HOPF MONOIDS, PERMUTOHEDRAL TANGENT CONES, AND GENERALIZED RETARDED FUNCTIONS

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Abstract. The commutative Hopf monoid of set compositions is a fundamental Hopf monoid internal to vector species, having Bosonic Fock image the Hopf algebra of quasisymmetric functions. We construct a geometric realization of this Hopf monoid over the adjoint braid arrangement, which identifies the monomial basis with signed characteristic functions of open permutohedral tangent cones. We show that the indecomposable quotient is obtained by identifying functions which differ only on hyperplanes, so that the resulting Lie coalgebra consists of functions on chambers of the adjoint braid arrangement. These functions are characterized by the Steinmann relations of axiomatic QFT, demonstrating an equivalence between the Steinmann relations, tangent cones to (generalized) permutohedra, and having algebraic structure in species. Our results give the pure mathematical interpretation of a classical construction in axiomatic QFT. We show that generalized time-ordered functions correspond to the cocommutative Hopf monoid of set compositions, and generalized retarded functions correspond to its primitive part.

Contents

Introduction

The theory of species was developed by André Joyal as a method for analyzing combinatorial structures in terms of generating functions [Joy81], [Joy86]. See also [BLL98], [BD01]. Formally, a species is a functor on the category of finite sets and bijections, often valued in sets, called set species, or vector spaces, called vector species. Up to isomorphism, a species is a sequence of objects such that the nth object is equipped with an action of the Weyl group of type A_{n-1}. Species come to life by considering their handful of richly interacting monoidal products, which vertically categorify familiar operations on formal power series.

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In particular, we can take the Day convolution of set species or vector species with respect to the disjoint union of finite sets. This categorifies the Cauchy product of power series. Hopf theory in species concerns Hopf monoids defined internal to set species and vector species, using the Day convolution as the tensor product. In the case of Hopf monoids in vector species, we also have internal Lie algebras, Lie coalgebras, universal enveloping algebras, and universal coenveloping algebras. Many famous graded Hopf algebras are obtained from Hopf monoids in vector species by generalized Fock space constructions [AM10, Part III]. However, these constructions are forgetful, which makes species nicer to work with than graded vector spaces.

Operads are also species, however operads are monoids which use plethysm, a monoidal product that categorifies the composition of power series. There is an equivalent description of Hopf theory in species in terms of left (co)actions of the (co)operads $\mathbf{Com}^{(*)}$, $\mathbf{Ass}^{(*)}$, $\mathbf{Lie}^{(*)}$ [AM10, Appendix B.5].

For the foundations of Hopf theory in species, see the work of Aguiar and Mahajan [AM10], [AM13] and references therein, in particular [Bar78], [Joy86], [Sto93]. The reflection hyperplane arrangement of type A, called the braid arrangement, provides a consistent geometric interpretation of the theory, which motivates and facilitates the development of the theory over generic hyperplane arrangements [AM17]. In this paper, we stay in type A, but we extend the geometric interpretation to the adjoint[1] of the braid arrangement. The adjoint braid arrangement lives in the dual root space, and consists of hyperplanes which are spanned by coroots. This hyperplane arrangement has several names. It is known as the restricted all-subset arrangement [KTT11], [KTT12], [BMM⁺12], the resonance arrangement [CJM11], [Cav16], [BBT18], [GMP19], and the root arrangement [LMPS19]. Its spherical representation is called the Steinmann planet by physicists [BG67], [Eps16].

