ALL 2-TRANSITIVE GROUPS HAVE THE EKR-MODULE PROPERTY

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ABSTRACT. We prove that every 2-transitive group has a property called the *EKR-module property*. This property gives a characterization of the maximum intersecting sets of permutations in the group. Specifically, the characteristic vector of any maximum intersecting set in a 2-transitive group is the linear combination of the characteristic vectors of the stabilizers of a points and their cosets. We also consider when the derangement graph of a 2-transitive group is connected and when a maximum intersecting set is a subgroup or a coset of a subgroup.

1. Introduction

The Erdős-Ko-Rado (EKR) Theorem [9] is a major result in extremal set theory. This famous result gives the size and the structure of the largest collection of pairwise intersecting k-subsets from an n-set. The Erdős-Ko-Rado Theorem has been generalized in many different ways. One generalization is to show that a version of the theorem holds for different objects. To date, a version of the EKR theorem has been shown to hold for the following objects: k-subsets of an n-set [3, 9, 25], integer sequences [21], k-dimensional subspaces of an n-dimensional vector space over a finite field [11], signed sets [6], partitions [16] and perfect matchings [11], as well as many other objects.

The commonality relating these results is that a largest set of (pairwise) intersecting objects must be a set of objects that intersect in a "canonical" way. For example, a largest set of intersecting k-sets is the collection of all k-sets that contain a common point. A largest set of intersecting k-subspaces is the set of all subspaces that contain a common 1-dimensional subspace. Similarly, a largest set of intersecting perfect matchings is the collection of all perfect matchings that contain a fixed pair. In all of these cases, the objects are sets of elements and two objects are said to intersect if they contain a common element. And for all the cases named above, a largest set of intersecting objects is the collection of all objects that contain a fixed element—these are the canonical intersecting sets.

In general, whenever we have objects formed from elements we can ask "what is the size and structure of a largest set of intersecting objects?". If a largest intersecting set must be a canonical intersecting set, then we say that a version of the EKR theorem holds.

In this paper we consider permutations. Two permutations $g, h \in \mathrm{Sym}(n)$ *intersect* if there exists an $i \in \{1, \dots, n\}$ with $i^g = i^h$. (Here a permutation g is the object, and the elements that form it are the pairs (i, j) where $i^g = j$).

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Let G be a subgroup of $\mathrm{Sym}(n)$. Clearly the stabilizer in G of a point, or the coset of a stabilizer of a point, is an intersecting set of permutations. These sets are denoted by

$$S_{i,j} = \{g \in G \,|\, i^g = j\},$$

where $i, j \in \{1, \dots, n\}$ and we call them the *canonical intersecting sets*.

We say that such a group G has the *EKR property* if the largest intersecting sets, which we shall call *maximum intersecting sets*, are the canonical intersecting sets. The group G is further said to have the *strict-EKR property* if the canonical intersecting sets are the only maximum intersecting sets. (Note that these properties depend on the group action.) Many specific groups have been shown to have either the strict-EKR property, or the EKR property [1, 17, 18, 20, 24]. One of the most general results is the following which is equivalent to every 2-transitive group has the EKR property.

Theorem 1.1 (Theorem 1.1 [19])**.** *Let G be a finite 2-transitive permutation group on the set $\{1, \dots, n\}$. A maximum intersecting set in G has cardinality $|G|/n$.*

There are 2-transitive permutation groups that do not have the strict-EKR property, for example $PGL_n(q)$ has the strict-EKR property if and only if $n = 2$ [24]. In this paper we consider a property that lies between the EKR property, and the strict-EKR property; this property is called the *EKR-module property*. The EKR-module property was first defined in [20], the definition we give here is slightly different, but equivalent. First we need some notation.

The *regular module* of G is the (complex) vector space with basis G. We can think of its elements as vectors of length $|G|$. For any $S \leq G$ define the *characteristic vector* of S to be the vector with entry 1 in position g if $g \in S$ and 0 otherwise; this vector is denoted by v_S. We denote the characteristic vector of $S_{i,j}$ by $v_{i,j}$.

A group G has the EKR-module property if, for any maximum intersecting set of permutations S in G, the characteristic vector v_S is a linear combination of the vectors $v_{i,j}$ with $i, j \in \{1, \dots, n\}$. Like the EKR property and strict-EKR property, this is a property of the group action. The main result of this paper is the following.

Theorem 1.2. *Any 2-transitive group has the EKR-module property.*

We feel that this is the most general statement for all 2-transitive group, in the context of EKR-type results. This result also gives information about the structure of the maximum intersecting sets in a 2-transitive group, this is described in detail in Section 7.

2. Background

In this paper we only consider 2-transitive permutation groups, so throughout this paper G is assumed to be a 2-transitive group acting faithfully on a set X of size n. For each such group we let χ_G denote the permutation character of this 2-transitive action. Since G is 2-transitive, χ_G is the sum of the trivial character (denoted 1_G) and an irreducible character which we will denote by ψ_G.

Let $\mathbb{C}[G]$ be the complex group algebra. The regular module can be identified with the vector space $\mathbb{C}[G]$ and given the structure of a left $\mathbb{C}[G]$-module by left multiplication. Thus, $\mathbb{C}[G]$ also becomes identified with a subalgebra of the $|G| \times |G|$-matrices.

For any irreducible character ϕ of G, let E_ϕ to be the $|G| \times |G|$-matrix with the (g, h)-entry equal to $\frac{\phi(1)}{|G|}\phi(hg^{-1})$. Then $E_\phi \in \mathbb{C}[G]$ is the primitive central idempotent corresponding to ϕ. We call the image E_ϕ (considered as a linear operator on $\mathbb{C}[G]$) the ϕ-module. It is an ideal of $\mathbb{C}[G]$ of dimension $\phi(1)^2$. For the trivial representation the central idempotent is $E_{1_G} = \frac{1}{|G|}J$, where J is the all ones matrix. We set $E_{\chi_G} = E_{1_G} + E_{\psi_G}$ and define the χ_G-module to be the image of E_{χ_G}, an ideal of dimension $1 + (n-1)^2$ in $\mathbb{C}[G]$. This leads to an equivalent definition of the EKR-module property, from which its name originates.

