A Survey on Varieties Generated by Small Semigroups and a Companion Website

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November 15, 2019

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*Supported by the Fundação para a Ciência e Tecnologia (Portuguese Foundation for Science and Technology) through the project CEMAT-CIÊNCIAS UID/Multi/04621/2013, and through project Hilbert's 24th problem PTDC/MHC-FIL/2583/2014.

†Supported by the Fundação para a Ciência e Tecnologia (Portuguese Foundation for Science and Technology) through the project CEMAT-CIÊNCIAS UID/Multi/04621/2013

Abstract

Abstract The aim of this paper is to provide an atlas of identity bases for varieties generated by small semigroups and groups. To help the working mathematician easily find information, we provide a companion website that runs in the background automated reasoning tools, finite model builders, and GAP, so that the user has an automatic *intelligent* guide on the literature.

This paper is mainly a survey of what is known about identity bases for semigroups or groups of small orders, and we also mend some gaps left unresolved by previous authors. For instance, we provide the first complete and justified list of identity bases for the varieties generated by a semigroup of order up to 4, and the website contains the list of varieties generated by a semigroup of order up to 5.

The website also provides identity bases for several types of semigroups or groups, such as bands, commutative groups, and metabelian groups. On the inherently non-finitely based finite semigroups side, the website can decide if a given finite semigroup possesses this property or not. We provide some other functionalities such as a tool that outputs the multiplication table of a semigroup given by a C-presentation, where C is any class of algebras defined by a set of first order formulas.

The companion website can be found here
`http://sgv.pythonanywhere.com`
Please send any comments/suggestions to `jj.araujo@fct.unl.pt`

1 Introduction

We assume familiarity with the general theory of varieties, semigroups, and groups. For general references, we suggest the monographs of Almeida [1], Burris and Sankappanavar [7], Howie [20], McKenzie *et al*. [53], and H. Neumann [57].

Studying the lattice of varieties of semigroups is an old area of research, but given its complexity, this topic remains very active up to the present and certainly will continue into the foreseeable future. There are several very good surveys, such as Evans [13], Shevrin *et al.* [70], and Vernikov [86], that allow the reader to become familiar with the main results and problems; our goal is different. We aim at a living survey powered by a companion computational tool that helps the working mathematician finding either new results or locate old ones in the literature.

As an illustration, suppose that we are researchers in some area of mathematics who, for some reason, need to investigate semigroups satisfying the implication

$$xy \approx yx \implies x \approx y,$$

objects we might call *anti-commutative semigroups*. To understand their properties, we could use GAP to find some small models, as for example, the semigroup U_1 in Table 1. At a certain point, we observe that all elements of U_1 are idempotents—such a semigroup satisfies the idempotency identity $x^2 \approx x$ and is commonly called a *band*—and searching for varieties of bands we find a reference [14] that contains the lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands, as shown in Figure 1.

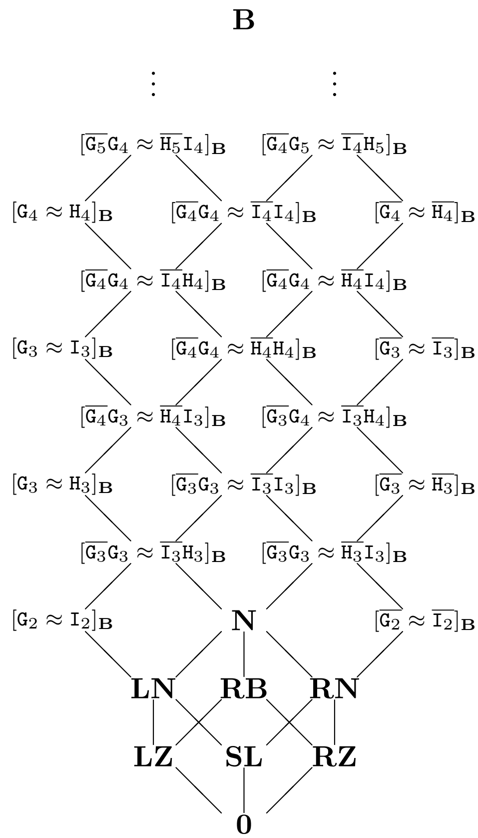
U_1	1	2	3	4
1	1	1	3	3
2	2	2	4	4
3	1	1	3	3
4	2	2	4	4

U_2	1	2	3	4	5
1	1	1	1	1	5
2	1	2	3	4	5
3	1	3	4	2	5
4	1	4	2	3	5
5	1	5	5	5	5

Table 1: The semigroups U_1 and U_2

Again we could use GAP to see that our semigroup U_1 violates the identities $xy \approx x$ and $xy \approx y$ but satisfies the identity $xyx \approx x$. Therefore the variety $\operatorname{var}\{U_1\}$ generated by U_1 is contained in the variety of bands defined by the identity $xyx \approx x$—the variety $\mathbf{RB}$ of *rectangular bands*—but is excluded from its two maximal subvarieties $\mathbf{LZ}$ and $\mathbf{RZ}$, whence $\operatorname{var}\{U_1\} = \mathbf{RB}$. Now an easy exercise shows that a semigroup is anti-commutative if and only if it satisfies the identity $xyx \approx x$, and from here we get access to an enormous amount of literature on our original object. The key steps in the above process were the observation that U_1 is a band and the complete knowledge of the lattice of varieties of bands.

Now suppose that we are working with a different theory and our test semigroup is U_2 in Table 1. Since U_2 is not a band, there is no general lattice, similar to Figure 1, that allows us to repeat what we have done with U_1. It turns out that the variety $\operatorname{var}\{U_2\}$ is defined by the identities



$$\mathbf{B} = [x^2 \approx x]$$
$$\mathbf{N} = [xyzx \approx xzyx]_{\mathbf{B}}$$
$$\mathbf{LN} = [xyz \approx xzy]_{\mathbf{B}}$$
$$\mathbf{RN} = [xyz \approx zyx]_{\mathbf{B}}$$
$$\mathbf{SL} = [xy \approx yx]_{\mathbf{B}}$$
$$\mathbf{RB} = [xyx \approx x]$$
$$\mathbf{LZ} = [xy \approx x]$$
$$\mathbf{RZ} = [xy \approx y]$$
$$\mathbf{0} = [x \approx y]$$

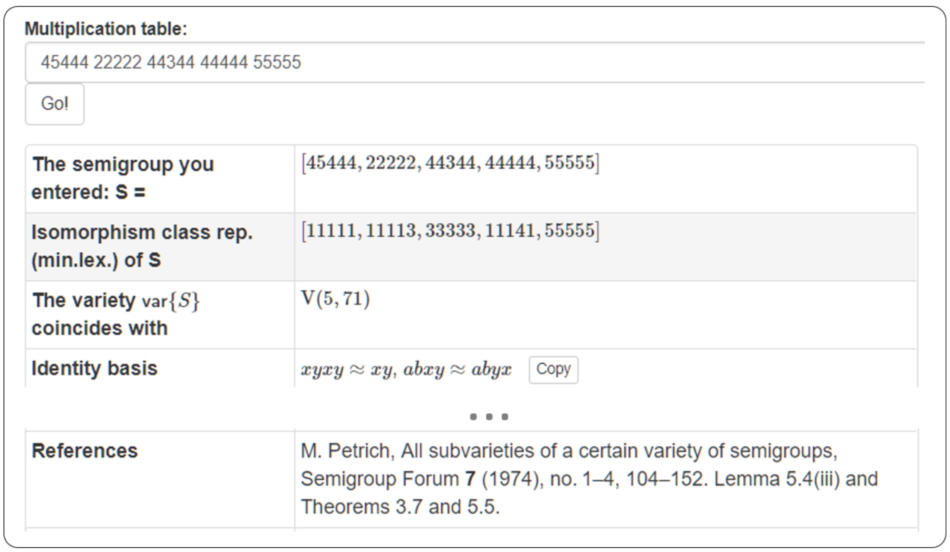
Figure 1: The lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands, where $[\mathbf{u} \approx \mathbf{v}]_{\mathbf{B}} = \mathbf{B} \cap [\mathbf{u} \approx \mathbf{v}]$ and details on the words $\mathtt{G}_n, \mathtt{H}_n, \mathtt{I}_n, \overline{\mathtt{G}_n}, \overline{\mathtt{H}_n}, \overline{\mathtt{I}_n}$ are given in Subsection 2.4.

$\{x^4 \approx x,\, xyx \approx yx^2\}$, but only a substantial search would allow us to locate a reference [77, Proposition 3.16].

In general, given a semigroup S of order up to 6, there is still a good chance that information on the variety var$\{S\}$ and its subvarieties can be found in the literature, since such varieties have received much attention over the years [12, 30–39, 42, 46, 47, 54, 77, 89, 92], especially in the investigation of the finite basis problem for small semigroups [10, 11, 40, 43, 48, 59, 63, 64, 76, 79, 81–83, 93]. The first goal of this survey is to provide such information, but we go far beyond that. The overall aim is to provide a survey on identity bases defining varieties generated by finite semigroups and set up a companion website, running GAP and automated reasoning tools in the background, that will be continuously updated to better assist the working mathematician. Resources provided by the present survey and the website so far are described as follows.

(a) Identity bases, and corresponding proofs or references, for all varieties generated by a semigroup of order up to 4. This survey is the first source providing this information.

(b) Identity bases for many varieties generated by semigroups of higher orders, including all semigroups of order 5, the proofs of which will be disseminated elsewhere.


Multiplication table:

45444 22222 44344 44444 55555

Go!

The semigroup you entered: S =	$[45444, 22222, 44344, 44444, 55555]$
Isomorphism class rep. (min.lex.) of S	$[11111, 11113, 33333, 11141, 55555]$
The variety var$\{S\}$ coincides with	$\mathbf{V}(5, 71)$
Identity basis	$xyxy \approx xy$, $abxy \approx abyx$ Copy

• • •

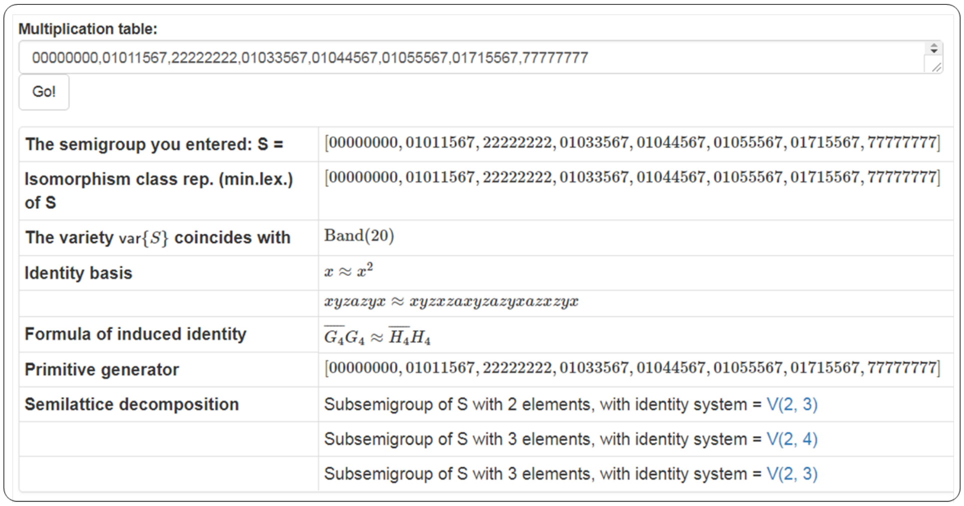
References	M. Petrich, All subvarieties of a certain variety of semigroups, Semigroup Forum **7** (1974), no. 1–4, 104–152. Lemma 5.4(iii) and Theorems 3.7 and 5.5.

Figure 2: Companion website: example of a reference given for an order 5 semigroup

(c) Identity bases for all varieties generated by a group that has abelian normal and factor subgroups N and G/N such that $\gcd(|N|, |G/N|) = 1$.

(d) For some classes of semigroups, including bands and some classes of groups, the website finds identity bases for varieties generated by arbitrarily large finite models; see Figure 3 on page 7.

(e) For a given finite semigroup S, the companion website gives bibliographic information about the variety var$\{S\}$, its prime decomposi-



Multiplication table:

00000000,01011567,22222222,01033567,01044567,01055567,01715567,77777777

Go!

The semigroup you entered: S =	[00000000, 01011567, 22222222, 01033567, 01044567, 01055567, 01715567, 77777777]
Isomorphism class rep. (min.lex.) of S	[00000000, 01011567, 22222222, 01033567, 01044567, 01055567, 01715567, 77777777]
The variety var$\{S\}$ coincides with	Band(20)
Identity basis	$x \approx x^2$
	$xyzazyx \approx xyzxzaxyzazyxazxzyx$
Formula of induced identity	$\overline{G_4}G_4 \approx \overline{H_4}H_4$
Primitive generator	[00000000, 01011567, 22222222, 01033567, 01044567, 01055567, 01715567, 77777777]
Semilattice decomposition	Subsemigroup of S with 2 elements, with identity system = V(2, 3)
	Subsemigroup of S with 3 elements, with identity system = V(2, 4)
	Subsemigroup of S with 3 elements, with identity system = V(2, 3)

Figure 3: Companion website: the variety generated by an order 8 band

tion, varieties that cover it, and a generator for var$\{S\}$ of minimal order; see Figure 4 on page 34.

(f) The *vector* of a semigroup S of order n, denoted by $\vec{v}(S)$, is the vector of dimension n^2 that is formed by concatenating the n rows of the Cayley table of S. For example, the vector $\vec{v}(J)$ of the semigroup J in Table 2 is $[1,1,1,1,1,1,1,2,3]$; it is unambiguous, and in fact clearer, to only use commas to separate different rows, that is,

$$\vec{v}(J) = [111, 111, 123].$$

The isomorphic copies of a given semigroup can then be lexicographically ordered as vectors; for example, the semigroup J' in Table 2 is isomorphic to J, but since

$$\vec{v}(J) = [111, 111, 123] <_{\text{lex}} [333, 123, 333] = \vec{v}(J'),$$

we place J before J'. The companion website finds the smallest element in each isomorphism class and this is the standard form of the output; of course, this is an expensive feature that can only be applied to semigroups of relatively small order (up to 11). See Figure 5 on page 35.

(g) For any finitely generated variety $\mathbf{V}$, there exist only finitely many

J	1	2	3
1	1	1	1
2	1	1	1
3	1	2	3

J'	1	2	3
1	3	3	3
2	1	2	3
3	3	3	3

Table 2: The semigroups J and J'

non-isomorphic generators of minimal order, say $S_1, S_2, \ldots, S_k$ with

$$\vec{v}(S_1) <_{\text{lex}} \vec{v}(S_2) <_{\text{lex}} \cdots <_{\text{lex}} \vec{v}(S_k).$$

Then S_1 is called the *primitive generator* of $\mathbf{V}$. Each variety has a label $\mathrm{V}(n, k)$, where n is the order of its primitive generator S and k is the number of primitive semigroups of order n for other varieties that lexicographically precede S. For example, the variety $\mathrm{var}\{J\}$ is defined by the identities $\{x^2a \approx xa, xy^2 \approx yx^2\}$ and is labeled $\mathrm{V}(3, 3)$, meaning that its primitive generator has order 3—which happens to be J—and there are two semigroups of order 3 with vectors preceding $\vec{v}(J)$ that are primitive generators for two other distinct varieties, namely $\mathrm{V}(3, 1)$ and $\mathrm{V}(3, 2)$. See Figure 6 on page 35.

(h) In many cases, the website provides a presentation for the primitive generator of the given variety. Conversely, the user can introduce a semigroup as a semigroup presentation in any variety or quasi-variety. For instance, Kiselman [26] considered the semigroup with the presentation

$$\left\langle c, \ell, m \;\middle|\; \begin{array}{l} c^2 = c,\ \ell^2 = \ell,\ m^2 = m,\ c\ell c = \ell c,\ \ell c\ell = \ell c, \\ cmc = mc,\ mcm = mc,\ \ell m\ell = m\ell,\ m\ell m = m\ell \end{array} \right\rangle$$

while investigating some operators in convexity theory. In less than a second the website shows that this semigroup has 17 elements as a semigroup presentation (as shown in Kiselman [26]), 7 elements as a band presentation, and 3 elements as a left cancellative semigroup presentation, etc. See Figure 7 on page 36.

(i) The website does not provide an identity basis for the variety generated by the Kiselman semigroup of order 17 computed above. However, it will say that the Kiselman semigroup of order 7 generates the variety of semilattices whose primitive generator is the chain of length two. If a given semigroup S of arbitrarily finite order generates a variety whose primitive generator has order 5 or less, then the website will automatically provide an identity basis for the variety var$\{S\}$. See Figure 8 on page 37.

(j) In addition to presentations, the user can input semigroups by giving the Cayley table, with several formats and on different sets that one can define, or by introducing identification numbers in the GAP libraries of small groups or small semigroups. See Figure 9 on page 38.

(k) The companion website also provides information on dual varieties or self-dual ones when applicable. Recall that a variety of semigroups is *self-dual* if it is closed under anti-isomorphism.

(l) Let $\mathbf{E}_n$ denote the variety of unary semigroups defined by the identities

$$xx^* \approx x^*x, \quad x(x^*)^2 \approx x^*, \quad x^{n+1}x^* \approx x^n.$$

Then the proper inclusions $\mathbf{E}_1 \subset \mathbf{E}_2 \subset \mathbf{E}_3 \subset \cdots$ hold, and for any finite semigroup S, there exist a unary operation * on S and some minimal $n \geq 1$ such that $(S, {}^*)$ is a unary semigroup in $\mathbf{E}_n$; the companion website finds this natural number n. For more information on the operation * and the varieties $\mathbf{E}_n$, see Subsection 2.6 and Shevrin [69].

(m) A semilattice Y is a partially ordered set in which every pair $i, j \in Y$ of elements has a greatest lower bound $i \wedge j$, called the *meet* of i and j. A semigroup S is a *semilattice of semigroups* if there exist a semilattice $(Y, \leq)$ and a family $\{S_i\}_{i \in Y}$ of semigroups indexed by Y such that $S = \bigcup_{i \in Y} S_i$ and $S_i S_j \subseteq S_{i \wedge j}$. Every semigroup can be decomposed as a semilattice of semigroups $\{S_i\}_{i \in Y}$ with each S_i being semilattice indecomposable [73]. Based on results from Tamura [74], the companion website finds the largest semilattice decomposition of a given semigroup S into semilattice indecomposable semigroups $\{S_i\}_{i \in Y}$, and provides the variety generated by each S_i. This tool can be used on a relatively large semigroup S, even when we cannot determine an identity basis for the variety var$\{S\}$.

(n) Let Σ be some given first order theory. The companion website can find all the varieties $\mathbf{V}$ in the database such that $\Sigma \vdash \mathbf{V}$ or $\mathbf{V} \models \Sigma$.

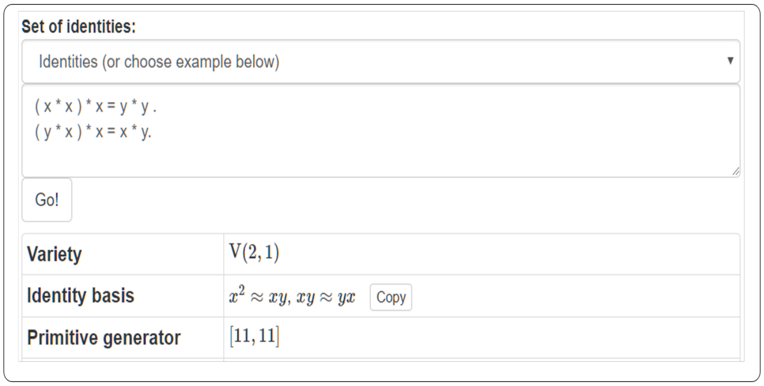


Figure 11: Companion website: a set of identities entered by the user is found to be equivalent to an identity basis for a variety in the database

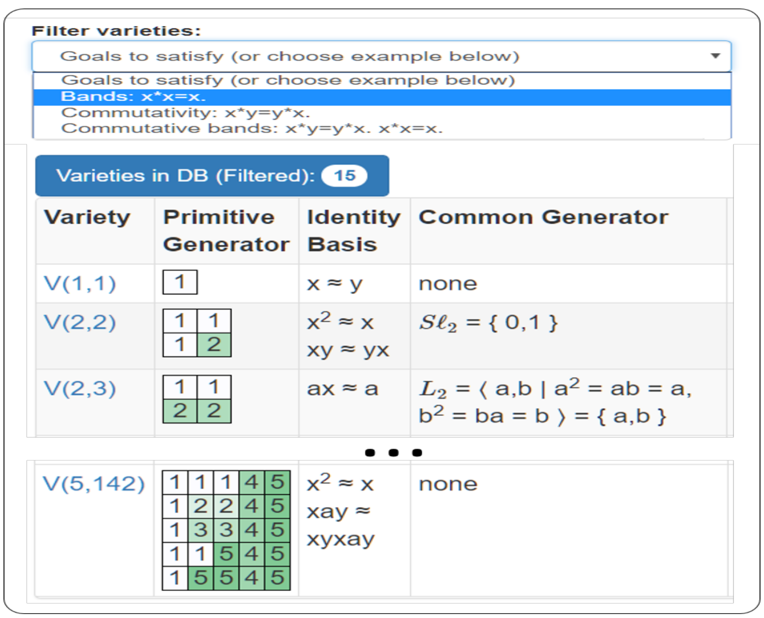


Figure 12: Companion website: finding all varieties in the database that satisfy some given conditions

(o) The companion website can provide conjectures for the variety generated by a large semigroup, using an algorithm that gave the correct result on all semigroups up to order 5. However, given the computational cost of this algorithm, anyone interested should first contact one of the authors.

Let S be the semigroup with universe $\{1, \ldots, 5\}$ and whose Cayley table has the following rows: $11111, 11111, 11113, 44444, 12345$. The variety generated by S is defined by the identities:

$$x^3 = x^2 \quad x^2yx = xyx \quad xyxz = x^2yz \quad xyz^2x = x^2yz^2.$$

Our algorithm produced the following candidate:

$$x^3 = x^2 \quad x^2yx = xyx \quad xyxz = x^2yz \quad xy^2x = x^2y^2.$$

It is easy to see that the two sets are equivalent and hence our candidate base is in fact a base for the variety generated by S. Note that the two bases differ only on the last identity, with the elegance prize going to the one found by the computer.

(p) In particular, the companion website can give some information about user's conjectures. Suppose that we have a semigroup S and guess that a certain set Σ of identities is an identity basis for $\mathrm{var}\{S\}$. Then the website will try to see if $\mathrm{var}\{S\}$ is in the database; if yes, it will try to prove if the stored identity basis is equivalent to the given one and return the result; it is very unlikely that no result is returned in such a case. If $\mathrm{var}\{S\}$ does not belong to the database, then the website will try to find identities holding in S, but not provable from Σ. If some are found, then the result is returned. Otherwise, the user's conjecture is returned as a reasonable one.

(q) We will keep the website updated with new discovered results in order to have a state of the art tool assisting the work of mathematicians.

In Section 2, we give some background material on varieties of groups and of semigroups, the lattice of varieties of bands, varieties with infinitely many subvarieties, an infinite chain of varieties of epigroups, and semilattice decompositions of semigroups. Section 3 is dedicated to a survey of some known results on varieties generated by small groups; it consists of mostly old material and we collect the main results here to highlight the gaps waiting to be filled. It is our conviction that the topic was more or less *abandoned*, not because everything was too easy, but exactly the opposite. Given the classification of finite simple groups, perhaps it is time for group theorists to start looking into varieties of groups again. In addition, for experts in semigroup theory, it might be useful to know to which varieties of groups belong the maximal subgroups (the $\mathcal{H}$-classes) of the semigroup. Section 4 deals with varieties generated by semigroups of order 5, and also treats the case of inherently non-finitely based finite semigroups. Section 5 introduces the features of the companion website and explains how to use it. Section 6 provides the database of varieties generated by semigroups of orders up to 4. Then we have a section on problems, and three appendix sections providing justifications of results in Section 6.

2 Preliminaries

2.1 Isomorphic semigroups and lexicographic minimum

Two algebras A and B of the same type are said to be *isomorphic*, indicated by $A \cong B$, if there exists an isomorphism between them. The relation $\cong$ is an equivalence relation on any class of algebras of the same type. Occasionally, given a finite algebra A, it is practical to have a canonical representative of the equivalence class $[A]_{\cong}$. For a semigroup S, an obvious choice for the representative of the class $[S]_{\cong}$ is the semigroup whose vector lexicographically precedes the vectors of all other semigroups in $[S]_{\cong}$. For instance, consider the semigroup

$$P = \langle a, b \mid ab = a,\ ba = 0,\ b^2 = b \rangle = \{0, a, b\}.$$

Then there are six semigroups on the set $\{1, 2, 3\}$ that are isomorphic to P, as shown in Table 3. Since $\vec{v}(S_1) \leq_{\text{lex}} \vec{v}(S_i)$ for all $i \neq 1$, the semigroup S_1 is the representative of the class $[P]_{\cong}$.

S_1	1	2	3
1	1	1	1
2	1	1	2
3	1	1	3

S_2	1	2	3
1	1	1	1
2	1	2	1
3	1	3	1

S_3	1	2	3
1	1	2	2
2	2	2	2
3	3	2	2

S_4	1	2	3
1	1	3	3
2	2	3	3
3	3	3	3

S_5	1	2	3
1	2	2	1
2	2	2	2
3	2	2	3

S_6	1	2	3
1	3	1	3
2	3	2	3
3	3	3	3

Table 3: Semigroups isomorphic to P

The *dual* of a semigroup S, denoted by $\overleftarrow{S}$, is the semigroup obtained from S by reversing its operation, that is, for any $a, b \in \overleftarrow{S} = S$, the product ab in $\overleftarrow{S}$ is equal to the product ba in S. The Cayley table of $\overleftarrow{S}$ is obtained simply by transposing the Cayley table of S. For instance, the semigroup $\overleftarrow{S_1}$ is isomorphic to the semigroup J in Table 2. The *dual* of a variety $\mathbf{V}$ is the variety $\overleftarrow{\mathbf{V}} = \{\overleftarrow{S} \mid S \in \mathbf{V}\}$. A variety $\mathbf{V}$ is *self-dual* if $\mathbf{V} = \overleftarrow{\mathbf{V}}$.

Two semigroups S and T are *equivalent* if either $S \cong T$ or $\overleftarrow{S} \cong T$. In the GAP package Smallsemi, semigroups are stored up to equivalence but not up to isomorphism, a decision not without some disadvantages. In this paper, unless otherwise stated, we work with semigroups up to isomorphism.

2.2 Varieties of semigroups

The variety generated by an algebra A, denoted by $\text{var}\{A\}$, is the smallest class of algebras of the same type containing A that is closed under the formation of homomorphic images, subalgebras, and arbitrary direct products. Since a variety $\text{var}\{A\}$ coincides with the class of all algebras that satisfy the identities of A, two algebras generate the same variety if and only if they satisfy the same identities. It is clear that if A and B are isomorphic algebras, then $\text{var}\{A\} = \text{var}\{B\}$; however, the converse does not hold in general, even if the algebras A and B have the same order. For example, the dihedral group D_4 and the quaternion group Q are groups of order 8 that generate the same variety [91], but they are not isomorphic.

Up to isomorphism, the number of semigroups of order up to five is 2,133 [96, A027851], while the number of varieties generated by these semigroups is only 218.

A *identity basis* for a variety $\mathbf{V}$ is a set of identities holding in $\mathbf{V}$ from which all other identities of $\mathbf{V}$ can be deduced. A variety is *finitely based* if it possesses a finite identity basis. Since a semigroup satisfies the same identities as the variety it generates, it is unambiguous to define an *identity basis* for a semigroup S to be an identity basis for $\text{var}\{S\}$, and say that S is *finitely based* whenever $\text{var}\{S\}$ is finitely based. Every variety generated by a semigroup of order at most 5 is finitely based, but up to isomorphism, precisely four semigroups of order 6 are non-finitely based [44]; see Subsection 4.2

2.3 Varieties of groups

For a general reference on varieties of groups, we recommend the monograph of H. Neumann [57]. Unlike what happens in semigroups, every variety generated by a finite group has a finite identity basis, and in group theory, every finite set of identities is equivalent to a single identity. Therefore every variety generated by a finite group can be defined by a single identity. We will see a similar phenomenon in the variety of bands below. More details on varieties of groups can be found in Section 3.

2.4 The lattice of varieties of bands

A description of the lattice $\mathscr{L}(\mathbf{B})$ of varieties of bands can be found in Birjukov [3], Fennemore [14], Gerhard [15], Gerhard and Petrich [17], and Howie [20]; see Figure 1. At the very top of the lattice is the variety $\mathbf{B} = [x^2 \approx x]$ of all bands. In the lower region is the sublattice $\mathscr{L}(\mathbf{N})$ of $\mathscr{L}(\mathbf{B})$ consisting of eight varieties:

$$\begin{aligned}
\mathbf{N} &= [xyzx \approx xzyx]_{\mathbf{B}}, && \text{normal bands;}\\
\mathbf{LN} &= [xyz \approx xzy]_{\mathbf{B}}, && \text{left normal bands;}\\
\mathbf{RN} &= [xyz \approx yxz]_{\mathbf{B}}, && \text{right normal bands;}\\
\mathbf{SL} &= [xy \approx yx]_{\mathbf{B}}, && \text{semilattices;}\\
\mathbf{RB} &= [xyx \approx x], && \text{rectangular bands;}\\
\mathbf{LZ} &= [xy \approx x], && \text{left zero bands;}\\
\mathbf{RZ} &= [xy \approx y], && \text{right zero bands;}\\
\mathbf{0} &= [x \approx y], && \text{trivial bands.}
\end{aligned}$$

The remaining varieties in the lattice $\mathscr{L}(\mathbf{B})$ are defined by identities that are formed by the words $\{\mathtt{G}_n, \mathtt{H}_n, \mathtt{I}_n \mid n \geq 2\}$ inductively defined as follows:

$$\begin{aligned}
&\mathtt{G}_2 = x_2x_1, && \mathtt{H}_2 = x_2, && \mathtt{I}_2 = x_2x_1x_2,\\
\text{and}\quad &\mathtt{G}_n = x_n\overline{\mathtt{G}_{n-1}}, && \mathtt{H}_n = \mathtt{G}_nx_n\overline{\mathtt{H}_{n-1}}, && \mathtt{I}_n = \mathtt{G}_nx_n\overline{\mathtt{I}_{n-1}}, \qquad \text{for all } n \geq 3,
\end{aligned}$$

where $\overline{X}$ is the word X written in reverse. For example,

$$[\mathtt{G}_3 \approx \mathtt{H}_3]_{\mathbf{B}} = [x_3x_1x_2 \approx x_3x_1x_2x_3x_2,\ x^2 \approx x].$$

By simple inspection of the identities in Figure 1, it is clear that the varieties in column 3 are self-dual, the varieties in columns 1 and 5 are dual to each other, and the varieties in columns 2 and 4 are dual to each other.

