arXiv:1911.04795v1 [math.CO] 12 Nov 2019

A study on the fixed points of the γ function

A. Frosini [*] G. Palma [†] E. Pergola [*] S. Rinaldi [†]

13th November 2019

Abstract

Recently a permutation on Dyck paths, related to the chip firing game, was introduced and studied by Barnabei et al. [1]. It is called γ-operator, and uses symmetries and reflections to relate Dyck paths having the same length. The study of the fixed points of γ was carried on in [1], where the authors provided a characterization of these objects, leaving the problem of their enumeration open. In this paper, using tools from combinatorics of words, we determine new combinatorial properties of the fixed points of γ. Then we present an algorithm, denoted by **GenGammaPath**(t), which receives as input an array $t = (t_0, \ldots, t_k)$ of positive integers and generates all the elements of F_γ with degree k.

KEYWORDS: DYCK PATHS, ENUMERATIVE COMBINATORICS, GENERATING FUNCTIONS

1 Introduction

The recent study of the Riemann-Roch Theorem for graphs presented in details by Baker and Norine in [2] gave rise to a series of side researches that overflew the main stream of graph theory, touching combinatorics on words and theory of formal languages, as well. An equivalent presentation can be provided in terms of the chip firing game played on a graph $G = (E(G), V(G))$: let us consider an initial configuration where at each vertex $v \in V(G)$ is assigned an integer number f_v of coins both positive, and negative. The objective of the game is to reach a positive configuration $f = (f_1, \ldots, f_n)$, i.e. a configuration where all the vertices have a nonnegative

*Università di Firenze, Dipartimento di Matematica e Informatica, Viale Morgagni 65, 50134 Firenze, Italy

†Università di Siena, Dipartimento di Ingegneria dell'Informazione e Scienze Matematiche, Via Roma 56, 53100 Siena, Italy

number of coins, by using a sequence of two types of moves that, for each vertex, consist either in borrowing one coin from each of its neighbors or in giving one coin to each of its neighbors. Such a configuration is called a *winning configuration*, and each sequence of moves which leads to such a configuration is a *winning strategy*. Let $g = |E(G)|+|V(G)|-1$, Theorem 1.9 in [2] states that

Theorem 1.1. *Let $N = deg(D)$ be the total number of coins present in the graph G at any step of the game.*

1. If $N \geq g$, then there is always a winning strategy.

2. If $N < g$, then there is always an initial configuration for which no winning strategy exists.

In [6], the authors considered a restriction of the game to complete graphs, and they studied the notion of rank of the winning configurations. Basing on Theorem 1.1, they provided a useful characterization of the winning configurations in terms of parking configurations. Let us recall that, in a complete graph, a sequence $(f_1, \ldots, f_{n-1})$ is the initial sequence of a parking configuration if and only if, after reordering, it is a weakly increasing sequence $(g_1, \ldots, g_{n-1})$ such that for each i it holds $g_i < i$.

In this paper we are concerned with an alternative characterization of the winning configurations of complete graphs as fixed points of an operator, called γ-operator, that geometrically acts on Dyck paths [1]. As one can expect, Dyck paths may be related to parking configurations in the sense that each length k prefix of their coding words contains a number of descending steps that is upper bounded by $k - 1$.

The authors of [1] present three permutations on Dyck words. The first one, α, is related to the Baker and Norine theorem on graphs, the second one, β is the symmetry, and the third one, γ, is the composition of these two. The fixed points of α and β are not difficult to characterize, and the studies of [1] concentrate on the characterization of the fixed points of γ, showing combinatorial properties of its cycles.

In Section 2 we recall basic definitions and known results, mainly from [1]. In Section 3 we provide a new characterization of the fixed points of γ, in terms of combinatorial properties (and in particular some recursive decomposition) of the words encoding them. The objective of this characterization is that of obtaining an algorithm for the generation of these fixed points. In Section 4, we first define the notion of *degree* of a fixed point of γ, then we write down an algorithm, denoted by **GenGammaPath**(t), which receives as input an array $t = (t_0, \dots, t_{k-1})$ of positive integers and generates all the elements of F_γ with degree k.

In the final section we investigate the relation between the degree and the length of an element of F_γ. We believe that the algorithm **GenGammaPath** can be used in some further research for exhaustive generation of the fixed points of γ, and also to study the generating function of these objects according to their length.

2 Definitions and preliminary results

Let w be a word on the free monoid $\Sigma^\star$, where $\Sigma = \{a, b\}$. As usual, let $|w|$ denote the length of w, i.e. the number of its letters, and let $|w|_a$ and $|w|_b$ denote the number of the occurrences of the letters a and b in w, respectively. Furthermore, to each word $w \in \Sigma^\star$, we associate the integer number $\delta(w) = |w|_a - |w|_b$. A word u is a *prefix* of w if, for some v we have $w = uv$; in this case v is said to be a *suffix* of w. Two words w and w' are *conjugate* if there are words u, v such that $w = uv$ and $w' = vu$.

Dyck words are an almost ubiquitous family of words which show natural connections with a huge quantity of problems in different scientific areas: more importantly for us, in [6] it is shown a strict connections between Dyck words and parking configurations on complete graphs.

Definition 1. A word $w \in \Sigma^\star$ of length $2n$ is a Dyck word if and only if $\delta(w) = 0$ and, for each prefix v of w, it holds $\delta(v) \geq 0$.