At the very heart of Hopf theory in vector species is the cocommutative Hopf monoid of set compositions $\mathbf{\Sigma}$, together with its dual, the commutative Hopf monoid of set compositions $\mathbf{\Sigma}^*$. More familiar objects are perhaps the Bosonic Fock images of $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$, which are noncommutative symmetric functions $\mathbf{NSym}$ and quasisymmetric functions $\mathbf{QSym}$ respectively [AM10, Proposition 17.1]. In this paper, we geometrically realize $\mathbf{\Sigma}^*$ over the adjoint braid arrangement by identifying the monomial basis with signed characteristic functionals of open permutohedral tangent cones. We denote this realization by $\check{\mathbf{\Sigma}}^*$. A construction related to this appears in [MNT13, Section 5] for the full Fock image of $\mathbf{\Sigma}^*$, which is word quasisymmetric functions $\mathbf{WQSym}$, also called noncommutative quasisymmetric functions $\mathbf{NCQSym}$. This graded Hopf algebra has been studied in several places, e.g. [NT06], [PR06], [BZ09]. There is a more classical geometric realization of $\mathbf{\Sigma}^*$ over the braid arrangement, where the monomial basis is identified with characteristic functions of relatively open faces. We denote this realization by $\hat{\mathbf{\Sigma}}^*$. The realizations $\check{\mathbf{\Sigma}}^*$ and $\hat{\mathbf{\Sigma}}^*$ are dual in the sense of polyhedral algebras [BP99, Theorem 2.7]. To express this duality, we introduce the cone basis of $\mathbf{\Sigma}^*$, which is the image of set compositions under the standard homomorphism $\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*$, where $\mathbf{O}$ is preposets. The geometric realizations of the cone basis are characteristic functions/functionals of closed convex cones.

As we show, the beauty of the adjoint realization $\check{\mathbf{\Sigma}}^*$ is that its indecomposable quotient is obtained by simply restricting functionals to chambers. We denote the resulting Lie coalgebra by $\check{\mathbf{\Gamma}}^*$. It is isomorphic to the dual of the free Lie algebra on the positive exponential species $\mathbf{\Gamma} = \mathcal{L}ie(\mathbf{E}^*_+)$ [AM10, Section 11.9]. The adjoint analog of this construction, i.e. the restriction

[1] in the sense of [AM17, Section 1.9.2]

of $\hat{\mathbf{\Sigma}}^*$ to Weyl chambers, is a geometric realization of the commutative Hopf monoid of linear orders $\mathbf{L}^*$.

Let $\mathbf{L}^\vee$ denote the species of formal linear combinations of chambers of the adjoint braid arrangement. The Steinmann relations are certain four term linear relations on $\mathbf{L}^\vee$, which appear in the foundations of axiomatic quantum field theory (QFT) [Ste60b], [Ste60a], [Str75, p. 827-828]. Let a Steinmann functional be a linear functional on the vector space $\mathbf{L}^\vee[I]$, i.e. the value of $\mathbf{L}^\vee$ on a finite set I, which respects the Steinmann relations. In [LNO19], it was shown that Steinmann functionals (which were denoted there by $\mathbf{\Gamma}^*$) form a Lie coalgebra in species, with cobracket the discrete differentiation of functionals across hyperplanes. Moreover, the Steinmann relations are necessary for such a Lie coalgebra structure. In this paper, we show that $\check{\mathbf{\Gamma}}^*$ is exactly this Lie coalgebra of Steinmann functionals. Since $\check{\mathbf{\Gamma}}^*$ is equivalently the span of characteristic functionals of (generalized) permutohedral tangent cones, this result is clearly closely related to [AA17, Theorem 6.1]. Dually, we obtain a geometric realization of the Lie algebra $\mathbf{\Gamma}$ as the quotient of $\mathbf{L}^\vee$ by the Steinmann relations, where the Lie structure is the action of (semisimple) Lie elements on faces [LNO19, Section 4.2].