Lemma 2.1. *A 2-transitive group G has the EKR-module property if and only if the characteristic vector for any maximum intersecting set is in the χ_G-module. Equivalently, G has the EKR-module property if $E_{\chi_G} v_S = v_S$ for any maximum intersecting set S.*

Proof. This follows from two results from [1]. First, Lemma 4.1 of [1] states that if G is 2-transitive, then every $v_{i,j}$ is in the χ_G-module. Lemma 4.2 of the same paper states that the vectors $v_{i,j}$ are a spanning set for the module. □

We also state a simply corollary of this lemma that gives the result in a format that can be more convenient.

Corollary 2.2. *If a 2-transitive group G has the EKR-module property then for any maximum intersecting set S,*

$$E_{\psi} v_S = v_S - \frac{1}{n}\mathbf{1}. \tag{2.1}$$

Proof. From Theorem 1.1, if S is a maximum intersecting set, then $E_{\mathbf{1}} v_s = \frac{1}{n}\mathbf{1}$ where $\mathbf{1}$ denotes the all-ones vector. Then Lemma 2.1 implies the equation. □

A common approach to EKR theorems is to convert the problem to a graph problem, and the apply techniques from algebraic graph theory (see [13] for details and examples). This is the approach that we will use as well. The *derangement graph* of G is the graph with vertices the elements of G, in which two vertices are adjacent if they are not intersecting. The set of derangements in G (these are the permutations with no fixed points) is denoted by Der_G, and the derangement graph of G is denoted by Γ_G. The derangement graph is the Cayley graph on G with connection set Der_G. A coclique (or independent set) in Γ_G is equivalent to a set of intersecting permutations in G. Theorem 1.1 can be expressed as the size of a maximum coclique in Γ_G is $\frac{|G|}{n}$ for any 2-transitive group G.

Using this graph structure allows us to use results from graph theory. For example the clique-coclique bound (this is a well-known bound, for a proof see [13, Corollary 2.1.2]) easily translates to the following.

Lemma 2.3. *Let $\omega(\Gamma_G)$ denote the size of the largest clique in Γ_G, and $\alpha(\Gamma_G)$, the size of the largest coclique. Then*

$$\omega(\Gamma_G)\,\alpha(\Gamma_G) \leq |G|.$$

Further, if equality holds, then each maximum clique intersects each maximum coclique in exactly one vertex. □

Since the connection set of Γ_G is a normal subset in G, Γ_G is a normal Cayley graph and all of the eigenvalues can be calculated from the irreducible representations of G. The eigenvalue of Γ_G belonging to the irreducible representation ϕ of G is

$$\lambda_\phi = \frac{1}{\phi(1)} \sum_{d \in \mathrm{Der}_G} \phi(d).$$

This result is usually attributed to Babai [4], or Diaconis and Shahshahani [8]; a proof may be found in [13, Section 11.12]. The eigenvalue belonging to the trivial character is clearly $d_G := |\,\mathrm{Der}_G\,|$, and it is not difficult to see that the eigenvalue belonging to ψ is $-\frac{d_G}{n-1}$. Equation 2.1 implies that if a 2-transitive group G has the EKR-module property, then for any maximum coclique S

$$A(\Gamma_G)(v_S - \frac{1}{n}\mathbf{1}) = -\frac{d_G}{n-1}(v_S - \frac{1}{n}\mathbf{1})$$

(where $A(\Gamma_G)$ is the adjacency matrix of Γ_G).

In the next section we will prove Theorem 1.2 for 2-transitive groups in which the minimal normal subgroup is abelian. Section 4 we will prove the result for the groups in which the minimal normal subgroup is not abelian. We will consider when Γ_G is connected in Section 5. Section 6 considers when the maximum intersecting sets are groups or cosets of groups. In Section 7 we show that Theorem 1.2 gives information about the structure of the maximum intersecting sets. Finally we discuss some questions for further investigation in Section 8.

3. 2-transitive groups with a regular normal subgroup

In this section we consider 2-transitive permutation groups (G, X), with $|X| = n$, that have a regular normal subgroup N. In this case, N is an elementary abelian p-group for some prime p. Further, G is the semidirect product NG_x where G_x is the stabilizer of a point $x \in X$. In particular, G_x is a transversal of N in G and G_x is a coclique in Γ_G.

Proposition 3.1. *The elements in N form a clique of size n in Γ_G.*

Proof. Since N is regular, it has size n and every non-identity element is a derangement. For any distinct $n_1, n_2 \in N$, $n_1 n_2^{-1}$ is a non-identity element of N, and is a derangement. □

By the clique-coclique bound (Lemma 2.3), Proposition 3.1 implies that the size of a maximum coclique in Γ_G is bounded by $\frac{|G|}{n}$. Since G_x is a coclique of this size we have $\alpha(\Gamma_G) = \frac{|G|}{n}$. This shows that all of these groups have the EKR property. Further, any maximum coclique S in Γ_G intersects N (and any coset of N) in exactly one element. So any coclique S of maximum size is a transversal of N in G. This can also be seen since for any two distinct elements s and t of S, the element st^{-1} has a fixed point so does not belong to N.

The following is a well-known result that we state in this context.

Lemma 3.2. *Let $g = uh$ with $u \in N$ and $h \in G_x$. If g is G-conjugate to an element of G_x, then the following hold:*

(a) $g = uh$ can be conjugated to h by an element of N; and
(b) h is the unique N-conjugate of g in G_x.