The variety generated by a band B is the variety $\mathbf{V}$ of bands that satisfies both of the following properties: B belongs to $\mathbf{V}$ and B is excluded from every maximal subvariety of $\mathbf{V}$. When a semigroup S is entered into the companion website, there is a first test to check if S is a band. In the affirmative case, the website crawls up the lattice in Figure 1; the first identity satisfied by S defines the variety var$\{S\}$.

2.5 Varieties with infinitely many subvarieties

A variety that contains only finitely many subvarieties is said to be *small*. It easily follows from the well-known theorem of Oates and Powell [58] that every finite group generates a small variety of semigroups. But this result does not hold in general. A small counterexample is the monoid N_2^1 obtained by adjoining an identity element to the nilpotent semigroup $N_2 = \langle a \mid a^2 = 0 \rangle$ of order 2; see Figure 13 on page 40. Not only is the variety $\mathbf{N_2^1} = \mathrm{var}\{N_2^1\}$ not small [13], it is the only non-small variety among all varieties generated by a semigroup of order 3 or less; see Section 6.

As for the variety generated by a semigroup of order greater than 3, properties more extreme than being non-small can be satisfied. For instance, there exist

- semigroups of order 4 that generate varieties that are *finitely universal* [32] in the sense that their lattices of subvarieties each embeds all finite lattices;
- semigroups of order 6 that generate varieties with continuum many subvarieties [12, 22].

All examples of varieties with continuum many subvarieties discovered so far are also finitely universal. It is unknown if there exists a variety with continuum many subvarieties that is not finitely universal. Refer to Shevrin *et al*. [70] for a survey of results regarding other properties satisfied by lattices of varieties.

Given a finite semigroup, it is of natural interest to determine if it generates a small variety. Whether or not smallness of a variety is decidable remains open, but some special case has been found. Recall that an identity of the form

$$x_1 x_2 \cdots x_n \approx x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(n)},$$

where π is some nontrivial permutation on $\{1, 2, \ldots, n\}$, is called a *permutation identity*, while a nontrivial identity of the form

$$x_1 x_2 \cdots x_n \approx \mathbf{w}$$

that is not a permutation identity is said to be *diverse*.

Proposition 2.1 (Malyshev [51])**.** *Any variety that satisfies some permutation identity and some diverse identity is small.*

2.6 Epigroups

Let S be a semigroup. An element $a \in S$ is an *epigroup element* if there exists an integer $n \geq 1$ such that a^n belongs to a subgroup of S, that is, the $\mathcal{H}$-class H_{a^n} of a^n is a group; if $n = 1$, then a is said to be *completely regular*. If we denote by e the identity element of H_{a^n}, then ae is in H_{a^n} and we define the *pseudo-inverse* a' of a by $a' = (ae)^{-1}$, where $(ae)^{-1}$ denotes the inverse of ae in the group H_{a^n} [69, Subsection 2.1]. An *epigroup* is a semigroup consisting entirely of epigroup elements, and a *completely regular semigroup* is a semigroup whose elements are all completely regular. The important fact for us is that all finite semigroups are examples of epigroups. Following Petrich and Reilly [62] for completely regular semigroups and Shevrin [69] for epigroups, it is now customary to consider an epigroup or a completely regular semigroup $(S, \cdot)$ as a *unary* semigroup $(S, \cdot, ')$, where $x \mapsto x'$ is the map sending each element to its pseudo-inverse.

For any semigroup S, let $\mathrm{Epi}(S)$ denote the set of all epigroup elements of S and let $\mathrm{Epi}_n(S)$ denote the subset of $\mathrm{Epi}(S)$ consisting of elements of index bounded by n. Then the inclusions

$$\mathrm{Epi}_1(S) \subseteq \mathrm{Epi}_2(S) \subseteq \cdots \subseteq \bigcup_{n\geq 1} \mathrm{Epi}_n(S) = \mathrm{Epi}(S)$$

hold, where $\mathrm{Epi}_1(S)$ consists of completely regular elements of S, and $\mathrm{Epi}(S) = S$ if and only if S is an epigroup.

For any $a \in \mathrm{Epi}_n(S)$, let e_a denote the identity element of the group H_{a^n}. Then $ae_a = e_a a$ is in H_{a^n} and the definition of *pseudo-inverse* introduced above leads to a characterization of the epigroup elements of the semigroup: $a \in \mathrm{Epi}(S)$ if and only if there exist some $n \geq 1$ and some (necessarily unique) element $a' \in S$ such that

$$a'aa' = a', \quad aa' = a'a, \quad a^{n+1}a' = a^n; \tag{2.1}$$

see Shevrin [69, Section 2]. If a is an epigroup element, then so is a' with $a'' = aa'a$. The element a'' is always completely regular and $a''' = a'$. A standard notation in finite semigroup theory is to write $a^\omega = aa'$ for an epigroup element a; see, for example, Almeida [1]. Then

$$a^\omega = a''a' = a'a'', \quad (a')^\omega = (a'')^\omega = a^\omega,$$

and more generally, for any $m \geq 1$,

$$a^\omega = (aa')^m = (a')^m a^m = a^m (a')^m.$$

For each $n \geq 1$, the class $\mathbf{E}_n$ consisting of all epigroups S such that $S = \mathrm{Epi}_n(S)$ is a variety; in particular, $\mathbf{E}_1$ is the class of completely regular semigroups. The chain $\mathbf{E}_1 \subset \mathbf{E}_2 \subset \mathbf{E}_3 \subset \cdots$ of varieties has the following property [69]: for any variety $\mathbf{V}$ of epigroups, there exists a smallest $n \geq 1$ such that $\mathbf{V} \subseteq \mathbf{E}_n$. Given a finite semigroup S, the companion website finds the smallest n such that $S \in \mathbf{E}_n$. This gives some occasionally useful information about the given semigroup, but of course it does not match knowing an identity basis for $\mathrm{var}\{S\}$.

2.7 Semilattice decompositions of semigroups

There are many ways that a semigroup can be decomposed into smaller subsemigroups, for example, direct products, subdirect products, and Zappa–Szép extensions. Some has the property that each component cannot be further decomposed using the same tool, in which case the decomposition is said to be *atomic*. An obvious example of atomic decompositions for finite algebras is the direct product decomposition as, resorting on an argument similar to the one used to prove that every natural number is a product of prime numbers, we can easily show that every finite algebra can be decomposed in a direct product of directly indecomposable algebras. Finding atomic decompositions of infinite semigroups is more difficult; according to Bogdanović *et al*. [5], there are only five known atomic decompositions of general semigroups: semilattice decompositions [73], ordinal decomposition [50], U-decomposition [68], orthogonal decomposition [6], and the general subdirect decomposition whose atomicity was proved by Birkhoff.

Here we will concentrate on semilattice decompositions of semigroups. We saw above that a semilattice is a commutative band. It is easy to prove that every semilattice Y induces a partially ordered set in which every pair $i, j \in Y$ of elements has a meet $i \wedge j$; conversely, every such partially ordered set induces a semilattice. Therefore, the term *semilattice* is commonly used to refer to a commutative band or a partially ordered set admitting meet of every pair of elements. In this subsection it is more convenient to use it in the latter sense.

A semigroup S is a *semilattice of semigroups* if there exist a semilattice $(Y, \leq)$ and a collection $\{S_i\}_{i\in Y}$ of semigroups indexed by Y such that $S = \bigcup_{i\in Y} S_i$ and $S_iS_j \subseteq S_{i\wedge j}$. Every semigroup can be decomposed as a semilattice $\{S_i\}_{i\in Y}$ of semigroups S_i that are semilattice indecomposable [73].

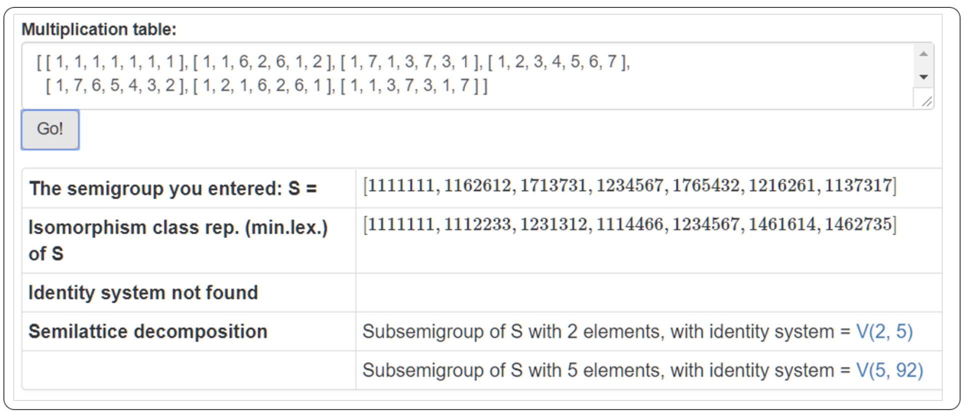
In Tamura [74], two equivalent ways of finding the smallest semilattice congruence are provided. For any semigroup S, let S^1 denote the smallest monoid containing S, that is,

$$S^1 = \begin{cases} S & \text{if } S \text{ is not a monoid,} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$$

Then the smallest semilattice decomposition of S is the smallest partition containing the sets

$$\{(x, y) \in S^1 \times S^1 \,|\, \{xy, yx, xyx\}\}.$$

The companion website finds the largest semilattice decomposition of a given semigroup S into semilattice indecomposable semigroups $\{S_i\}_{i \in Y}$, and provides the variety generated by each S_i. This tool can be used on a semigroup S of relatively large order, even when we cannot determine an identity basis for the variety var$\{S\}$.



Multiplication table:

[[1, 1, 1, 1, 1, 1, 1], [1, 1, 6, 2, 6, 1, 2], [1, 7, 1, 3, 7, 3, 1], [1, 2, 3, 4, 5, 6, 7], [1, 7, 6, 5, 4, 3, 2], [1, 2, 1, 6, 2, 6, 1], [1, 1, 3, 7, 3, 1, 7]]

Go!

The semigroup you entered: S =	[1111111, 1162612, 1713731, 1234567, 1765432, 1216261, 1137317]
Isomorphism class rep. (min.lex.) of S	[1111111, 1112233, 1231312, 1114466, 1234567, 1461614, 1462735]
Identity system not found	
Semilattice decomposition	Subsemigroup of S with 2 elements, with identity system = V(2, 5)
	Subsemigroup of S with 5 elements, with identity system = V(5, 92)

Figure 14: Companion website: semilattice decomposition of an order 7 semigroup

3 Varieties of groups

The theory of varieties of groups differs from that of semigroups in several ways, which will be briefly mentioned here. In particular, after a decade of activity, the monograph [57] of H. Neumann was published; this is still the best reference for the subject. Also, the notation used in H. Neumann [57] became standard among group theorists: we will point out some of the differences. In particular, varieties of groups are typically denoted by Fraktur capital letters, such as $\mathfrak{A}$ for the variety of abelian groups; following the usage established earlier, we will use bold-face letters such as **A** instead.

3.1 The basics

As briefly noted in Section 2.3, every group identity can be put into the form $\mathbf{w} \approx 1$, where $\mathbf{w}$ is a word in the variables and their inverses. We can regard $\mathbf{w}$ as an element of the free group $F(X)$ over a countable set X of variables. The identities satisfied by a variety $\mathbf{V}$ form a *fully invariant subgroup* of $F(X)$, one mapped into itself by all endomorphisms of the group. Thus there is a bijection between varieties of groups and fully invariant subgroups of $F(X)$.

Each finite nontrivial group with finite exponent $e \geq 2$ satisfies the identity $x^e \approx 1$ and so also the identity $x^{e-1} \approx x^{-1}$. Therefore any identity of a finite group is equivalent to one of the form $\mathbf{w} \approx 1$, where $\mathbf{w}$ is a semigroup word. In fact, a more specific result holds. Recall that a *commutator word* is an element of the derived subgroup of the free group. Alternatively, a commutator word can be described as one in which the sum of the exponents of every variable is 0.

Theorem 3.1 (B. H. Neumann [56]). *Every identity of a finite group with exponent e is equivalent to $\{x^e \approx 1, \mathbf{w} \approx 1\}$ for some commutator word $\mathbf{w}$.*

A *factor* of a group G is a quotient of a subgroup of G, that is, H/K where $K \trianglelefteq H \leq G$; it is *proper* unless $H = G$ and $K = 1$. A *chief factor* is one where $K \trianglelefteq G$ and H/K is a minimal normal subgroup of G/K; a *composition factor* is a factor H/K, when H and K subnormal in G (that is, terms in a descending series in which each term is normal in its predecessor) and K is a maximal normal subgroup of H.

If A and B are subgroups of G, then $[A, B]$ is the subgroup generated by the commutators in $\{[a, b] \mid a \in A, b \in B\}$. The *lower central series* is the descending series $G = G_1 > G_2 > \cdots$ with $G_{i+1} = [G_i, G]$; G is *nilpotent of class c* if $G_{c+1} = 1$ (and c is minimal subject to this). The *derived series* is the descending series $G = G^{(0)} > G^{(1)} > \cdots$ with $G^{(i+1)} = [G^{(i)}, G^{(i)}]$; G is *solvable of derived length ℓ* if $G^{(\ell)} = 1$ (and ℓ is minimal subject to this).

The *product* $\mathbf{UV}$ of varieties $\mathbf{U}$ and $\mathbf{V}$ consists of all groups G which are *extensions* of a group $H \in \mathbf{U}$ by a group $K \in \mathbf{V}$, that is, G has a normal subgroup isomorphic to H with quotient isomorphic to K. The product of two varieties is a variety, and the product operation is associative. But product varieties are not usually generated by finite groups.

Theorem 3.2 (Šmel'ken [71]). *A product of three or more nontrivial varieties is not generated by a finite group. A product $\mathbf{UV}$ is generated by some finite group if and only if $\mathbf{U}$ and $\mathbf{V}$ have coprime exponents, $\mathbf{U}$ is nilpotent, and $\mathbf{V}$ is abelian.*

The variety $\mathbf{UV}$ has an identity basis of the form $\mathbf{u}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) \approx 1$, where $\mathbf{u}(x_1, x_2, \ldots, x_n) \approx 1$ is an identity of $\mathbf{U}$ and each $\mathbf{v}_i \approx 1$ is an identity of $\mathbf{V}$. (Note that, even if for some cases we can do better, usually all identities of $\mathbf{V}$ are needed, not just an identity basis.)

Many further results about varieties of groups are known, but the interest of the present survey lies in those that are finitely generated.

The most important result about varieties of finite groups is the *Oates–Powell Theorem*, asserting that, for any finite group G, the variety $\mathrm{var}\{G\}$ is finitely based. Actually it is a little stronger. A variety of groups is *Cross* if it is finitely based, finitely generated, and small. (Recall that a variety of algebras of any type—in particular, groups—is *finitely generated* if it is generated by one of its finite algebras.)

Theorem 3.3 (Oates and Powell [58])**.** *The variety generated by any finite group is Cross.*

A group is *critical* if it does not lie in the variety generated by all of its proper factors. It is known that, if two non-isomorphic critical groups generate the same variety, then they have abelian monoliths. Hence non-isomorphic finite simple groups generate different varieties.

3.2 Abelian groups

The structure of varieties generated by abelian groups is very simple. The class $\mathbf{A}$ of all abelian groups is the variety defined by the identity $[x, y] \approx 1$; for each integer $m \geq 1$, the class $\mathbf{A}_m$ of abelian groups of exponent m is the variety defined by the commutator identity and the identity $x^m \approx 1$. Hence the lattice of varieties of abelian groups is isomorphic to the set of positive integers ordered by divisibility, with a top element added. We remark that GAP includes commands `IsAbelian` and `Exponent`, so these conditions are easily checked.

Inclusions in the other direction are more problematic. For sufficiently large m, there are uncountably many varieties of groups covering $\mathbf{A}_m$ [21,29].

3.3 Metabelian groups

A group is *metabelian* if it lies in the product variety $\mathbf{AA}$, that is, it has an abelian normal subgroup with abelian quotient. Among small groups, many are metabelian; for example, 1,005 of the 1,048 groups of order up to 100 are metabelian. The smallest non-metabelian groups are the groups S_4 and $\mathrm{SL}(2, 3)$ of order 24.

A finite metabelian group lies in the variety $\mathbf{A}_m\mathbf{A}_n$ for some $m, n \geq 1$. The smallest subgroup of a group G whose quotient is abelian of exponent dividing n is generated by the nth powers and commutators in G, so the variety $\mathbf{A}_m\mathbf{A}_n$ is defined by the identities

$$x^{mn} \approx [x, y]^m \approx [x^n, y^n] \approx \big[x^n, [y, z]\big] \approx \big[[x, y], [z, w]\big] \approx 1.$$

However, finding an identity basis for individual finite metabelian groups is more difficult.

Higman [19] showed that for each prime p and $n \geq 1$, the proper subvarieties of $\mathbf{A}_p\mathbf{A}_n$ containing $\mathbf{A}_{pn}$ are characterized by an identity of the form

$$[x^n, y^{d_1}, y^{d_2}, \ldots, y^{d_k}] \approx 1,$$

where $d_1 > d_2 > \cdots > d_k \geq 1$ are divisors of n such that d_i does not divide d_j whenever $i > j$.

As an example which we will examine later, consider the subvariety $\operatorname{var}\{A_4\}$ of $\mathbf{A}_2\mathbf{A}_3$. The only possible Higman identity is $[x^3, y] \approx 1$, which does not hold in A_4. Therefore $\operatorname{var}\{A_4\} = \mathbf{A}_2\mathbf{A}_3$.

H. Neumann [57, p.179] quotes a generalization of this, an unpublished result of C. H. Houghton according to which, assuming that $\gcd(m, n) = 1$, any such variety lies between $\mathbf{A}_{rs}$ and $\mathbf{A}_r\mathbf{A}_s$ for some $r, s \geq 1$ such that r divides m and s divides n. Moreover, such a variety is defined by identities of the form

$$[x^s, y^{d_1}, \ldots, y^{d_k}]^t \approx 1,$$

where t is a divisor of r and $d_1 > d_2 > \cdots > d_k \geq 1$ are divisors of n such that d_i does not divide d_j whenever $i > j$.

Houghton did not publish the proof of his result. The proof, and a generalization that determines when the equality $\operatorname{var}\{A\}\operatorname{var}\{B\} = \operatorname{var}\{A \operatorname{wr} B\}$ holds for abelian groups A and B, can be found in Mikaelian [55].

There are also some results for the case when the condition $\gcd(m, n) = 1$ is relaxed.

For an example, consider `SmallGroup(12,1)` in GAP with presentation

$$\langle a, b \,|\, a^3 = 1,\, b^4 = 1,\, b^{-1}ab = a^2 \rangle.$$

Clearly, this group lies in $\mathbf{A}_3\mathbf{A}_4$ (as $\gcd(m, n) = 1$, this group can be handled with Higman's Theorem), and the possible Higman identities are $[x^4, y] \approx 1$ and $[x^4, y^2] \approx 1$. It is readily shown that the second is satisfied but the first is not. Adding $[x^4, y^2] \approx 1$ to the identity basis we see that the identity

$[x^4, y^4] \approx 1$ is now redundant and can be discarded. Further reductions are possible, but we do not strive for the simplest identity basis.

A result of Kovács [27] describes the variety generated by a finite dihedral group. We have restated his theorem in a way which is more useful for us.

Theorem 3.4 (Kovács). *Let D_{2n} denote the dihedral group of order $2n$, where $n = 2^d m$ and m is odd.*

(a) If $d \leq 1$, then $\mathrm{var}(D_{2n}) = \mathbf{A}_m\mathbf{A}_2$.

(b) If $m = 1$ and $d > 2$, then $\mathrm{var}(D_{2n}) = \mathbf{A}_{2^{d-1}}\mathbf{A}_2 \cap \mathbf{N}_d$, where $\mathbf{N}_d$ is the variety of nilpotent groups of class at most d.

(c) If $m > 1$ and $d > 2$, then $\mathrm{var}(D_{2n}) = \mathrm{var}(D_{2m}, D_{2^{d+1}})$.

Now it follows from our general remarks on metabelian groups that an identity basis for $\mathbf{A}_n\mathbf{A}_2$is given by $x^{2n} = [x^2, y^2] = 1$. (For a group lies in this variety if and only if the squares commute and have orders dividing n.) An identity basis for $\mathbf{N}_d$ is given by the *left-normed commutator* $[x_1, x_2, \ldots, x_{d+1}] = 1$ (this means $[[\ldots[[x_1, x_2], x_3], \ldots], x_{d+1}] = 1$). Given varieties $\mathbf{V}$ and $\mathbf{W}$, an identity basis for $\mathbf{V} \cap \mathbf{W}$ consists of the union of the identity bases for $\mathbf{V}$ and $\mathbf{W}$. Finally, the identities of $\mathrm{var}(G, H)$ consist of all products of an identity for G and an identity for H. So the identities for varieties of dihedral groups can be described explicitly.

3.4 Other groups

Apart from the above, results about particular finite groups are fairly scarce. Cossey and Macdonald [8] and Cossey *et al.* [9] found explicit identity bases for the varieties $\mathrm{var}\{G\}$, where $G \in \{S_4, A_5, \mathrm{PSL}(2, 7)\}$; they also found identities that hold in $\mathrm{PSL}(2, p^m)$ with prime p, but without proof that these identities form an identity basis. In the case $p = 2$, an identity basis was found by Southcott [72].

Such cases are best dealt with by database lookup.

Description for the identities of the groups $\mathrm{SL}(2, q)$ in some cases—when $q = 9$ or $q = p^m$ for some odd prime $p \not\equiv \pm 1 \pmod{16}$ and odd $m \geq 1$—are also available. In these cases, the identities are of the form $[\mathbf{w}, x] \approx 1$ and $\mathbf{w}^2 \approx 1$, where $\mathbf{w} \approx 1$ ranges over an identity basis for $\mathrm{PSL}(2, q)$ and x is a variable not occurring in $\mathbf{w}$.

In particular, this result holds for $\mathrm{SL}(2, 3)$ and $\mathrm{PSL}(2, 3) \cong A_4$, where identities of the latter group have been described in Subsection 3.3.

3.5 Non-metabelian groups of order 24

As noted earlier, S_4 and SL(2, 3) are the only non-metabelian groups of order 24. An identity basis for the variety var$\{S_4\}$ can be found in Cossey *et al.* [9]:

$$x^{12} \approx \big((x^3y^3)^4[x^3, y^6]^3\big)^3 \approx [x^2, y^2]^2 \approx [x, y]^6 \approx [x^6, y^6] \approx \big[[x, y]^3, y^3, y^2\big] \approx 1.$$

The goal of this subsection is to describe the subvarieties of the varieties var$\{S_4\}$ and var$\{\mathrm{SL}(2, 3)\}$, and to show that their proper subvarieties are all metabelian.

Lemma 3.5. *Let G be any non-abelian group in* var$\{S_3\}$. *Then G has a subgroup isomorphic to S_3.*

Proof. We know that G' is a nontrivial elementary abelian 3-group while G/G' is an abelian group that is a direct product of elementary abelian 2-groups and 3-groups. Since G is non-abelian, there must be elements $a, b \in G$ that fail to commute. We consider various cases, assuming that there is no subgroup isomorphic to S_3 and aiming for a contradiction. Note that any two elements of order 3 commute, since $[x^2, y^2] \approx 1$ is an identity of S_3.

- a and b have order 2. Then $\langle a, b\rangle$ is a dihedral group of order 6 or 12 and so contains a subgroup isomorphic to S_3. So we may assume that involutions commute.
- a has order 2 and b has order 3. Then $c = b^a$ is another element of order 3 and c commutes with b. Since $(bc^{-1})^a = cb^{-1} = (bc^{-1})^{-1}$, the subgroup $\langle a, b\rangle$ is isomorphic to S_3. Hence we can assume that elements of prime orders commute.
- a has order 2 or 3 and b has order 6. Then a commutes with b^2 and b^3, and so with b.
- a and b have order 6. Then a^2 and a^3 both commute with b, so that a and b commute.

The proof is thus complete. □

Theorem 3.6. *Let G be any critical group in* var$\{S_4\}$ *that is not metabelian. Then* var$\{G\}$ = var$\{S_4\}$.

Proof. Let N be the verbal subgroup of G corresponding to the identities of var$\{S_3\}$, that is, the subgroup generated by values in G of the identities of S_3. Then N is an elementary abelian 2-group, and it is nontrivial because $1 \neq G'' \leq N$. Further, G/N belongs to $\mathrm{var}\{S_3\}$.

If G/N is abelian, then $G' \leq N$, so that the contradiction $G'' = 1$ is deduced. Therefore G/N is non-abelian. Further, G/N has order divisible by 3, since otherwise G is a 2-group; but 2-groups in $\mathrm{var}\{S_4\}$ belong to $\mathrm{var}\{D_8\}$ and so are metabelian. Therefore by Lemma 3.5, the group G/N must contain a subgroup K isomorphic to S_3.

Moreover, such a subgroup in G/N cannot centralize N. For if it did, then $C_G(N)$ (and hence G) would have a normal 3-subgroup; but G is critical and therefore monolithic (it contains a unique minimal normal subgroup, which is a 2-group) [57, 51.32].

An orbit of K on N has order at most 6, and so generates a subgroup of order at most 2^6. We show there must be such a subgroup of order 2^2. First, consider the action of an element of order 3 in K; let $\{x_1, x_2, x_3\}$ be an orbit. The subgroup $\langle x_1, x_2, x_3\rangle$ has order 2^2 or 2^3; in the latter case, the subgroup $\langle x_1x_2, x_2x_3, x_3x_1\rangle$ has order 2^2.

If such a subgroup $\{1, y_1, y_2, y_3\}$ of order 2^2 is invariant under an element t of order 2 in K, our claim is proved; so suppose not. Let $z_i = y_i^t$ where $i = 1, 2, 3$. Then the group generated by the ys and zs has order 2^4 and is invariant under S_3. We can assume that conjugation by an element u of order 3 in K induces the permutation $(y_1, y_2, y_3)(z_1, z_3, z_2)$ (since t inverts u). Then the subgroup $\langle y_1z_1, y_2z_3, y_3z_2\rangle$ has order 2^2 and is S_3-invariant.

Now the group generated by K together with this K-invariant subgroup of N is isomorphic to S_4, and belongs to $\mathrm{var}\{G\}$. So $\mathrm{var}\{S_4\} \subseteq \mathrm{var}\{G\}$, and we have equality as required. $\square$

Corollary 3.7. *Any proper subvariety of* $\mathrm{var}\{S_4\}$ *is metabelian.*

The analogous result for $\mathrm{SL}(2, 3)$ is similar but easier to establish. We have noted in Subsection 3.4 that the identities of $\mathrm{SL}(2, 3)$ have the form $[\mathbf{w}, x] \approx \mathbf{w}^2 \approx 1$, where $\mathbf{w} \approx 1$ ranges over the identities of A_4 and x is a variable not in $\mathbf{w}$.

Theorem 3.8. *Let G be any critical group in* $\mathrm{var}\{\mathrm{SL}(2, 3)\}$ *that is not metabelian. Then* $\mathrm{var}\{G\} = \mathrm{var}\{\mathrm{SL}(2, 3)\}$.

Proof. The preliminary result, that a non-abelian group in $\mathrm{var}\{A_4\}$ contains a subgroup isomorphic to A_4, is proved similarly to the analogous result for S_3.

Now let $G \in \mathrm{var}\{\mathrm{SL}(2, 3)\}$ and suppose that G is critical and not metabelian. Then G'' is an elementary abelian 2-group and is contained in $Z(G)$, so all its subgroups are normal in G. Since G is monolithic, we find that $|G''| = 2$. Now G/G'' has a subgroup isomorphic to A_4, and it is easy to see that this lifts to a subgroup of G isomorphic to $\mathrm{SL}(2,3)$. $\square$

Corollary 3.9. *Any proper subvariety of* $\mathrm{var}\{\mathrm{SL}(2,3)\}$ *is metabelian.*

3.6 Toward an explicit bound

It follows from Theorem 3.3—the Oates–Powell Theorem—that the variety generated by a finite group is finitely based and small. Can explicit bounds for the orders of critical groups in such a variety be extracted from the proof of this result?

The proof of the Oates–Powell Theorem rests on three lemmas, of which the third concerns the class $\mathbf{C}(e,m,c)$ of finite groups G such that

- G has exponent dividing e;
- the order of any chief factor of G is at most m;
- the nilpotency class of any nilpotent factor of G is at most c.

Then $\mathbf{C}(e,m,c)$ is a class of finite groups in a variety, whence if $G \in \mathbf{C}(e,m,c)$ then every critical group in $\mathrm{var}\{G\}$ belongs to $\mathbf{C}(e,m,c)$.

Lemma 3.10 (H. Neumann [57, 52.23])**.** *The class* $\mathbf{C}(e,m,c)$ *contains only a finite number of (non-isomorphic) critical groups.*

Lemma 3.11 (H. Neumann [57, 52.5])**.** *If* $G \in \mathbf{C}(e,m,c)$ *is critical and has non-abelian monolith, then* $|G| \leq m!$.

The abelian monolith case is much harder. Neumann [57] says:

> If a bound for the index of $\Phi(G)$ in G is found, then a bound for $|G|$ can be derived. For, since $\Phi(G)$ consists of all non-generators of G, the number of elements needed to generate G can be at most $|G/\Phi(G)|$. But from bounds for the number of generators of G and the index of $\Phi(G)$ in G, one obtains a bound for the number of generators of $\Phi(G)$ by means of Schreier's formula. As $\Phi(G)$ is nilpotent, of class at most c and exponent dividing e, this leads to a bound for the order of $\Phi(G)$, and so for the order of G.

Suppose that we can show that $|G/\Phi(G)| \leq b$. Then G has at most $\log_2 b$ generators, so our bound for the number of generators of $\Phi(G)$ is $(b-1)\log_2 b + 1$, or in broad brush terms, $d \leq b \log b$. This gives a bound for the order of $\Phi(G)$ which is roughly $e^{d+d^2+\cdots+d^c}$, since the lower central factors are generated by commutators.