The polyhedral algebras $\hat{\mathbf{\Sigma}}^*[I]$ and $\check{\mathbf{\Sigma}}^*[I]$ are also studied in [Ear17], where the quotients corresponding to the Lie cooperad $\mathbf{Lie}^*$, $\mathbf{L}^*$, and $\mathbf{\Gamma}^*$ are considered in relation to the cone basis, and a certain second basis. The c-basis of $\mathbf{\Gamma}^*$, which is defined in Section 2.4, is the image of this second basis. By the 'symmetry' of a function, let us mean the degree to which is it constant in the direction of one dimensional flats. On the braid arrangement, by either quotienting out codimensions and then symmetry, or symmetry and then codimensions, we obtain the following commutative square,

$$\begin{tikzcd}
\hat{\mathbf{\Sigma}}^* \arrow[r, two heads, "\text{codim}"] \arrow[d, two heads, "\text{sym}"'] & \hat{\mathbf{L}}^* \arrow[d, two heads, "\text{sym}"] \\
\hat{\mathbf{\Gamma}}^* \arrow[r, two heads, "\text{codim}"'] & \mathbf{Lie}^*
\end{tikzcd}$$

On the adjoint braid arrangement, by either quotienting out codimensions and then symmetry, or symmetry and then codimensions, we obtain the following commutative square,

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\check{\mathbf{\Sigma}}^* \arrow[r, two heads, "\text{sym}"] \arrow[d, two heads, "\text{codim}"'] & \check{\mathbf{L}}^* \arrow[d, two heads, "\text{codim}"] \\
\check{\mathbf{\Gamma}}^* \arrow[r, two heads, "\text{sym}"'] & \mathbf{Lie}^*
\end{tikzcd}$$

In this paper, we only consider the quotients by codimensions.

Axiomatic QFT. Our results give the pure mathematical interpretation of a classical construction in axiomatic QFT. Let I be a finite set, and let $\mathcal{X}$ be a time-oriented Lorentzian manifold. In [Eps16], a linear system of generalized time-ordered functions is defined to be a linear map $\mathbf{t}_I$ on $\mathbf{\Sigma}[I]$ into distributions on the space of configurations $I \to \mathcal{X}$, satisfying certain physically motivated properties. Let $\mathcal{U} : \mathbf{\Gamma} \hookrightarrow \mathbf{\Sigma}$ be the universal enveloping map. We show that the composite $\mathbf{t} \circ \mathcal{U}$ is naturally the association of generalized retarded functions to chambers of the adjoint braid arrangement, as defined in e.g. [Ara61, Section 9], [Eps16, Section 2.2]. Thus, generalized retarded functions are the image of the primitive part of $\mathbf{\Sigma}$ in distributions.

It is known that graded Hopf algebras encode combinatorial aspects of QFT [CK99], [EFK05], [FGB05], [Mor06]. For a connection between causal perturbation theory and the Connes-Kreimer Hopf algebras of rooted trees, see [BK05]. The species analogs of the Connes-Kreimer Hopf algebras, denoted $\overrightarrow{\mathbf{F}}$ and $\mathbf{F}$, have been defined by Aguiar and Mahajan [AM10, Section 13.3]. The algebras $\overrightarrow{\mathbf{F}}$ and $\mathbf{F}$ are connected with the algebras we consider in this paper in the following diagram.

$$
\begin{tikzcd}
\overrightarrow{\mathbf{F}} \arrow[r, hook] \arrow[d, two heads] & \mathbf{L}\times\mathbf{P} \arrow[d, two heads] & & & \\
\mathbf{F} \arrow[r, hook] & \mathbf{P} \arrow[r] & \mathbf{\Sigma}^* \arrow[r, two heads, "\text{quotient out}", "\text{codims}"'] & \mathbf{\Gamma}^* & \\
& & \mathbf{\Sigma} & \mathbf{\Gamma} \arrow[l, hook', "\text{primitive}"', "\text{elements}"] & \mathbf{L}^\vee \arrow[l, two heads, "\text{Steinmann}"', "\text{relations}"]
\end{tikzcd}
$$

Here, $\mathbf{P}$ is the species of posets, and $\mathbf{P} \to \mathbf{\Sigma}^*$ is the restriction of the map $\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*$ from Section 2.3. The monoidal product $\times$ is the categorification of the Hadamard product of power series.