By definition, for each Dyck word w of length $2n$, it holds $|w|_a = |w|_b = n$.

Let us consider the two sets of words A_n and D_n defined as follows: A_n contains any word w of length $2n+1$ such that $|w|_a = n$ and $|w|_b = n+1$, while the set D_n is the set of Dyck words followed by a single occurrence of b. By definition, we have that, with $n > 0$, $D_n \subset A_n$. A non trivial connection between these two sets is established by the so called *Cycle Lemma*, illustrated in [8], which can be stated as follows:

Lemma 2.1. ***(Cycle Lemma)*** *Let w be an element of A_n. Then w admits a unique factorization $w = uv$ such that the conjugate word $w' = vu$ belongs to D_n.*

The Cycle Lemma states that the conjugacy relation induces a partition on A_n into equivalence classes whose minimal lexicographical representatives are exactly the elements of D_n.

2.1 Three permutations on the set D_n

In this section, unless otherwise specified, we borrow notation and definitions from [1]. We present two involutions α and β on D_n, whose composition gives a permutation, called γ, on which our study will be focused.

Few more definitions are needed: given a word $w = w_1 w_2 \dots w_m$, its *complement* $\overline{w}$ is the word $\overline{w}_1 \overline{w}_2 \dots \overline{w}_m$, where $\overline{w}_i$, with $1 \leq i \leq m$, exchanges the letter a with b and viceversa, its *mirror* $\widetilde{w}$ is the word $w_m \dots w_2 w_1$, and its symmetric $Sym(w)$ is the complement of its mirror, i.e. the word $\overline{w}_m \dots \overline{w}_1$.

The involution α. The function α, introduced in [7], maps an element w of D_n onto the unique conjugate of $\widetilde{w}$ that belongs to D_n. By the cycle lemma, we know that there is exactly one such element. As an example, let the word $w = aabbaababaabbbb$ be an element of D_n. By definition, $\widetilde{w} = bbbbaababaabbaa$ and its conjugate is $\alpha(w) = aababaabbaa\ bbbb$. In [1] the authors show that α is indeed an involution on D_n.

The involution β. The function β maps any element $w = w_1 w_2 \dots w_{2n+1}$ of D_n onto the word obtained by applying the symmetry operator to its first $2n$ letters, i.e. $\beta(w) = Sym(w_1 \dots w_{2n})\, \overline{w}_{2n+1}$. The fact that β is an involution is immediate, since it realizes the central symmetry of the first $2n$ elements of w, as one can check by considering $w = aabbaababaabbbb \in D_n$. Then $\beta(w) = \overline{bbbaababaabbaa}\, b = aaabbababbaabb\, b$ that still belongs to D_n.

The permutation γ. Let w be a word of D_n. The *principal prefix* (resp. *principal suffix*) of w is the shortest prefix (resp. suffix) u of w such that $\delta(u)$ is maximal. Now, the mapping γ is defined as the composition of α and β. Formally, with $w \in D_n$, we have: $\gamma(w) = \alpha(\beta(w))$. By definition, γ acts on a word $w = u\ b$ of D_n, and provides the unique word w' in D_n that is the conjugate of $\overline{u}\, b$. It is easy to check that, if $w = u\, v\, b$, then $\gamma(w) = \overline{v}\, b\, \overline{u}$, with u being the principal prefix of w. The application of γ to the word $w = aabbaababaabbbb$ gives

$$\gamma(w) = \alpha(\beta(w)) = \alpha(aaabbababbaabb\, b) = aaa\, bbbaabbababb.$$

The mapping γ is a permutation of the words of D_n. Actually γ determines a partition of the words of D_n into classes, or *cycles*, that contain all the words that can be obtained by iterated applications of γ. Again in [1], it was proved that each cycle induced by γ has odd cardinality; on the other hand, there are no results concerning enumeration of the elements of the cycles with respect to their length.

Dyck words can be naturally represented as lattice paths commonly known under the name of *Dyck paths*. They are paths in the first quadrant which begin at the origin, end at $(0, 2n)$ and use North-East steps $(1, 1)$ (*rise* steps) and South-West steps $(1, -1)$ (*fall* steps). The correspondence between Dyck words and Dyck paths is obtained coding rise (resp. fall) steps with the letter a (resp. b). To understand the coding, see for instance the example in Fig. 1, which depicts the path associated with a word in D_n. In a Dyck path, the *level* of a point of the path is its ordinate; furthermore we call *peak* and *valley* any occurrence in the related word of the sequence ab and ba, respectively. We observe that the three mappings α, β and γ defined above can be easily described in a graphical way using the path representation of Dyck words. Such a representation also helps us check some of the properties of the fixed points of α, β and γ, which will be studied in this paper. So, from now on, we will use the word representation and the path representation of Dyck words indifferently.

2.2 The fixed points of α and β

The following non trivial property, proved in [1], provides the characterization of the fixed points of α:

Proposition 1. *The word $w \in D_n$ is a fixed point for α if and only if w is the concatenation of two palindromes. Moreover w has a unique decomposition as concatenation of two palindromes.*

Figure 1, (a) shows a fixed point of α; the (unique) factorization of the path in two palindromes is pointed out. Note that, for each $w \in D_n$, the only conjugate $w' \in D_n$ of $\widetilde{w}$ is such that $\widetilde{w} = uv$, $w' = vu$ and the point of u with the lowest ordinate is precisely its last point. Concerning the involution β, each one of its fixed points can be represented by a path $w = w_1 \dots w_{2n}\, b$ of D_n such that the prefix of length $2n$ is vertically symmetric, i.e. $w = w_1 \dots w_n\, Sym(w_1 \dots w_n)\, b$, as shown in Fig. 1 (b).