A small improvement is possible. If $\Phi(G)$ is not a p-group, then it is the direct product of its Sylow p-subgroups, each of which contains a nontrivial normal subgroup of G, contradicting the fact that G is monolithic. So we can replace e in the above bound by the largest prime divisor of e.

Continuing, the proof considers a series

$$\Phi(G) < F < C < G,$$

and shows that $|G/C| \leq (m!)^c$ and $|F/\Phi(G)| \leq m^c$, while $|C/F| \leq (m!)^t$, where $t \leq 1 + ce(m!)$. The bound for b is the product of these numbers.

Even for very moderate values of e, m, and c, the resulting bound is going to be rather large!

4 The database of varieties generated by small semigroups

4.1 The library of varieties generated by a semigroup of order up to 5

We produced a database containing all the semigroups up to order 5 and an identity basis for the variety generated by each of them. All the proofs regarding semigroups up to order 4 appear (or are referred to) in this paper. The proofs regarding semigroups of order 5 will be published elsewhere.

4.2 Non-finitely based varieties generated by a semigroup of order 6

Every variety generated by a semigroup of order up to 5 is finitely based [40, 80, 82]. Among all varieties generated by a semigroup of order 6, precisely four are non-finitely based [43, 48]; these varieties are generated by the following semigroups:

- the monoid B_2^1 obtained from the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0,\ aba = a,\ bab = b \rangle = \{0, a, b, ab, ba\};$$

n	Number of semigroups of order n, up to equivalence	Number of semigroups of order n, up to isomorphism	Number of varieties with a primitive generator of order n
1	1	1	1
2	4	5	5
3	18	24	14
4	126	188	53
5	1,160	1,915	145
6	15,973	28,634	At least 461
7	836,021	1,627,672	Unknown
8	1,843,120,128	3,684,030,417	Unknown
9	52,989,400,714,478	105,978,177,936,292	Unknown

Table 4: Some numerical data

- the monoid A_2^1 obtained from the 0-simple semigroup

$$A_2 = \langle a, b \mid a^2 = aba = a,\ bab = b,\ b^2 = 0 \rangle = \{0, a, b, ab, ba\};$$

- the semigroup A_2^g obtained by adjoining a new element g to A_2 with $g^2 = 0$ and $gA_2 = A_2 g = \{g\}$;
- the $\mathscr{J}$-trivial semigroup

$$L_3 = \langle a, b \mid a^2 = a,\ b^2 = b,\ aba = 0 \rangle = \{0, a, b, ab, ba, bab\}.$$

The Cayley tables of these semigroups are given in Table 5; refer to Lee *et al*. [44] for more historical information on their discovery.

Besides the four non-finitely based semigroups of order six, many other non-finitely based finite semigroups have been discovered since the 1970s; see the survey by Volkov [88]. But explicit identity bases have not been found for varieties generated by most of these semigroups because the task is neither necessary (in establishing the non-finite basis property) nor trivial. Nevertheless, explicit identity bases are available for a few non-finitely based varieties.

Proposition 4.1 (Jackson [24, Proposition 4.1]). *The identities*

$$x^4 \approx x^3, \quad x^3 y \approx y x^3, \quad x^2 y x \approx x^3 y, \quad x y x^2 \approx x^3 y, \quad xyxzx \approx x^3 yz,$$
$$\Big(\prod_{i=1}^{m} x_i\Big)\Big(\prod_{i=m}^{1} x_i\Big) y^2 \approx y^2 \Big(\prod_{i=1}^{m} x_i\Big)\Big(\prod_{i=m}^{1} x_i\Big), \quad m = 1, 2, 3, \ldots$$

B_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	1	3	1
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	1	6	1

A_2^1	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	2	2	3
3	1	2	3	2	3	3
4	1	1	1	4	4	6
5	1	2	3	4	5	6
6	1	4	6	4	6	6

A_2^g	1	2	3	4	5	6
1	1	1	1	1	1	6
2	1	1	1	2	3	6
3	1	2	3	2	3	6
4	1	1	1	4	5	6
5	1	4	5	4	5	6
6	6	6	6	6	6	1

L_3	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	1	1	1	1	2
3	1	1	1	1	1	3
4	1	1	2	1	4	2
5	1	1	3	1	5	3
6	1	2	2	4	4	6

Table 5: Non-finitely based semigroups of order 6

constitute an identity basis for a non-finitely based variety generated by a certain semigroup of order 211.

Proposition 4.2 (Lee and Volkov [47, Section 1])**.** *For each* $n \geq 2$, *the identities*

$$x^{n+2} \approx x^2, \quad (xy)^{n+1}x \approx xyx, \quad xyxzx \approx xzxyx,$$
$$\left(\prod_{i=1}^{m} x_i^n\right)^3 \approx \left(\prod_{i=1}^{m} x_i^n\right)^2, \quad m = 2, 3, 4, \ldots$$

constitute an identity basis for the non-finitely based variety $\operatorname{var}\{A_2, \mathbb{Z}_n\}$*. In particular,* $\operatorname{var}\{A_2, \mathbb{Z}_2\} = \operatorname{var}\{A_2^g\}$.

Proposition 4.3 (Lee [41, Corollary 3.5])**.** *For each* $n \geq 1$, *the identities*

$$x^{n+2} \approx x^2, \quad x^{n+1}yx^{n+1} \approx xyx, \quad xhykxty \approx yhxkytx,$$
$$x\left(\prod_{i=1}^{m}(y_ih_iy_i)\right)x \approx x\left(\prod_{i=m}^{1}(y_ih_iy_i)\right)x, \quad m = 2, 3, 4, \ldots$$

constitute an identity basis for the non-finitely based variety $\operatorname{var}\{L_3, \mathbb{Z}_n\}$.

4.3 Inherently non-finitely based finite semigroups

The *finite basis problem*—first posed by Tarski [75] in the 1960s as a decision problem—questions which finite algebras are finitely based. This problem is undecidable for general algebras [52] but remains open for finite semigroups. In contrast, it is decidable if a finite semigroup S is *inherently non-finitely based* in the sense that every locally finite variety containing S is non-finitely based. This result follows from the work of Sapir [63, 64], a description of which requires the following concepts:

- the *period* of a semigroup S is the least number d such that S satisfies the identity $x^{m+d} \approx x^m$ for some $m \geq 1$;
- the *upper hypercenter* of a group G, denoted by $\Gamma(G)$, is the last term in the upper central series of G;
- a word $\mathbf{w}$ is an *isoterm* for a semigroup S if S violates every nontrivial identity of the form $\mathbf{w} \approx \mathbf{w}'$;
- the *Zimin words* $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$ are words over the variables $\{x_1, x_2, x_3, \ldots\}$ defined inductively by $\mathbf{z}_1 = x_1$ and $\mathbf{z}_{k+1} = \mathbf{z}_k x_{k+1} \mathbf{z}_k$ for each $k \geq 1$.

Theorem 4.4 (Sapir [66, Theorem 3.6.34]). (i) *A finite semigroup S is inherently non-finitely based if and only if there exists some idempotent $e \in S$ such that the submonoid eSe of S is inherently non-finitely based.*

(ii) *A finite monoid M with period d is inherently non-finitely based if and only if there exist $a \in M$ and an idempotent $e \in MaM$ such that the elements eae and $ea^{d+1}e$ do not belong to the same coset of the maximal subgroup M_e of M containing e with respect to the upper hypercenter $\Gamma(M_e)$.*

(iii) *A finite semigroup S is inherently non-finitely based if and only if the Zimin words $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_m$, where $m = |S|^3$, are isoterms for S.*

The non-finitely based semigroups A_2^g and L_3 are not inherently non-finitely based because they satisfy the identities $\mathbf{z}_2 \approx x_1(x_2x_1)^3$ and $\mathbf{z}_2 \approx x_1x_2x_1^2$, respectively. On the other hand, the semigroups A_2^1 and B_2^1 are inherently non-finitely based since all Zimin words are isoterms [64, Lemma 3.7]. It follows that a finite semigroup S is inherently non-finitely based if either $A_2^1 \in \mathrm{var}\{S\}$ or $B_2^1 \in \mathrm{var}\{S\}$. Observe that the condition in Theorem 4.4(ii) can hold in a trivial way, namely when eae or $ea^{d+1}e$ does not belong to M_e, so that both elements do not belong to the same coset of M_e. This is the case for B_2^1; see, for example, Volkov and Gol'berg [90, observation after Proposition 1].

For certain finite monoids M, the condition $B_2^1 \in \mathrm{var}\{M\}$ is not only sufficient, but also necessary for M to be inherently non-finitely based.

Lemma 4.5. *Let M be any finite monoid that satisfies the identity $x^{2n} \approx x^n$ for some $n \geq 2$. Suppose that M satisfies at least one of the following four conditions*: $|M| \leq 55$, *M is regular, the idempotents of M form a submonoid, and all subgroups of M are nilpotent. Then the following conditions are equivalent*:

(a) *M is inherently non-finitely based*;

(b) $B_2^1 \in \mathrm{var}\{M\}$;

(c) *M violates the identity*

$$\big((xy)^n(yx)^n(xy)^n\big)^n \approx (xy)^n. \tag{4.1}$$

Proof. (a) $\Leftrightarrow$ (b): This holds by Jackson [23, Theorems 1.4 and 2.2] and Sapir [63, Theorem 2].
(c) $\Rightarrow$ (b): If M violates the identity (4.1), then $B_2 \in \mathrm{var}\{M\}$ by Sapir and Suhanov [67, Theorem 1], so that $B_2^1 \in \mathrm{var}\{M\}$ by Jackson [25, Lemma 1.1].
(b) $\Rightarrow$ (c): It is routinely verified that B_2^1 violates the identity (4.1). Therefore if M satisfies the identity (4.1), then $B_2^1 \notin \mathrm{var}\{M\}$. $\square$

There is yet another method to check if a finite monoid is inherently non-finitely based. For each $n \geq 2$, define the words $[x,y]_1^n, [x,y]_2^n, [x,y]_3^n, \ldots$ over $\{x,y\}$ inductively by $[x,y]_1^n = x^{n-1}y^{n-1}xy$ and $[x,y]_{k+1}^n = \big[[x,y]_k^n, y\big]_1^n$ for each $k \geq 1$. Then for any variety $\mathbf{V}$ generated by a finite semigroup that satisfies the identity $x^{2n} \approx x^n$, the subsequence $\{[x,y]_{k!}^n\}$ converges in the $\mathbf{V}$-free semigroup over $\{x,y\}$; let $[x,y]_\infty^n$ denote the limit of this subsequence [87, Subsection 4.4].

Lemma 4.6 (Volkov [87, Proposition 4.4])**.** *Let M be any finite monoid that satisfies the identity $x^{2n} \approx x^n$ for some $n \geq 2$. Then M is inherently non-finitely based if and only if it violates either* (4.1) *or*

$$[\mathbf{e}z\mathbf{e}, (\mathbf{e}y\mathbf{e})^{n-1}\mathbf{e}y^{n+1}\mathbf{e}]_\infty^n \approx \mathbf{e} \quad \textit{with } \mathbf{e} = (xyzt)^n.$$

The companion website checks if an input finite semigroup S is inherently non-finitely based in the following manner. Suppose that $e_1, e_2, \ldots, e_r$ are all the idempotents of S. Then by Theorem 4.4(i), it suffices to check if some submonoid $M_i = e_iSe_i$ of S is inherently non-finitely based; this can be achieved by applying Theorem 4.4(ii). As this is the most general result, the website can handle semigroups of order higher than 55; if the semigroup is inherently non-finitely based, then the website provides the relevant information such as the hypercenter. The website also allows the user to check if a semigroup is inherently non-finitely based with Lemma 4.5. Results on isoterms are computationally demanding and hence are not used.

Refer to the surveys by Volkov [87, 88] for more information on inherently non-finitely based semigroups and the finite basis problem for finite semigroups in general.

Based on results in this subsection, a description of inherently non-finitely based semigroups of order up to 9 is possible. For this purpose, the semigroup A_2^1 and B_2^1, together with those given in Tables 6–8, are required.

U_7	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3
3	1	2	3	1	1	3	1
4	4	4	4	4	4	4	4
5	4	4	4	4	5	5	7
6	1	2	3	4	5	6	7
7	4	5	7	4	4	7	4

V_7	1	2	3	4	5	6	7
1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3
3	1	2	3	1	2	3	3
4	4	4	4	4	4	4	4
5	4	4	4	4	5	5	7
6	1	2	3	4	5	6	7
7	4	5	7	4	5	7	7

W_7	1	2	3	4	5	6	7
1	1	1	1	1	5	5	5
2	1	2	1	2	5	5	7
3	1	1	3	3	5	6	5
4	1	2	3	4	5	6	7
5	5	5	5	5	1	1	1
6	5	6	5	6	1	1	3
7	5	5	7	7	1	2	1

Table 6: The semigroups U_7, V_7, and W_7

U_8	1	2	3	4	5	6	7	8
1	1	1	1	1	1	1	1	1
2	1	1	1	1	2	2	3	4
3	1	2	3	4	3	4	4	4
4	4	4	4	4	4	4	4	4
5	1	2	3	4	5	6	7	8
6	4	4	4	4	6	6	7	8
7	4	6	7	8	7	8	8	8
8	8	8	8	8	8	8	8	8

V_8	1	2	3	4	5	6	7	8
1	1	1	1	1	5	5	7	7
2	1	2	1	2	5	5	7	8
3	1	1	3	3	5	6	7	7
4	1	2	3	4	5	6	7	8
5	5	5	5	5	7	7	1	1
6	5	6	5	6	7	7	1	3
7	7	7	7	7	1	1	5	5
8	7	7	8	8	1	2	5	5

Table 7: The semigroups U_8 and V_8

Since the semigroups in Tables 6–8 are monoids, it is routinely checked by Lemma 4.5 that they are all inherently non-finitely based. With the exception of V_7 and U_8, each of these semigroups is isomorphic to its dual.

Proposition 4.7. *Let S be any inherently non-finitely based semigroup of order 9 or less.*

(i) *If $|S| \leq 6$, then S is isomorphic to one of the semigroups A_2^1 and B_2^1.*

U_9	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	2	2	3	4
3	1	2	3	4	4	3	4	4	4
4	4	4	4	4	4	4	4	4	4
5	5	5	5	5	5	5	5	5	5
6	1	2	3	4	5	6	7	8	9
7	5	5	5	5	5	7	7	8	9
8	5	7	8	9	9	8	9	9	9
9	9	9	9	9	9	9	9	9	9

V_9	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	6	6	6	6
2	1	1	1	2	2	6	6	6	7
3	3	3	3	3	3	8	8	8	8
4	1	2	3	4	5	6	7	8	9
5	3	3	3	5	5	8	8	8	9
6	1	1	1	6	1	6	6	6	6
7	1	2	1	7	1	6	7	6	6
8	3	3	3	8	3	8	8	8	8
9	3	5	3	9	3	8	9	8	8

W_9	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	6	6	6	6
2	1	1	1	2	2	6	6	6	7
3	3	3	3	3	3	8	8	8	8
4	1	2	3	4	5	6	7	8	9
5	3	3	3	5	5	8	8	8	9
6	1	1	1	6	1	6	6	6	6
7	1	2	1	7	2	6	7	6	7
8	3	3	3	8	3	8	8	8	8
9	3	5	3	9	5	8	9	8	9

Table 8: The semigroups U_9, V_9, and W_9

(ii) *If $|S| = 7$, then either S contains A_2^1 or B_2^1 as a subsemigroup or S is isomorphic to one of the semigroups U_7, V_7, $\overleftarrow{V}_7$, and W_7.*

(iii) *If $|S| = 8$, then either S contains a proper subsemigroup that is inherently non-finitely based or S is isomorphic to one of the semigroups U_8, $\overleftarrow{U}_8$, and V_8.*

(iv) *If $|S| = 9$ and S satisfies the identity $x^4 \approx x^2$, then either S contains a proper subsemigroup that is inherently non-finitely based or S is isomorphic to one of the semigroups U_9, $\overleftarrow{U}_9$, V_9, and W_9.*

It is long and well known that the semigroups A_2^1 and B_2^1 of order 6 are the smallest inherently non-finitely based semigroups. GAP's package SmallSemi contains all the semigroups of order up to 8 and hence we could routinely run the algorithm outlined after Lemma 4.6.

To find inherently non-finitely based semigroups of order 9, we used the following algorithm (which in fact uses different results and computations to double check Proposition 4.7 parts (ii) and (iii)):

(a) Use Mace4 [97] to generate all monoids of orders 6–9 that satisfy the identity $x^4 \approx x^2$ but violate the identity (4.1), thus resorting to Lemma 4.5; this led to 457,745 semigroups.

(b) Use Isofilter to discard isomorphic copies; this led to 7,625 semigroups which are all inherently non-finitely based, but many of which contain proper subsemigroups that are inherently non-finitely based.

(c) Use GAP's SmallSemi to discard the semigroups of order $n \in \{7, 8, 9\}$ that contain a proper subsemigroup that is inherently non-finitely based; this left us with the semigroups in Tables 6–8.

GAP Smallsemi:

Order: 5 | Sequence: 67 | Go!

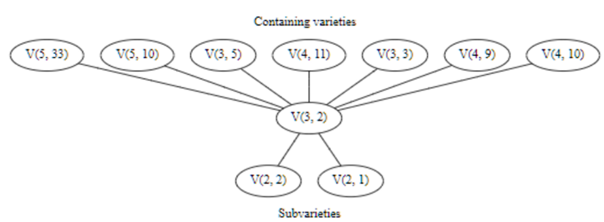
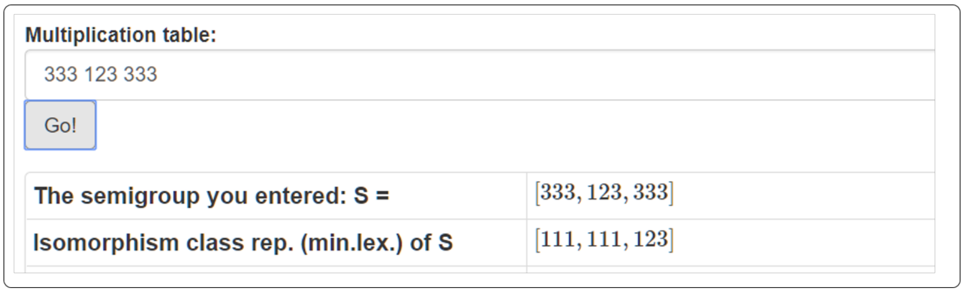
The semigroup you entered: S =	$[11111, 11111, 11111, 11111, 11115]$
Isomorphism class rep. (min.lex.) of S	$[11111, 11111, 11111, 11111, 11115]$
The variety $\mathsf{var}\{S\}$ **coincides with**	$\mathbf{V}(3, 2)$
Identity basis	$x^2a \approx xa,\ xy \approx yx$ Copy
Primitive generator	$[111, 111, 113]$
Common generator	none
Prime decomposition	V(2,1) $\vee$ V(2,2) $= \mathsf{var}\{N_2\} \vee \mathsf{var}\{S\ell_2\}$
Identities defining maximal subvarieties	(1) $x^2 \approx x$
	(2) $x^2 \approx xy$
Maximal subvarieties	(1) $\mathbf{V}(3, 2) \cap [x^2 \approx x] =$ V(2,2) $= \mathsf{var}\{S\ell_2\}$
	(2) $\mathbf{V}(3, 2) \cap [x^2 \approx xy] =$ V(2,1) $= \mathsf{var}\{N_2\}$
Other information on $\mathbf{V}(3, 2)$	Self-dual
	Cross with 4 subvarieties
References	T. Evans, The lattice of semigroup varieties, Semigroup Forum **2** (1971), no. 1, 1–43. Figure 3.
Some inclusions of varieties	

Figure 4: Companion website: example of information displayed if the identity basis for the variety generated by the given semigroup is found



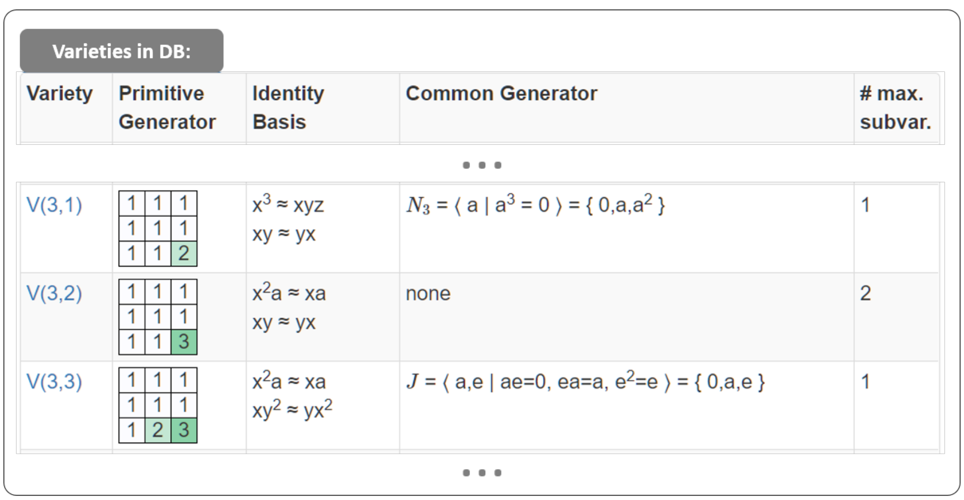
Multiplication table:

333 123 333

Go!

The semigroup you entered: S =	$[333, 123, 333]$
Isomorphism class rep. (min.lex.) of S	$[111, 111, 123]$

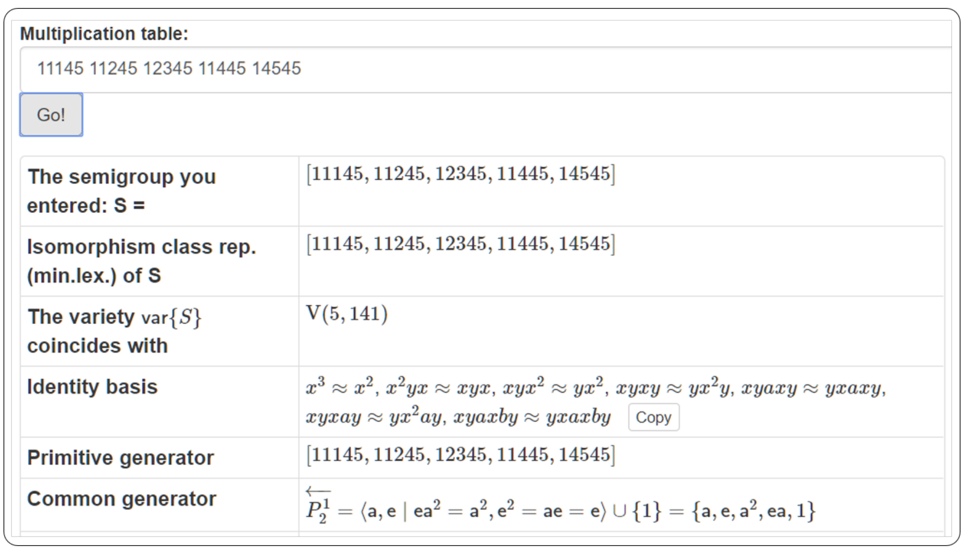
Figure 5: Companion website: computation of the smallest element in the isomorphism class of [333,123,333]



Varieties in DB:

Variety	Primitive Generator	Identity Basis	Common Generator	# max. subvar.
...				
V(3,1)	1 1 1 1 1 1 1 1 2	$x^3 \approx xyz$ $xy \approx yx$	$N_3 = \langle a \mid a^3 = 0 \rangle = \{ 0, a, a^2 \}$	1
V(3,2)	1 1 1 1 1 1 1 1 3	$x^2a \approx xa$ $xy \approx yx$	none	2
V(3,3)	1 1 1 1 1 1 1 2 3	$x^2a \approx xa$ $xy^2 \approx yx^2$	$J = \langle a, e \mid ae=0, ea=a, e^2=e \rangle = \{ 0, a, e \}$	1
...				

Figure 6: Companion website: varieties V(3, 1), V(3, 2), and V(3, 3) in the database



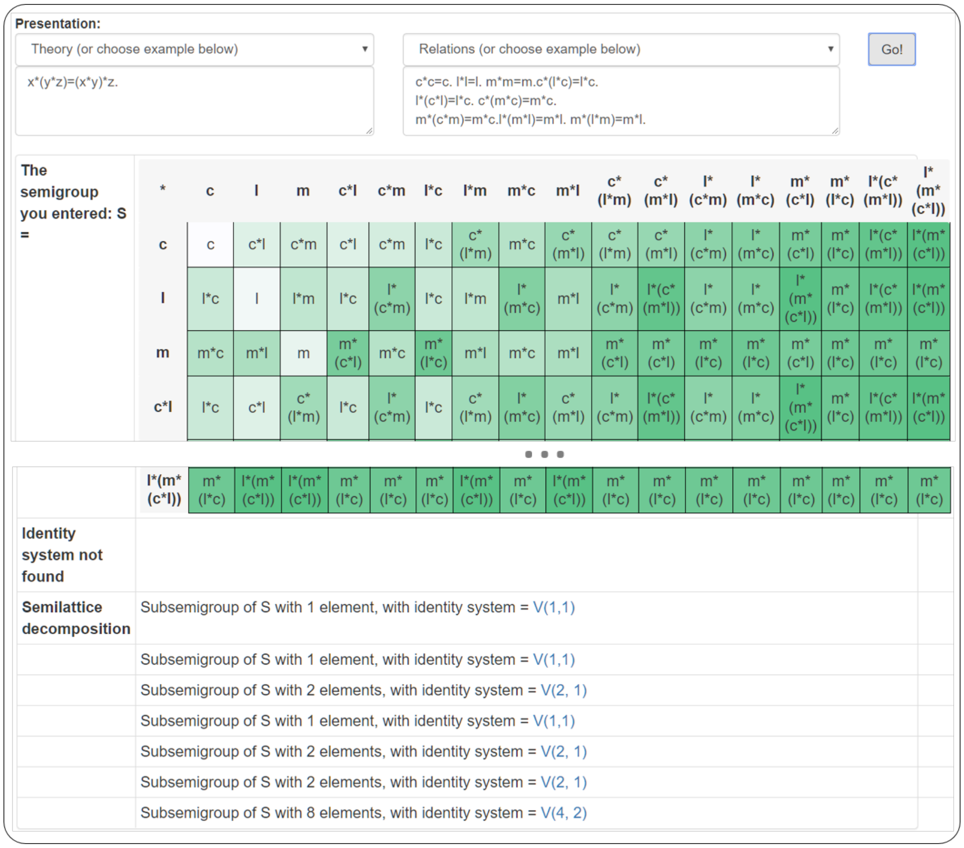
Multiplication table:

11145 11245 12345 11445 14545

Go!

The semigroup you entered: S =	$[11145, 11245, 12345, 11445, 14545]$
Isomorphism class rep. (min.lex.) of S	$[11145, 11245, 12345, 11445, 14545]$
The variety $\mathsf{var}\{S\}$ **coincides with**	$\mathrm{V}(5, 141)$
Identity basis	$x^3 \approx x^2$, $x^2yx \approx xyx$, $xyx^2 \approx yx^2$, $xyxy \approx yx^2y$, $xyaxy \approx yxaxy$, $xyxay \approx yx^2ay$, $xyaxby \approx yxaxby$ Copy
Primitive generator	$[11145, 11245, 12345, 11445, 14545]$
Common generator	$\overleftarrow{P_2^1} = \langle \mathsf{a}, \mathsf{e} \mid \mathsf{ea}^2 = \mathsf{a}^2, \mathsf{e}^2 = \mathsf{ae} = \mathsf{e} \rangle \cup \{1\} = \{\mathsf{a}, \mathsf{e}, \mathsf{a}^2, \mathsf{ea}, 1\}$

Figure 7: Companion website: example of a presentation provided


Presentation:

Theory (or choose example below)

x*(y*z)=(x*y)*z.

Relations (or choose example below)

c*c=c. l*l=l. m*m=m.c*(l*c)=l*c.
l*(c*l)=l*c. c*(m*c)=m*c.
m*(c*m)=m*c.l*(m*l)=m*l. m*(l*m)=m*l.

Go!