Proof. By hypothesis there exists $a \in G$ such that $a^{-1}ga \in G_x$. We may write $a = mk$, where $m \in N$ and $k \in G_x$. Then $k^{-1}m^{-1}(uh)mk \in G_x$, so $m^{-1}(uh)m \in kG_xk^{-1} = G_x$.

As G_x is a transversal of N in G, two elements of G_x with the same image in G/N must be equal. Therefore the only possible N-conjugate of uh in G_x is h. So $m^{-1}uhm = h$ and both parts of the lemma are proved. □

Let S be a maximum coclique in Γ_G, for any elements $s, t \in S$ (including $s = t$), write $st^{-1} = uh$ with $u \in N$ and $h \in G_x$. As uh has a fixed point, it is G-conjugate to an element of G_x, hence N-conjugate to h by Lemma 3.2. If we fix t and let s run over S, then each element $h \in G_x$ is obtained in this way exactly once, since St^{-1} is also a transversal of N in G. These observations will allow us, in the next lemma, to generalize to arbitrary cocliques a calculation that was made for canonical cocliques in [1, Lemma 4.1].

Let ψ be the irreducible character of G of degree $n-1$ from the 2-transitive action.

Lemma 3.3. *Let S be a coclique and $y \in G$.*

$$\sum_{s\in S} \psi(sy^{-1}) = \begin{cases} |G_x| & \text{if } y \in S, \\ \frac{-|G_x|}{n-1} & \text{if } y \notin S. \end{cases} \tag{3.1}$$

Proof. First suppose that $y \in S$. Write $sy^{-1} = uh$, where $u \in N$ and $h \in G_x$. We know from the previous lemma that sy^{-1} is G-conjugate to h, and so $\psi(sy^{-1}) = \psi(h)$. Moreover, as s runs over S we obtain each $h \in G_x$ once, so

$$\sum_{s\in S} \psi(sy^{-1}) = \sum_{h\in G_x} \psi(h) = |G_x|.$$

Next suppose $y \notin S$. Since S is a transversal of N in G, we can write $y = mt$, with $t \in S$ and m a nonidentity element of N. Suppose $st^{-1} = uh$, where $u \in N$ and $h \in G_x$. By Lemma 3.2, there exists $v \in N$ such that $v(st^{-1})v^{-1} = v(uh)v^{-1} = h$. Then

$$\psi(sy^{-1}) = \psi(st^{-1}m^{-1}) = \psi(vst^{-1}m^{-1}v^{-1}) = \psi(vst^{-1}v^{-1}m^{-1}) = \psi(hm^{-1}).$$

Here we used the fact that N is abelian. Moreover, the transversal property of St^{-1} means that, as s runs over S, each element of G_x is conjugate to st^{-1} for exactly one s. Hence

$$\sum_{s\in S} \psi(sy^{-1}) = \sum_{h\in G_x} \psi(hm^{-1}). \tag{3.2}$$

Note that the right-hand side does not depend on S. This allows us to proceed as in the proof of [1, Lemma 4.1]. The right-hand side of (3.2) is the sum of ψ over a coset of G_x that is not equal G_x. By the 2-transitivity of G the value of this sum is the same for all cosets of G_x other than G_x itself. Then, since $\sum_{g\in G} \psi(g) = 0$ and $\sum_{g\in G_x} \psi(g) = |G_x|$, it follows that

$$\sum_{h\in G_x} \psi(hm^{-1}) = -\frac{|G_x|}{n-1}.$$

□

As in [1], the sum computed in the Equation 3.1 is the coefficient of y when the element $\frac{|G|}{\psi(1)} E_\psi v_S \in \mathbb{C}[G]$ is expressed in the group basis. It follows as in [1], that

$$E_\psi(v_S - \frac{1}{n}\mathbf{1}) = v_S - \frac{1}{n}\mathbf{1}, \tag{3.3}$$

which shows that v_S lies in the 2-sided ideal of $\mathbb{C}[G]E_\psi$ of $\mathbb{C}[G]$. This shows that G has the EKR-module property, so Theorem 1.2 holds for any 2-transitive group with a regular normal subgroup.

4. 2-transitive groups of almost simple type

In this section we consider the 2-transitive groups that do not have a regular abelian normal subgroup N; these are the 2-transitive groups of almost simple type. In this section, we assume that G is such a group and $K \trianglelefteq G$ is the minimal nonabelian normal subgroup of G. These groups are listed in Table 1. With the exception of $G = \mathrm{Ree}(3)$, for each of these groups the subgroup K is 2-transitive. The eigenvalues of the group Ree(3) can all be directly calculated, and $\psi_{\mathrm{Ree}(3)}$ is the only irreducible character affording the minimal eigenvalue. Thus Ree(3) has the EKR-module property. So we will restrict to the case where K is 2-transitive.

We will show if K has the EKR module property, then G also has the EKR-module property. Then we will prove that each of these groups, the minimal normal subgroup has the EKR-module property.

We assume that G and K are both acting on an n-set. We denote character from this 2-transitive action of G by χ_G, and χ_K is the representation of K for its 2-transitive action. Similarly, we use ψ_G and ψ_K for the irreducible character of degree $n-1$ that is a component of χ_G and χ_K.

Lemma 4.1. *Let G be a 2-transitive group. If S is a maximum coclique in Γ_G, then $v_s - \frac{1}{n}$ is a $\frac{-d_G}{n-1}$-eigenvector of $A(\Gamma_G)$.*

Proof. From Theorem 1.1, the size of S is $\frac{|G|}{n}$.