Tropical Toric Geometry. The geometric interpretation of $\mathbf{\Sigma}$ over the braid arrangement identifies the dual monomial basis, called the $\mathtt{H}$-basis, with faces [AM10, Chapter 10]. This is how the usual geometric interpretation of the Tits product is obtained, which is the action of $\mathbf{\Sigma}$ on itself by Hopf powers. See also the geometric interpretation of Lie and Zie elements, which correspond to the embeddings $\mathbf{Lie} \hookrightarrow \mathbf{\Gamma} \hookrightarrow \mathbf{\Sigma}$, and their generalization to generic hyperplane arrangements [AM17].

We can alternatively identify the $\mathtt{H}$-basis with faces of the permutohedron, as follows. The type A root space, which we denote by $\mathbf{T}^I$, is naturally a tropical algebraic torus, in fact the dequantization of the maximal torus of $\mathrm{PGL}_I(\mathbb{C})$. Let *tropical permutohedral space* $\mathbb{T}\mathfrak{p}^I$ be the toric compactification of $\mathbf{T}^I$ with respect to the braid arrangement fan (see e.g. [Mey11, Chapter 1] for the construction of tropical toric varieties). If we view $\mathbf{T}^I$ as the space of configurations $I \to \mathbb{R}$ modulo translations, then $\mathbb{T}\mathfrak{p}^I$ adds configurations where particles may be separated by infinite distances (or 'times'). The compactifications $\mathbb{T}\mathfrak{p}^I$ form a cocommutative Hopf monoid in set species $\mathbb{T}\mathfrak{p}$, with multiplication induced by embedding facets of the permutohedron, and comultiplication induced by quotienting the permutohedron in the direction of fundamental weights. These operations are well-defined on torus orbits, and the Hopf monoid of set compositions Σ is recovered as the quotient of $\mathbb{T}\mathfrak{p}$ by the torus action, $\Sigma[I] = \mathbb{T}\mathfrak{p}^I/\mathbf{T}^I$. Moreover, the geometric realization of $\mathbf{\Sigma}^*$ as functions on $\mathbf{T}^I$ may be extended by continuity to functions on the compactification $\mathbb{T}\mathfrak{p}^I$, and the commutative Hopf algebraic structure of $\mathbf{\Sigma}^*$ for these functions is exactly that which is induced by the cocommutative Hopf monoid structure of the underlying space $\mathbb{T}\mathfrak{p}$. In particular, the comultiplication of $\mathbf{\Sigma}^*$ is the restriction of functions to boundary facets.

Structure. This paper has five sections. In Section 1, we describe combinatorial gadgets that index aspects of the type A hyperplane arrangements. In particular, we introduce transitive families, which index cones of the adjoint braid arrangement. In Section 2, we define the algebras in species which feature in this paper. We construct several bases of the indecomposable quotient of $\mathbf{\Sigma}^*$. In Section 3, we describe the two geometric realizations of $\mathbf{\Sigma}^*$. In Section 4, we prove our main results. We show that the indecomposable quotient of $\mathbf{\Sigma}^*$ naturally lives on the chambers of the adjoint braid arrangement, has cobracket discrete differentiation across hyperplanes, and is characterized by the Steinmann relations. In Section 5, we describe the connection with QFT, and show that generalized retarded functions correspond to the primitive part of $\mathbf{\Sigma}$.

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1. Combinatorial Background

We recall species, set partitions, set compositions, preposets, maximal unbalanced families, and we introduce transitive families, which simultaneously generalize preposets and maximal unbalanced families.