The semigroup you entered: S =

*	c	l	m	c*l	c*m	l*c	l*m	m*c	m*l	c*(l*m)	c*(m*l)	l*(c*m)	l*(m*c)	m*(c*l)	m*(l*c)	l*(c*(m*l))	l*(m*(c*l))
c	c	c*l	c*m	c*l	c*m	l*c	c*(l*m)	m*c	c*(m*l)	c*(l*m)	c*(m*l)	l*(c*m)	l*(m*c)	m*(c*l)	m*(l*c)	l*(c*(m*l))	l*(m*(c*l))
l	l*c	l	l*m	l*c	l*(c*m)	l*c	l*m	l*(m*c)	m*l	l*(c*m)	l*(c*(m*l))	l*(c*m)	l*(m*c)	l*(m*(c*l))	m*(l*c)	l*(c*(m*l))	l*(m*(c*l))
m	m*c	m*l	m	m*(c*l)	m*c	m*(l*c)	m*l	m*c	m*l	m*(c*l)	m*(c*l)	m*(l*c)	m*(l*c)	m*(c*l)	m*(l*c)	m*(l*c)	m*(l*c)
c*l	l*c	c*l	c*(l*m)	l*c	l*(c*m)	l*c	c*(l*m)	l*(m*c)	c*(m*l)	l*(c*m)	l*(c*(m*l))	l*(c*m)	l*(m*c)	l*(m*(c*l))	m*(l*c)	l*(c*(m*l))	l*(m*(c*l))
• • •																	
l*(m*(c*l))	m*(l*c)	l*(m*(c*l))	l*(m*(c*l))	m*(l*c)	m*(l*c)	m*(l*c)	l*(m*(c*l))	m*(l*c)	l*(m*(c*l))	m*(l*c)	m*(l*c)	m*(l*c)	m*(l*c)	m*(l*c)	m*(l*c)	m*(l*c)	m*(l*c)

Identity system not found	
Semilattice decomposition	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 8 elements, with identity system = V(4, 2)

Figure 8: Companion website: Kiselman semigroup entered as a presentation

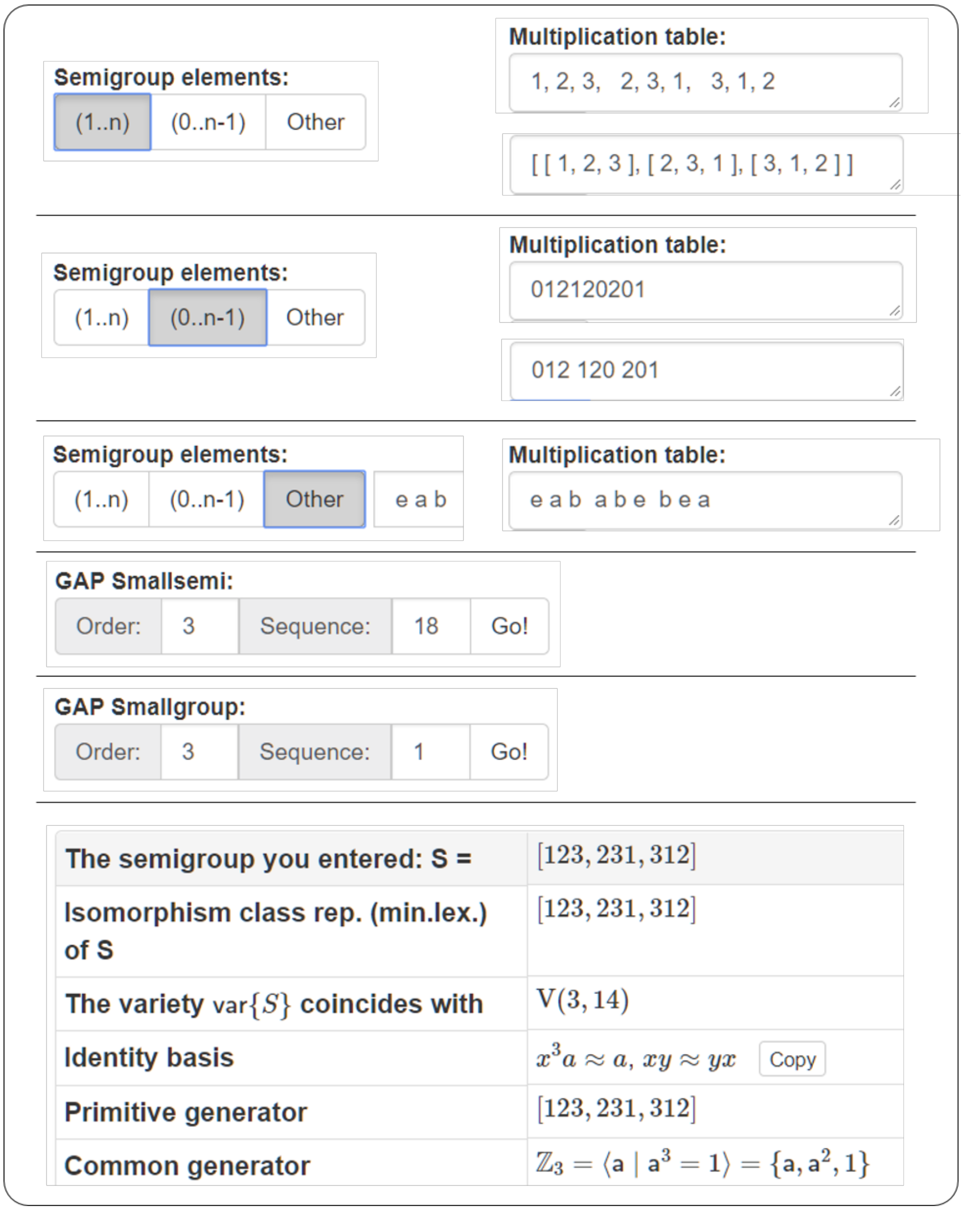
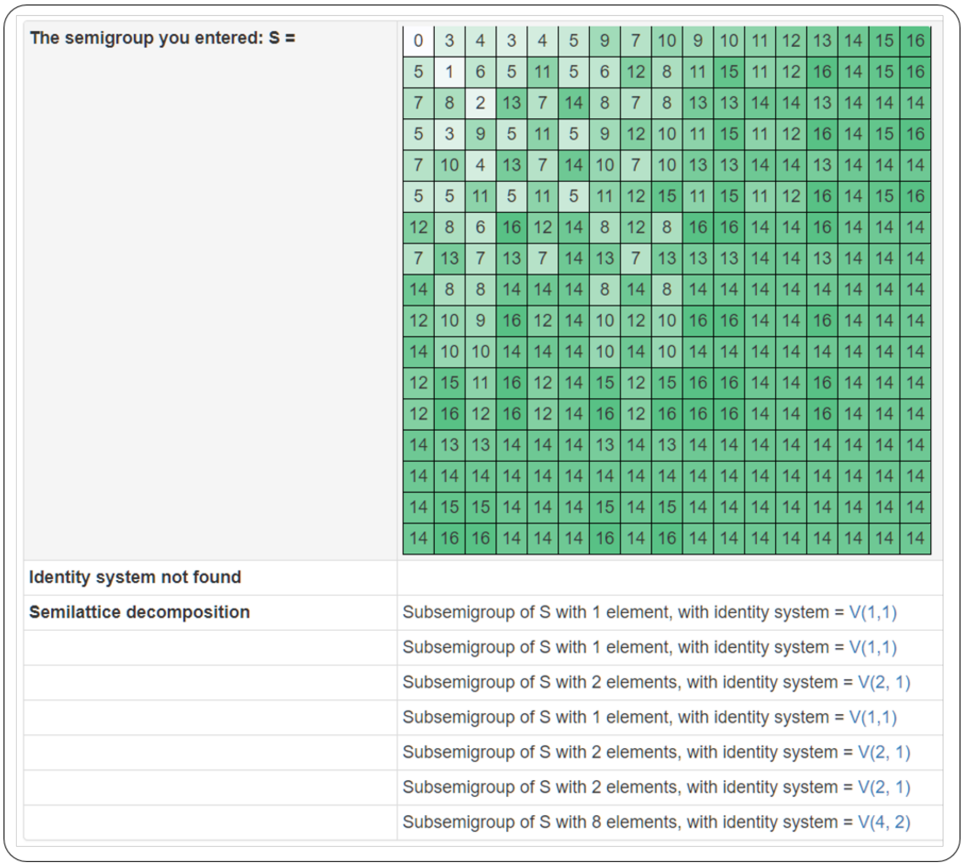


Figure 9: Companion website: examples of input alternatives for semigroup [123,231,312]



The semigroup you entered: S =

0	3	4	3	4	5	9	7	10	9	10	11	12	13	14	15	16
5	1	6	5	11	5	6	12	8	11	15	11	12	16	14	15	16
7	8	2	13	7	14	8	7	8	13	13	14	14	13	14	14	14
5	3	9	5	11	5	9	12	10	11	15	11	12	16	14	15	16
7	10	4	13	7	14	10	7	10	13	13	14	14	13	14	14	14
5	5	11	5	11	5	11	12	15	11	15	11	12	16	14	15	16
12	8	6	16	12	14	8	12	8	16	16	14	14	16	14	14	14
7	13	7	13	7	14	13	7	13	13	13	14	14	13	14	14	14
14	8	8	14	14	14	8	14	8	14	14	14	14	14	14	14	14
12	10	9	16	12	14	10	12	10	16	16	14	14	16	14	14	14
14	10	10	14	14	14	10	14	10	14	14	14	14	14	14	14	14
12	15	11	16	12	14	15	12	15	16	16	14	14	16	14	14	14
12	16	12	16	12	14	16	12	16	16	16	14	14	16	14	14	14
14	13	13	14	14	14	13	14	13	14	14	14	14	14	14	14	14
14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14	14
14	15	15	14	14	14	15	14	15	14	14	14	14	14	14	14	14
14	16	16	14	14	14	16	14	16	14	14	14	14	14	14	14	14

Identity system not found	
Semilattice decomposition	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 1 element, with identity system = V(1,1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 2 elements, with identity system = V(2, 1)
	Subsemigroup of S with 8 elements, with identity system = V(4, 2)

Figure 10: Companion website: example of semitalattice decomposition of a order 17 semigroup

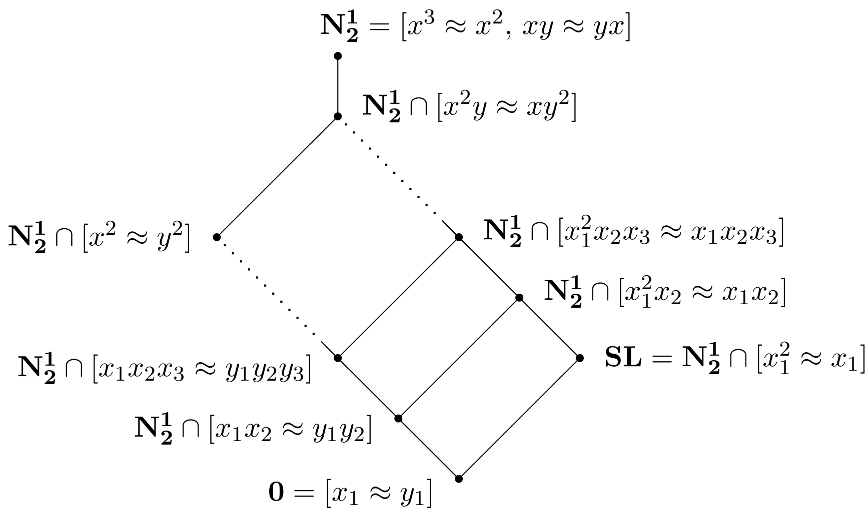


Figure 13: The lattice of subvarieties of $\mathbf{N_2^1} = \mathrm{var}\{N_2^1\}$

5 The companion webpage

In this section we will give some brief details on the architecture of the website.

5.1 Multiplication table

A very flexible data entry tool was developed to allow the input of a multiplication table of a semigroup S. By default the elements of the semigroup are assumed to be $1, 2, \ldots, \mathrm{N}$. This is convenient to use the multiplication tables coming from GAP. Some other computational tools use the elements $0, 1, \ldots, \mathrm{N}-1$, and this can also be used, along with sets on different (given) elements.

The entries of the Cayley table can be separated by commas or spaces, and optionally can include [·] to bound each line and/or the full multiplication table. If the elements are all single-digit, all or part of the separators can be omitted. For instance, all input strings below can be used as input for the same multiplication table:

1 1 1 1 1 1 1 1 2	space separated
1,1,1,1,1,1,1,1,2	comma separated
1, 1, 1, 1, 1, 1, 1, 1, 2	mixed commas and spaces
[1, 1, 1, 1, 1, 1, 1, 1, 2]	"[" and "]" enclosed
[[1, 1, 1], [1, 1, 1], [1, 1, 2]]	GAP syntax
111 111 112	separators omitted (only for single digit elements)
111111112	separators omitted (only for single digit elements)

Using the GAP syntax option, it is possible to copy a multiplication table from GAP and paste it here. For example, we can just copy and paste the output of GAP coming from the following command:

```
gap> MultiplicationTable(SmallGroup(5,1));
[ [ 1, 2, 3, 4, 5 ], [ 2, 3, 4, 5, 1 ], [ 3, 4, 5, 1, 2 ], [ 4, 5, 1, 2, 3 ], [ 5, 1, 2, 3, 4
] ]
```

The number of the multiplication table entries must be a perfect square, otherwise an error will be returned. Only semigroups will be accepted, so the associativity property is checked by default.

Semigroups up to order 100 are accepted, but the representative in the isomorphism class of S, whose vector $\vec{v}(S)$ is lexicographically the least, will only be computed in case the order of S is 10 or less.

5.2 Finding the least semigroup of its isomorphism class

Finding the semigroup S in its isomorphism class whose vector $\vec{v}(S)$ is the least lexicographically is not necessary to access the main tools available on the website; however, it is much more convenient and an essential part of the way we name varieties.

An obvious algorithm would be to give to some model builder, such as Mace4, the Cayley table of the semigroup and ask for all the isomorphic models in the same underlying set. This gives a list of vectors that we only need to order.

We decided to use our own algorithm that proved to deliver the result for semigroups of order up to 10 in less than a second, and that we now outline.

5.2.1 The presentation to semigroup algorithm

Input: order, mtable: order and multiplication table of a semigroup.

Output: minlex: multiplication table of the least (lexicographically) semigroup isomorphic to the given semigroup.

```
routine Minlex (order mtable):
minlex = mtable
for i in 1 to order
  newElem[i] = i
for x in order-permutations of order
  for i = 1 to order
    newElem[x[i]] = i
  equal = True
  stop = False
  smaller = False
  if newElem[mtable[x[1]][x[1]]] = 1
    for l = 1 to order
      for c = 1 to order
        e = newElem[mtable[x[l]][x[c]]]
        e0 = minlex[l][c]
        if equal = True
```

```
            if e > e0
              stop = True
              exit for loop
            else if e < e0
              equal = False
              menor = True
          a1[l][c] = e
        if stop = True
          exit for loop
    if smaller == True
      minlex = a1
return minlex
```

5.3 Generating a semigroup from a given presentation

The presentation tool finds the multiplication table from a presentation. One of the distinctive features of this tool is that it allows to define infinitely many different presentations (semigroups, bands, etc.) defined as varieties or quasi-varieties. The presentation (both theory and relations) must be written in Prover9 syntax. A presentation has two ingredients: the theory and some relations between the generators.
To specify the identities that define the theory and the relations, a subset of Prover9 syntax is used:

- Variables (with names started by “u”, “v”, “w”, “x”, “y” and “z”). No variables will be allowed at the *relations* window;
- Constants (with names started with a $0-9$, $a-s$, or $A-Z$);
- Binary operation character $*$;
- Equal sign $=$;
- Parentheses (and);
- Each identity must end with a final mark.

Examples:

Consider the following example presentations, and how to enter the corresponding theory:

Presentation	Theory	Relations
$\langle a, e \mid ea^2 = a^2, e^2 = ae = e\rangle$ $= \{a, e, a^2, ea\}$	$x * (y * z) = (x * y) * z.$	$(e * a) * a = a * a.$ $e * e = a * e.$ $a * e = e.$
$\langle a \mid a^5 = 1\rangle$ $= \{a, a^2, a^3, a^4, 1\}$	$x * (y * z) = (x * y) * z.$ $x * 1 = x.\ 1 * x = x.$	$(((a * a) * a) * a) * a = 1.$
$\langle a, e \mid ae = 0, ea = a, e^2 = e\rangle$ $\cup \{1\} = \{0, a, e, 1\}$	$x * (y * z) = (x * y) * z.$ $x * 0 = 0.\ 0 * x = 0.$ $x * 1 = x.\ 1 * x = x.$	$a * e = 0.$ $e * a = a.$ $e * e = e.$

The tool will try to close the multiplication table, but if more than 20 elements are reached, an error will be returned.

Entering a semigroup as a presentation (or using given identities to find or filter varieties) demands the use of an automated theorem prover (in this site Prover/Mace4), something usually very expensive (in time). Therefore a strategy to limit calls and also to speed-up the use of Prover9/Mace4 was implemented (see Table 11).

5.4 Finding an identity basis for a finitely generated variety

Let $\mathbf{V}$ be any finitely generated variety. Then the number of maximal subvarieties of $\mathbf{V}$ is some positive integer $k \geq 1$; see Lee *et al.* [45, Proposition 4.1]. Let $\mathbf{M}_1, \mathbf{M}_2, \dots, \mathbf{M}_k$ be these maximal subvarieties. By maximality, each $\mathbf{M}_i$ can be defined within $\mathbf{V}$ by some identity μ_i. If $k \geq 2$, then $\mathbf{V} = \mathbf{M}_i \vee \mathbf{M}_j$ for all distinct i and j; otherwise, $\mathbf{V}$ has a unique maximal subvariety and is said to be *prime*. It follows that each finitely generated variety is either prime or a join of some of its prime subvarieties.

Now it is clear that for any finite semigroup S, the equality $\operatorname{var}\{S\} = \mathbf{V}$ holds if and only if $S \in \mathbf{V}$ and $S \notin \mathbf{M}_i$ for all i. However, if the variety $\mathbf{V}$ is finitely based and a finite identity basis Σ is available, then the equality $\operatorname{var}\{S\} = \mathbf{V}$ holds whenever $S \models \Sigma$ and $S \not\models \mu_i$ for all i. Therefore the identity system $(\Sigma; \mu_1, \mu_2, \dots, \mu_k)$, called a *Bas-Max system* for $\mathbf{V}$, provides an easily verifiable sufficient condition to check if a finite semigroup generates $\mathbf{V}$. Presently, the website database contains Bas-Max systems for all of the following varieties:

(a) varieties with a primitive generator of order up to 4;

#	Step	Description
1	Presentation	User enters a presentation in Prover9/ Mace4 format (both the theory and relations).
2	Normalization	User formulas are normalized to a internal notation and ordering rules, to increase cache's hit rate.
3	Presentation cache (SQL)	If a similar presentation (in normalized notation) is recorded in SQL, its result will be used.
4	Proofs cache (user session)	If the user had requested other similar proofs during the session, the results are used to reduce the number of proofs.
5	Proofs cache (SQL)	If all users had requested other similar proofs recorded in SQL, theirs result will be used to speed the process.
6	Launch Prover9/Mace4	Launched at the same time, but the first to find a proof or counterexample (respectively) stops the other.

Table 11: Presentations algorithm

(b) proper subvarieties of Cross varieties in (a);

(c) varieties with a primitive generator of order 5.

Now when a semigroup S entered into the website is shown to generate a variety $\mathbf{V}$ via its Bas-Max system $(\Sigma; \mu_1, \mu_2, \ldots, \mu_k)$, then besides the identity basis Σ for $\text{var}\{S\}$, other important information, such as the primitive generator for $\mathbf{V}$, any decomposition of $\mathbf{V}$ into a join of its prime subvarieties, and the number of subvarieties of $\mathbf{V}$, will also be displayed by the website.

Bas-Max systems for varieties in (a) and (b), together with the aforementioned properties, will be listed in Section 6, while their proofs will be given in the appendix sections. Justification of the Bas-Max systems for varieties in (c) will be disseminated elsewhere.

The website will be regularly updated with newly established Bas-Max systems for varieties.

5.5 Testing for equivalent identity bases

Suppose we have a finite set Σ of identities and would like to know information about the variety $[\Sigma]$ of semigroups, such as the primitive generator for $[\Sigma]$ and the varieties covered by $[\Sigma]$. If this variety happens to be in our database, then many of these information is available. The question is how do we identify $[\Sigma]$ with a variety in the database. A tool was developed that will, by specifying one or more identities in Prover9 format, retrieve the variety whose identity basis is equivalent to Σ.

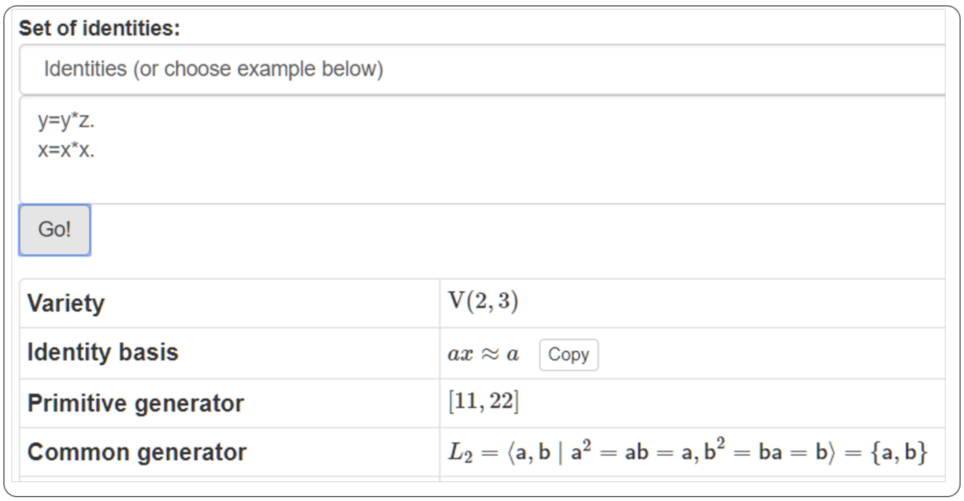


Figure 15: Companion website: example of testing for equivalent identity basis

5.6 Filtering varieties using conditions

Suppose we have some property and want to check which varieties in the database satisfy the property. This can be done on the website. To specify the identities, a subset of Prover9 syntax is used. Only variables (with names started by $u - z$, the operation character $*$, the equal sign $=$, parentheses (and), and final mark).

It is not necessary to specify associativity.

The automatic theorem prover Prover9 and its accompanying program Mace4 that look for counterexamples will run simultaneously to check if the identity basis for each variety in the database implies the identities provided.

There exist four options to invoke:

Option	Prover9/Mace4 status
Proofs	• All varieties for which a proof was found by Prover9 within 1 second.
No countermodels	• The varieties for which a proof was found by Prover9 within 1 second plus: • The varieties where a proof was not found by Prover9 within 1 second but Mace4 also didn't found a countermodel within 1 second.
Countermodels	• The varieties for which a countermodel was found by Mace4 within 1 second;
No proofs	• The varieties for which a countermodel was found by Mace4 within 1 second, plus: • The varieties for which a countermodel was not found by Mace4 within 1 second, but also a proof was not found by Prover9 within 1 second.

It is possible to apply successive filters to the sets of varieties obtained.

5.7 Obtaining lattices of varieties

A tool was developed to obtain a lattice of a set of varieties created with the filtering tool.

It is also possible to filter the list of varieties by leaving only the maximal varieties.

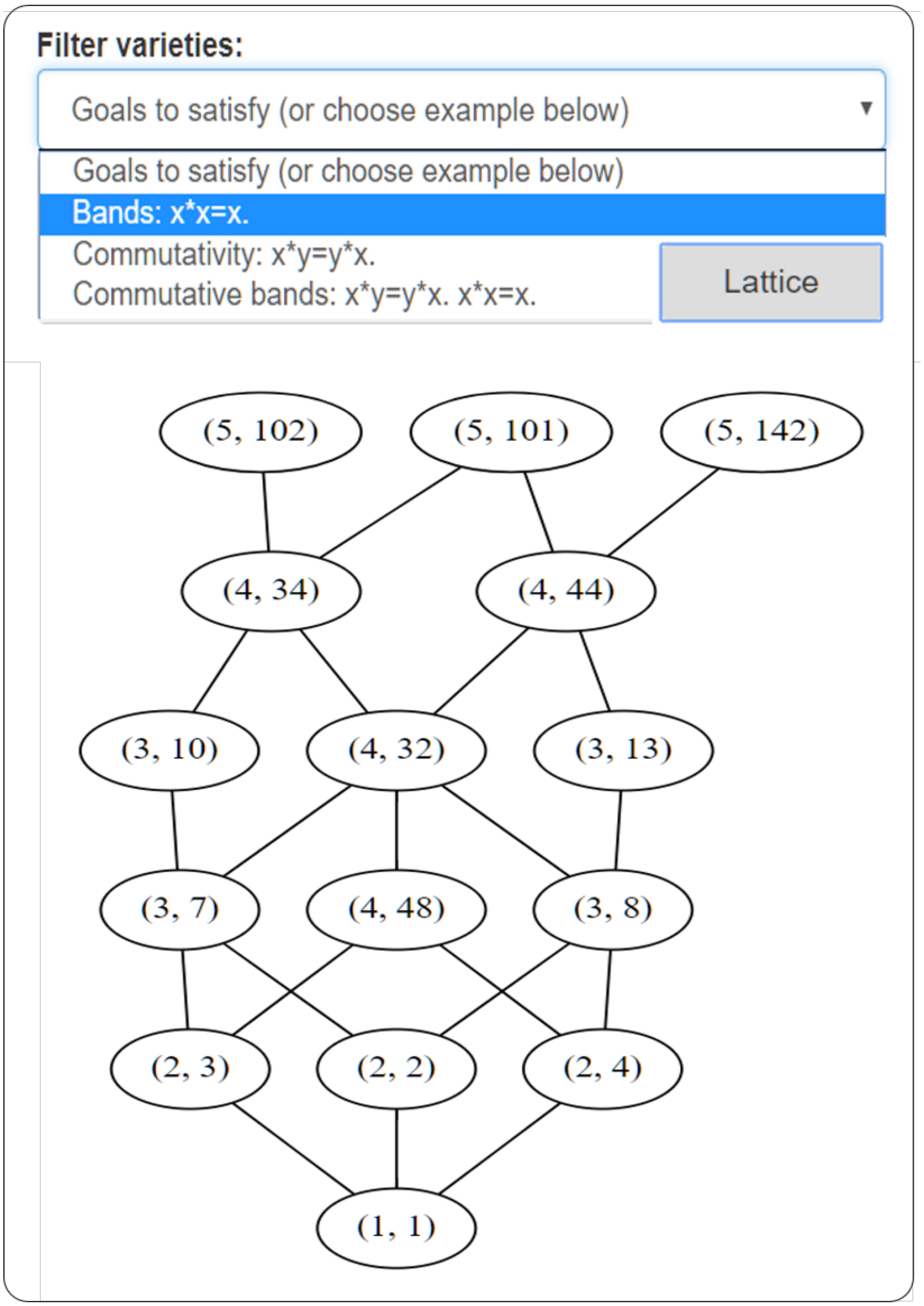


Figure 16: Companion website: obtaining the lattice of varieties generated by bands up to order 5

5.8 Extending the database: finding identity bases for new varieties

Suppose we have identity bases for all varieties generated by a semigroup of order $n-1$ and we want to find an identity basis for the variety generated some semigroup S of order n. If S does not belong to any variety generated by a semigroup of order less than n, then $\operatorname{var}\{S\}$ is a new variety and we want to find an identity basis for it. The website has a tool to try to find candidates of identities that can form an identity basis for $\operatorname{var}\{S\}$. The first thing it does is to check, based on results from Subsection 4.3, if S is inherently non-finitely based. If the semigroup S is not inherently non-finitely based, then the website searches, in some *ad hoc* intelligent ways, for candidates of identities of S to form an identity basis for $\operatorname{var}\{S\}$. Of course, if S happens to be non-finitely based, then the process will not terminate. But if we are lucky, then the website will produce a natural conjecture for an identity basis Σ for $\operatorname{var}\{S\}$. The variety defined by Σ coincides with $\operatorname{var}\{S\}$ if the conjecture is correct, and properly contains $\operatorname{var}\{S\}$ otherwise. We checked this procedure against all varieties generated by semigroups of order up to 5 and in every case, the procedure gave an identity basis equivalent to the known one.

6 Varieties generated by small semigroups

As mentioned in Subsection 5.4, the present section lists Bas-Max systems for all varieties generated by a semigroup of order up to 4 and for some that are their proper subvarieties. Important information such as primitive generators, decompositions into joins of prime subvarieties, and number of subvarieties are also given. To this end, the semigroups in Tables 13–15 play a crucial role; these semigroups are primitive generators for the varieties they generate, which are in fact precisely all prime varieties generated by a semigroup of order up to 4.

N_2	1	2
1	1	1
2	1	1

$S\ell_2$	1	2
1	1	1
2	1	2

LZ_2	1	2
1	1	1
2	2	2

RZ_2	1	2
1	1	2
2	1	2

$\mathbb{Z}_2$	1	2
1	1	2
2	2	1

Table 13: Primitive generators of prime varieties generated by a semigroups of order 2

Some well-known semigroups in Tables 13–15 are the semilattice $S\ell_2$ of order 2, the left zero band LZ_2 of order 2, the right zero band RZ_2 of order 2,

N_3	1	2	3
1	1	1	1
2	1	1	1
3	1	1	2

J	1	2	3
1	1	1	1
2	1	1	1
3	1	2	3

$\overleftarrow{J}$	1	2	3
1	1	1	1
2	1	1	2
3	1	1	3

N_2^1	1	2	3
1	1	1	1
2	1	1	2
3	1	2	3

LZ_2^1	1	2	3
1	1	1	1
2	1	2	3
3	3	3	3

RZ_2^1	1	2	3
1	1	1	3
2	1	2	3
3	1	3	3

$\mathbb{Z}_3$	1	2	3
1	1	2	3
2	2	3	1
3	3	1	2

Table 14: Primitive generators of all prime varieties generated by a semigroups of order 3

the monogenic nilpotent semigroup

$$N_n = \langle a \mid a^n = 0 \rangle = \{a, a^2, \ldots, a^{n-1}, 0\}$$

of order n, and the cyclic group

$$\mathbb{Z}_n = \langle a \mid a^n = 1 \rangle = \{a, a^2, \ldots, a^{n-1}, 1\}$$

of order n. Recall that for any semigroup S, the smallest monoid containing S is denoted by S^1, and the dual of S is denoted by $\overleftarrow{S}$.

In the remainder of the section, information on 88 varieties are grouped by the order of their primitive generators and given below in four subsections; these varieties are named Variety `N`, or simply $\mathbf{V}_{\mathtt{N}}$, where $\mathtt{N} \in \{1, 2, \ldots, 88\}$. Proofs and references for all results are deferred to the appendix sections.

To illustrate how information on each variety can be read, consider Variety 43 in Subsection 6.3, repeated here for reader convenience.

Variety 43 (Subsection C.3)**.**

(Gen) [1111, 1112, 3333, 1214]

(Bas) $x^3 \approx x^2$, $axy \approx ayx$

(Max) $x^2y^2 \approx y^2x^2$; $a^2x^2 \approx a^2x$

(Dec) $\mathrm{var}\{LZ_2\} \vee \mathrm{var}\{N_2^1\}$

(Sub) Countably infinite

F_4	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	2	1

G_4	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	2	1

N_4	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	2	3

N_3^1	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	1	2	3
4	1	2	3	4

B_0	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	1
4	1	1	1	4

A_0	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	2
4	1	1	1	4

J^1	1	2	3	4
1	1	1	1	1
2	1	1	1	2
3	1	2	3	3
4	1	2	3	4

P_2	1	2	3	4
1	1	1	1	1
2	1	1	1	3
3	3	3	3	3
4	4	4	4	4

$\overleftarrow{J^1}$	1	2	3	4
1	1	1	1	1
2	1	1	2	2
3	1	1	3	3
4	1	2	3	4

O_2	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	3	3	3	3
4	3	4	1	2

$\overleftarrow{O_2}$	1	2	3	4
1	1	1	3	3
2	1	2	3	4
3	1	3	3	1
4	1	4	3	2

$\overleftarrow{P_2}$	1	2	3	4
1	1	1	3	4
2	1	1	3	4
3	1	1	3	4
4	1	3	3	4

$\mathbb{Z}_4$	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	3	4	2	1
4	4	3	1	2

Table 15: Primitive generators of all prime varieties generated by a semigroups of order 4

The vector of the primitive generator of the variety $\mathbf{V}_{43}$ is given in (Gen). The two identities in (Bas) form an identity basis for $\mathbf{V}_{43}$, while each identity in (Max) defines within $\mathbf{V}_{43}$ a maximal subvariety; in other words, the identities in (Bas) and (Max) form a Bas-Max system for $\mathbf{V}_{43}$. The join in (Dec) is a decomposition of $\mathbf{V}_{43}$ into the join of the prime subvarieties var$\{LZ_2\}$ and var$\{N_2^1\}$. As indicated in (Sub), the variety $\mathbf{V}_{43}$ has countably infinitely many subvarieties. All these results regarding $\mathbf{V}_{43}$ are established in Subsection C.3.