Since S is a maximum coclique and Γ_G is d_G-regular, the number of edges between vertices in S and vertices in $V(\Gamma_G)\backslash S$ is $d_G|S|$. So the quotient graph of Γ_G with the partition $\{S, V(\Gamma_G)\backslash S\}$ is

$$\begin{bmatrix} 0 & d_G \\ d_G\left(\frac{|S|}{|G|-|S|}\right) & d_G\left(1-\frac{|S|}{|G|-|S|}\right) \end{bmatrix}.$$

The eigenvalues of this quotient graph are d_G and $-\frac{d_G}{n-1}$. These eigenvalues interlace the eigenvalues of Γ_G. Further, d_G is the eigenvalue of Γ_G afforded by the trivial representation and $-\frac{d_G}{n-1}$ is the eigenvalue afforded by ψ_G. Since the eigenvalues of the quotient graph are eigenvalues of the graph, the interlacing is tight. This means that $\{S, G\backslash S\}$ is an equitable partition [14, Lemma 9.6.1]. So each vertex in $G\backslash S$ is adjacent to exactly $d_G\frac{|S|}{|G|-|S|}$ vertices in S and $d_G\left(1-\frac{|S|}{|G|-|S|}\right)$ vertices not in S. By direct calculation of $A(\Gamma_G)(v_S - \frac{1}{n})$, the vector $v_S - \frac{1}{n}$ is a $-\frac{d_G}{n-1}$-eigenvector of Γ_G. □

Lemma 4.2. *Suppose H and G are 2-transitive groups with $H \lneq G$. Then there exist derangements in G that are not in H.*

Proof. We have

$$\sum_{g\in G}\chi_G(g) = |G| \qquad \text{and} \qquad \sum_{h\in H}\chi_H(h) = |H|,$$

so

$$\sum_{x\in G\setminus H}\chi_G(x) = |G\setminus H|. \tag{4.1}$$

Suppose $\mathrm{Der}_G \subseteq H$. Then $\chi_G(x) \geq 1$ for all $x \in G\setminus H$ so, by (4.1), we must have $\chi_G(x) = 1$ and $\psi_G(x) = 0$ for all $x \in G\setminus H$.

Since G and H both act 2-transitively, both ψ_G and its restriction to H are irreducible characters. We have

$$\sum_{g\in G}\psi_G(g)^2 = |G| \qquad \text{and} \qquad \sum_{h\in H}\psi_H(h)^2 = |H|. \tag{4.2}$$

so

$$\sum_{x\in G\setminus H}\psi_G(x)^2 = |G\setminus H|. \tag{4.3}$$

Therefore, there exists $x \in G\setminus H$, with $\psi(x) \neq 0$. This contradiction completes the proof. □

Theorem 4.3. *Let G be a 2-transitive group with minimal nonabelian normal subgroup K. Assume K is 2-transitive and that ψ_K is the unique character of K affording the least eigenvalue $-\frac{d_K}{n-1}$ of Γ_K. Then for any maximum coclique S of Γ_G, $v_S - \frac{1}{n}\mathbf{1}$ is in the ψ_G-module.*

Proof. Assume that S is any maximum coclique of Γ_G. Since G is 2-transitive, by Theorem 1.1 G has the EKR property, so the size of S is $\frac{|G|}{n}$. By Lemma 4.1, $v_S - \frac{1}{n}\mathbf{1}$ is a $-\frac{d_G}{n-1}$-eigenvector of $A(G)$.

Since K is a subgroup of G, the graph Γ_G contains $[G : K]$ copies of Γ_K as a subgraph. Let A be the adjacency matrix for the $[G : K]$ copies of Γ_K. This is a weighted adjacency matrix for Γ_G where the edge $\{\sigma, \pi\}$ is weighted by one if $\sigma\pi^{-1}$ is in the intersection of the derangements of G and K (so $\sigma\pi^{-1}$ is a derangement in K), and zero otherwise.

The matrix A is the adjacency matrix for the disjoint union of $[G : K]$ copies of Γ_K (this means that $A = I_{[G:K]} \otimes A(\Gamma_K)$). Further, if $\{x_1K, x_2K, \ldots, x_{[G:K]}K\}$ is a set of coset representatives for G/K, then each $S_i = S \cap x_iK$ is a coclique of size $\frac{|K|}{n}$ and each $v_{S_i} - \frac{1}{n}\mathbf{1}$ is a $-\frac{d_K}{n-1}$-eigenvector for A. This means that $v_S - \frac{1}{n}\mathbf{1}$ is a $-\frac{d_K}{n-1}$-eigenvector for A.

The eigenvalues of A are the same as the eigenvalues of Γ_K, but the multiplicities of the eigenvalues for A are equal to the multiplicities of Γ_K multiplied by $[G : K]$. In particular, the eigenvalue $-\frac{d_K}{n-1}$ has multiplicity $[G : K](n-1)^2$ in A.

Consider the induced character $\mathrm{ind}_G(1_K)$, this is the sum of irreducible characters

$$1_K^G = \phi_1 + \phi_2 + \cdots + \phi_\ell,$$

where $\phi_1 = 1_G$. Then each of $\phi_i\psi_G$ is an irreducible character of G. The eigenvalue of A afforded by each of these characters is $-\frac{d_K}{n-1}$. Thus the $-\frac{d_K}{n-1}$ eigenspace of A is exactly the span of these modules. So the vector $v_S - \frac{1}{n}\mathbf{1}$ is in the span of these modules.

Next we will use the fact that $v_S - \frac{1}{n}\mathbf{1}$ is also a $-\frac{d_G}{n-1}$-eigenvector for the adjacency matrix of Γ_G to show that it is entirely contained in the $\phi_1\psi_G$-module.

