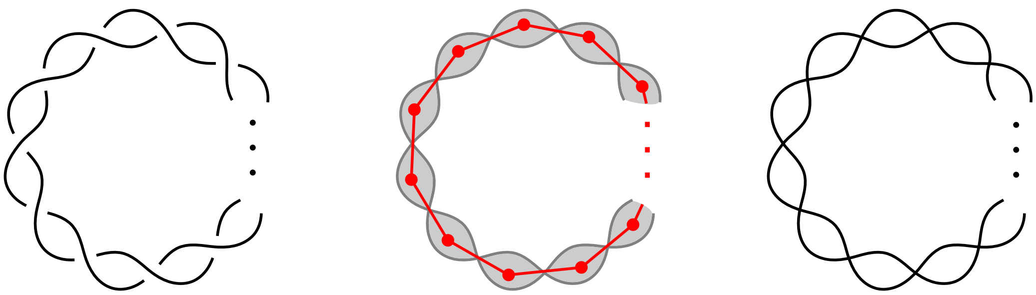
$$K_n(A,B,d) = \frac{(A+Bd)^n + (d^2-1)\,A^n}{d}. \tag{1}$$

The following triangles are in *The On-Line Encyclopedia of Integer Sequences* [4], and consist of the coefficients in the expansion of $gK_n(A,B,d)$ for some values of $A,B,d$ and $g$ (see Table 1, Table 2 and Table 3).

- Row $n$ in A123192 is generated by

$$x^{|3n-2|}K_n\left(x,x^{-1},-x^{-2}-x^2\right) = \begin{cases} \dfrac{(x^8+x^4+1)\,x^{4n}+(-1)^n x^4}{x^8+x^4} & \textit{if } n\geq 1;\\ -x^4-1 & \textit{if } n=0. \end{cases} \tag{2}$$

The interpretation for the choice of $A,B,d$ and $g$ is given in section 2 .



(a) (b) (c)

Figure 1: (a) $(2, n)$-torus knot diagram, (b) $n$-cycle graph and its "medial graph", (b) $(2, n)$-torus knot shadow diagram.

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | −1 | 0 | 0 | 0 | −1 | | | | | | | | | | | | |
| 1 | 0 | 0 | 0 | 0 | −1 | | | | | | | | | | | | |
| 2 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | | | | | | | | |
| 3 | 0 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | | | | | |
| 4 | −1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 |

Table 1: First 5 rows in A123192.

- Row $n$ in A137396 is generated by the chromatic polynomial of the $n$-cycle graph.

$$x^{\frac{1}{2}} K_n \left(-1, x^{\frac{1}{2}}, x^{\frac{1}{2}}\right) = (x-1)^n + (x-1)(-1)^n. \tag{3}$$

In this note, we consider the $n$-cycle graph to be the planar graph associated with the $(2, n)$-torus knot diagram (Figure 1 (b)).

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | | | | | | | |
| 2 | 0 | −1 | 1 | | | | | |
| 3 | 0 | 2 | −3 | 1 | | | | |
| 4 | 0 | −3 | 6 | −4 | 1 | | | |
| 5 | 0 | 4 | −10 | 10 | −5 | 1 | | |
| 6 | 0 | −5 | 15 | −20 | 15 | −6 | 1 | |
| 7 | 0 | 6 | −21 | 35 | −35 | 21 | −7 | 1 |

Table 2: First 7 rows in A137396.

- Row $n$ in A300453 is generated by

$$xK_n(1, 1, x) = (x+1)^n + x^2 - 1. \tag{4}$$

We referred to the polynomial in (4) as "generating polynomial" [3]. This is, in fact, the expression of the bracket evaluated at the shadow diagram (Figure 1 (c)).

| $n \setminus k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | | | | | |
| 1 | 0 | 1 | 1 | | | | | |
| 2 | 0 | 2 | 2 | | | | | |
| 3 | 0 | 3 | 4 | 1 | | | | |
| 4 | 0 | 4 | 7 | 4 | 1 | | | |
| 5 | 0 | 5 | 11 | 10 | 5 | 1 | | |
| 6 | 0 | 6 | 16 | 20 | 15 | 6 | 1 | |
| 7 | 0 | 7 | 22 | 35 | 35 | 21 | 7 | 1 |

Table 3: First 8 rows in A300453.

We show in the next section the connection between these polynomials.

# 2 Construction and interpretation

## 2.1 Bracket polynomial

The bracket polynomial for the knot diagram $K$ is defined by

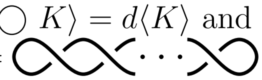
$$\langle K \rangle = K(A, B, d) = \sum_{s} \langle K|s \rangle d^{|s|-1}, \tag{5}$$

where $\langle K|s \rangle$ denotes the product of the splitting variables ($A$ and $B$) associated with the state $s$, and $|s|$ denotes the number of circles (or loops) in $s$.

Formula (5) can also be expressed as

- $\langle K \rangle = A\langle K' \rangle + B\langle K'' \rangle$,
- $\langle \bigcirc \bigcirc \cdots \bigcirc \rangle = d^{k-1}$ (disjoint union of $k$ circles),

where $K'$ and $K''$ are obtained from $K$ by performing $A$ and $B$ splits at a given crossing in $K$, see Figure 2. Last formula reads as well as $\langle \bigcirc\ K \rangle = d\langle K \rangle$ and $\langle \bigcirc \rangle = 1$.