For another example, consider Variety 78 in Subsection 6.4.

Variety 78 (Zhang and Luo [92, Variety **C** in Figure 4]; Figure 20)**.**

(Gen) $[11111, 11113, 11133, 11144, 11155]$

(Bas) $ax^2 \approx ax$, $xyx \approx x^2y$, $a^2xy \approx a^2yx$

(Max) $axy \approx ayx$

(Dec) None

(Sub) 11

The vector of the primitive generator of the variety $\mathbf{V}_{78}$ is given in (Gen). The three identities in (Bas) form an identity basis for $\mathbf{V}_{78}$, while the identity in (Max) define the unique maximal subvariety within $\mathbf{V}_{78}$. Since $\mathbf{V}_{78}$ has only one maximal subvariety, it is prime and cannot be decomposed into a join of two or more prime subvarieties, as indicated by "None" in (Dec). The number 11 in (Sub) is the number of subvarieties of $\mathbf{V}_{78}$. Justification of the all these results regarding $\mathbf{V}_{78}$ can be found in Zhang and Luo [92, Variety **C** in Figure 4]. For any variety with finitely many subvarieties, its lattice of subvarieties is given in Section B. Specifically, the lattice of subvarieties of $\mathbf{V}_{78}$ can be found in Figure 20.

6.1 Varieties with primitive generator of order 2

Variety 1 (Evans [13, Figure 3])**.**

(Gen) $[11, 11] = N_2$

(Bas) $x^2 \approx xy$, $xy \approx yx$

(Max) $x \approx y$

(Dec) None

(Sub) 2

Variety 2 (Evans [13, Figure 3])**.**

(Gen) $[11, 12] = S\ell_2$

(Bas) $x^2 \approx x$, $xy \approx yx$

(Max) $x \approx y$

(Dec) None

(Sub) 2

Variety 3 (Evans [13, Figure 3]).

(Gen) $[11, 22] = LZ_2$

(Bas) $ax \approx a$

(Max) $x \approx y$

(Dec) None

(Sub) 2

Variety 4 (Evans [13, Figure 3]).

(Gen) $[12, 12] = RZ_2$

(Bas) $xa \approx a$

(Max) $x \approx y$

(Dec) None

(Sub) 2

Variety 5 (Lee *et al*. [45, Proposition 5.4]).

(Gen) $[12, 21] = \mathbb{Z}_2$

(Bas) $x^2a \approx a$, $xy \approx yx$

(Max) $x \approx y$

(Dec) None

(Sub) 2

6.2 Varieties with primitive generator of order 3

Variety 6 (Tishchenko [78, Variety $\mathbf{CN}_3$ on page 439]; Figures 22, 23, 24, or 27).

- (Gen) $[111, 111, 112] = N_3$
- (Bas) $x^3 \approx xyz$, $xy \approx yx$
- (Max) $x^3 \approx x^2$
- (Dec) None
- (Sub) 4

Variety 7 (Evans [13, Figure 3]; Figures 17, 19, 20, 23, or 25).

- (Gen) $[111, 111, 113]$
- (Bas) $x^2a \approx xa$, $xy \approx yx$
- (Max) $x^2 \approx x$; $x^2 \approx xy$
- (Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{S\ell_2\}$
- (Sub) 4

Variety 8 (Zhang and Luo [92, Variety **D** in Figure 2]; Figures 17, 19, 20, or 25).

- (Gen) $[111, 111, 123] = J$
- (Bas) $x^2a \approx xa$, $xy^2 \approx yx^2$
- (Max) $xy \approx yx$
- (Dec) None
- (Sub) 5

Variety 9 (Evans [13, Figure 3]; Figures 19, 20, or 22).

- (Gen) $[111, 111, 333]$
- (Bas) $x^2 \approx xy$
- (Max) $x^2 \approx x$; $xy \approx yx$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{LZ_2\}$

(Sub) 4

Variety 10 (Zhang and Luo [92, Variety **E** in Figure 2]; Figures 17, 19, 20, or 25)**.**

(Gen) $[111, 112, 113] = \overleftarrow{J}$

(Bas) $ax^2 \approx ax$, $x^2y \approx y^2x$

(Max) $xy \approx yx$

(Dec) None

(Sub) 5

Variety 11 (Subsection C.1)**.**

(Gen) $[111, 112, 123] = N_2^1$

(Bas) $x^3 \approx x^2$, $xy \approx yx$

(Max) $x^2y \approx xy^2$

(Dec) None

(Sub) Countably infinite

Variety 12 (Gerhard and Petrich [16, Variety `LNB` in Section 2]; Figures 18, 19, 20, or 21)**.**

(Gen) $[111, 121, 333]$

(Bas) $x^2 \approx x$, $axy \approx ayx$

(Max) $xy \approx x$; $xy \approx yx$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{LZ_2\}$

(Sub) 4

Variety 13 (Gerhard and Petrich [16, Variety `RNB` in Section 2]; Figures 18, 19, 20, or 21)**.**

(Gen) $[111, 123, 123]$

(Bas) $x^2 \approx x$, $xya \approx yxa$

(Max) $xy \approx y$; $xy \approx yx$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{RZ_2\}$

(Sub) 4

Variety 14 (Subsection B.9)**.**

(Gen) $[111, 123, 132]$

(Bas) $x^3 \approx x$, $xy \approx yx$

(Max) $x^2 \approx x$; $x^2y \approx y$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{\mathbb{Z}_2\}$

(Sub) 4

Variety 15 (Gerhard and Petrich [16, Variety LRB in Section 2]; Figures 18, 19, 20, or 21)**.**

(Gen) $[111, 123, 333] = LZ_2^1$

(Bas) $x^2 \approx x$, $xyx \approx xy$

(Max) $axy \approx ayx$

(Dec) None

(Sub) 5

Variety 16 (Evans [13, Figure 3]; Figures 19, 20, or 22)**.**

(Gen) $[113, 113, 113]$

(Bas) $x^2 \approx yx$

(Max) $x^2 \approx x$; $xy \approx yx$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{RZ_2\}$

(Sub) 4

Variety 17 (Subsection B.9)**.**

(Gen) $[113, 113, 331]$

(Bas) $x^2ab \approx ab$, $xy \approx yx$

(Max) $x^3 \approx x$; $x^3 \approx x^2$

(Dec) $\text{var}\{N_2\} \vee \text{var}\{\mathbb{Z}_2\}$

(Sub) 4

Variety 18 (Gerhard and Petrich [16, Variety `RRB` in Section 2]; Figures 18, 19, 20, or 21)**.**

(Gen) $[113, 123, 133] = RZ_2^1$

(Bas) $x^2 \approx x$, $xyx \approx yx$

(Max) $xya \approx yxa$

(Dec) None

(Sub) 5

Variety 19 (Lee *et al.* [45, Proposition 5.4])**.**

(Gen) $[123, 231, 312] = \mathbb{Z}_3$

(Bas) $x^3a \approx a$, $xy \approx yx$

(Max) $x \approx y$

(Dec) None

(Sub) 2

6.3 Varieties with primitive generator of order 4

Variety 20 (Tishchenko [78, Variety $\mathbf{N}_{3,2}$ on page 439]; Figures 17 or 22)**.**

(Gen) $[1111, 1111, 1111, 1121] = F_4$

(Bas) $x^2 \approx yzt$

(Max) $xy \approx yx$

(Dec) None

(Sub) 4

Variety 21 (Tishchenko [78, Variety $\mathbf{N}_3$ on page 438]; Figure 22)**.**

(Gen) $[1111, 1111, 1111, 1122]$

(Bas) $x^3 \approx yzt$

(Max) $x^3 \approx x^2$; $xy \approx yx$

(Dec) $\mathrm{var}\{N_3\} \vee \mathrm{var}\{F_4\}$

(Sub) 6

Variety 22 (Tishchenko [78, Variety $\mathbf{CN}_{3,2}$ on page 439] ; Figures 17, 22, 23, 24, or 27)**.**

(Gen) $[1111, 1111, 1112, 1121] = G_4$

(Bas) $x^2 \approx xyz$, $xy \approx yx$

(Max) $x^2 \approx xy$

(Dec) None

(Sub) 3

Variety 23 (Lee *et al*. [45, Condition A8]; Figure 27)**.**

(Gen) $[1111, 1111, 1112, 1123] = N_4$

(Bas) $x^4 \approx xyzt$, $x^2y \approx xy^2$, $xy \approx yx$

(Max) $x^4 \approx x^3$

(Dec) None

(Sub) 8

Variety 24 (Zhang and Luo [92, Variety $\mathbf{D} \vee \mathbf{E}$ in Figure 2]; Figure 17)**.**

(Gen) $[1111, 1111, 1113, 1214]$

(Bas) $x^3 \approx x^2$, $xyx \approx x^2y^2$, $xyx \approx y^2x^2$, $ax^2b \approx axb$

(Max) $xyx \approx x^2y$; $xyx \approx yx^2$

(Dec) $\mathrm{var}\{J\} \vee \mathrm{var}\{\overleftarrow{J}\}$

(Sub) 13

Variety 25 (Subsection C.2)**.**

(Gen) $[1111, 1111, 1113, 1234]$

(Bas) $x^3 \approx x^2$, $x^2y^2 \approx y^2x^2$, $xya \approx yxa$

(Max) $x^2y \approx yx^2$; $xy^2 \approx yx^2$

(Dec) $\text{var}\{J\} \vee \text{var}\{N_2^1\}$

(Sub) Countably infinite

Variety 26 (Lee *et al.* [45, Proposition 6.14]; Figure 23).

(Gen) $[1111, 1111, 1121, 1114]$

(Bas) $x^2ab \approx xab$, $xy \approx yx$

(Max) $x^3 \approx x^2$; $x^3 \approx y^3$

(Dec) $\text{var}\{S\ell_2\} \vee \text{var}\{N_3\}$

(Sub) 8

Variety 27 (Tishchenko [78, Variety $\mathbf{L}_{1,3}$ on page 438]; Figure 22).

(Gen) $[1111, 1111, 1121, 4444]$

(Bas) $x^3 \approx xyz$

(Max) $x^3 \approx x^2$; $x^3 \approx y^3$

(Dec) $\text{var}\{LZ_2\} \vee \text{var}\{N_3\}$

(Sub) 10

Variety 28 (Evans [13, Figure 3]; Figures 19 or 20).

(Gen) $[1111, 1111, 1131, 4444]$

(Bas) $x^2a \approx xa$, $ax^2 \approx ax$, $axy \approx ayx$

(Max) $x^2 \approx x$; $x^2 \approx xy$; $xy \approx yx$

(Dec) $\text{var}\{N_2\} \vee \text{var}\{S\ell_2\} \vee \text{var}\{LZ_2\}$

(Sub) 8

Variety 29 (Evans [13, Figure 3]; Figures 19 or 20).

(Gen) $[1111, 1111, 1134, 1134]$

(Bas) $x^2a \approx xa$, $ax^2 \approx ax$, $xya \approx yxa$

(Max) $x^2 \approx x$; $x^2 \approx yx$; $xy \approx yx$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{RZ_2\}$

(Sub) 8

Variety 30 (Subsection B.9)**.**

(Gen) $[1111, 1111, 1134, 1143]$

(Bas) $x^3a \approx xa$, $xy \approx yx$

(Max) $x^3 \approx x$; $x^3 \approx x^2$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{\mathbb{Z}_2\}$

(Sub) 8

Variety 31 (Zhang and Luo [92, Variety $\mathbf{L^1} \vee \mathbf{N}$ in Figure 5]; Figures 19 or 20)**.**

(Gen) $[1111, 1111, 1134, 4444]$

(Bas) $x^2a \approx xa$, $ax^2 \approx ax$, $xyx \approx xy$

(Max) $x^2 \approx x$; $axy \approx ayx$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{LZ_2^1\}$

(Sub) 10

Variety 32 (Zhang and Luo [92, Variety $\mathbf{D} \vee \mathbf{L}$ in Figure 4]; Figure 19)**.**

(Gen) $[1111, 1111, 1231, 4444]$

(Bas) $x^2a \approx xa$, $axy^2 \approx ayx^2$

(Max) $ax^2 \approx ax$; $xy^2 \approx yx^2$

(Dec) $\mathrm{var}\{LZ_2\} \vee \mathrm{var}\{J\}$

(Sub) 10

Variety 33 (Dual of Variety 41; Figure 20)**.**

(Gen) $[1111, 1111, 1234, 1234]$

(Bas) $x^2a \approx xa$, $xya \approx yxa$

(Max) $ax^2 \approx ax$; $xy^2 \approx yx^2$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{J\}$

(Sub) 10

Variety 34 (Subsection B.9)**.**

(Gen) $[1111, 1111, 1234, 1243]$

(Bas) $x^3a \approx xa$, $x^2y^2 \approx y^2x^2$, $xya \approx yxa$

(Max) $x^3 \approx x^2$; $xy \approx yx$

(Dec) $\mathrm{var}\{\mathbb{Z}_2\} \vee \mathrm{var}\{J\}$

(Sub) 10

Variety 35 (Edmunds [11, Semigroup $\mathtt{S(4,11)}$ on page 70]; Figure 19)**.**

(Gen) $[1111, 1111, 1234, 4444]$

(Bas) $x^2a \approx xa$, $xyx \approx xy^2$

(Max) $ax^2 \approx ax$; $axy^2 \approx ayx^2$

(Dec) $\mathrm{var}\{J\} \vee \mathrm{var}\{LZ_2^1\}$

(Sub) 13

Variety 36 (Subsection C.2)**.**

(Gen) $[1111, 1112, 1113, 1134]$

(Bas) $x^3 \approx x^2$, $x^2y^2 \approx y^2x^2$, $axy \approx ayx$

(Max) $x^2y \approx yx^2$; $x^2y \approx y^2x$

(Dec) $\mathrm{var}\{\overleftarrow{J}\} \vee \mathrm{var}\{N_2^1\}$

(Sub) Countably infinite

Variety 37 (Subsection C.1)**.**

(Gen) $[1111, 1112, 1123, 1234] = N_3^1$

(Bas) $x^4 \approx x^3$, $xy \approx yx$

(Max) $x^3y^2 \approx x^2y^3$

(Dec) None

(Sub) Countably infinite

Variety 38 (Subsection C.5)**.**

(Gen) $[1111, 1112, 1231, 1114] = B_0$

(Bas) $x^3 \approx x^2$, $x^2yx^2 \approx yxy$, $x^2y^2 \approx y^2x^2$

(Max) $a^2x^2b^2 \approx a^2xb^2$

(Dec) None

(Sub) Countably infinite

Variety 39 (Subsection C.5)**.**

(Gen) $[1111, 1112, 1232, 1114] = A_0$

(Bas) $x^3 \approx x^2$, $x^2yx^2 \approx yxy$

(Max) $x^2y^2 \approx y^2x^2$

(Dec) None

(Sub) Countably infinite

Variety 40 (Subsection C.6)**.**

(Gen) $[1111, 1112, 1233, 1234] = J^1$

(Bas) $x^3 \approx x^2$, $x^2y^2 \approx y^2x^2$, $xyx \approx yx^2$

(Max) $x^2ya^2 \approx yx^2a^2$

(Dec) None

(Sub) Countably infinite

Variety 41 (Zhang and Luo [92, Variety $\mathbf{E} \vee \mathbf{L}$ in Figure 4]; Figure 20)**.**

(Gen) $[1111, 1112, 3333, 1114]$

(Bas) $ax^2 \approx ax$, $axy \approx ayx$

(Max) $x^2a \approx xa$; $x^2y \approx y^2x$

(Dec) $\text{var}\{LZ_2\} \vee \text{var}\{\overleftarrow{J}\}$

(Sub) 10

Variety 42 (Edmunds [11, Semigroup S(4, 25) on page 70]; Figure 20).

(Gen) $[1111, 1112, 3333, 1134]$

(Bas) $ax^2 \approx ax$, $xyx \approx x^2y$

(Max) $x^2a \approx xa$; $a^2xy \approx a^2yx$

(Dec) $\text{var}\{\overleftarrow{J}\} \vee \text{var}\{LZ_2^1\}$

(Sub) 14

Variety 43 (Subsection C.3).

(Gen) $[1111, 1112, 3333, 1214]$

(Bas) $x^3 \approx x^2$, $axy \approx ayx$

(Max) $x^2y^2 \approx y^2x^2$; $a^2x^2 \approx a^2x$

(Dec) $\text{var}\{LZ_2\} \vee \text{var}\{N_2^1\}$

(Sub) Countably infinite

Variety 44 (Subsection C.4).

(Gen) $[1111, 1112, 3333, 1234]$

(Bas) $x^3 \approx x^2$, $xyx \approx x^2y$

(Max) $a^2x^2 \approx a^2x$; $a^2x^2y^2 \approx a^2y^2x^2$

(Dec) $\text{var}\{N_2^1\} \vee \text{var}\{LZ_2^1\}$

(Sub) Countably infinite

Variety 45 (Tishchenko [78, Variety $\mathbf{L}_{2,2}$ on page 438]; Figure 22).

(Gen) $[1111, 1113, 3333, 4444] = P_2$

(Bas) $abx \approx ab$

(Max) $x^2 \approx xy$

(Dec) None

(Sub) 5

Variety 46 (Dual of Variety 40).

(Gen) $[1111, 1122, 1133, 1234] = \overleftarrow{J}^1$

(Bas) $x^3 \approx x^2$, $x^2y^2 \approx y^2x^2$, $xyx \approx x^2y$

(Max) $a^2x^2y \approx a^2yx^2$

(Dec) None

(Sub) Countably infinite

Variety 47 (Dual of Variety 32; Figure 19).

(Gen) $[1111, 1122, 1134, 1134]$

(Bas) $ax^2 \approx ax$, $x^2ya \approx y^2xa$

(Max) $x^2a \approx xa$; $x^2y \approx y^2x$

(Dec) $\text{var}\{RZ_2\} \vee \text{var}\{\overleftarrow{J}\}$

(Sub) 10

Variety 48 (Dual of Variety 34).

(Gen) $[1111, 1122, 1134, 1143]$

(Bas) $ax^3 \approx ax$, $x^2y^2 \approx y^2x^2$, $axy \approx ayx$

(Max) $x^3 \approx x^2$; $xy \approx yx$

(Dec) $\text{var}\{\mathbb{Z}_2\} \vee \text{var}\{\overleftarrow{J}\}$

(Sub) 10

Variety 49 (Subsection C.3).

(Gen) $[1111, 1122, 1234, 1234]$

(Bas) $x^3 \approx x^2$, $xya \approx yxa$

(Max) $x^2y^2 \approx y^2x^2$; $x^2a^2 \approx xa^2$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{N_2^1\}$

(Sub) Countably infinite

Variety 50 (Subsection C.1)**.**

(Gen) $[1111, 1122, 1234, 1243]$

(Bas) $x^4 \approx x^2$, $xy \approx yx$

(Max) $x^3 \approx x^2$; $x^3y \approx xy^3$

(Dec) $\mathrm{var}\{\mathbb{Z}_2\} \vee \mathrm{var}\{N_2^1\}$

(Sub) Countably infinite

Variety 51 (Gerhard and Petrich [16, Variety `NB` in Section 2]; Figure 18)**.**

(Gen) $[1111, 1214, 3333, 1214]$

(Bas) $x^2 \approx x$, $axya \approx ayxa$

(Max) $xyx \approx x$; $xyx \approx xy$; $xyx \approx yx$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{LZ_2\} \vee \mathrm{var}\{RZ_2\}$

(Sub) 8

Variety 52 (Petrich [60, Lemma 7.3(vii)]; Figure 21)**.**

(Gen) $[1111, 1214, 3333, 1412]$

(Bas) $x^3 \approx x$, $axy \approx ayx$

(Max) $x^2 \approx x$; $xy \approx yx$; $ax^2 \approx a$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{LZ_2\} \vee \mathrm{var}\{\mathbb{Z}_2\}$

(Sub) 8

Variety 53 (Gerhard and Petrich [16, Variety `LQNB` in Section 2]; Figure 18)**.**

(Gen) $[1111, 1234, 1234, 4444]$

(Bas) $x^2 \approx x$, $xyxa \approx xya$

(Max) $xyx \approx xy$; $axya \approx ayxa$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{LZ_2^1\}$

(Sub) 10

Variety 54 (Tishchenko [77, Variety $\mathbf{V}_2$ on page 111]; Figure 21)**.**

(Gen) $[1111, 1234, 1324, 4444]$

(Bas) $x^3 \approx x$, $xyx \approx x^2y$

(Max) $x^2 \approx x$; $axy \approx ayx$

(Dec) $\mathrm{var}\{\mathbb{Z}_2\} \vee \mathrm{var}\{LZ_2^1\}$

(Sub) 10

Variety 55 (Subsection B.9)**.**

(Gen) $[1111, 1234, 1342, 1423]$

(Bas) $x^4 \approx x$, $xy \approx yx$

(Max) $x^2 \approx x$; $x^3a \approx a$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{\mathbb{Z}_3\}$

(Sub) 4

Variety 56 (Tishchenko [77, Proposition 2.25]; Figure 21)**.**

(Gen) $[1111, 1234, 3333, 3412] = O_2$

(Bas) $x^3 \approx x$, $xyxy \approx xy^2x$

(Max) $xyx \approx x^2y$

(Dec) None

(Sub) 11

Variety 57 (Dual of Variety 27; Figure 22)**.**

(Gen) $[1114, 1114, 1124, 1114]$

(Bas) $x^3 \approx yzx$

(Max) $x^3 \approx x^2$; $x^3 \approx y^3$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{N_3\}$

(Sub) 10

Variety 58 (Subsection B.8)**.**

(Gen) $[1114, 1114, 1124, 4441]$

(Bas) $x^2abc \approx abc$, $xy \approx yx$

(Max) $x^4 \approx x^2$; $x^4 \approx x^3$

(Dec) $\mathrm{var}\{\mathbb{Z}_2\} \vee \mathrm{var}\{N_3\}$

(Sub) 8

Variety 59 (Dual of Variety 31; Figures 19 or 20)**.**

(Gen) $[1114, 1114, 1134, 1144]$

(Bas) $x^2a \approx xa$, $ax^2 \approx ax$, $xyx \approx yx$

(Max) $x^2 \approx x$; $xya \approx yxa$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{RZ_2^1\}$

(Sub) 10

Variety 60 (Dual of Variety 42; Figure 20)**.**

(Gen) $[1114, 1114, 1234, 1144]$

(Bas) $x^2a \approx xa$, $xyx \approx yx^2$

(Max) $ax^2 \approx ax$; $xya^2 \approx yxa^2$

(Dec) $\mathrm{var}\{J\} \vee \mathrm{var}\{RZ_2^1\}$

(Sub) 14

Variety 61 (Dual of Variety 35; Figure 19)**.**

(Gen) $[1114, 1124, 1134, 1144]$

(Bas) $ax^2 \approx ax$, $xyx \approx y^2x$

(Max) $x^2a \approx xa$; $x^2ya \approx y^2xa$

(Dec) $\operatorname{var}\{\overleftarrow{J}\} \vee \operatorname{var}\{RZ_2^1\}$

(Sub) 13

Variety 62 (Subsection C.4)**.**

(Gen) $[1114, 1124, 1234, 1144]$

(Bas) $x^3 \approx x^2$, $xyx \approx yx^2$

(Max) $x^2a^2 \approx xa^2$; $x^2y^2a^2 \approx y^2x^2a^2$

(Dec) $\operatorname{var}\{N_2^1\} \vee \operatorname{var}\{RZ_2^1\}$

(Sub) Countably infinite

Variety 63 (Gerhard and Petrich [16, Variety `RQNB` in Section 2]; Figure 18)**.**

(Gen) $[1114, 1224, 1334, 1444]$

(Bas) $x^2 \approx x$, $axyx \approx ayx$

(Max) $xyx \approx yx$; $axya \approx ayxa$

(Dec) $\operatorname{var}\{LZ_2\} \vee \operatorname{var}\{RZ_2^1\}$

(Sub) 10

Variety 64 (Dual to Variety 52; Figure 21)**.**

(Gen) $[1114, 1234, 1234, 4441]$

(Bas) $x^3 \approx x$, $xya \approx yxa$

(Max) $x^2 \approx x$; $x^2a \approx a$; $xy \approx yx$

(Dec) $\operatorname{var}\{S\ell_2\} \vee \operatorname{var}\{RZ_2\} \vee \operatorname{var}\{\mathbb{Z}_2\}$

(Sub) 8

Variety 65 (Dual of Variety 54; Figure 21)**.**

(Gen) $[1114, 1234, 1324, 1444]$

(Bas) $x^3 \approx x$, $xyx \approx yx^2$

(Max) $x^2 \approx x$; $xya \approx yxa$

(Dec) $\mathrm{var}\{\mathbb{Z}_2\} \vee \mathrm{var}\{RZ_2^1\}$

(Sub) 10

Variety 66 (Dual of Variety 56; Figure 21).

(Gen) $[1133, 1234, 1331, 1432] = \overleftarrow{O_2}$

(Bas) $x^3 \approx x$, $xyxy \approx yx^2y$

(Max) $xyx \approx yx^2$

(Dec) None

(Sub) 11

Variety 67 (Gerhard and Petrich [16, Variety `Rec B` in Section 2]; Figure 18).

(Gen) $[1133, 2244, 1133, 2244]$

(Bas) $xyx \approx x$

(Max) $xy \approx x$; $xy \approx y$

(Dec) $\mathrm{var}\{LZ_2\} \vee \mathrm{var}\{RZ_2\}$

(Sub) 4

Variety 68 (Tishchenko [77, Variety $\mathbf{A}_2 \vee \mathbf{L}_1$ on page 108]; Figure 21).

(Gen) $[1133, 2244, 3311, 4422]$

(Bas) $ax^2 \approx a$, $axy \approx ayx$

(Max) $x^2 \approx x$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{LZ_2\} \vee \mathrm{var}\{\mathbb{Z}_2\}$

(Sub) 4

Variety 69 (Dual of Variety 45; Figure 22).

(Gen) $[1134, 1134, 1134, 1334] = \overleftarrow{P_2}$

(Bas) $xab \approx ab$

(Max) $x^2 \approx yx$

(Dec) None

(Sub) 5

Variety 70 (Subsection B.9)**.**

(Gen) $[1134, 1134, 3341, 4413]$

(Bas) $x^3ab \approx ab$, $xy \approx yx$

(Max) $x^4 \approx x$; $x^3 \approx x^2$

(Dec) $\mathrm{var}\{N_2\} \vee \mathrm{var}\{\mathbb{Z}_3\}$

(Sub) 4

Variety 71 (Dual of Variety 68; Figure 21)**.**

(Gen) $[1234, 1234, 3412, 3412]$

(Bas) $x^2a \approx a$, $xya \approx yxa$

(Max) $x^2 \approx x$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{\mathbb{Z}_2\}$

(Sub) 4

Variety 72 (Lee *et al*. [45, Proposition 5.4]; Figure 26)**.**

(Gen) $[1234, 2143, 3421, 4312] = \mathbb{Z}_4$

(Bas) $x^4a \approx a$, $xy \approx yx$

(Max) $x^3 \approx x$

(Dec) None

(Sub) 3

6.4 Some varieties with primitive generator of order greater than 4

Variety 73 (Zhang and Luo [92, Variety $\mathbf{F} \vee \mathbf{S}$ in Figure 2]; Figure 17)**.**

(Gen) $[11111, 11111, 11111, 11141, 11211]$

(Bas) $x^3 \approx x^2$, $x^2ab \approx xab$, $xya \approx yxa$, $axy \approx ayx$

(Max) $x^2y \approx x^2$; $xy \approx yx$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{F_4\}$

(Sub) 8

Variety 74 (Tishchenko [78, Variety $\mathbf{V}_{1,3}$ on page 439]; Figure 22)**.**

(Gen) $[11111, 11111, 11111, 11211, 55555]$

(Bas) $x^2 \approx xyz$

(Max) $x^2 \approx xy$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{LZ_2\} \vee \mathrm{var}\{G_4\}$

(Sub) 7

Variety 75 (Dual of Variety 78; Figure 20)**.**

(Gen) $[11111, 11111, 11111, 11345, 13345]$

(Bas) $x^2a \approx xa$, $xyx \approx yx^2$, $xya^2 \approx yxa^2$

(Max) $xya \approx yxa$

(Dec) None

(Sub) 11

Variety 76 (Zhang and Luo [92, Variety $\mathbf{G} \vee \mathbf{S}$ in Figure 2]; Figures 17 or 23)**.**

(Gen) $[11111, 11111, 11112, 11141, 11211]$

(Bas) $x^3 \approx x^2$, $x^2ab \approx xab$, $xy \approx yx$

(Max) $x^2a \approx xa$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{S\ell_2\} \vee \mathrm{var}\{G_4\}$

(Sub) 6

Variety 77 (Tishchenko [78, Variety $\mathbf{L}_{2,3}$ in Proposition 3.1]; Figure 22).

(Gen) $[11111, 11111, 11214, 44444, 55555]$

(Bas) $xyx \approx xyz$

(Max) $x^3 \approx x^2$; $x^3 \approx xyx$

(Dec) $\mathrm{var}\{N_3\} \vee \mathrm{var}\{P_2\}$

(Sub) 13

Variety 78 (Zhang and Luo [92, Variety **C** in Figure 4]; Figure 20).

(Gen) $[11111, 11113, 11133, 11144, 11155]$

(Bas) $ax^2 \approx ax$, $xyx \approx x^2y$, $a^2xy \approx a^2yx$

(Max) $axy \approx ayx$

(Dec) None

(Sub) 11

Variety 79 (Gerhard and Petrich [16, Variety `RB` in Section 2]; Figure 18).

(Gen) $[11111, 12125, 33333, 12345, 12155]$

(Bas) $x^2 \approx x$, $xyxzx \approx xyzx$

(Max) $axyx \approx ayx$; $xyxa \approx xya$

(Dec) $\mathrm{var}\{LZ_2^1\} \vee \mathrm{var}\{RZ_2^1\}$

(Sub) 13

Variety 80 (Dual of Variety 74; Figure 22).