Consider

$$\lambda_{\phi_i\psi_G} = \frac{1}{(n-1)\phi_i(1)} \sum_{d\in \mathrm{Der}_G} \phi_i(d)\psi_G(d) = \frac{-1}{(n-1)\phi_i(1)} \sum_{d\in \mathrm{Der}_G} \phi_i(d).$$

By Lemma 4.2 there are derangements in G that are not in K, so some d we have $\phi_i(d) \neq 1$. So, if $\phi_i \neq 1_G$, then

$$\frac{1}{\phi_i(1)} \sum_{d\in \mathrm{Der}_G} \phi_i(d) < \frac{1}{\phi_i(1)} \sum_{d\in \mathrm{Der}_G} \phi_i(1) = d_G.$$

So no $\phi_i\psi_G$ affords $-\frac{d_G}{n-1}$ as an eigenvector, other than $\phi_i = 1_G$. Since $v_S - \frac{1}{n}\mathbf{1}$ is both a $\frac{-d_G}{n-1}$ eigenvector and in the $\phi_i\psi_G$-modules, it must be in the ψ_G-module. □

The classification of finite simple groups has allowed for the complete classification the finite 2-transitive groups. Below is Table 1 from [19] which lists the finite 2-transitive groups of almost simple type (this table was extracted from [5, page 197]).

Proposition 4.4. *For $n \geq 5$ the least eigenvalue of $\Gamma_{\mathrm{Alt}(n)}$ is given by $\psi_{\mathrm{Alt}(n)}$ and no other representations, and the largest eigenvalue is given by the trivial character and no other.*

Proof. The number of derangements in $\mathrm{Alt}(n)$ is known [22, Sequence A003221], and for $n \geq 5$ we have

$$\begin{aligned} d_{\mathrm{Alt}(n)} &= \frac{n!}{2}\sum_{i=0}^{n-2}(-1)^i\frac{1}{i!} + (-1)^{n-1}(n-1) \\ &\geq \frac{n!}{2}(1 - 1 + \frac{1}{2} - \frac{1}{6}) = \frac{n!}{6} \end{aligned}$$

Line	Group K	Degree	Condition on G	Remarks
1	$\mathrm{Alt}(n)$	n	$\mathrm{Alt}(n) \leq G \leq \mathrm{Sym}(n)$	$n \geq 5$
2	$\mathrm{PSL}_m(q)$	$\frac{q^m-1}{q-1}$	$\mathrm{PSL}_m(q) \leq G \leq \mathrm{P\Gamma L}_m(q)$	$m \geq 2$, $(m,q) \neq (2,2),(2,3)$
3	$\mathrm{Sp}_{2m}(2)$	$2^{m-1}(2^m-1)$	$G = K$	$m \geq 3$
4	$\mathrm{Sp}_{2m}(2)$	$2^{m-1}(2^m+1)$	$G = K$	$m \geq 3$
5	$\mathrm{PSU}_3(q)$	q^3+1	$\mathrm{PSU}_3(q) \leq G \leq \mathrm{P\Gamma U}_3(q)$	$q \neq 2$
6	$\mathrm{Sz}(q)$	q^2+1	$\mathrm{Sz}(q) \leq G \leq \mathrm{Aut}(\mathrm{Sz}(q))$	$q = 2^{2m+1}$, $m > 0$
7	$\mathrm{Ree}(q)$	q^3+1	$\mathrm{Ree}(q) \leq G \leq \mathrm{Aut}(\mathrm{Ree}(q))$	$q = 3^{2m+1}$, $m > 0$
8	M_n	n	$M_n \leq G \leq \mathrm{Aut}(M_n)$	$n \in \{11,12,22,23,24\}$, M_n Mathieu group, $G = K$ or $n = 22$
9	M_{11}	12	$G = K$	
10	$\mathrm{PSL}_2(11)$	11	$G = K$	
11	$\mathrm{Alt}(7)$	15	$G = K$	
12	$\mathrm{PSL}_2(8)$	28	$G = \mathrm{P\Sigma L}_2(8) \cong \mathrm{Ree}(3)$	
13	HS	176	$G = K$	HS Higman-Sims group
14	Co_3	276	$G = K$	Co_3 third Conway group

Table 1. Finite 2-transitive groups of almost simple type

Using Lemma 2.4 from [19], if the character $\phi \neq \psi_{\mathrm{Alt}}$ of the alternating group affords the minimum eigenvalue of $\Gamma_{\mathrm{Alt}(n)}$, then

$$\phi(1) \leq (n-1)\left(\frac{|\operatorname{Alt}(n)|}{d_{\mathrm{Alt}(n)}} - 2\right)^{\frac{1}{2}}.$$

Since

$$(n-1)\left(\frac{|\operatorname{Alt}(n)|}{d_{\mathrm{Alt}(n)}} - 2\right)^{\frac{1}{2}} \leq (n-1)\left(\frac{n!}{2}\left(\frac{n!}{6}\right)^{-1} - 2\right)^{\frac{1}{2}} = n-1,$$

any character giving the minimal eigenvalue must have dimension no more than $n-1$. Since the only representations with degree no more than $n-1$ are the trivial representation and ψ_{Alt}, it follows that ψ_{Alt} is the unique irreducible representation affording the minimum eigenvalue. Note that this also implies that only the trivial representation gives the largest eigenvalue. □

Theorem 4.5. *The minimal groups K of each type in Table 1 have χ_K as the only irreducible character that gives the eigenvalue $-\frac{d_K}{n-1}$.*

Proof. The previous result shows this holds for $\mathrm{Alt}(n)$. For $\mathrm{PSL}_2(q)$, this fact can be read off the tables in Simpson and Frame [23], for $\mathrm{PSL}_3(q)$ it is in [18, Table 5], and for $\mathrm{PSL}_m(q)$ with $m \geq 4$ it is stated in [19, Proposition 8.3]. For the groups in lines 3 and 4, $Sp_{2m}(2)$ this result is from [19, Proposition 9.1] for $m \geq 7$. For $\mathrm{PSU}_3(q)$ this is from [20, Table 5 and Table 6]. For $Sz(q)$ the result is given in [19, Proposition 4.1] and for $Ree(q)$ this is [19, Proposition 5.1]. The eigenvalues of the Mathieu groups are given in [1, Lemma 5.1]. For all the other finite groups all the eigenvalues can be calculated from the character table, and only χ_K gives the eigenvalue $-\frac{d_K}{n-1}$. □

5. Connected derangement graphs

Consider the example of a Frobenius group G with Frobenius kernel N and Frobenius complement H. In this case, the cosets of N are cliques in the derangement graph of G. In fact, the derangement graph is exactly the disjoint union of these cliques. Since any transversal of N is a coclique, as long as $|H| > 2$, there are non-canonical cocliques of the form $H \setminus \{h\} \cup \{hu\}$, where $h \in H$ and $u \in N$ are nonidentity elements. The Frobenius groups are a family of 2-transitive groups that do not satisfy the strict-EKR property. Further the non-canonical independent sets just described are neither subgroups, nor cosets of subgroups.