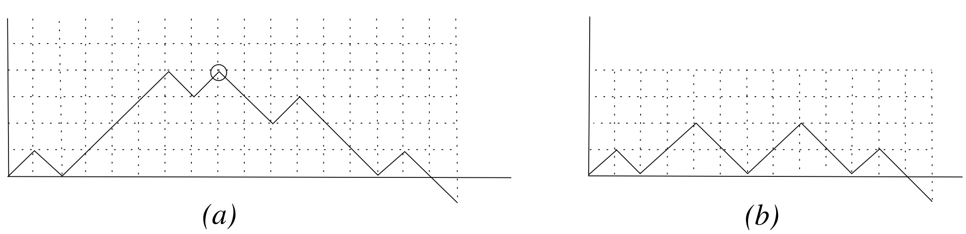

Figure 1: Two examples of fixed points of the involutions α, (a), and β, (b). The decomposition in two palindromes is highlighted.

So, let us consider the class $F_n \subset D_n$ of the fixed points of γ of length $2n+1$, and let $F_\gamma = \cup_{n=1}^{\infty} F_n$.

In what follows we will indicate with w_i the generic path (in the case in which w_i indicates the i-th component of the word it will be properly specified).

3 Combinatorial properties of the fixed points of γ

We observe that a generic element $w \in F_\gamma$, being a fixed point of β, can be written as $w = u\,a\,v\,b\,Sym(u)\,b$ with $u\,a$ being its principal prefix, and $b\,Sym(u)\,b$ its principal suffix.

Proposition 2. *Let $w = u\,a\,v\,b\,Sym(u)\,b$ be an element of F_n; the following statements hold:*

- *(i) $\delta(u\,a) = M$, where M is the maximal level of w, and $\delta(v) = 0$;*
- *(ii) u and $u\,a\,v$ are palindromes;*
- *(iii) if $v = \lambda$, the empty word, then $w = a^n\,b^{n+1}$, where the power notation x^y stands for the usual concatenation of the string x with itself y times;*
- *(iv) if $v \neq \lambda$, then v can be decomposed as $v = v_1\,a\,v_2$, with v_1 and v_2 palindromes, and v_2 possibly empty. Furthermore, it holds $u = v_2(av)^t$, with $t \geq 0$.*

Proof. The proofs of (i) and (ii) are straightforward, and follow from the fact that w is a fixed point of both β and α. To prove (iii) we observe that $v = \lambda$ implies that $u\,a$ and u are both palindromes, by statement (ii), so $u = a^{n-1}$. Finally, the properties in (iv) are direct consequences of the palindromicity of u and $u\,a\,v$: let $v \neq \lambda$, it holds

$u\,a\,v = \quad \tilde{v}\,a\ \ldots$, by palindromicity of $u\,a\,v$;

$u\,a\,v = \quad \ldots\ a\,v\,a\,v$, by palindromicity of u;

$u\,a\,v = \quad \tilde{v}\,a\,\tilde{v}\,a \ldots$, by palindromicity of $u\,a\,v$;

$\vdots \qquad \vdots \qquad \vdots$

$u\,a\,v = \quad (\tilde{v}\,a)^t\,\tilde{v}_2 = v_2\,(a\,v)^t$.

for some decomposition of v as $v_1\, a\, v_2$, with v_2 possibly empty. Hence, the two different ways of decomposing $u\,a\,v$ directly lead to $v_2 = \widetilde{v}_2$. Finally, by the symmetry of v, we also deduce that $v_1 = \widetilde{v}_1$, as desired. □

Remark 1. Let us now analyze the form of a generic element of F_n, with m_1 (resp. m_2) being the level of the last (resp. first) step of v_1 (resp.v_2). Since, by assumption, the first and the last points of v are at the same maximal level M, and v_1 and v_2 are palindromes, then the levels of the points of v_1 lie between m_1 and M, and those of v_2 between m_2 and M. Observe in fact that, if a step of v_1, resp. v_2, was below m_1, resp. m_2, then, due to palindromicity, the symmetric step is above M, which is a contradiction. As a consequence $m_1 = m_2 + 1$, since v_1 and v_2 are connected by a rise step a. We recall that v is symmetric since it is the central part of w, that is a fixed point of β. Two cases arise: either $Sym(v_1)$ is a suffix of v_2 or viceversa. In the first case, v_2 reaches the level m_1 that is lower than m_2 and this is a contradiction, so necessarily we have that $Sym(a\, v_2)$ is a prefix of v_1. □

Corollary 1. *Let $w = u\,a\,v\,b\,Sym(u)\,b$ be an element of F_n. Then the word ua can be decomposed into two factors u_1 and u_2, such that either $u_2\,\bar{u}_1$ or $\bar{u}_2\,u_1$ is palindrome.*

The proof follows acting as in Proposition 2, after observing that the roles of u and v can be exchanged and a suitable decomposition can be obtained by alternating palindromicity of $u\,a\,v$ and symmetry of $u\,a\,v\,b\,Sym(u)$.