For example, $\langle T_n \rangle = (A + Bd)^n$, where $T_n :=$  (with $n$ half-twists).

Indeed, we have

$\langle$  $\rangle = A\,\langle$  $\rangle + B\,\langle$  $\rangle$

$$\begin{aligned} &= A\langle T_{n-1} \rangle + B\langle \bigcirc\ T_{n-1} \rangle` \\ &= (A + Bd)\,\langle T_{n-1} \rangle \\ &= \cdots \\ &= (A + Bd)^n. \end{aligned}$$

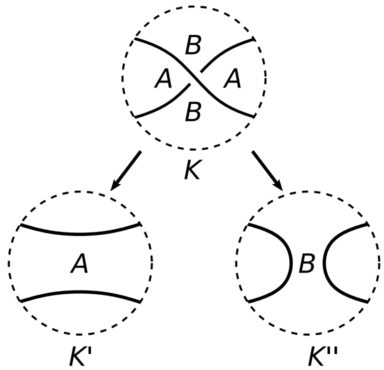



Figure 2: $A$-split and $B$-split.

Now, formula (2) is straightforward if we notice that,

$$\langle K_n \rangle = A \langle K_{n-1} \rangle + B \langle T_{n-1} \rangle, \ \textit{with} \ \langle K_0 \rangle = d. \tag{6}$$

Finally, set $B = A^{-1}$ and $d = -A^{-2} - A^2$ so that the bracket is invariant under Reidemeister moves II and III [1, p. 31–33]. For $n = 0, 1, \ldots, 5$ the corresponding bracket polynomial reads

$$\begin{aligned}
\langle K_0 \rangle &= -A^2 - \frac{1}{A^2} \\
\langle K_1 \rangle &= -A^3 \\
\langle K_2 \rangle &= -A^4 - \frac{1}{A^4} \\
\langle K_3 \rangle &= -A^5 - \frac{1}{A^3} + \frac{1}{A^7} \\
\langle K_4 \rangle &= -A^6 - \frac{1}{A^2} + \frac{1}{A^6} - \frac{1}{A^{10}} \\
\langle K_5 \rangle &= -A^7 - \frac{1}{A} + \frac{1}{A^5} - \frac{1}{A^9} + \frac{1}{A^{13}}.
\end{aligned}$$

The $n$-th row polynomial of triangle in A123192 is then obtained by multiplying the bracket $\langle K_n \rangle$ by $A^{|3n-2|}$.

## 2.2 Chromatic polynomial

In the present framework, the splits of type $A$ and $B$ may be regarded in terms of graph as the "contraction" and "deletion" operations, respectively, as shown in Figure 3. Kaufman refers to the resulting states as "chromatic states" [1, p. 353–358].

Let $n \geq 1$, and let $G(K_n, x) = xK(A, B, x) = \sum_s \langle K|s \rangle x^{|s|}$. By (3), we can rewrite (5) as

$$G(K_n, x) = \sum_s (-1)^{A(s)} \left( x^{\frac{1}{2}} \right)^{B(s)} \left( x^{\frac{1}{2}} \right)^{|s|}, \tag{7}$$

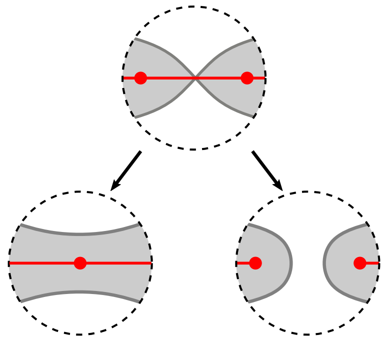


Figure 3: Edge contraction and deletion.

where $A(s)$ and $B(s) = n - A(s)$ are the number of $A$-splits and $B$-splits in the state $s$, respectively. Furthermore, we have $|s| = 2$ if $B(s) = 0$, and $|s| = B(s)$ otherwise [3]. Hence

- $G(K_1, x) = (-1)^1 \left(x^{\frac{1}{2}}\right)^0 \left(x^{\frac{1}{2}}\right)^2 + (-1)^0 \left(x^{\frac{1}{2}}\right)^1 \left(x^{\frac{1}{2}}\right)^1 = 0;$
- and for $n \geq 2$,

$$\begin{aligned} G(K_n, x) &= \sum_s (-1)^{n-B(s)} x^{\frac{1}{2}(B(s)+|s|)} \\ &= (-1)^n x + \sum_{|s|\geq 1} (-1)^{n-B(s)} x^{B(s)} \\ &= \sum_s (-1)^{i(s)} x^{B(s)}, \end{aligned}$$

where $i(s)$ is the number of "interior vertices" in the chromatic state $s$ [1, p. 358] and $B(s)$ matches the number of shaded components in $s$ [3] (with $i(s) = 1$ if $|s| = 1$, and $i(s) = n - B(s)$ otherwise).

## 2.3 Generating polynomial

Now, what if we evaluate the bracket polynomial at the shadow diagram? Recall that a shadow is a knot diagram without under or over-crossing information. In such case, it is natural to set $A = B = 1$ in (1). The bracket becomes

$$K_n(1, 1, d) = \frac{(d+1)^n + d^2 - 1}{d},$$

and formula (4) implies

$$xK_n(1, 1, x) = \sum_s x^{|s|} = \sum_k s(n, k)x^k,$$

where $s(n, k)$ is the number of states having exactly $k$ circles [3].

# References

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