In this section we will consider other groups that have a disconnected derangement; this occurs exactly when the derangements do not generate the group.

Lemma 5.1. *Suppose G contains a proper 2-transitive subgroup H. Then G is generated by $H \cup \mathrm{Der}_G$. In particular, if H is generated by Der_H, then G is generated by Der_G.*

Proof. Suppose for a contradiction that the subgroup M of G generated by $H \cup \mathrm{Der}_G$ is proper. Then we may apply Lemma 4.2 to the group G and the subgroup M, to obtain a derangement outside M. This is a contradiction and hence $M = G$. The last statement of the lemma follows immediately. $\square$

For all the groups G in Table 1, with the exception of Ree(3), this corollary applies. Proposition 4.4 implies that the derangement graph for the Alternating group is connected. The fact that the minimal groups in lines 2-7 of Table 1 have a connected derangement graph can be read from [19] (with results from [15] for lines 3 and 4). The groups in lines 8-11 and 13-14 are finite, and the eigenvalues of the derangement graphs for the minimal group can be directly calculated and individually checked. With these facts, we have the following corollary.

Corollary 5.2. *With the exception of* Ree(3) *(isomorphic to* $\mathrm{P\Sigma L}_2(8)$ *with its action on 28 points), the derangement graph for any 2-transitive group of almost-simple type is connected.*

Proof. $\mathrm{PSL}_2(8)$ is a subgroup with index 3 in $\mathrm{P\Sigma L}_2(8)$. Every element in $\mathrm{P\Sigma L}_2(8)$ that is not in $\mathrm{PSL}_2(8)$ has order 3, 6 or 9 and $\psi_{\mathrm{P\Sigma L}_2(8)}$ vanishes on these points. So all derangement of $\mathrm{P\Sigma L}_2(8)$ are in $\mathrm{PSL}_2(8)$. $\square$

Next we focus on the 2-transitive groups G with a regular normal subgroup N. We begin with an immediate consequence of the fact that Der_G is a union of conjugacy classes.

Lemma 5.3. *Let G be a 2-transitive finite permutation group, with a regular normal subgroup N. If $G/N \cong G_x$ is a simple group and there are derangements outside N, then the derangement graph of G is connected.* $\square$

Fix an element x, from the set on which G acts, and let $H = G_x$ be its stabilizer. Then, by definition of the regular normal subgroup, there is a map $N \to X$ defined by $u \mapsto u(x)$ that is an isomorphism of N sets where N acts on itself by left multiplication. This is also an isomorphism of H-sets where H acts on N by conjugation. That is to say, for all $h \in H$ and $u \in N$ we have $h(u(x)) = (huh^{-1})(x)$.

Under this identification of N with X, the action of G on X is equivalent to an action of G on N given as follows. Each element of G has the unique form mh for $m \in N$ and $h \in H$. Then $mh(u) = m(huh^{-1})$ for all $u \in N$. We will make use of this G-action on N in the following lemmas.

Lemma 5.4. *Let G be a 2-transitive finite permutation group with a regular normal subgroup N and point stabilizer H. Then for $h \in H$, the coset Nh contains a derangement if and only if h centralizes a nonidentity element of N.*

Proof. Consider the map $f_h : N \to N$ defined by

$$f_h(u) = huh^{-1}u^{-1}.$$

Then h centralizes a nonidentity element of N if and only if f_h is not injective, which in turn is equivalent to f_h not being surjective.

Suppose f_h is not surjective, and let $m \in N$ be an element not in the image of f_h. We claim that $m^{-1}h$ is a derangement. Here we use the identification of X with N described above. Supposed $m^{-1}h$ is not a derangement, then is has a fixed point. So

$$u = (m^{-1}h)(u) = m^{-1}huh^{-1} \tag{5.1}$$

and it follows that $f_h(u) = m$, a contradiction. Thus if f_h is not surjective then Nh contains a derangement.

Conversely, if f_h is surjective, then for every $m \in N$, there exists $u \in N$ such that $f_h(u) = m^{-1}$. This equation can be written as $mhuh^{-1} = u$, that is $(mh)(u) = u$. Thus every element of Nh has a fixed point. □

Theorem 5.5. *Let G be a 2-transitive finite permutation group with a regular normal subgroup N and point stabilizer $H = G_x$. Then the subgroup of G generated by Der_G is equal to the subgroup generated by N and the two-point stabilizers H_y, for $y \neq x$.*

Proof. Let M be the subgroup of G generated by Der_G. Then $N \subseteq M$. By Lemma 5.4, a coset Nh, with $h \in H$ contains a derangement if and only if h centralizes a nonidentity element of N. In this case, the whole coset Nh will be contained in M since N is contained in M. Thus, M is equal to the subgroup generated by those cosets Nh for which h centralizes a nonidentity element of N.

As the conjugation action of H on N is isomorphic to the permutation action of H on X, an element h centralizes a nonidentity element of N if and only if h lies in H_y for some $y \in X$, $y \neq x$. This completes the proof. □

Proposition 5.6. *Let G be a 2-transitive finite permutation group, with a regular normal subgroup N. Then G is a Frobenius group if and only if $\mathrm{Der}_G = N \setminus \{1\}$.*

Proof. If G is a Frobenius group then it is immediate that $\mathrm{Der}_G = N \setminus \{1\}$.