As an example, the path in Fig. 2 has a decomposition of the form $(u\,a\,\bar{u}\,b)^t\,u\,a\,Sym(\tilde{u}_1)\,\tilde{u}_1\,(b\,\bar{u}\,a\,u)^t$, with $t = 0$. So, it turns out that $\tilde{u}_1 = ba$, $Sym(\tilde{u}_1) = \bar{u}_1 = ba$, $u_2 = abaababaa$ and the sub-word $u_2\,\bar{u}_1 = abaababaa\,ba$ is palindrome.

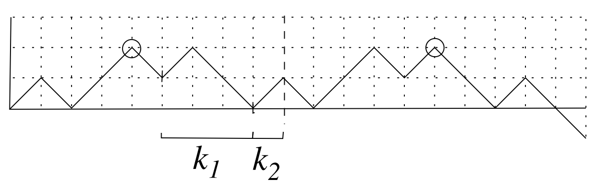

Figure 2: A fixed point of γ where $u = v_2$, and the decomposition of Corollary 1.

4 The algorithm for the construction of F_γ elements

As a consequence of Corollary 1, a generic Dyck word w is an element of F_γ if and only if

1. w is symmetric;

2. $w\,b$ is uniquely decomposable in two palindromes π_1 and π_2.

A direct consequence of (i) of Proposition 2 is that the decomposition of $w\,b$ in the two palindromes π_1 and π_2 can be obtained by first defining the points M and M', where M (resp. M') is the leftmost (resp. rightmost) point of w with greatest ordinate.
The path running from $(0,0)$ to M' is π_1, while the one from M' to the end of the path is π_2.
Figure 10 shows a path where the points M, M', and the paths π_1 and π_2 have been highlighted.

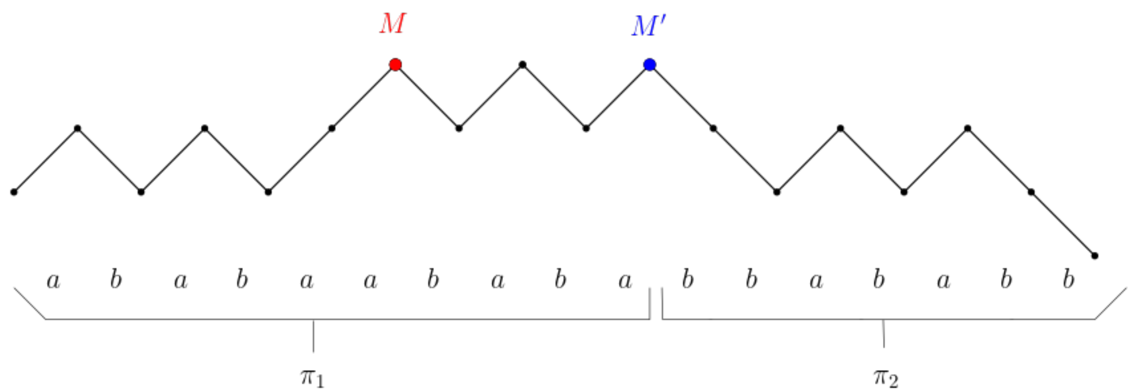

Figure 3: A path where M, M', π_1 and π_2 have been highlighted.

Our aim in this section is to provide a recursive description of the elements of F_γ. Given $w \in F_\gamma$ we first show that the following Proposition holds:

Proposition 3. *Let $w \in F_\gamma$ and M and M' defined as before. Then w can be uniquely decomposed as $x\,z\,Sym(x)$, where z is the subpath running from M to M' and $\widetilde{z} \in F_\gamma$.*

Proof. To prove that $\widetilde{z} \in F_\gamma$ is equivalent to showing that:

1. z is symmetric,

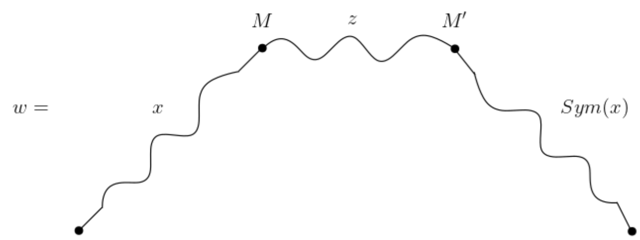

Figure 4: Decomposition of the path w

2. bz can be uniquely decomposed in two palindromes.

The first claim immediately follows from the fact that $w \in F_\gamma$ and therefore it is symmetric. Indeed M_1 and M_2 are two points one symmetric to the other and then the subpath whose extremes are M_1 and M_2 is symmetric.
Let us now prove 2. Since $z \neq \lambda$ (i.e. λ is the empty path) and $w \in F_\gamma$, then it follows from (iv) of Proposition 2 that z can be decomposed as $z = v_1\, a\, v_2$, v_2 possibly empty and v_1 and v_2 are palindromes.
Since v_2 is palindrome, $\widetilde{v_2}$ is palindrome too and then $\pi_2 = b\, \widetilde{v_2}\, b$ is palindrome. Moreover, $\pi_1 = \widetilde{v_1}$ is palindrome, since v_1 is a palindrome as well. Therefore π_1 and π_2 are palindromes.

□

We present an example of the previous decomposition.

Example 1. The path w represented in Figure 10 can be uniquely decomposed in xzy, where

$$x = ababaa$$

$$z = baba$$

$$y = bbababb$$

and $\beta(z) = baba \in F_\gamma$.