(Gen) $[11115, 11115, 11115, 11215, 11115]$

(Bas) $x^2 \approx yzx$

(Max) $x^2 \approx yx$; $x^2 \approx y^2$

(Dec) $\mathrm{var}\{RZ_2\} \vee \mathrm{var}\{G_4\}$

(Sub) 7

Variety 81 (Dual of Variety 77; Figure 22)**.**

(Gen) $[11145, 11145, 11245, 11145, 11445]$

(Bas) $xyx \approx zyx$

(Max) $x^3 \approx x^2$; $x^3 \approx xyx$

(Dec) $\mathrm{var}\{N_3\} \vee \mathrm{var}\{\overleftarrow{P_2}\}$

(Sub) 13

Variety 82 (Zhang and Luo [92, Variety $\mathbf{D} \vee \mathbf{F}$ in Figure 2]; Figure 17)**.**

(Gen) $[111111, 111111, 111111, 111111, 111211, 113116]$

(Bas) $x^3 \approx x^2$, $xy^2 \approx yx^2$, $ax^2b \approx axb$

(Max) $x^2a \approx xa$; $x^2y \approx xy^2$

(Dec) $\mathrm{var}\{J\} \vee \mathrm{var}\{G_4\}$

(Sub) 10

Variety 83 (Zhang and Luo [92, Variety $\mathbf{E} \vee \mathbf{F}$ in Figure 2]; Figure 17)**.**

(Gen) $[111111, 111111, 111111, 111114, 112111, 111116]$

(Bas) $x^3 \approx x^2$, $x^2y \approx y^2x$, $ax^2b \approx axb$

(Max) $ax^2 \approx ax$; $x^2y \approx xy^2$

(Dec) $\mathrm{var}\{\overleftarrow{J}\} \vee \mathrm{var}\{G_4\}$

(Sub) 10

Variety 84 (Tishchenko [78, Variety $\mathbf{V}_{2,3}$ on page 439]; Figure 22)**.**

(Gen) $[111111, 111111, 111111, 112115, 555555, 666666]$

(Bas) $x^3 \approx x^2$, $xyx \approx xyz$

(Max) $xyx \approx x^2$; $xyx \approx xy$

(Dec) $\mathrm{var}\{G_4\} \vee \mathrm{var}\{P_2\}$

(Sub) 9

Variety 85 (Dual of Variety 84; Figure 22)**.**

(Gen) $[111156, 111156, 111156, 112156, 111156, 111556]$

(Bas) $x^3 \approx x^2$, $xyx \approx zyx$

(Max) $xyx \approx x^2$; $xyx \approx yx$

(Dec) $\mathrm{var}\{G_4\} \vee \mathrm{var}\{\overleftarrow{P_2}\}$

(Sub) 9

Variety 86 (Mel'nik [54, Variety B_{24} in Figure 3]; Figure 27)**.**

(Gen) $[1111111, 1111111, 1111112, 1111121, 1111122, 1112235, 1121254]$

(Bas) $x^3 \approx xyzt$, $x^2y \approx xy^2$, $xy \approx yx$

(Max) $x^2y \approx x^3$

(Dec) None

(Sub) 7

Variety 87 (Mel'nik [54, Variety B_{26} in Figure 3]; Figure 27)**.**

(Gen) $[1111\,1111, 1111\,1111, 1111\,1112, 1111\,1121, 1111\,1211, 1111\,2134, 1112\,1315, 1121\,1451]$

(Bas) $x^2 \approx xyzt$, $xy \approx yx$

(Max) $x^2 \approx xyz$

(Dec) None

(Sub) 4

Variety 88 (Mel'nik [54, Variety B_{25} in Figure 3]; Figure 27)**.**

(Gen) $[1111\,1111, 1111\,1111, 1111\,1112, 1111\,1121, 1111\,1211, 1111\,2134, 1112\,1315, 1121\,1452]$

(Bas) $x^2y \approx xyzt$, $xy \approx yx$

(Max) $x^3 \approx xyz$; $x^3 \approx x^2$

(Dec) $\mathrm{var}\{N_3\} \vee \mathbf{V}_{87}$

(Sub) 6

7 Problems

In this section we propose a number of problems that are naturally prompted by the results in this paper.

Problem 7.1. Identify all varieties generated by a semigroup of order 6.

Regarding groups we propose the following problems.

Problem 7.2. Given a finite group G, find good bounds for the following:

(a) the number of critical groups in $\mathrm{var}\{G\}$;

(b) the order of the largest critical group in $\mathrm{var}\{G\}$;

(c) the number of subvarieties of $\mathrm{var}\{G\}$;

(d) the number of varieties covered by $\mathrm{var}\{G\}$.

Solve the same problems for the class $\mathbf{C}(e, m, c)$ introduced in Subsection 3.6.

A Basic results on identities of some semigroups

The present section establishes some background equational results that are required in Sections B and C. For more information on universal algebra, refer to the monograph of Burris and Sankappanavar [7].

Words are formed over some countably infinite set $\mathscr{X}$ of variables. An *identity* is an expression $\mathbf{u} \approx \mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in \mathscr{X}^+$. An identity $\mathbf{u} \approx \mathbf{v}$ is *nontrivial* if $\mathbf{u} \neq \mathbf{v}$. A semigroup S *satisfies* an identity $\mathbf{u} \approx \mathbf{v}$ if for any substitution $\varphi : \mathscr{X} \to S$, the elements $\varphi(\mathbf{u})$ and $\varphi(\mathbf{v})$ of S are equal; otherwise, S *violates* $\mathbf{u} \approx \mathbf{v}$. An identity $\mathbf{u} \approx \mathbf{v}$ is *deducible* from some identity $\mathbf{u}' \approx \mathbf{v}'$ if there exist some substitution $\varphi : \mathscr{X} \to \mathscr{X}^+$ and some words $\mathbf{p}, \mathbf{q} \in \mathscr{X}^*$ such that $\mathbf{u} = \mathbf{p}\big(\varphi(\mathbf{u}')\big)\mathbf{q}$ and $\mathbf{v} = \mathbf{p}\big(\varphi(\mathbf{v}')\big)\mathbf{q}$. An identity $\mathbf{u} \approx \mathbf{v}$ is *deducible* from some set Σ of identities if there exists some sequence

$$\mathbf{u} = \mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_m = \mathbf{v}$$

of words where each identity $\mathbf{w}_i \approx \mathbf{w}_{i+1}$ is deducible from some identity in Σ.

For any word $\mathbf{w}$,

- the *head* of $\mathbf{w}$, denoted by $\mathsf{h}(\mathbf{w})$, is the first variable occurring in $\mathbf{w}$;
- the *tail* of $\mathbf{w}$, denoted by $\mathsf{t}(\mathbf{w})$, is the last variable occurring in $\mathbf{w}$;
- the *initial part* of $\mathbf{w}$, denoted by $\mathsf{ini}(\mathbf{w})$, is the word obtained by retaining the first occurrence of each variable in $\mathbf{w}$;
- the *content* of $\mathbf{w}$, denoted by $\mathsf{con}(\mathbf{w})$, is the set of variables occurring in $\mathbf{w}$;
- the number of occurrences of a variable x in $\mathbf{w}$ is denoted by $\mathsf{occ}(x, \mathbf{w})$.

Lemma A.1. *Let* $\mathbf{u} \approx \mathbf{v}$ *be any identity. Then*

(i) LZ_2 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v})$;

(ii) LZ_2^1 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{v})$;

(iii) N_3 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if either*

$$|\mathbf{u}|, |\mathbf{v}| \geq 3 \quad \textit{or} \quad \mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{v}) \textit{ for all } x \in \mathscr{X};$$

(iv) N_n^1 *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if for all* $x \in \mathscr{X}$, *either*

$$\mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{v}) < n \quad \textit{or} \quad \mathsf{occ}(x, \mathbf{u}), \mathsf{occ}(x, \mathbf{v}) \geq n;$$

(v) $\mathbb{Z}_n$ *satisfies* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{occ}(x, \mathbf{u}) \equiv \mathsf{occ}(x, \mathbf{v}) \pmod{n}$ *for all* $x \in \mathscr{X}$.

Proof. These results are well-known and easily verified. For instance, see Petrich and Reilly [62, Theorem V.1.9] for parts (i) and (ii) and Almeida [1, Lemma 6.1.4] for parts (iv) and (v). □

Lemma A.2. *Let* $\mathbf{W}$ *be any variety that satisfies the identity*

$$x^{n+k} \approx x^n \tag{A.1}$$

for some $n \geq 2$ *and* $k \geq 1$. *Suppose that* $N_n^1 \notin \mathbf{W}$. *Then* $\mathbf{W}$ *satisfies the identity*

$$(x^n y)^{n-1+k} x^n \approx (x^n y)^{n-1} x^n. \tag{A.2}$$

Proof. By assumption, the variety $\mathbf{W}$ satisfies some identity $\alpha : \mathbf{u} \approx \mathbf{v}$ that is violated by the semigroup N_n^1. In view of Lemma A.1(iv), generality is not lost by assuming the existence of some variable $y \in \mathscr{X}$ such that $\mathsf{occ}(y, \mathbf{u}) = r < n$ and $\mathsf{occ}(y, \mathbf{v}) = s > r$. Then

$$\mathbf{u} = \mathbf{u}_0 y \mathbf{u}_1 y \mathbf{u}_2 \cdots y \mathbf{u}_r \quad \text{and} \quad \mathbf{v} = \mathbf{v}_0 y \mathbf{v}_1 y \mathbf{v}_2 \cdots y \mathbf{v}_s$$

for some $\mathbf{u}_i, \mathbf{v}_j \in \mathscr{X}^*$ such that $y \notin \mathsf{con}(\mathbf{u}_i \mathbf{v}_j)$. Let φ denote the substitution that maps y to $x^n y$ and every other variable to x^k. Then since

$$(x^n y)^r x^n \overset{(A.1)}{\approx} \big(\varphi(\mathbf{u})\big) x^n \overset{\alpha}{\approx} \big(\varphi(\mathbf{v})\big) x^n \overset{(A.1)}{\approx} (x^n y)^s x^n,$$

the variety $\mathbf{W}$ satisfies the identity $(x^n y)^r x^n \approx (x^n y)^s x^n$. It follows that $\mathbf{W}$ satisfies the identity $\beta : (x^n y)^{n-1} x^n \approx (x^n y)^{n-1+t} x^n$ for some $t \geq 1$. Since

$$\begin{aligned}(x^n y)^{n-1} x^n &\overset{\beta}{\approx} (x^n y)^{n-1+t} x^n \overset{\beta}{\approx} (x^n y)^{n-1+2t} x^n \overset{\beta}{\approx} \cdots \\ &\overset{\beta}{\approx} (x^n y)^{n-1+kt} x^n \overset{(A.1)}{\approx} (x^n y)^{n-1+k} x^n,\end{aligned}$$

the variety $\mathbf{W}$ also satisfies the identity (A.2). □

Lemma A.3 ([18, Lemma 7])**.** *The semigroup* J *satisfies an identity* $\mathbf{u} \approx \mathbf{v}$ *if and only if* $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ *and either of the following conditions holds*:

(i) $\mathsf{occ}\big(\mathsf{t}(\mathbf{u}), \mathbf{u}\big) = \mathsf{occ}\big(\mathsf{t}(\mathbf{v}), \mathbf{v}\big) = 1$ *with* $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v})$;

(ii) $\mathsf{occ}\big(\mathsf{t}(\mathbf{u}), \mathbf{u}\big), \mathsf{occ}\big(\mathsf{t}(\mathbf{v}), \mathbf{v}\big) \geq 2$.

Lemma A.4. *Let* $\mathbf{W}$ *be any variety that satisfies the identity*

$$x^{2n} \approx x^n \tag{A.3}$$

for some $n \geq 2$. *Suppose that* $J \notin \mathbf{W}$. *Then* $\mathbf{W}$ *satisfies one of the identities*

$$(x^n y)^{n+1} \approx x^n y, \tag{A.4}$$

$$x^n y x^n \approx x^n y. \tag{A.5}$$

Proof. By assumption, the variety $\mathbf{W}$ satisfies an identity $\alpha : \mathbf{u} \approx \mathbf{v}$ that is violated by the semigroup J. It is well known and easily shown that if $\mathsf{con}(\mathbf{u}) \neq \mathsf{con}(\mathbf{v})$, then the identity $(x^n y)^n x^n \approx x^n$ is deducible from the identities $\{(\text{A.3}), \mathbf{u} \approx \mathbf{v}\}$ and so is satisfied by the variety $\mathbf{W}$, whence $\mathbf{W}$ also satisfies the identity (A.4). Therefore assume that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$. By Lemma A.3, there are two cases.

Case 1: $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v}) = y$ with $\mathsf{occ}(y, \mathbf{u}) = 1$ and $\mathsf{occ}(y, \mathbf{v}) = m \geq 2$. Then

$$\mathbf{u} = \mathbf{w}_0 y \quad \text{and} \quad \mathbf{v} = \mathbf{w}_1 y \mathbf{w}_2 y \cdots \mathbf{w}_m y$$

for some $\mathbf{w}_i \in \mathscr{X}^*$ such that $y \notin \mathsf{con}(\mathbf{w}_i)$. Let φ denote the substitution that maps y to $x^n y$ and every other variable to x^n. Then

$$x^n y \overset{(\text{A.3})}{\approx} x^n\big(\varphi(\mathbf{u})\big) \overset{\alpha}{\approx} x^n\big(\varphi(\mathbf{v})\big) \overset{(\text{A.3})}{\approx} (x^n y)^m,$$

so that $\mathbf{W}$ satisfies the identity $\beta : x^n y \approx (x^n y)^{\ell+1}$ with $\ell = m - 1$. Since

$$x^n y \overset{\beta}{\approx} (x^n y)^{\ell+1} \overset{\beta}{\approx} (x^n y)^{2\ell+1} \overset{\beta}{\approx} \cdots \overset{\beta}{\approx} (x^n y)^{n\ell+1} \overset{(\text{A.3})}{\approx} (x^n y)^{n+1},$$

the variety $\mathbf{W}$ also satisfies the identity (A.4).

Case 2: $\mathsf{t}(\mathbf{u}) = y \neq z = \mathsf{t}(\mathbf{v})$ with $\mathsf{occ}(y, \mathbf{u}) = 1$ and $\mathsf{occ}(z, \mathbf{v}) \geq 1$. The assumption $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v})$ implies that $\mathsf{occ}(y, \mathbf{v}) = m \geq 1$. Then

$$\mathbf{u} = \mathbf{w}_0 y \quad \text{and} \quad \mathbf{v} = \mathbf{w}_1 y \mathbf{w}_2 y \cdots \mathbf{w}_m y \mathbf{w}_{m+1} z$$

for some $\mathbf{w}_i \in \mathscr{X}$ such that $y \notin \mathsf{con}(\mathbf{w}_i)$. Let φ denote the substitution in Case 1. Then

$$x^n y \overset{(\text{A.3})}{\approx} x^n\big(\varphi(\mathbf{u})\big) \overset{\alpha}{\approx} x^n\big(\varphi(\mathbf{v})\big) \overset{(\text{A.3})}{\approx} (x^n y)^m x^n,$$

so that $\mathbf{W}$ satisfies the identity $\gamma : x^n y \approx (x^n y)^m x^n$. Since

$$x^n(y x^n) \overset{\gamma}{\approx} \big(x^n(y x^n)\big)^m x^n \overset{(\text{A.3})}{\approx} (x^n y)^m x^n \overset{\gamma}{\approx} x^n y,$$

the variety $\mathbf{W}$ also satisfies the identity (A.5). □

Lemma A.5. *A variety that contains only finitely based subvarieties, contains at most countably many subvarieties.*

Proof. Up to renaming of variables, there can only be countably many finite sets of identities. □

B Some finite lattices of varieties

B.1 Subvarieties of $\mathbf{V}_{24} = \mathrm{var}\{J, \overleftarrow{J}\}$

Proposition B.1 (Zhang and Luo [92, Figure 2])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{24} = \mathrm{var}\{J, \overleftarrow{J}\}$ *are*

$$\begin{array}{lll} \mathbf{V}_1 = \mathrm{var}\{N_2\}, & \mathbf{V}_2 = \mathrm{var}\{S\ell_2\}, & \mathbf{V}_7 = \mathrm{var}\{N_2, S\ell_2\}, \\ \mathbf{V}_8 = \mathrm{var}\{J\}, & \mathbf{V}_{10} = \mathrm{var}\{\overleftarrow{J}\}, & \mathbf{V}_{20} = \mathrm{var}\{F_4\}, \\ \mathbf{V}_{22} = \mathrm{var}\{G_4\}, & \mathbf{V}_{73} = \mathrm{var}\{S\ell_2, F_4\}, & \mathbf{V}_{76} = \mathrm{var}\{S\ell_2, G_4\}, \\ \mathbf{V}_{82} = \mathrm{var}\{J, G_4\}, & \mathbf{V}_{83} = \mathrm{var}\{\overleftarrow{J}, G_4\}. & \end{array}$$

(ii) *The lattice* $\mathscr{L}(\mathbf{V}_{24})$ *is given in Figure 17.*

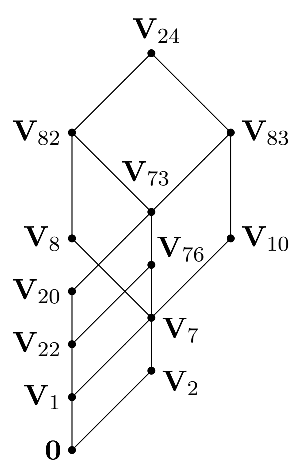


Figure 17: The lattice $\mathscr{L}(\mathbf{V}_{24})$

B.2 Subvarieties of $\mathbf{V}_{79} = \mathrm{var}\{LZ_2^1, RZ_2^1\}$

Proposition B.2 (Gerhard and Petrich [16, Section 2])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{79} = \mathrm{var}\{LZ_2^1, RZ_2^1\}$ *are*

$$\begin{array}{lll} \mathbf{V}_2 = \mathrm{var}\{S\ell_2\}, & \mathbf{V}_3 = \mathrm{var}\{LZ_2\}, & \mathbf{V}_4 = \mathrm{var}\{RZ_2\}, \\ \mathbf{V}_{12} = \mathrm{var}\{S\ell_2, LZ_2\}, & \mathbf{V}_{13} = \mathrm{var}\{S\ell_2, RZ_2\}, & \mathbf{V}_{15} = \mathrm{var}\{LZ_2^1\}, \\ \mathbf{V}_{18} = \mathrm{var}\{RZ_2^1\}, & \mathbf{V}_{51} = \mathrm{var}\{S\ell_2, LZ_2, RZ_2\}, & \mathbf{V}_{53} = \mathrm{var}\{RZ_2, LZ_2^1\}, \\ \mathbf{V}_{63} = \mathrm{var}\{LZ_2, RZ_2^1\}, & \mathbf{V}_{67} = \mathrm{var}\{LZ_2, RZ_2\}. & \end{array}$$

(ii) *The lattice* $\mathscr{L}(\mathbf{V}_{79})$ *is given in Figure 18.*

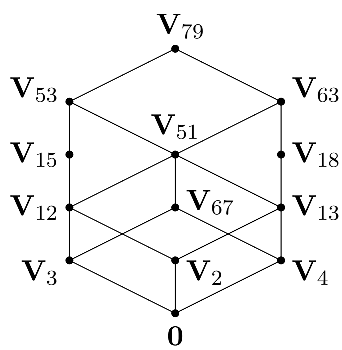


Figure 18: The lattice $\mathscr{L}(\mathbf{V}_{79})$

B.3 Subvarieties of $\mathbf{V}_{35} = \mathrm{var}\{J, LZ_2^1\}$ and $\mathbf{V}_{61} = \mathrm{var}\{\overleftarrow{J}, RZ_2^1\}$

Proposition B.3 (Zhang and Luo [92, Subvarieties of **A** in Figure 5])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{35} = \mathrm{var}\{J, LZ_2^1\}$ *are*

$$\begin{array}{lll} \mathbf{V}_1 = \mathrm{var}\{N_2\}, & \mathbf{V}_2 = \mathrm{var}\{S\ell_2\}, & \mathbf{V}_3 = \mathrm{var}\{LZ_2\}, \\ \mathbf{V}_7 = \mathrm{var}\{N_2, S\ell_2\}, & \mathbf{V}_8 = \mathrm{var}\{J\}, & \mathbf{V}_9 = \mathrm{var}\{N_2, LZ_2\}, \\ \mathbf{V}_{12} = \mathrm{var}\{S\ell_2, LZ_2\}, & \mathbf{V}_{15} = \mathrm{var}\{LZ_2^1\}, & \mathbf{V}_{28} = \mathrm{var}\{N_2, S\ell_2, LZ_2\}, \\ \mathbf{V}_{31} = \mathrm{var}\{N_2, LZ_2^1\}, & \mathbf{V}_{32} = \mathrm{var}\{LZ_2, J\}. & \end{array}$$

(ii) *The proper nontrivial subvarieties of* $\mathbf{V}_{61} = \text{var}\{\overleftarrow{J}, RZ_2^1\}$ *are*

$$\mathbf{V}_1 = \text{var}\{N_2\}, \quad \mathbf{V}_2 = \text{var}\{S\ell_2\}, \quad \mathbf{V}_4 = \text{var}\{RZ_2\},$$
$$\mathbf{V}_7 = \text{var}\{N_2, S\ell_2\}, \quad \mathbf{V}_{10} = \text{var}\{\overleftarrow{J}\}, \quad \mathbf{V}_{13} = \text{var}\{S\ell_2, RZ_2\},$$
$$\mathbf{V}_{16} = \text{var}\{N_2, RZ_2\}, \quad \mathbf{V}_{18} = \text{var}\{RZ_2^1\}, \quad \mathbf{V}_{29} = \text{var}\{N_2, S\ell_2, RZ_2\},$$
$$\mathbf{V}_{47} = \text{var}\{RZ_2, \overleftarrow{J}\}, \quad \mathbf{V}_{59} = \text{var}\{N_2, RZ_2^1\}.$$

(iii) *The lattices* $\mathscr{L}(\mathbf{V}_{35})$ *and* $\mathscr{L}(\mathbf{V}_{61})$ *are given in Figure 19.*

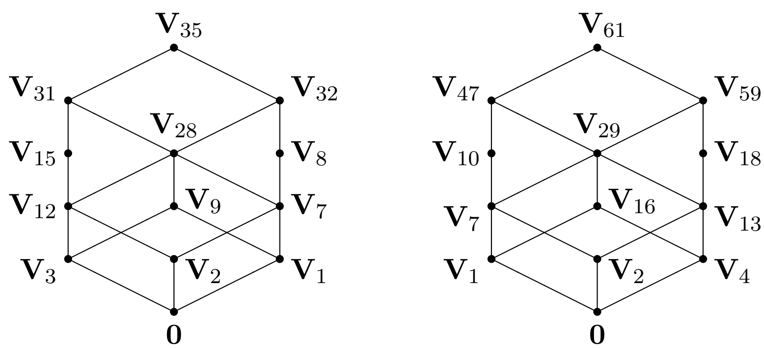


Figure 19: The lattices $\mathscr{L}(\mathbf{V}_{35})$ and $\mathscr{L}(\mathbf{V}_{61})$

B.4 Subvarieties of $\mathbf{V}_{42} = \text{var}\{\overleftarrow{J}, LZ_2^1\}$ and $\mathbf{V}_{60} = \text{var}\{J, RZ_2^1\}$

Proposition B.4 (Zhang and Luo [92, Subvarieties of **B** in Figure 5])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{42} = \text{var}\{\overleftarrow{J}, LZ_2^1\}$ *are*

$$\mathbf{V}_1 = \text{var}\{N_2\}, \quad \mathbf{V}_2 = \text{var}\{S\ell_2\}, \quad \mathbf{V}_3 = \text{var}\{LZ_2\},$$
$$\mathbf{V}_7 = \text{var}\{N_2, S\ell_2\}, \quad \mathbf{V}_9 = \text{var}\{N_2, LZ_2\}, \quad \mathbf{V}_{10} = \text{var}\{\overleftarrow{J}\},$$
$$\mathbf{V}_{12} = \text{var}\{S\ell_2, LZ_2\}, \quad \mathbf{V}_{15} = \text{var}\{LZ_2^1\}, \quad \mathbf{V}_{28} = \text{var}\{N_2, S\ell_2, LZ_2\},$$
$$\mathbf{V}_{31} = \text{var}\{N_2, LZ_2^1\}, \quad \mathbf{V}_{41} = \text{var}\{LZ_2, \overleftarrow{J}\},$$
$$\mathbf{V}_{78} = \text{var}\{[11111, 11113, 11133, 11144, 11155]\}.$$

(ii) *The proper nontrivial subvarieties of* $\mathbf{V}_{60} = \text{var}\{J, RZ_2^1\}$ *are*

$$\mathbf{V}_1 = \text{var}\{N_2\}, \quad \mathbf{V}_2 = \text{var}\{S\ell_2\}, \quad \mathbf{V}_4 = \text{var}\{RZ_2\},$$

$\mathbf{V}_7 = \mathrm{var}\{N_2, S\ell_2\}$, $\mathbf{V}_8 = \mathrm{var}\{J\}$, $\mathbf{V}_{13} = \mathrm{var}\{S\ell_2, RZ_2\}$,
$\mathbf{V}_{16} = \mathrm{var}\{N_2, RZ_2\}$, $\mathbf{V}_{18} = \mathrm{var}\{RZ_2^1\}$, $\mathbf{V}_{29} = \mathrm{var}\{N_2, S\ell_2, RZ_2\}$,
$\mathbf{V}_{33} = \mathrm{var}\{RZ_2, J\}$, $\mathbf{V}_{59} = \mathrm{var}\{N_2, RZ_2^1\}$,
$\mathbf{V}_{75} = \mathrm{var}\{[11111, 11111, 11111, 11345, 13345]\}$.

(iii) *The lattices* $\mathscr{L}(\mathbf{V}_{42})$ *and* $\mathscr{L}(\mathbf{V}_{60})$ *are given in Figure 20.*

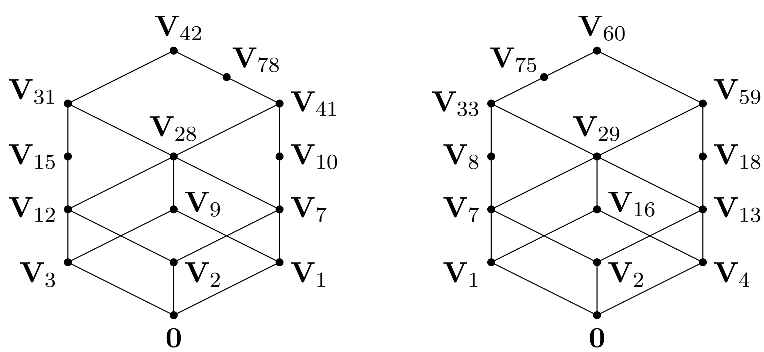


Figure 20: The lattices $\mathscr{L}(\mathbf{V}_{42})$ and $\mathscr{L}(\mathbf{V}_{60})$

B.5 Subvarieties of $\mathbf{V}_{56} = \mathrm{var}\{O_2\}$ and $\mathbf{V}_{66} = \mathrm{var}\{\overleftarrow{O_2}\}$

Proposition B.5 ([77, Figure 7])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{56} = \mathrm{var}\{O_2\}$ *are*

$\mathbf{V}_2 = \mathrm{var}\{S\ell_2\}$, $\mathbf{V}_3 = \mathrm{var}\{LZ_2\}$, $\mathbf{V}_5 = \mathrm{var}\{\mathbb{Z}_2\}$,
$\mathbf{V}_{12} = \mathrm{var}\{S\ell_2, LZ_2\}$, $\mathbf{V}_{14} = \mathrm{var}\{S\ell_2, \mathbb{Z}_2\}$, $\mathbf{V}_{15} = \mathrm{var}\{LZ_2^1\}$,
$\mathbf{V}_{52} = \mathrm{var}\{S\ell_2, LZ_2, \mathbb{Z}_2\}$, $\mathbf{V}_{54} = \mathrm{var}\{\mathbb{Z}_2, LZ_2^1\}$, $\mathbf{V}_{68} = \mathrm{var}\{LZ_2, \mathbb{Z}_2\}$.

(ii) *The proper nontrivial subvarieties of* $\mathbf{V}_{66} = \mathrm{var}\{\overleftarrow{O_2}\}$ *are*

$\mathbf{V}_2 = \mathrm{var}\{S\ell_2\}$, $\mathbf{V}_4 = \mathrm{var}\{RZ_2\}$, $\mathbf{V}_5 = \mathrm{var}\{\mathbb{Z}_2\}$,
$\mathbf{V}_{13} = \mathrm{var}\{S\ell_2, RZ_2\}$, $\mathbf{V}_{14} = \mathrm{var}\{S\ell_2, \mathbb{Z}_2\}$, $\mathbf{V}_{18} = \mathrm{var}\{RZ_2^1\}$,
$\mathbf{V}_{64} = \mathrm{var}\{S\ell_2, RZ_2, \mathbb{Z}_2\}$, $\mathbf{V}_{65} = \mathrm{var}\{\mathbb{Z}_2, RZ_2^1\}$, $\mathbf{V}_{71} = \mathrm{var}\{RZ_2, \mathbb{Z}_2\}$.