Suppose that G is not a Frobenius group. Then there is a nonidentity element $h \in H$ that centralizes a nonidentity element of N. Then by Lemma 5.4, the coset Nh contains a derangement. □

Corollary 5.7. *Let G be a 2-transitive finite permutation group, with a regular normal subgroup N. Then G is a Frobenius group if and only if Γ_G is the union of disjoint complete graphs.*

Proof. It is not hard to see that if G is a Frobenius group, then Γ_G is the union of complete graphs on n vertices, see [2, Theorem 3.6] for details. If Γ_G is the union of disjoint complete graphs then, since a point stabilizer is a coclique of size $|G|/n$, no complete subgraph has more than n vertices. In particular, the identity element can have no more than $n-1$ neighbors. However the set of neighbors of the identity element is Der_G, which contains $N \setminus \{1\}$, a set of size $n-1$. Thus, $\mathrm{Der}_G = N \setminus \{1\}$, and by Proposition 5.6 G is a Frobenius group. □

There are many 2-transitive groups with a regular normal subgroup that are not Frobenius groups and have disconnected derangement graphs. For example, as we shall see, the groups $\mathrm{A\Gamma L}_1(p^e)$, for $p > 2$ and $e \geq 2$, are 2-transitive groups with a disconnected derangement graphs, and further examples may be found among their subgroups. Each of these groups have the EKR-property, the EKR-module property, but not the strict-EKR property. Further, for each of these groups there are maximum cocliques that are neither subgroups, not cosets of subgroups.

Proposition 5.8. *If $p > 2$ is prime and $e \geq 2$ then $\mathrm{A\Gamma L}_1(p^e)$ is a 2-transitive group with a disconnected derangement graph.*

Proof. Let N be the regular normal subgroup of $\mathrm{A\Gamma L}_1(p^e)$ (these are the translations of the form $x \mapsto x + b$ with $b \in \mathbb{F}_{p^e}$). The two point stabilizers of $\mathrm{A\Gamma L}_1(p^e)$ all have order e and are generated by transformations of the form $x \mapsto a^{(p-1)}x^p + b$ where $a, b \in \mathbb{F}_q$ and $a \neq 0$. These permutations do not generate all of $\mathrm{A\Gamma L}_1(p^e)$. □

6. Non-canonical Cocliques that are cosets of subgroups

In this section we describe examples of noncanonical cocliques in which the derangement graph is connected. These come from considering non-canonical cocliques that are subgroups in 2-transitive finite permutation groups G with a regular normal subgroup N.

Since any coclique in Γ_G must be a transversal of N, any subgroup that is a non-canonical coclique must be complementary to N, but not conjugate to G_x. The G-conjugacy classes of subgroups that are complementary to N, are classified by the first cohomology group $H^1(G_x, N)$, where N is viewed as an G_x-module by conjugation. The trivial element of $H^1(G_x, N)$ corresponds to the G-conjugacy class of G_x and, if $H^1(G_x, N)$ is not trivial, each nontrivial element corresponds to a G-conjugacy class of *nonstandard complements*, by which we mean subgroups complementary to N, but not G-conjugate to G_x.

The following is a necessary and sufficient condition for a nonstandard complement to be a maximum coclique in Γ_G.

Lemma 6.1. *A complement K to N in $G = NG_x$ is a coclique in Γ_G if and only if every element of K is G-conjugate to an element of G_x.*

Proof. Assume that K is a complement to N that is a coclique in Γ_G. Since $1 \in K$, each element of K must have a fixed point (as it intersects with the identity element). Thus any element of K lies in a point stabilizer and is G-conjugate to an element of G_x.

Conversely, if each element of a subgroup K is G-conjugate to an element of G_x, then every element has a fixed point. So for any $h, k \in K$, the element $hk^{-1} \in K$ has a fixed point which implies that K is a coclique. □

Using the notation of the previous proof, let $g \in K$ and let g_p be its p-part. It follows from the injectivity of the restriction of $H^1(\langle g \rangle, N) \to H^1(\langle g_p \rangle, N)$ (see [7, Ch.XII, Theorem 10.1]) that we may replace the condition in Lemma 6.1 that every element of K be G-conjugate to an element of H, by the same condition on p-elements only.

Theorem 6.2. *For $e \geq 2$, the group $ASL_2(2^e)$ of affine transformations of $X = \mathbb{F}_{2^e}^2$ does not have the strict-EKR property.*

Proof. Let $G = \mathrm{ASL}_2(2^e)$ be the group of affine transformations of $X = \mathbb{F}_{2^e}^2$ generated by the linear group $H = \mathrm{SL}_2(2^e)$ (this is the stabilizer of the zero vector) and the group $N = \mathbb{F}_{2^e}^2$ of translations, where $uh : x \mapsto hx + u$, for $x \in X$, $h \in H$ and $u \in N$. It is well known that $H^1(H, N) \cong \mathbb{F}_{2^e}$ when $e \geq 2$ [10, Lemma 14.7].

If $K \neq H$ is a non-standard complement, then K and H are isomorphic. This implies that either all elements of K are either involutions or have odd order. Thus there is a single K-conjugacy class of involutions. The involutions in K are of the form ut, where $t \in H$ and $u \in C_N(t)$.