We now describe the following algorithm that generates all and only the paths of F_γ:

GenGammaPath(t)

Input: an array $t = (t_0, \ldots, t_n)$ of nonnegative integers, with $t_0 > 0$.
Output: a path $w \in F_\gamma$ of degree n.

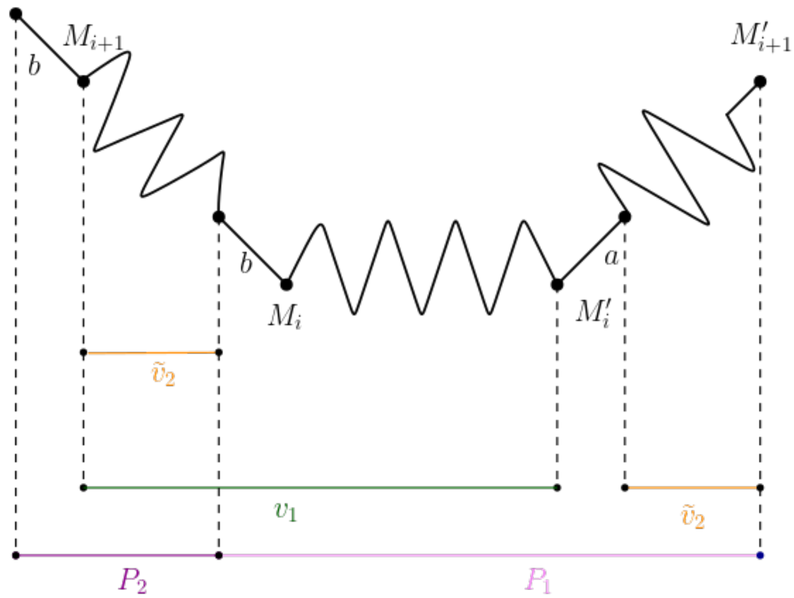

Figure 5: A part of a path

Part A Let $n = 2m$.
Initial step: Let $u_0 = a^{t_0-1}$ and $w_0 = a^{t_0}\, b^{t_0}$
For $i = 1, 2, \ldots, 2m$ **(Basic step):**

- If i is even then:
  $u_i = Sym(u_{i-1})(a\, w_{i-1})^{t_i}$ and $w_i = u_i\, a\, w_{i-1}\, b\, Sym(u_i)$;

- If i is odd then:
  $u_i = Sym(u_{i-1})(b\, w_{i-1})^{t_i}$ and $w_i = u_i\, b\, w_{i-1}\, a\, Sym(u_i)$;

The algorithm returns the word w_{2m}, and $2m$ is said the *degree of* w.

Part B Let $n = 2m + 1$.
Initial step: Let $u_0 = b^{t_0-1}$ and $w_0 = b^{t_0}\, a^{t_0}$
For $i = 1, 2, \ldots, 2m + 1$ **(Basic step):**

- If i is odd then:
  $u_i = Sym(u_{i-1})(a\, w_{i-1})^{t_i}$ and $w_i = u_i\, a\, w_{i-1}\, b\, Sym(u_i)$

- If i is even then:
  $u_i = Sym(u_{i-1})(b\, w_{i-1})^{t_i}$ and $w_i = u_i\, b\, w_{i-1}\, a\, Sym(u_i)$

The algorithm returns the word w_{2m+1}, and $2m+1$ is said the *degree of* w.

We present an example of application of **GenGammaPath**(t) to generate a path of degree 2.

Example 2. Let us consider the path depicted in Figure 6:
Let us show how to generate w using the algorithm **GenGammaPath**(t),

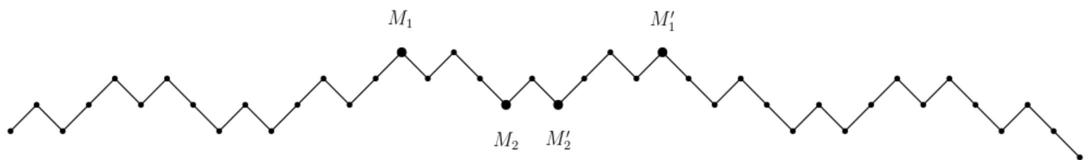

Figure 6: A path $w \in F_\gamma$ and its decomposition as w_0, w_1, w_2

with $t = (1, 1, 1)$.

Step 0. Since $t_0 = 1$, then $u_0 = \lambda$, and $w_0 = a^{t_0} b^{t_0} = ab$.

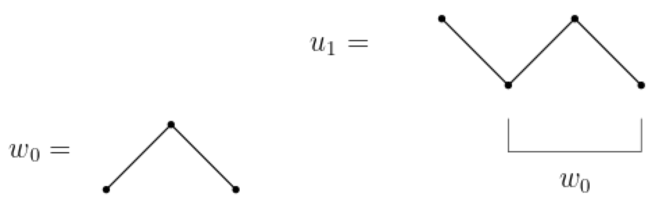

Figure 7: The path w_0, on the left, and the path u_1, on the right.