(iii) *The lattices* $\mathscr{L}(\mathbf{V}_{56})$ *and* $\mathscr{L}(\mathbf{V}_{66})$ *are given in Figure 21.*

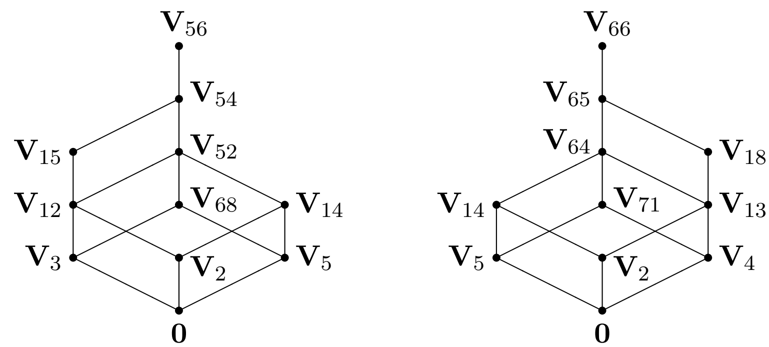


Figure 21: The lattices $\mathscr{L}(\mathbf{V}_{56})$ and $\mathscr{L}(\mathbf{V}_{66})$

B.6 Subvarieties of $\mathbf{V}_{77} = \mathrm{var}\{N_3, P_2\}$ and $\mathbf{V}_{81} = \mathrm{var}\{N_3, \overleftarrow{P_2}\}$

Proposition B.6 (Tishchenko [78, Figure 1])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{77} = \mathrm{var}\{N_3, P_2\}$ *are*

$$\begin{array}{lll} \mathbf{V}_1 = \mathrm{var}\{N_2\}, & \mathbf{V}_3 = \mathrm{var}\{LZ_2\}, & \mathbf{V}_6 = \mathrm{var}\{N_3\}, \\ \mathbf{V}_9 = \mathrm{var}\{N_2, LZ_2\}, & \mathbf{V}_{20} = \mathrm{var}\{F_4\}, & \mathbf{V}_{21} = \mathrm{var}\{N_3, F_4\}, \\ \mathbf{V}_{22} = \mathrm{var}\{G_4\}, & \mathbf{V}_{27} = \mathrm{var}\{LZ_2, N_3\}, & \mathbf{V}_{45} = \mathrm{var}\{P_2\}, \\ \mathbf{V}_{74} = \mathrm{var}\{LZ_2, G_4\}, & \mathbf{V}_{84} = \mathrm{var}\{G_4, P_2\}. & \end{array}$$

(ii) *The proper nontrivial subvarieties of* $\mathbf{V}_{81} = \mathrm{var}\{N_3, \overleftarrow{P_2}\}$ *are*

$$\begin{array}{lll} \mathbf{V}_1 = \mathrm{var}\{N_2\}, & \mathbf{V}_4 = \mathrm{var}\{RZ_2\}, & \mathbf{V}_6 = \mathrm{var}\{N_3\}, \\ \mathbf{V}_{16} = \mathrm{var}\{N_2, RZ_2\}, & \mathbf{V}_{20} = \mathrm{var}\{F_4\}, & \mathbf{V}_{21} = \mathrm{var}\{N_3, F_4\}, \\ \mathbf{V}_{22} = \mathrm{var}\{G_4\}, & \mathbf{V}_{57} = \mathrm{var}\{RZ_2, N_3\}, & \mathbf{V}_{69} = \mathrm{var}\{\overleftarrow{P_2}\}, \\ \mathbf{V}_{80} = \mathrm{var}\{RZ_2, G_4\}, & \mathbf{V}_{85} = \mathrm{var}\{G_4, \overleftarrow{P_2}\}. & \end{array}$$

(iii) *The lattices* $\mathscr{L}(\mathbf{V}_{77})$ *and* $\mathscr{L}(\mathbf{V}_{81})$ *are given in Figure 22.*

B.7 Subvarieties of $\mathbf{V}_{26} = \mathrm{var}\{S\ell_2, N_3\}$

Lemma B.7 ([85, Lemma 1.3])**.** *Let* $\mathbf{V}$ *be any variety such that* $S\ell_2 \notin \mathbf{V}$.

(i) *The lattice* $\mathscr{L}(\mathbf{V})$ *is isomorphic to the interval*

$$\mathscr{I} = [\mathrm{var}\{S\ell_2\}, \mathrm{var}\{S\ell_2\} \vee \mathbf{V}].$$

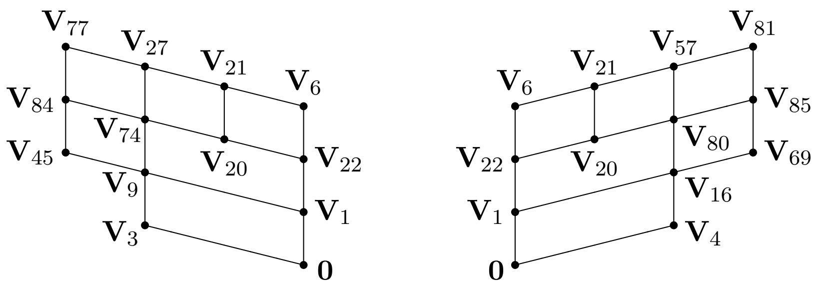


Figure 22: The lattices $\mathscr{L}(\mathbf{V}_{77})$ and $\mathscr{L}(\mathbf{V}_{81})$

(ii) *The lattice* $\mathscr{L}(\text{var}\{S\ell_2\} \vee \mathbf{V})$ *is isomorphic to the direct product*

$$\mathscr{L}(\text{var}\{S\ell_2\}) \times \mathscr{L}(\mathbf{V}).$$

Consequently, $\mathscr{L}(\text{var}\{S\ell_2\} \vee \mathbf{V})$ *is the disjoint union of* $\mathscr{L}(\mathbf{V})$ *and* $\mathscr{I}$*.*

Proposition B.8. *The lattice* $\mathbf{V}_{26} = \text{var}\{S\ell_2, N_3\}$ *is given in Figure 23.*

Proof. By Proposition B.6, the subvarieties of $\mathbf{V}_6 = \text{var}\{N_3\}$ constitute the chain $\mathbf{0} \subset \mathbf{V}_1 \subset \mathbf{V}_{22} \subset \mathbf{V}_6$. Since $\mathbf{V}_{26} = \text{var}\{S\ell_2\} \vee \text{var}\{N_3\}$, the result follows from Lemma B.7. □

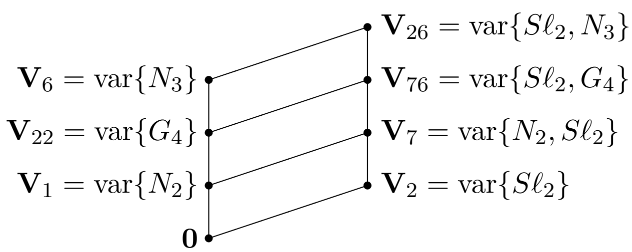


Figure 23: The lattice $\mathscr{L}(\mathbf{V}_{26})$

B.8 Subvarieties of $\text{var}\{N_3, \mathbb{Z}_n\}$

Lemma B.9. *Let* $n \geq 1$ *be any integer.*

(i) *The lattice* $\mathscr{L}(\text{var}\{N_3, \mathbb{Z}_n\})$ *is isomorphic to the direct product*

$$\mathscr{L}(\text{var}\{N_3\}) \times \mathscr{L}(\text{var}\{\mathbb{Z}_n\}).$$

Consequently, $\mathscr{L}(\mathrm{var}\{N_3, \mathbb{Z}_n\})$ is the disjoint union of the intervals

$$\mathscr{I}_d = [\mathrm{var}\{\mathbb{Z}_d\}, \mathrm{var}\{N_3, \mathbb{Z}_d\}],$$

where d ranges over all divisors of n.

(ii) *The interval $\mathscr{I}_d$ coincides with the chain*

$$\mathrm{var}\{\mathbb{Z}_d\} \subset \mathrm{var}\{N_2, \mathbb{Z}_d\} \subset \mathrm{var}\{G_4, \mathbb{Z}_d\} \subset \mathrm{var}\{N_3, \mathbb{Z}_d\}.$$

Proof. (i) This follows from Vernikov [84, Proposition 2].

(ii) This follows from part (i) since by Figure 23, the lattice $\mathscr{L}(\mathrm{var}\{N_3\})$ coincides with the chain $\mathbf{0} \subset \mathrm{var}\{N_2\} \subset \mathrm{var}\{G_4\} \subset \mathrm{var}\{N_3\}$. □

Proposition B.10. *For any prime $p \geq 2$, the lattice $\mathscr{L}(\mathrm{var}\{N_3, \mathbb{Z}_p\})$ is given in Figure 24.*

Proof. This follows from Lemma B.9. □

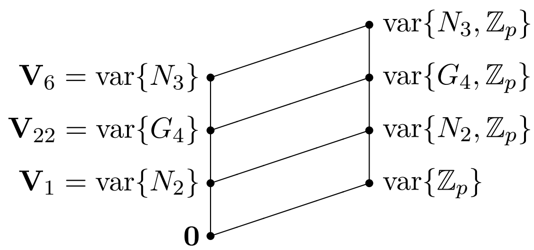


Figure 24: The lattice $\mathscr{L}(\mathrm{var}\{N_3, \mathbb{Z}_p\})$ with prime $p \geq 2$

Proposition B.11. *Let $n \geq 2$ be any integer. Then the identities*

$$x^n abc \approx abc, \tag{B.1a}$$

$$xy \approx yx \tag{B.1b}$$

constitute an identity basis for the variety $\mathrm{var}\{N_3, \mathbb{Z}_n\}$.

Proof. It is routinely checked that the identities (B.1) are satisfied by the variety $\mathrm{var}\{N_3, \mathbb{Z}_n\}$. Therefore it remains to show that any nontrivial identity $\mathbf{u} \approx \mathbf{v}$ satisfied by $\mathrm{var}\{N_3, \mathbb{Z}_n\}$ is deducible from (B.1). By Lemma A.1 parts (iii) and (v), the following properties hold:

(a) either $|\mathbf{u}|, |\mathbf{v}| \geq 3$ or $\mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{v})$ for all $x \in \mathscr{X}$;

(b) $\mathsf{occ}(x, \mathbf{u}) \equiv \mathsf{occ}(x, \mathbf{v}) \pmod{n}$ for all variables x.

If $\mathsf{occ}(x, \mathbf{u}) = \mathsf{occ}(x, \mathbf{v})$ for all $x \in \mathscr{X}$, then it is clear that the identity $\mathbf{u} \approx \mathbf{v}$ is deducible from (B.1b). Therefore suppose that $|\mathbf{u}|, |\mathbf{v}| \geq 3$. Generality is not lost by assuming that $\mathsf{con}(\mathbf{u}) = \{x_1, x_2, \ldots, x_k\}$ and $\mathsf{con}(\mathbf{v}) \backslash \mathsf{con}(\mathbf{u}) = \{y_1, y_2, \ldots, y_m\}$ for some $k \geq 1$ and $m \geq 0$. Let $e_i = \mathsf{occ}(x_i, \mathbf{u})$, so that $\sum_{i=1}^{k} e_i = |\mathbf{u}| \geq 3$. By (b), there exist $r_i, s_j \geq 1$ such that $\mathsf{occ}(x_i, \mathbf{v}) = e_i + r_i n \geq 0$ and $\mathsf{occ}(y_j, \mathbf{v}) = s_j n \geq 0$. Let $r_i' \geq 1$ be any integer such that $r_i + r_i' \geq 1$. Then

$$\begin{aligned}
\mathbf{v} &\stackrel{\text{(B.1a)}}{\approx} \left(\prod_{i=1}^{k} x_i^{r_i' n}\right)\mathbf{v} && \text{since } |\mathbf{v}| \geq 3 \\
&\stackrel{\text{(B.1b)}}{\approx} \left(\prod_{i=1}^{k} x_i^{r_i + r_i'} \prod_{i=1}^{m} y_i^{s_i}\right)^{n} \prod_{i=1}^{k} x_i^{e_i} \\
&\stackrel{\text{(B.1a)}}{\approx} \prod_{i=1}^{k} x_i^{e_i} && \text{since } \sum_{i=1}^{k} e_i \geq 3 \\
&\stackrel{\text{(B.1b)}}{\approx} \mathbf{u}.
\end{aligned}$$

□

Proposition B.12. *Let $n \geq 2$ be any integer. Then the identities*

$$x^n abc \approx abc, \quad xy \approx yx, \quad x^{n+2} \approx x^2 \tag{B.2}$$

constitute an identity basis for the variety $\operatorname{var}\{G_4, \mathbb{Z}_n\}$.

Proof. Let $\mathbf{W}$ denote the variety defined by the identities (B.2). Then it is routinely checked that the inclusions $\operatorname{var}\{G_4, \mathbb{Z}_n\} \subseteq \mathbf{W} \subseteq \operatorname{var}\{N_3, \mathbb{Z}_n\}$ hold. But the semigroup N_3 violates the last identity in (B.2), so that $\mathbf{W} \neq \operatorname{var}\{N_3, \mathbb{Z}_n\}$. Therefore $\mathbf{W} = \operatorname{var}\{G_4, \mathbb{Z}_n\}$ by Lemma B.9(ii). □

B.9 Subvarieties of $\operatorname{var}\{J, \mathbb{Z}_p\}$ and $\operatorname{var}\{S\ell_2, \mathbb{Z}_{p^2}\}$

Lemma B.13 ([65, Part (b) of the main theorem]). *Let $\mathbf{G}$ be any periodic variety generated by a group. Then each subvariety of* $\operatorname{var}\{J\} \vee \mathbf{G}$ *is the join of some subvariety of $\mathbf{G}$ with some of the following varieties*:

$$\mathbf{0}, \quad \mathbf{V}_1 = \operatorname{var}\{N_2\}, \quad \mathbf{V}_2 = \operatorname{var}\{S\ell_2\}, \quad \mathbf{V}_8 = \operatorname{var}\{J\}.$$

Proposition B.14. *Let $p \geq 2$ be any prime.*

(i) *The lattice* $\mathscr{L}(\mathrm{var}\{J, \mathbb{Z}_p\})$ *is given in Figure 25.*

(ii) *The lattice* $\mathscr{L}(\mathrm{var}\{S\ell_2, \mathbb{Z}_{p^2}\})$ *is given in Figure 26.*

Proof. This follows from Lemma B.13. □

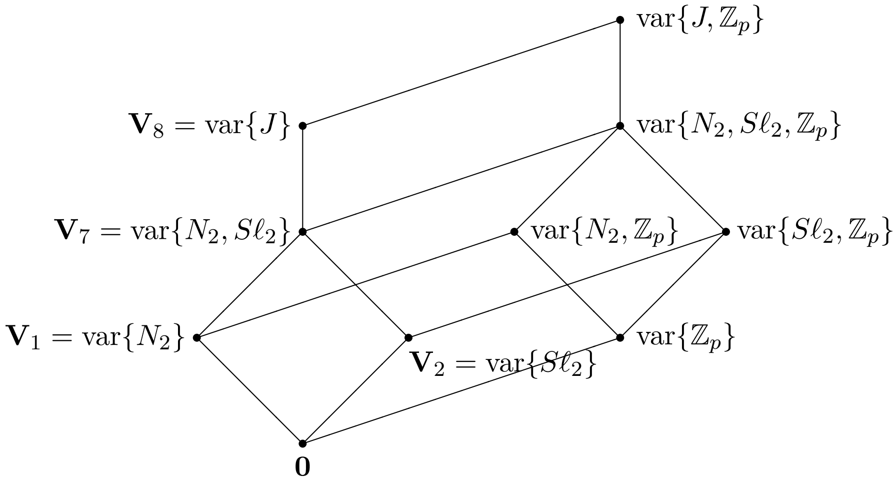


Figure 25: The lattice $\mathscr{L}(\mathrm{var}\{J, \mathbb{Z}_p\})$ with prime $p \geq 2$

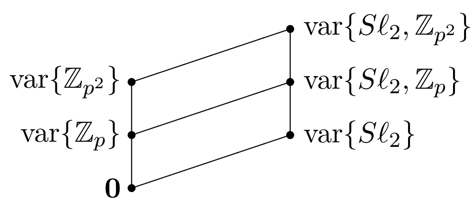


Figure 26: The lattice $\mathscr{L}(\mathrm{var}\{S\ell_2, \mathbb{Z}_{p^2}\})$ with prime $p \geq 2$

Proposition B.15. *Let* $n \geq 2$ *be any integer.*

(i) *The identities*

$$x^{n+1}a \approx xa, \tag{B.3a}$$
$$x^{m_1}y^{m_2} \approx y^{m_2}x^{m_1}, \quad m_1, m_2 \geq 2, \tag{B.3b}$$

$$xya \approx yxa. \tag{B.3c}$$

constitute an identity basis for the variety var$\{J, \mathbb{Z}_n\}$.

(ii) *The identities*

$$x^{n+1}a \approx xa, \quad x^2y^2 \approx y^2x^2, \quad xya \approx yxa$$

also constitute an identity basis for the variety var$\{J, \mathbb{Z}_n\}$.

Proof. (i) It is routinely checked that the identities (B.3) are satisfied by the variety var$\{J, \mathbb{Z}_n\}$. Therefore it remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by var$\{J, \mathbb{Z}_n\}$ is deducible from (B.3). By Lemma A.3, generality is not lost by assuming that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v}) = \{x_1, x_2, \ldots, x_m\}$, so that $e_i = \mathsf{occ}(x_i, \mathbf{u}) \geq 1$ and $f_i = \mathsf{occ}(x_i, \mathbf{v}) \geq 1$. Then $e_i \equiv f_i \pmod{n}$ by Lemma A.1(v). By Lemma A.3, there are two cases.
CASE 1: $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v}) = x_k$ with either $e_k = f_k = 1$ or $e_k, f_k \geq 2$. Then

$$\begin{aligned}
\mathbf{u} &\overset{(B.3c)}{\approx} \left(\prod_{i \neq k} x_i^{e_i}\right) x_k^{e_k} \\
&\overset{(B.3a)}{\approx} \left(\prod_{i \neq k} x_i^{f_i}\right) x_k^{f_k} \quad \text{since } e_i \equiv f_i \ (\text{mod} n) \\
&\overset{(B.3c)}{\approx} \mathbf{v}.
\end{aligned}$$

CASE 2: $\mathsf{t}(\mathbf{u}) = x_k$ and $\mathsf{t}(\mathbf{v}) = x_\ell$ with $k < \ell$ and $e_k, f_\ell \geq 2$. Choose any integer $g_i > \max\{e_i, f_i\}$ such that $g_i \equiv e_i \equiv f_i \pmod{n}$. Then

$$\begin{aligned}
\mathbf{u} &\overset{(B.3c)}{\approx} \left(\prod_{i \neq k,\ell} x_i^{e_i}\right) x_\ell^{e_\ell} x_k^{e_k} \\
&\overset{(B.3a)}{\approx} \left(\prod_{i \neq k,\ell} x_i^{g_i}\right) x_\ell^{g_\ell} x_k^{g_k} \quad \text{since } g_i \equiv e_i \pmod{n} \text{ and } e_k \geq 2 \\
&\overset{(B.3b)}{\approx} \left(\prod_{i \neq k,\ell} x_i^{g_i}\right) x_k^{g_k} x_\ell^{g_\ell} \\
&\overset{(B.3a)}{\approx} \left(\prod_{i \neq k,\ell} x_i^{f_i}\right) x_k^{f_k} x_\ell^{f_\ell} \quad \text{since } g_i \equiv f_i \pmod{n} \text{ and } f_\ell \geq 2 \\
&\overset{(B.3c)}{\approx} \mathbf{v}.
\end{aligned}$$

(ii) It suffices to show that the identities (B.3b) are deducible from the identities $\alpha : x^2y^2 \approx y^2x^2$ and $\beta : xya \approx yxa$. Write $m_i = 2p_i + r_i$ where $p_i \geq 1$ and $r_i \in \{0, 1\}$. Then

$$x^{m_1}y^{m_2} \overset{\beta}{\approx} y^{r_2}x^{r_1}x^{2p_1}y^{2p_2} \overset{\alpha}{\approx} y^{r_2}x^{r_1}y^{2p_2}x^{2p_1} \overset{\beta}{\approx} y^{m_2}x^{m_1}. \qquad \square$$

Proposition B.16 ([60, Lemma 7.3 and Diagram 8])**.** *Let $n \geq 2$ be any integer.*

(i) *The identities*

$$x^{n+1}a \approx xa, \quad xy \approx yx$$

constitute an identity basis for the variety $\mathrm{var}\{N_2, S\ell_2, \mathbb{Z}_n\}$.

(ii) *The identities*

$$x^n ab \approx ab, \quad xy \approx yx$$

constitute an identity basis for the variety $\mathrm{var}\{N_2, \mathbb{Z}_n\}$.

(iii) *The identities*

$$x^{n+1} \approx x, \quad xy \approx yx$$

constitute an identity basis for the variety $\mathrm{var}\{S\ell_2, \mathbb{Z}_n\}$.

B.10 Subvarieties of $\mathbf{V}_{23} = \mathrm{var}\{N_4\}$

Proposition B.17 (Mel'nik [54, Subvarieties of B_{23} in Figure 3])**.**

(i) *The proper nontrivial subvarieties of* $\mathbf{V}_{23} = \mathrm{var}\{N_4\}$ *are*

$$\mathbf{V}_1 = \mathrm{var}\{N_2\}, \qquad \mathbf{V}_6 = \mathrm{var}\{N_3\}, \qquad \mathbf{V}_{22} = \mathrm{var}\{G_4\},$$

$$\mathbf{V}_{86} = \mathrm{var}\left\{\begin{array}{l}[1111111, 1111111, 1111112, 1111121, 1111122, 1112235,\\ 1121254]\end{array}\right\},$$

$$\mathbf{V}_{87} = \mathrm{var}\left\{\begin{array}{l}[1111\,1111, 1111\,1111, 1111\,1112, 1111\,1121, 1111\,1211,\\ 1111\,2134, 1112\,1315, 1121\,1451]\end{array}\right\},$$

$$\mathbf{V}_{88} = \mathrm{var}\left\{\begin{array}{l}[1111\,1111, 1111\,1111, 1111\,1112, 1111\,1121, 1111\,1211,\\ 1111\,2134, 1112\,1315, 1121\,1452]\end{array}\right\}.$$

(ii) *The lattice $\mathscr{L}(\mathbf{V}_{23})$ is given in Figure 27.*

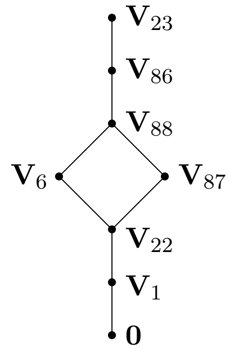


Figure 27: The lattice $\mathscr{L}(\mathbf{V}_{23})$

C Some varieties with infinitely many subvarieties

C.1 The variety $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$

Proposition C.1. *Let $p \geq 1$ and $n \geq 2$ be any integers.*

(i) *The identities*

$$x^{n+p} \approx x^n, \tag{C.1a}$$

$$xy \approx yx \tag{C.1b}$$

constitute an identity basis for the variety $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$.

(ii) *The variety* $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$ *contains countably infinitely many subvarieties.*

Proof. (i) It is routinely checked that the identities (C.1) are satisfied by the variety $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$. Hence it remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$ is deducible from (C.1). Generality is not lost by assuming that $\mathbf{u}, \mathbf{v} \in \{x_1, x_2, \ldots, x_m\}^*$ with $e_i = \mathsf{occ}(x_i, \mathbf{u})$ and $f_i = \mathsf{occ}(x_i, \mathbf{v})$. Then it follows from Lemma A.1 parts (iv) and (v) that for each i,

(a) either $e_i = f_i < n$ or $e_i, f_i \geq n$;

(b) $e_i \equiv f_i \pmod{p}$.

If $e_i \neq f_i$ for some i, then $e_i, f_i \in \{n + rp \,|\, r \geq 0\}$ by (a) and (b), whence the identity $x^{e_i} \approx x^{f_i}$ is deducible from (C.1a). It follows that

$$\mathbf{u} \overset{\text{(C.1b)}}{\approx} \prod_{i=1}^{m} x_i^{e_i} \overset{\text{(C.1a)}}{\approx} \prod_{i=1}^{m} x_i^{f_i} \overset{\text{(C.1b)}}{\approx} \mathbf{v}$$

(ii) Any variety of commutative semigroups is finitely based [59]. Hence by Lemma A.5, the variety $\mathrm{var}\{\mathbb{Z}_p, N_n^1\}$ contains countably many subvarieties. The result then holds since the subvariety $\mathrm{var}\{N_2^1\}$ of $\mathrm{var}\{\mathbb{Z}_p, N_n^1\}$ contains infinitely many subvarieties [13, Figure 5(b)]. □

Corollary C.2. *Let $n \geq 2$ be any integer.*

(i) *The identities*

$$x^{n+1} \approx x^n, \tag{C.2a}$$

$$xy \approx yx \tag{C.2b}$$

constitute an identity basis for the variety $\mathrm{var}\{N_n^1\}$.

(ii) *The variety* $\mathrm{var}\{N_n^1\}$ *contains countably infinitely many subvarieties.*

Lemma C.3 (Lee *et al.* [45, Proposition 5.10])**.** *Each proper subvariety of* $\mathrm{var}\{N_n^1\}$ *satisfies the identity*

$$x^n y^{n-1} \approx x^{n-1} y^n. \tag{C.3}$$

Proposition C.4. *Let $n \geq 2$ be any integer.*

(i) *The variety* $\mathrm{var}\{(\text{C.2}), (\text{C.3})\}$ *is the only maximal subvariety of* $\mathrm{var}\{N_n^1\}$.

(ii) *The variety* $\mathrm{var}\{(\text{C.2}), (\text{C.3})\}$ *is not finitely generated.*

Proof. (i) This follows from Corollary C.2(i) and Lemma C.3.

(ii) It is easily seen that the variety $\mathrm{var}\{(\text{C.2}), (\text{C.3})\}$ violates the identity

$$x_1 x_2 \cdots x_m y^n \approx x_1 x_2 \cdots x_m y^{n-1} \tag{C.4}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in the variety $\mathrm{var}\{(\text{C.2}), (\text{C.3})\}$ satisfies the identity (C.4) for all $m \geq n|S|$. Choose any elements $a_1, a_2, \ldots, a_m, b \in S$. Then the list $a_1, a_2, \ldots, a_m$ contains some element $a \in S$ at least n times, due to the magnitude of m. Therefore $a_1 a_2 \cdots a_m \overset{\text{(C.2b)}}{=} sa^n$ for some $s \in S$, whence

$$\begin{aligned} a_1 a_2 \cdots a_m b^n &\overset{\text{(C.2b)}}{=} sa^n b^n \overset{\text{(C.3)}}{=} sa^{n+1} b^{n-1} \\ &\overset{\text{(C.2a)}}{=} sa^n b^{n-1} \overset{\text{(C.2b)}}{=} a_1 a_2 \cdots a_m b^{n-1}. \end{aligned}$$

□

Lemma C.5. *Let $p \geq 2$ be any prime and $n \geq 2$ be any integer. Then each proper subvariety of* $\mathrm{var}\{\mathbb{Z}_p, N_n^1\}$ *satisfies one of the following identities*:

$$x^{n-1+p}y^{n-1} \approx x^{n-1}y^{n-1+p}, \tag{C.5}$$

$$x^{n+1} \approx x^n. \tag{C.6}$$

Proof. Let $\mathbf{W}$ be any proper subvariety of $\mathrm{var}\{\mathbb{Z}_p, N_n^1\}$. Then either $\mathbb{Z}_p \notin \mathbf{W}$ or $N_n^1 \notin \mathbf{W}$. First suppose that $N_n^1 \notin \mathbf{W}$. Then it follows from Lemma A.2 that the variety $\mathbf{W}$ satisfies the identity $\alpha : (x^n y)^{n-1+p}x^n \approx (x^n y)^{n-1}x^n$. Let $r \geq 1$ be such that $n^2 + r \equiv n \pmod{p}$. Then since

$$\begin{aligned} x^{n-1+p}y^n &\overset{\text{(C.1a)}}{\approx} x^{n-1+p}y^{n^2+r} = x^{n-1+p}(y^n)^n y^r \\ &\overset{\text{(C.1a)}}{\approx} x^{n-1+p}(y^n)^{n+p}y^r \overset{\text{(C.1b)}}{\approx} (y^n x)^{n-1+p}y^n y^r \\ &\overset{\alpha}{\approx} (y^n x)^{n-1}y^n y^r \overset{\text{(C.1b)}}{\approx} x^{n-1}y^{n^2+r} \overset{\text{(C.1a)}}{\approx} x^{n-1}y^n, \end{aligned}$$

it follows that $\mathbf{W}$ satisfies the identity $\beta : x^{n-1+p}y^{n-1+p} \approx x^{n-1}y^{n-1+p}$. But since

$$\begin{aligned} x^{n-1}y^{n-1+p} &\overset{\beta}{\approx} x^{n-1+p}y^{n-1+p} \overset{\text{(C.1b)}}{\approx} y^{n-1+p}x^{n-1+p} \\ &\overset{\beta}{\approx} y^{n-1}x^{n-1+p} \overset{\text{(C.1b)}}{\approx} x^{n-1+p}y^{n-1}, \end{aligned}$$

the variety $\mathbf{W}$ also satisfies the identity (C.5).

It remains to assume that $\mathbb{Z}_p \notin \mathbf{W}$. Then by Lemma A.1(v), the variety $\mathbf{W}$ satisfies an identity $\gamma : \mathbf{u} \approx \mathbf{v}$ with $\mathsf{occ}(x, \mathbf{u}) \not\equiv \mathsf{occ}(x, \mathbf{v}) \pmod{p}$ for some variable $x \in \mathscr{X}$. Generality is not lost with the assumption that $e \equiv \mathsf{occ}(x, \mathbf{u}) \pmod{p}$ and $f \equiv \mathsf{occ}(x, \mathbf{v}) \pmod{p}$ with $0 \leq e < f \leq p - 1$. Let φ denote the substitution that fixes x and maps every other variable to x^p. Then

$$x^{n+e} \overset{\text{(C.1b)}}{\approx} \big(\varphi(\mathbf{u})\big)x^n \overset{\gamma}{\approx} \big(\varphi(\mathbf{v})\big)x^n \overset{\text{(C.1b)}}{\approx} x^{n+f},$$

so that the variety $\mathbf{W}$ satisfies the identity $\delta : x^{n+e} \approx x^{n+f}$. Since

$$x^n \overset{\text{(C.1a)}}{\approx} x^{n+e}x^{p-e} \overset{\delta}{\approx} x^{n+f}x^{p-e} \overset{\text{(C.1a)}}{\approx} x^{n+f-e},$$

the variety $\mathbf{W}$ satisfies the identity $\varepsilon : x^n \approx x^{n+\ell}$ for some $\ell \geq 1$. Since p is prime, there exists some $m \geq 1$ such that $m\ell \equiv 1 \pmod{p}$. Therefore

$$x^n \overset{\varepsilon}{\approx} x^{n+\ell} \overset{\varepsilon}{\approx} x^{n+2\ell} \overset{\varepsilon}{\approx} \cdots \overset{\varepsilon}{\approx} x^{n+m\ell} \overset{\text{(C.1a)}}{\approx} x^{n+1},$$

so that the variety $\mathbf{W}$ satisfies the identity (C.6). □

Proposition C.6. *For any prime $p \geq 2$ and integer $n \geq 2$, let*

$$\mathbf{U} = \operatorname{var}\{(C.1), (C.5)\} \quad \textit{and} \quad \mathbf{V} = \operatorname{var}\{(C.1), (C.6)\}.$$

Then

(i) $\mathbf{U}$ *and* $\mathbf{V}$ *are precisely all maximal subvarieties of* $\operatorname{var}\{\mathbb{Z}_p, N_n^1\}$;

(ii) $\mathbf{U}$ *is not finitely generated*;

(iii) $\mathbf{V} = \operatorname{var}\{N_n^1\}$.