If we regard N as a $\mathbb{F}_{2^e}$-vector space and $t \in H$ as a linear map, then $C_N(t) = \mathrm{Ker}(t-1)$, and a simple calculation shows that $\mathrm{Ker}(t-1) = \mathrm{Im}(t-1)$. It follows that for any $u \in N$ there exists $m \in N$ such that $u = t^{-1}mtm^{-1}$, so $ut = tu = mtm^{-1}$

is conjugate to $t \in H$. As every odd order element of K is conjugate to an element of H, the Schur-Zassenhaus Theorem, Lemma 6.1 shows that K (and its cosets) are non-canonical cocliques in Γ_G. □

Example 6.3. For an explicit example, let $e = 2$ and α be a primitive element of $\mathbb{F}_4$. We can think of $\mathrm{ASL}_2(4)$ as the subgroup of $\mathrm{SL}_3(4)$ consisting of matrices of the block form

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix}$$

where $A \in \mathrm{SL}_2(4)$ and $v \in \mathbb{F}_4^2$.

Consider the elements

$$t = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad s = \begin{bmatrix} 0 & 1 & 0 \\ 1 & \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.1}$$

of orders 2, 2 and 5 respectively.

The standard complement H is generated by the elements t and s, while tu and s generate a nonstandard complement. It is interesting to note that for this value of q that this non-standard complement is the stabilizer of a maximum arc in $\mathbb{F}_4^2$.

Many other examples of non-canonical cocliques arising from nonstandard complements can be found. However, it is not always the case that a nonstandard complement will yield a non-canonical coclique in the derangement graph, as it may fail to satisfy the hypotheses of Lemma 6.1, as in the following example.

Example 6.4. Let $G = \mathrm{AGL}_3(2) = NH$, with $H = \mathrm{GL}_3(2)$ and $N = \mathbb{F}_2^3$, acting on $X = \mathbb{F}_2^3$ by affine transformations $uh : x \mapsto hx + u$, for $x \in X$, $h \in H$ and $u \in N$. We can view G as the subgroup of $GL_4(2)$ consisting of matrices of the following block form

$$\begin{bmatrix} A & v \\ 0 & 1 \end{bmatrix}$$

where $A \in \mathrm{GL}_3(2)$ and $v \in \mathbb{F}_2^3$.

Consider the elements

$$a = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad u = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad s = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.2}$$

of orders 2, 2 and 7 respectively.

The standard complement H is generated by the elements a and s, while au and s generate a nonstandard complement. If au were G-conjugate to any element of H, it would be conjugate under N to some element of H, as $G = NH$, and that element would have to have the same image as an in G/N. Thus au would be conjugate to a. However they are not conjugate in G, as $a - 1$ and $au - 1$ have different ranks. So there are no subgroups that are also non-canonical coclique. This particular group can be shown directly to have the strict-EKR property using the method described in [1].

7. Inner Distributions

We can get more information about the structure of maximum cocliques using Theorem 1.2 by considering the *conjugacy class scheme* on the group G. This is the association scheme that has the elements of G as its vertices and one class for each conjugacy class of G. Two elements $g, h \in G$ are adjacent in a class if hg^{-1} is in the corresponding conjugacy class. The matrices in this association scheme are indexed by the conjugacy classes (these are denoted by A_c), and the idempotents are indexed by the irreducible representations of G (these are denoted by E_ϕ).

Let S be any maximum intersecting set in G. Let v_S denote the characteristic vector of S. Then the *inner distribution* of S is the sequence

$$\left(\frac{v_S^T A_c v_S}{|S|}\right)_c$$

taken over the conjugacy classes c of G. This gives a count of how many pairs of elements in S are i-related in the association scheme. The *dual distribution* is defined to be the sequence

$$\left(\frac{v_S^T E_\phi v_S}{|S|}\right)_\phi$$

taken over the irreducible representations ϕ of G.

Lemma 2.1 implies for any maximum intersecting set S in G that $v_S^T E_\phi v_S = 0$, unless $\phi = 1_G$ or $\phi = \psi_G$. From the comments following Lemma 2.1, we have

$$\frac{v_S^T E_{1_G} v_S}{|S|} = \frac{|S|^2}{|G||S|} = \frac{1}{n}$$

and

$$\frac{v_S^T E_{\psi_G} v_S}{|S|} = 1 - \frac{1}{n}.$$

It is known (see [13, Theorem 3.5.1]) that in any association scheme the following equation holds.

$$\sum_c \frac{v_S^T A_c v_S}{|S|} A_c = \sum_\phi \frac{v_S^T E_\phi v_S}{|S|} E_\phi.$$

In particular, for any maximum intersecting set in G

$$\sum_c \frac{v_S^T A_c v_S}{|S|} A_c = \frac{1}{n} E_{1_G} + \left(1 - \frac{1}{n}\right) E_{\psi_G}.$$

In the conjugacy class association scheme the sets $\{A_c\}$ and $\{E_\phi\}$ both form a basis and the matrix of eigenvalues for the association scheme is a change of basis matrix. This implies that the inner distribution for S can be found by multiplying the dual distribution with the matrix of eigenvalues, which leads to the next result.

Lemma 7.1. *Let G be a 2-transitive group and let S be any maximum intersecting set in G. Then S has the same inner distribution as the stabilizer of a point.* □

8. Further Work

There have been many papers looking at specific groups to determine the structure of the maximum cocliques in the derangement graph. Theorem 1.2 gives a strong characterization of the maximum cocliques in any 2-transitive groups. We end with an open problem and a direction for further work.

Our only examples of groups that have non-canonical maximum cocliques in their derangement graphs, that are neither subgroups nor cosets, have the property that the derangement graphs are not connected. This leads to our remaining question.

Question 8.1. *Are there groups G, with connected derangement graphs, that have a maximum coclique that is neither a group nor a coset of a group?*

Finally, in this paper we only consider 2-transitive groups. The definition of the EKR-module property can be considered for any group, with the key difference being that, in general, the permutation module is not the sum of the trivial module and a single irreducible module. This situation will be more complicated, as there are transitive groups which satisfy neither the EKR property, nor the EKR-module property, nor the strict-EKR property. The first groups to consider are the rank 3 groups.

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