Step 1. Since $t_1 = 1$, then:
$u_1 = Sym(u_0)\,(b\,w_0)^{t_1} = \lambda(b\,w_0)^{t_1} = (bab)^{t_1} = bab$
$w_1 = u_1\,b\,w_0\,a\,Sym(u_1) = bab\,baba\,Sym(bab) = babbabaaba.$

Step 2. Since $t_2 = 1$, then

$u_2 = Sym(u_1)\,(a\,w_1)^{t_2} = Sym(bab)(ababbabaaba)^{t_2} = abaababbabaaba$
$w_2 = u_2\,a\,w_1\,b\,Sym(u_2) = abaababbabaabaababbabaababbabbabaababbab.$

Therefore, $w = abaababbabaabaababbabaababbabbabaababbabb.$

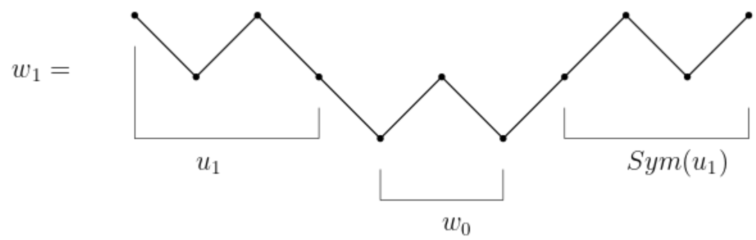

Figure 8: The factorization of w_1 in w.

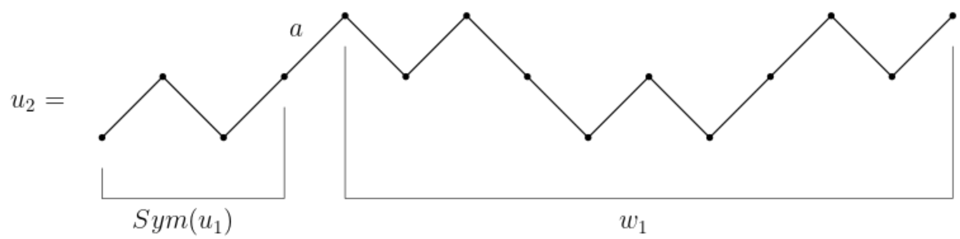

Figure 9: The factorization of u_2 in w.

The following theorem holds:

Theorem 4.1. *A path $w \in F_\gamma$ if and only if it can be obtained by the* ***GenGammaPath****(t).*

Proof. First of all we will prove that all the paths generated by **GenGammaPath**(t) are elements of F_γ, then we will show that all elements of F_γ are effectively generated by the **GenGammaPath**(t).
$(\Leftarrow)$ Given an element w generated by **GenGammaPath**(t), we prove by induction on the degree i of w that it belongs to F_γ.

Base step: If $i = 0$, then $w_0 = a^{t_0}b^{t_0}$, $t_0 \geq 1$.

The path w_0 is trivially symmetric and $w_0\, b$ is uniquely decomposable into two palindromes π_1 and π_2, hence $w_0 \in F_\gamma$.

Inductive step: Let us assume that our claim holds for $i = n$ and let us show that it also holds for $i = n + 1$.
So, let w_{n+1} be a path of degree $n + 1$ produced by **GenGammaPath**(t).
To prove that it belongs to F_γ we have to prove that:

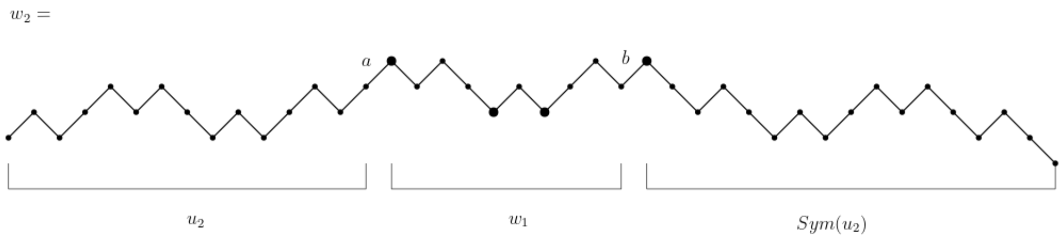

Figure 10: The factorization of w_2 in w.

1. w_{n+1} is symmetric,

2. $w_{n+1}\, b$ is uniquely decomposable into two palindromes π_1 and π_2.

Since the case in which $n + 1$ is odd is identical to the case in which $n + 2$ is even (we only have to exchange a and b and vice versa), we will treat, without loss of generality, only the case in which $n + 1$ is even.
By our construction, the path produced by **GenGammaPath**(t) is

$$w_{n+1} = u_{n+1}\ a\ w_n\ b\ Sym(u_{n+1})\,.$$

Moreover, we are assuming that it holds for n, then by the inductive argument we have that $w_n \in F_\gamma$. The we have:

1. w_{n+1} is trivially symmetric since:

   $$w_{n+1} = u_{n+1}\, a\, w_n\, b\, Sym(u_{n+1})\,,$$

   where $u_{n+1}\, a$ and $b\, Sym(u_{n+1})$ are symmetric to each by construction other and w_n is symmetric since $w_n \in F_\gamma$. Therefore w_{n+1} is symmetric since it is obtained from a symmetric path by attaching two symmetric pieces to the extremities.

2. Now we show that $w_{n+1}\ b$ is decomposable in two palindromes (for simplicity let us indicate $n+1$ by ℓ). More precisely, we will show that this decomposition is obtained as π_1 and π_2, where $\pi_1 = u_\ell\, a\, w_{\ell-1}$ and $\pi_2 = b\, Sym(u_\ell) b$.

   Let us start showing that π_2 is palindrome. We have that:

   $\pi_2 = b\, Sym(u_\ell) b$ is palindrome if and only if

   $Sym(u_\ell)$ is palindrome if and only if

   u_ℓ is palindrome if and only if

$Sym(u_{\ell-1})\,(a\,w_{\ell-1})^{t_\ell}$ is palindrome.