Proof. (i) Since $\mathbb{Z}_p$ satisfies $\{(C.1), (C.5)\}$ and violates (C.6), while N_n^1 satisfies $\{(C.1), (C.6)\}$ and violates (C.5), the varieties $\mathbf{U}$ and $\mathbf{V}$ are incomparable. The result then follows from Lemma C.5.

(ii) It is easily seen that the variety $\mathbf{U}$ violates the identity

$$x^{n-1+p} y_1 y_2 \cdots y_m \approx x^{n-1} y_1 y_2 \cdots y_m \tag{C.7}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in $\mathbf{U}$ satisfies the identity (C.7) for all $m \geq (n+p)|S|$. Choose any elements $a, b_1, b_2, \ldots, b_m \in S$. Then the list $b_1, b_2, \ldots, b_m$ contains some element $b \in S$ at least $n+p$ times, due to the magnitude of m. Therefore $b_1 b_2 \cdots b_m \overset{(C.1b)}{=} b^{n+p} s$ for some $s \in S$, whence

$$\begin{aligned} a^{n-1} b_1 b_2 \cdots b_m &\overset{(C.1b)}{=} a^{n-1} b^{n+p} s \overset{(C.5)}{=} a^{n-1+p} b^n s \\ &\overset{(C.1a)}{=} a^{n-1+p} b^{n+p} s \overset{(C.1b)}{=} a^{n-1+p} b_1 b_2 \cdots b_m. \end{aligned}$$

(iii) This follows from Corollary C.2(i). □

C.2 The varieties $\operatorname{var}\{J, N_n^1\}$ and $\operatorname{var}\{\overleftarrow{J}, N_n^1\}$

Proposition C.7. *Let $n \geq 2$ be any integer.*

(i) *The identities*

$$x^{n+1} \approx x^n, \tag{C.8a}$$
$$x^{m_1} y^{m_2} \approx y^{m_2} x^{m_2}, \quad m_1, m_2 \in \{2, 3, 4, \ldots\}, \tag{C.8b}$$
$$xya \approx yxa \tag{C.8c}$$

constitute an identity basis for the variety $\operatorname{var}\{J, N_n^1\}$.

(ii) *The identities*

$$x^{n+1} \approx x^n, \quad x^2y^2 \approx y^2x^2, \quad xya \approx yxa$$

also constitute an identity basis for the variety $\mathrm{var}\{J, N_n^1\}$.

(iii) *The variety* $\mathrm{var}\{J, N_n^1\}$ *contains countably infinitely many subvarieties.*

Proof. (i) It is routinely checked that the identities (C.8) are satisfied by the variety $\mathrm{var}\{J, N_n^1\}$. Hence it remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by $\mathrm{var}\{J, N_n^1\}$ is deducible from (C.8). By Lemma A.3, generality is not lost by assuming that $\mathsf{con}(\mathbf{u}) = \mathsf{con}(\mathbf{v}) = \{x_1, x_2, \ldots, x_m\}$, so that $e_i = \mathsf{occ}(x_i, \mathbf{u}) \geq 1$ and $f_i = \mathsf{occ}(x_i, \mathbf{v}) \geq 1$. Further, it follows from Lemma A.1(iv) that

(a) for each i, either $e_i = f_i < n$ or $e_i, f_i \geq n$.

There are two cases.
Case 1: $\mathsf{t}(\mathbf{u}) = \mathsf{t}(\mathbf{v}) = x_k$. Then

$$\begin{aligned}
\mathbf{u} &\overset{\text{(C.8c)}}{\approx} \Bigg(\prod_{i \neq k} x_i^{e_i}\Bigg) x_k^{e_k} \\
&\overset{\text{(C.8a)}}{\approx} \Bigg(\prod_{i \neq k} x_i^{f_i}\Bigg) x_k^{f_k} \quad \text{by (a)} \\
&\overset{\text{(C.8c)}}{\approx} \mathbf{v}.
\end{aligned}$$

Case 2: $\mathsf{t}(\mathbf{u}) = x_k$ and $\mathsf{t}(\mathbf{v}) = x_\ell$ with $k < \ell$. Then by (a) and Lemma A.3,

(b) $e_k, f_k, e_\ell, f_\ell \geq 2$.

Hence

$$\begin{aligned}
\mathbf{u} &\overset{\text{(C.8c)}}{\approx} \Bigg(\prod_{i \neq k,\ell} x_i^{e_i}\Bigg) x^{e_\ell} x^{e_k} \\
&\overset{\text{(C.8b)}}{\approx} \Bigg(\prod_{i \neq k,\ell} x_i^{e_i}\Bigg) x^{e_k} x^{e_\ell} \quad \text{by (b)} \\
&\overset{\text{(C.8a)}}{\approx} \Bigg(\prod_{i \neq k,\ell} x_i^{f_i}\Bigg) x^{f_k} x^{f_\ell} \quad \text{by (a)} \\
&\overset{\text{(C.8c)}}{\approx} \mathbf{v}.
\end{aligned}$$

(ii) As shown in the proof of Proposition B.15(ii), the identities (C.8b) are deducible from $x^2y^2 \approx y^2x^2$ and $xya \approx yxa$. The result thus follows from part (i).

(iii) Any finitely generated variety that satisfies the identity (C.8c) is finitely based [59]. Hence by Lemma A.5, the variety $\mathrm{var}\{J, N_n^1\}$ contains countably many subvarieties. The result then holds since the subvariety $\mathrm{var}\{N_2^1\}$ of $\mathrm{var}\{J, N_n^1\}$ contains infinitely many subvarieties [13, Figure 5(b)]. □

Lemma C.8. *Let $n \geq 2$ be any integer. Then each proper subvariety of $\mathrm{var}\{J, N_n^1\}$ satisfies one of the following identities:*

$$x^{n-1}y^n \approx y^{n-1}x^n, \tag{C.9}$$

$$x^n y \approx yx^n. \tag{C.10}$$

Proof. Let $\mathbf{W}$ be any proper subvariety of $\mathrm{var}\{J, N_n^1\}$. Then either $J \notin \mathbf{W}$ or $N_n^1 \notin \mathbf{W}$. First suppose that $N_n^1 \notin \mathbf{W}$. Then by Lemma A.2, the variety $\mathbf{W}$ satisfies the identity (A.2) with $k = 1$. Since

$$x^{n-1}y^n \overset{(\mathrm{C.8})}{\approx} (y^n x)^{n-1} y^n \overset{(\mathrm{A.2})}{\approx} (y^n x)^n y^n \overset{(\mathrm{C.8})}{\approx} x^n y^n,$$

the variety $\mathbf{W}$ satisfies the identity $\alpha : x^{n-1}y^n \approx x^n y^n$; since

$$x^{n-1}y^n \overset{\alpha}{\approx} x^n y^n \overset{(\mathrm{C.8b})}{\approx} y^n x^n \overset{\alpha}{\approx} y^{n-1}x^n,$$

it also satisfies the identity (C.9).

It remains to assume that $J \notin \mathbf{W}$, so that by Lemma A.4, the variety $\mathbf{W}$ satisfies one of the identities (A.4) and (A.5). Since

$$x^n y \overset{(\mathrm{A.4})}{\approx} (x^n y)^{n+1} \overset{(\mathrm{C.8})}{\approx} (x^n y)^{n+1} x^n \overset{(\mathrm{A.4})}{\approx} x^n y x^n \overset{(\mathrm{C.8})}{\approx} yx^n$$

$$\text{and} \quad x^n y \overset{(\mathrm{A.5})}{\approx} x^n y x^n \overset{(\mathrm{C.8})}{\approx} yx^n,$$

the variety $\mathbf{W}$ also satisfies the identity (C.10). □

Proposition C.9. *For any integer $n \geq 2$, let*

$$\mathbf{U} = \mathrm{var}\{(\mathrm{C.8}), (\mathrm{C.9})\} \quad \textit{and} \quad \mathbf{V} = \mathrm{var}\{(\mathrm{C.8}), (\mathrm{C.10})\}.$$

Then

(i) $\mathbf{U}$ *and* $\mathbf{V}$ *are precisely all maximal subvarieties of* $\mathrm{var}\{J, N_n^1\}$;

(ii) $\mathbf{U}$ *is not finitely generated*;

(iii) $\mathbf{V}$ *is not finitely generated.*

Proof. (i) Since the semigroup J satisfies $\{(\text{C.8}),(\text{C.9})\}$ and violates (C.10), while the semigroup N_n^1 satisfies $\{(\text{C.8}),(\text{C.10})\}$ and violates (C.9), the varieties $\mathbf{U}$ and $\mathbf{V}$ are incomparable. The result then follows from Lemma C.8.

(ii) It is easily seen that the variety $\mathbf{U}$ violates the identity

$$x^n y_1 y_2 \cdots y_m \approx x^{n-1} y_1 y_2 \cdots y_m \tag{C.11}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in $\mathbf{U}$ satisfies the identity (C.11) for all $m \geq (n+1)|S|$. Choose any $a, b_1, b_2, \ldots, b_m \in S$. Then the list $b_1, b_2, \ldots, b_m$ contains some element $b \in S$ at least $n+1$ times, due to the magnitude of m. Therefore $b_1 b_2 \cdots b_m \overset{(\text{C.8c})}{=} b^n sbt$ for some $s, t \in S^1$, whence

$$\begin{aligned} a^{n-1} b_1 b_2 \cdots b_m &\overset{(\text{C.8c})}{=} a^{n-1} b^n sbt \overset{(\text{C.8a})}{=} a^{n-1} b^n bsbt \overset{(\text{C.9})}{=} b^{n-1} a^n bsbt \\ &\overset{(\text{C.8c})}{=} a^n b^n sbt \overset{(\text{C.8c})}{=} a^n b_1 b_2 \cdots b_m. \end{aligned}$$

(iii) It is easily seen that the variety $\mathbf{V}$ violates the identity

$$x_1 x_2 \cdots x_m yz \approx x_1 x_2 \cdots x_m zy \tag{C.12}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in $\mathbf{V}$ satisfies the identity (C.12) for all $m \geq n|S|$. Choose any elements $a_1, a_2, \ldots, a_m, b, c \in S$. Then the list $a_1, a_2, \ldots, a_m$ contains some element $a \in S$ at least n times, due to the magnitude of m. Thus $a_1 a_2 \cdots a_m b \overset{(\text{C.8c})}{=} sa^n b$ and $a_1 a_2 \cdots a_m c \overset{(\text{C.8c})}{=} sa^n c$ for some $s \in S$, whence

$$\begin{aligned} a_1 a_2 \cdots a_m bc &\overset{(\text{C.8c})}{=} sa^n bc \overset{(\text{C.10})}{=} sbca^n \overset{(\text{C.8c})}{=} scba^n \\ &\overset{(\text{C.10})}{=} sa^n cb \overset{(\text{C.8c})}{=} a_1 a_2 \cdots a_m cb. \end{aligned}$$

□

Corollary C.10. *Let $n \geq 2$ be any integer. Then*

(i) *the identities*

$$x^{n+1} \approx x^n, \quad x^2 y^2 \approx y^2 x^2, \quad axy \approx ayx$$

constitute an identity basis for the variety $\text{var}\{\overleftarrow{J}, N_n^1\}$;

(ii) $\text{var}\{\overleftarrow{J}, N_n^1\}$ *contains countably infinitely many subvarieties*;

(iii) $\text{var}\{\overleftarrow{J}, N_n^1\}$ *contains precisely two maximal subvarieties.*

C.3 The varieties $\mathrm{var}\{LZ_2, N_n^1\}$ and $\mathrm{var}\{RZ_2, N_n^1\}$

Proposition C.11. *Let $n \geq 2$ be any integer.*

(i) *The identities*

$$x^{n+1} \approx x^n, \tag{C.13a}$$

$$axy \approx ayx. \tag{C.13b}$$

constitute an identity basis for the variety $\mathrm{var}\{LZ_2, N_n^1\}$.

(ii) *The variety* $\mathrm{var}\{LZ_2, N_n^1\}$ *contains countably infinitely many subvarieties.*

Proof. (i) It is routinely checked that the identities (C.13) are satisfied by the variety $\mathrm{var}\{LZ_2, N_n^1\}$. Therefore it remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by $\mathrm{var}\{LZ_2, N_n^1\}$ is deducible from the identities (C.13). Generality is not lost by assuming that $\mathbf{u}, \mathbf{v} \in \{x_1, x_2, \ldots, x_m\}^*$ with $e_i = \mathsf{occ}(x_i, \mathbf{u})$ and $f_i = \mathsf{occ}(x_i, \mathbf{v})$. By Lemma A.1 parts (i) and (iv),

(a) $\mathsf{h}(\mathbf{u}) = \mathsf{h}(\mathbf{v}) = x_k$ for some k;

(b) for each i, either $e_i = f_i < n$ or $e_i, f_i \geq n$.

Hence

$$\begin{aligned} \mathbf{u} &\overset{(\text{C.13b})}{\approx} x_k^{e_k} \prod_{i \neq k} x_i^{e_i} \\ &\overset{(\text{C.13a})}{\approx} x_k^{f_k} \prod_{i \neq k} x_i^{f_i} \quad \text{by (b)} \\ &\overset{(\text{C.13b})}{\approx} \mathbf{v}. \end{aligned}$$

(ii) See the proof of Proposition C.7(iii). □

Lemma C.12. *Let $n \geq 2$ be any integer. Then each proper subvariety of the variety* $\mathrm{var}\{LZ_2, N_n^1\}$ *satisfies one of the following identities*:

$$x^n y^n \approx y^n x^n, \tag{C.14}$$

$$a^n x^n \approx a^n x^{n-1}. \tag{C.15}$$

Proof. Let $\mathbf{W}$ be any proper subvariety of $\operatorname{var}\{LZ_2, N_n^1\}$. Then either $LZ_2 \notin \mathbf{W}$ or $N_n^1 \notin \mathbf{W}$. First suppose that $LZ_2 \notin \mathbf{W}$. Then the variety $\mathbf{W}$ satisfies the identity $\alpha : x^n(yx^n)^n \approx (yx^n)^n$ [45, Theorem 5.15]. Since

$$x^n y^n \overset{(C.13)}{\approx} x^n(yx^n)^n \overset{\alpha}{\approx} (yx^n)^n \overset{(C.13)}{\approx} y^n x^n,$$

the variety $\mathbf{W}$ satisfies the identity (C.14).

It remains to assume that $N_n^1 \notin \mathbf{W}$. Then by Lemma A.2, the variety $\mathbf{W}$ satisfies the identity (A.2) with $k = 1$. Since

$$a^n x^n \overset{(C.13)}{\approx} (a^n x)^n a^n \overset{(A.2)}{\approx} (a^n x)^{n-1} a^n \overset{(C.13)}{\approx} a^n x^{n-1},$$

the variety $\mathbf{W}$ satisfies the identity (C.15). □

Proposition C.13. *For any integer $n \geq 2$, let*

$$\mathbf{U} = \operatorname{var}\{(C.13), (C.14)\} \quad \textit{and} \quad \mathbf{V} = \operatorname{var}\{(C.13), (C.15)\}.$$

Then

(i) $\mathbf{U}$ *and* $\mathbf{V}$ *are the only maximal subvarieties of* $\operatorname{var}\{LZ_2, N_n^1\}$;

(ii) $\mathbf{U} = \operatorname{var}\{\overleftarrow{J}, N_2^1\}$ *if* $n = 2$;

(iii) $\mathbf{V}$ *is not finitely generated.*

Proof. (i) Since the semigroup LZ_2 satisfies $\{(C.13), (C.15)\}$ and violates (C.14), while the semigroup N_n^1 satisfies $\{(C.13), (C.14)\}$ and violates (C.15), the varieties $\mathbf{U}$ and $\mathbf{V}$ are incomparable. The result then follows from Lemma C.12.

(ii) This follows from the dual of Proposition C.7(ii).

(iii) It is easily seen that the variety $\mathbf{V}$ violates the identity

$$x_1 x_2 \cdots x_m y^n \approx x_1 x_2 \cdots x_m y^{n-1} \tag{C.16}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in $\mathbf{V}$ satisfies the identity (C.16) for all $m \geq (n+1)|S|$. Choose any $a_1, a_2, \ldots, a_m, b \in S$. Then the list $a_1, a_2, \ldots, a_m$ contains some element $a \in S$ at least $n+1$ times, due to the magnitude of m. Therefore $a_1 a_2 \cdots a_m \overset{(C.13b)}{=} sata^n$ for some $s, t \in S^1$, whence

$$a_1 a_2 \cdots a_m b^n \overset{(C.13b)}{=} sata^n b^n \overset{(C.15)}{=} sata^n b^{n-1} \overset{(C.13b)}{=} a_1 a_2 \cdots a_m b^{n-1}. \quad □$$

Corollary C.14. *Let $n \geq 2$ be any integer. Then*

(i) *the identities*

$$x^{n+1} \approx x^n, \quad xya \approx yxa$$

constitute an identity basis for the variety $\mathrm{var}\{RZ_2, N_n^1\}$;

(ii) $\mathrm{var}\{RZ_2, N_n^1\}$ *contains countably infinitely many subvarieties*;

(iii) $\mathrm{var}\{RZ_2, N_n^1\}$ *contains precisely two maximal subvarieties.*

C.4 The varieties $\mathrm{var}\{LZ_2^1, N_n^1\}$ and $\mathrm{var}\{RZ_2^1, N_n^1\}$

Proposition C.15. *Let $n \geq 2$ be any integer.*

(i) *The identities*

$$x^{n+1} \approx x^n, \tag{C.17a}$$

$$xyx \approx x^2 y. \tag{C.17b}$$

constitute an identity basis for the variety $\mathrm{var}\{LZ_2^1, N_n^1\}$.

(ii) *The variety* $\mathrm{var}\{LZ_2^1, N_n^1\}$ *contains countably infinitely many subvarieties.*

Proof. (i) It is routinely checked that the identities (C.17) are satisfied by the variety $\mathrm{var}\{LZ_2^1, N_n^1\}$. Therefore it remains to show that any identity $\mathbf{u} \approx \mathbf{v}$ satisfied by $\mathrm{var}\{LZ_2^1, N_n^1\}$ is deducible from the identities (C.17). In view of Lemma A.1(ii), generality is not lost by assuming that

(a) $\mathsf{ini}(\mathbf{u}) = \mathsf{ini}(\mathbf{v}) = \prod_{i=1}^m x_i$,

so that $e_i = \mathsf{occ}(x_i, \mathbf{u}) \geq 1$ and $f_i = \mathsf{occ}(x_i, \mathbf{v}) \geq 1$. By Lemma A.1(iv),

(b) for each i, either $e_i = f_i < n$ or $e_i, f_i \geq n$.

Hence

$$\begin{aligned} \mathbf{u} &\overset{(C.17b)}{\approx} \prod_{i=1}^m x_i^{e_i} && \text{by (a)} \\ &\overset{(C.17a)}{\approx} \prod_{i=1}^m x_i^{f_i} && \text{by (b)} \\ &\overset{(C.17b)}{\approx} \mathbf{v} && \text{by (a).} \end{aligned}$$

(ii) Any variety that satisfies the identity (C.17b) is finitely based [61]. Hence by Lemma A.5, the variety $\mathrm{var}\{LZ_2^1, N_n^1\}$ contains countably many subvarieties. The result then holds since the subvariety $\mathrm{var}\{N_2^1\}$ of $\mathrm{var}\{LZ_2^1, N_n^1\}$ contains infinitely many subvarieties [13, Figure 5(b)]. □

Lemma C.16. *Let $n \geq 2$ be any integer. Then each proper subvariety of the variety* $\mathrm{var}\{LZ_2^1, N_n^1\}$ *satisfies one of the following identities*:

$$a^n x^n y^n \approx a^n y^n x^n, \tag{C.18}$$

$$a^n x^n \approx a^n x^{n-1}. \tag{C.19}$$

Proof. Let $\mathbf{W}$ be any proper subvariety of $\mathrm{var}\{LZ_2^1, N_n^1\}$. Then either $LZ_2^1 \notin \mathbf{W}$ or $N_n^1 \notin \mathbf{W}$. First suppose that $LZ_2^1 \notin \mathbf{W}$. Then the variety $\mathbf{W}$ satisfies the identity $\alpha : a^n(xa^n)^n\big(ya^n(xa^n)^n\big)^n \approx a^n\big(ya^n(xa^n)^n\big)^n$ [45, Theorem 5.17]. Since

$$a^n x^n y^n \stackrel{(C.17)}{\approx} a^n(xa^n)^n\big(ya^n(xa^n)^n\big)^n \stackrel{\alpha}{\approx} a^n\big(ya^n(xa^n)^n\big)^n \stackrel{(C.17)}{\approx} a^n y^n x^n,$$

the variety $\mathbf{W}$ satisfies the identity (C.18).

It remains to assume that $N_n^1 \notin \mathbf{W}$. Then by Lemma A.2, the variety $\mathbf{W}$ satisfies the identity (A.2) with $k = 1$. Since

$$a^n x^n \stackrel{(C.17)}{\approx} (a^n x)^n a^n \stackrel{(A.2)}{\approx} (a^n x)^{n-1} a^n \stackrel{(C.17)}{\approx} a^n x^{n-1},$$

the variety $\mathbf{W}$ satisfies the identity (C.19). □

Proposition C.17. *For any integer $n \geq 2$, let*

$$\mathbf{U} = \mathrm{var}\{(C.17), (C.18)\} \quad \textit{and} \quad \mathbf{V} = \mathrm{var}\{(C.17), (C.19)\}.$$

Then

(i) $\mathbf{U}$ *and* $\mathbf{V}$ *are the only maximal subvarieties of* $\mathrm{var}\{LZ_2^1, N_n^1\}$;

(ii) $\mathbf{U}$ *is not finitely generated*;

(iii) $\mathbf{V}$ *is not finitely generated.*

Proof. (i) Since the semigroup LZ_2^1 satisfies $\{(C.17), (C.19)\}$ and violates (C.18), while the semigroup N_n^1 satisfies $\{(C.17), (C.18)\}$ and violates (C.19), the varieties $\mathbf{U}$ and $\mathbf{V}$ are incomparable. The result then follows from Lemma C.16.

(ii) It is easily seen that the variety $\mathbf{U}$ violates the identity

$$x_1 x_2 \cdots x_m y^n z^n \approx x_1 x_2 \cdots x_m z^n y^n \tag{C.20}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in the variety $\mathbf{U}$ satisfies the identity (C.20) for all $m \geq n|S|$. Choose any elements $a_1, a_2, \ldots, a_m, b, c \in S$. Then the list $a_1, a_2, \ldots, a_m$ contains some element $a \in S$ at least n times, due to the magnitude of m. Therefore $a_1a_2\cdots a_m \overset{(C.17b)}{=} sa^nt$ for some $s,t \in S^1$, whence

$$a_1a_2\cdots a_m b^nc^n \overset{(C.17b)}{=} sa^ntb^nc^n \overset{(C.17)}{=} sa^nta^nb^nc^n \overset{(C.18)}{=} sa^nta^nc^nb^n \\ \overset{(C.17)}{=} sa^ntc^nb^n \overset{(C.17b)}{=} a_1a_2\cdots a_m c^nb^n.$$

(iii) It is easily seen that the variety $\mathbf{V}$ violates the identity

$$x_1x_2\cdots x_my^n \approx x_1x_2\cdots x_my^{n-1} \tag{C.21}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in the variety $\mathbf{V}$ satisfies the identity (C.21) for all $m \geq n|S|$. Choose any elements $a_1, a_2, \ldots, a_m, b \in S$. Then the list $a_1, a_2, \ldots, a_m$ contains some element $a \in S$ at least n times, due to the magnitude of m. Therefore $a_1a_2\cdots a_m \overset{(C.17b)}{=} sa^nt$ for some $s,t \in S^1$, whence

$$a_1a_2\cdots a_m b^n \overset{(C.17b)}{=} sa^ntb^n \overset{(C.17)}{=} sa^nta^nb^n \overset{(C.19)}{=} sa^nta^nb^{n-1} \\ \overset{(C.17)}{=} sa^ntb^{n-1} \overset{(C.17b)}{=} a_1a_2\cdots a_m b^{n-1}.$$

□

Corollary C.18. *Let $n \geq 2$ be any integer. Then*

(i) *the identities*

$$x^{n+1} \approx x^n, \quad xyx \approx yx^2$$

constitute an identity basis for the variety $\mathrm{var}\{RZ_2^1, N_n^1\}$;

(ii) $\mathrm{var}\{RZ_2^1, N_n^1\}$ *contains countably infinitely many subvarieties*;

(iii) $\mathrm{var}\{RZ_2^1, N_n^1\}$ *contains precisely two maximal subvarieties.*

C.5 The varieties $\mathbf{V}_{38} = \mathrm{var}\{B_0\}$ and $\mathbf{V}_{39} = \mathrm{var}\{A_0\}$

Proposition C.19 (Edmunds [11, Semigroups $\mathtt{S}(4,21)$ and $\mathtt{S}(4,22)$ on page 70]; Lee [30])**.**

(i) *The identities*

$$x^3 \approx x^2, \quad x^2yx^2 \approx yxy, \quad x^2y^2 \approx y^2x^2$$

constitute an identity basis for the variety $\mathbf{V}_{38} = \mathrm{var}\{B_0\}$.

(ii) *The identities*

$$x^3 \approx x^2, \quad x^2yx^2 \approx yxy$$

constitute an identity basis for the variety $\mathbf{V}_{39} = \mathrm{var}\{A_0\}$.

(iii) *The varieties* $\mathrm{var}\{B_0\}$ *and* $\mathrm{var}\{A_0\}$ *each contains countably infinitely many subvarieties.*

Proposition C.20 (Lee [30, 31])**.**

(i) *The variety* $\mathrm{var}\{B_0\}$ *is the unique maximal subvariety of* $\mathrm{var}\{A_0\}$.

(ii) *The identities*

$$x^3 \approx x^2, \quad x^2yx^2 \approx yxy, \quad x^2y^2 \approx y^2x^2, \quad a^2x^2b^2 \approx a^2xb^2. \qquad \text{(C.22)}$$

constitute an identity basis for the unique maximal subvariety of $\mathrm{var}\{B_0\}$.

(iii) *The unique maximal subvariety of* $\mathrm{var}\{B_0\}$ *is not finitely generated.*

C.6 The varieties $\mathbf{V}_{40} = \mathrm{var}\{J^1\}$ and $\mathbf{V}_{46} = \mathrm{var}\{\overleftarrow{J^1}\}$

Proposition C.21.

(i) *The identities*

$$x^3 \approx x^2, \qquad \text{(C.23a)}$$

$$x^2y^2 \approx y^2x^2, \qquad \text{(C.23b)}$$

$$xyx \approx yx^2. \qquad \text{(C.23c)}$$

constitute an identity basis for the variety $\mathrm{var}\{J^1\}$.

(ii) *The variety* $\mathrm{var}\{J^1\}$ *contains countably infinitely many subvarieties.*

Proof. (i) See Edmunds [11, Semigroup $\mathtt{S}(4, 23)$ on page 70].

(ii) Any variety that satisfies the identity (C.23c) is finitely based [61]. Hence by Lemma A.5, the variety $\mathrm{var}\{J^1\}$ contains countably many subvarieties. The result then holds since the subvariety $\mathrm{var}\{N_2^1\}$ of $\mathrm{var}\{J^1\}$ contains infinitely many subvarieties [13, Figure 5(b)]. □

Lemma C.22. *Each proper subvariety of* $\mathrm{var}\{J^1\}$ *satisfies the identity*

$$x^2ya^2 \approx yx^2a^2. \qquad \text{(C.24)}$$

Proof. Let $\mathbf{W}$ be any proper subvariety of $\mathrm{var}\{J^1\}$, so that $J^1 \notin \mathbf{W}$. Then it follows from Almeida [1, Proposition 11.7.9] that $\mathbf{W}$ satisfies either (C.24) or $\alpha : x^2y^2 \approx xy^2$. Since

$$x^2ya^2 \stackrel{\alpha}{\approx} x^2y^2a^2 \stackrel{\text{(C.23b)}}{\approx} y^2x^2a^2 \stackrel{\alpha}{\approx} yx^2a^2,$$

the variety $\mathbf{W}$ always satisfies the identity (C.24). □

Proposition C.23. *Let* $\mathbf{U} = \mathrm{var}\{\text{(C.23)}, \text{(C.24)}\}$. *Then*

(i) $\mathbf{U}$ *is the unique maximal subvariety of* $\mathbf{V}_{40} = \mathrm{var}\{J^1\}$;

(ii) $\mathbf{U}$ *is not finitely generated.*

Proof. (i) This follows from Proposition C.21(i) and Lemma C.22.

(ii) It is easily seen that the variety $\mathbf{U}$ violates the identity

$$x^2yz_1z_2\cdots z_m \approx yx^2z_1z_2\cdots z_m \tag{C.25}$$

for any $m \geq 1$. Hence it suffices to show that each finite semigroup S in $\mathbf{U}$ satisfies the identity (C.25) for all $m > |S|$. Choose any elements $a, b, c_1, c_2, \ldots, c_m \in S$. The list $c_1, c_2, \ldots, c_m$ contains some element $c \in S$ twice, due to the magnitude of m. Therefore $c_1c_2\cdots c_m = s_1cs_2cs_3$ for some $s_i \in S^1$, whence

$$\begin{aligned} a^2bc_1c_2\cdots c_m = a^2bs_1cs_2cs_3 &\stackrel{\text{(C.23)}}{=} a^2bc^2s_1cs_2cs_3 \stackrel{\text{(C.24)}}{=} ba^2c^2s_1cs_2cs_3 \\ &\stackrel{\text{(C.23)}}{=} ba^2s_1cs_2cs_3 = ba^2c_1c_2\cdots c_m. \end{aligned}$$

□

Corollary C.24.

(i) *The identities*

$$x^3 \approx x^2, \quad x^2y^2 \approx y^2x^2, \quad xyx \approx x^2y.$$

constitute an identity basis for the variety $\mathbf{V}_{46} = \mathrm{var}\{\overleftarrow{J^1}\}$.

(ii) *The variety* $\mathrm{var}\{\overleftarrow{J^1}\}$ *contains countably infinitely many subvarieties.*

(iii) *The variety* $\mathrm{var}\{\overleftarrow{J^1}\}$ *contains a unique maximal subvariety.*

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