Therefore, let us show that $Sym(u_{\ell-1})\,(a\,w_{\ell-1})^{t_\ell}$ is palindrome.
We know that the building process has started with $w_0 = a^{t_0}b^{t_0}$ and we assume that $\ell > 0$ is even, then $\ell - 1$ is odd.
We recall thar $w_{\ell-1} = u_{\ell-1}bw_{\ell-2}aSym(u_{\ell-1})$. By inductive hypothesis $w_{\ell-1} \in F_\gamma$ then by (ii) of Proposition 2, $u_{\ell-1}$ and $u_{\ell-1}\,b\,w_{\ell-2}$ are palindromes. Let us consider some small values of t_ℓ: If $t_\ell = 0$, then

$$u_\ell = Sym(u_{\ell-1})\,(a\,w_{\ell-1})^0 = Sym(u_{\ell-1}).$$

Since $u_{\ell-1}$ is palindrome, also $Sym(u_{\ell-1})$ is palindrome.
Let us consider some small values of t_ℓ. If $t_\ell = 1$, then

$$\begin{aligned} u_\ell &= Sym(u_{\ell-1})(a\,w_{\ell-1}) \\ &= Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1}). \end{aligned}$$

Since $Sym(u_{\ell-1})$ is palindrome (indeed $u_{\ell-1}$ is palindrome) and $u_{\ell-1}bw_{\ell-2}$ is palindrome.
If $t_\ell = 2$, then

$$\begin{aligned} u_\ell &= Sym(u_{\ell-1})\,(a\,w_{\ell-2})^2 \\ &= Sym(u_{\ell-2})\,(a\,w_{\ell-1})\,(a\,w_{\ell-1}) \\ &= Sym(u_{\ell-2})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1})\,a\,u_{\ell-1}b\,w_{\ell-2}\,a\,Sym(u_{\ell-1}). \end{aligned}$$

The latter is palindrome indeed $Sym(u_{\ell-1})$ is palindrome (since $u_{\ell-1}$ is palindrome) and $u_{\ell-1}\,b\,w_{\ell-2}$ is palindrome.
In general, if ℓ is even:

$$u_\ell = Sym(u_{\ell-1})\,au_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,\ldots a,$$

$$Sym(u_{\ell-1})\,a\ldots u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1}).$$

The latter is palindrome indeed $Sym(u_{\ell-1})$ is palindrome (since $u_{\ell-1}$ is palindrome) and $u_{\ell-1}\,b\,w_{\ell-2}$ is palindrome.
Instead, if ℓ is odd:

$$u_\ell = Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\ldots u_{\ell-1}\,b,$$

$$w_{\ell-2}\ldots u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1})\,a\,u_{\ell-1}\,b\,w_{\ell-2}\,a\,Sym(u_{\ell-1}).$$

The latter is palindrome indeed $Sym(u_{\ell-1})$ is palindrome (since $u_{\ell-1}$ is palindrome) and $u_{\ell-1}\,b\,w_{\ell-2}$ is palindrome.

Therefore in both cases π_2 is palindrome.
Let us show that also $\pi_1 = u_\ell\, a\, w_{\ell-1}$ is palindrome.
Also in this case, by considering the smallest values of t_ℓ helps us in finding the proof.
If $t_\ell = 0$, then

$$\begin{aligned}\pi_1 &= Sym(u_{\ell-1})\ \lambda\ a\ u_{\ell-1} b\ w_{\ell-2}\ a\ Sym(u_{\ell-1}) \\ &= Sym(u_{\ell-1})\ a\ u_{\ell-1}\ b\ w_{\ell-2}\ a\ Sym(u_{\ell-1}).\end{aligned}$$

The latter is palindrome indeed $Sym(u_{\ell-1})$ is palindrome (since $u_{\ell-1}$ is palindrome) and $u_{\ell-1}\, b\, w_{\ell-2}$ is palindrome.
If $t_\ell = 1$, then

$$\pi_1 = u_\ell\ a\ w_{\ell-1} = Sym(u_{\ell-1})\ (a\ w_{\ell-1})\ a\ w_{\ell-1} = Sym(u_{\ell-1})(a\ w_{\ell-1})^2.$$

If $t_\ell = 2$, then

$$\begin{aligned}\pi_1 &= u_\ell\ a\ w_{\ell-1} \\ &= Sym(u_{\ell-1})\ (a\ w_{\ell-1})^2\ a\ w_{\ell-1} \\ &= Sym(u_{\ell-1})\ (a\ w_{\ell-1})\ (a\ w_{\ell-1})\ (a\ w_{\ell-1}) \\ &= Sym(u_{\ell-1})\ (a\ w_{\ell-1})^3.\end{aligned}$$

In general, given i,

$$\pi_1 = u_\ell\ a\ w_{\ell-1} = Sym(u_{\ell-1})(a\ w_{\ell-1})^{t_\ell}(a\ w_{\ell-1}) = Sym(u_{\ell-1})(a\ w_{\ell-1})^{t_{\ell+1}}.$$

Therefore, we can prove that π_1 is palindrome reasoning in an analogous way to what we did in the case of π_2.

($\Rightarrow$) Let us now prove that every path w that belongs to F_γ is obtained by **GenGammaPath**(t).
First of all, if w contains only one peak (resp. valley) then it is $w_0 = a^{t_0} b^{t_0}$ (resp. $w_0 = b^{t_0} a^{t_0}$) and therefore it has been produced by **GenGammaPath**(t).
Let us now assume that w contains more than one peak then, for reasons of symmetry, if it has an even (resp. odd) number of peaks then it has a valley (resp. peak) at the center of the path.
The two cases are identical, it is only a matter of exchanging the a and b and vice versa. So without loss of generality, we can only consider the case in which there is an odd number of peaks.

We are now ready to show that if $w \in F_\gamma$ then it is obtained by **GenGammaPath**(t).
Since $w \in F_\gamma$, for Proposition 3, w can be uniquely decomposed into $xzSym(x)$ where $\widetilde{z} \in F_\gamma$ and M (resp. M') is the highest leftmost (resp. rightmost) point of w.
Therefore $w = u\,a\,z\,b\,Sym(u)$. By definition, M is preceded by an up step, a, and M' is followed by a down step, b. From the Proposition 3, we get that z is a path of F_γ. Since w is obtained from z by attaching at the extremities of z, $\Omega = u\,a$ and $Sym(\Omega) = b\,Sym(u)$, then there exists an index i such that w is w_i and that z is w_{i-1}.
Since $Sym(u)$ is the symmetric of u as $w \in F_\gamma$ by hypothesis, then by definition of element belonging to F_γ, w is symmetric. Then w has the form of the paths produced by **GenGammaPath**(t).
To conclude that a generic element of F_γ is produced by **GenGammaPath**(t), it is enough to show that u is of the form $u_i = Sym(u_{i-1})(a\,w_{i-1})^{t_i}$.
Since $w_i \in F_\gamma$, by (ii) of Proposition 2, $u\,a\,w_{i-1}$ is palindrome and by (iv) of Proposition 2 w_{i-1} can be decomposed in $w_{i-1} = v_1\,a\,v_2$, where v_1 and v_2 are palindromes and v_2 is eventually void.
$u\,a\,w_{i-1}$ is palindrome, w_{i-1} is an inverted path and then $v_2 = Sym(u_{i-1})$. From the fact that $w \in F_\gamma$ and that $v \neq \lambda$, it follows that by (iv) of Proposition 2, $u = v_2\,(a\,v)^t$ with $t \geq 0$.
From the fact that $v_2 = Sym(u_{i-1})$ and that $u = v_2\,(a\,w_{i-1})^t$, it follows that $u = Sym(u_{i-1})(a\,w_{i-1})^{t_i}$.
Therefore every element of F_γ is obtained by **GenGammaPath**(t). $\square$

5 Conclusion and further works

The algorithm **GenGammaPath**(t) we have presented in the previous section produces a path $w = w(t)$ on input $t = (t_0, \ldots, t_n)$. A possible direction for further research is to use **GenGammaPath** to obtain a *Constant Amortized Time* (CAT) algorithm for the generation of the fixed points of γ. In order to prove that each of these paths is generated in constant amortized time we need to obtain more information about the enumeration of fixed points of γ according to their length. In particular, we expect to determine an explicit formula $p(n)$ that associates to an array t of length n the length of the path $w(t)$ obtained performing **GenGammaPath**(t). By some preliminary investigation we have that the following recurrence relation holds:

$$p(0) = 2t_0$$
$$p(n+1) = p(0) + p(n) + 2\sum_{i=0}^{n} t_{i+1}(p(i)+1)\,.$$

References

[1] M. Barnabei, F. Bonetti, N. Castronuovo, R. Cori, *Some permutations on Dyck words*, Theoretical Computer Science, 635, 51–63 (2016).

[2] M. Baker and S. Norine, *Riemann-Roch and Abel-Jacobi theory on a finite graph*, Advances in Mathematics, 215, 766–788 (2007).

[3] M. Bousquet-Mélou, Marko Petkovsek, Walks confined in a quadrant are not always D-finite, *Theor. Comput. Science* 307 (2003) 257-276.

[4] M. Bousquet-Mélou, A. Rechnitzer, The site-perimeter of bargraphs, *Adv. Appl. Math.*, 31 (2003) 86-112.

[5] Chomsky N., Schützenberger M.P.: The algebraic theory of context-free languages. Computer Programming and Formal Systems North-Holland Amsterdam (1963) 118–161.

[6] R. Cori and Y. Le Borgne, *The Riemann-Roch theorem for graphs and the rank in complete graphs*, arXiv 1308.5325 (2013).

[7] R. Cori and Y. Le Borgne *On Computation of Baker and Norine's rank on complete graphs*, Elec. J. Combin. 23 (2016), #P1.31.

[8] A. Dvoretsky, and T. Motzkin, *A problem of arrangements*, Duke Math. J., 14, 305–313 (1947).

[9] P. Flajolet, Analytic models and ambiguity of context free languages, *Theor. Comput. Sci.* 49 (1987) 283–309.

[10] A. J. Guttmann, Indicators of solvability for lattice models, *Disc. Math.* **217** (2000) 167–189.

[11] OEIS Foundation Inc., *The On-line Encyclopedia of Integer Sequences*, `http://oeis.org`, 2011.

[12] A. Rechnitzer, Haruspicy and anisotropic generating functions, *Adv. Appl. Math.*, 30 (2003) 228-257.

[13] A. Rechnitzer, Haruspicy 2: The self-avoiding polygon generating function is not D-finite, to appear in *J. Comb. Th., Series A.*