We model preposets over a finite set I as collections of order preserving functions $\mathbf{2} \hookrightarrow I$, where $\mathbf{2}$ is the ordinal with two elements. This gives the cospecies Π^*, L^*, Σ^*, and O. We then construct their adjoint analogs $\Pi^\vee$, $\mathrm{L}^\vee$, $\Sigma^\vee$, and $\mathrm{O}^\vee$, which are species whose elements are collections of functions $I \twoheadrightarrow \mathbf{2}$. The species of maximal unbalanced families is $\mathrm{L}^\vee$, and the species of transitive families is $\mathrm{O}^\vee$. A preposet is naturally a transitive family if it is modeled as the collection of order preserving functions $I \twoheadrightarrow \mathbf{2}$. The correspondence with aspects of the type A hyperplane arrangements will be as follows.

	flats*	chambers*	faces*	cones*
braid arrangement	Π^*	L^*	Σ^*	O
adjoint braid arrangement	$\Pi^\vee$	$\mathrm{L}^\vee$	$\Sigma^\vee$	$\mathrm{O}^\vee$

These combinatorial gadgets index the *characteristic functions* of the listed regions, hence the asterisks. For the close relationship between Hopf theory in species and the braid arrangement, see [AM10, Chapter 10]. There is no well-behaved duality on the nose, because the adjoint braid arrangement is less controlled than the braid arrangement. The duality is obtained by restricting the adjoint braid arrangement to its aspects which can be 'seen' by coroots.

	semisimple flats	pointed permutohedral cones	permutohedral cones	generalized permutohedral cones
adjoint braid arrangement	Π^*	L^*	Σ^*	O

The combinatorial gadgets do index regions of space in the restricted adjoint case. Their linearizations will then be functionals (functions of functions). A *semisimple flat* is a flat which is spanned by coroots. Such subspaces feature in the study of reflection length of $\widetilde{A}_{n-1}$ [LMPS19]. A generalized permutohedral cone is equivalently a cone which is generated by coroots. They are studied in [GMP19].

1.1. Species.

We briefly recall species, and Hopf monoids internal to species. References are [AM10], [AM13]. Let Set denote the category of sets, and let Vect denote the category of real vector spaces. Let S denote the monoidal category with objects finite sets, morphisms bijective functions, and monoidal product the restriction of the disjoint union of sets,

$$\mathsf{S} := (\mathrm{core}(\mathsf{finSet})\, ,\, \sqcup).$$

A *set (co)species* is a Set-valued (co)presheaf on S, and a *vector (co)species* is a Vect-valued (co)presheaf on S. The *linearization* of a set (co)species is its composition with the free functor $\mathsf{Set} \to \mathsf{Vect}$. The (co)presheaf categories of set (co)species and vector (co)species are symmetric monoidal categories when equipped with the Day convolution $\otimes_{\mathrm{Day}}$. Let a *(co)species Hopf monoid* be a Hopf monoid internal to set (co)species, and let a *(co)species Hopf algebra* be a Hopf monoid internal to vector (co)species. Often, we shall just say Hopf monoid and Hopf algebra.

In [AM10], [AM13], Aguiar and Mahajan explicitly consider just copresheafs on S. However, since the inversion of bijections is a dagger for S, the general theory of species is a copy of the general theory of cospecies.

1.2. **Partitions and Compositions.** Let $I \in \mathsf{S}$ be a finite set with cardinality $n \in \mathbb{N}$. A *partition* $P = (S_1 | \dots | S_k)$ of I is an unordered set of disjoint nonempty subsets $S_j \subseteq I$, called *blocks*, whose union is I. For partitions P and Q of I, if each block of Q is a subset of a block of P, we write $P \leq Q$.

A *composition* $F = (S_1, \dots, S_k)$ of I is an ordered sequence of disjoint nonempty subsets $S_j \subseteq I$, called *lumps*, whose union is I. We denote by Q_F the partition with blocks the lumps of F. For compositions F and G of I, if $Q_G \leq Q_F$, we write $G \leq F$. The *opposite* of $F = (S_1, \dots, S_k)$ is $\bar{F} := (S_k, \dots, S_1)$. For $S \subseteq I$, the *restriction* $F|_S$ is the composition of S defined by

$$F|_S := (S \cap S_1, \dots, S \cap S_k)_+$$

where '${}_+$' means 'delete any instances of the empty set'. Given compositions $G \leq F$ with $G = (S_1, \dots, S_k)$, let

$$l(F/G) := \prod_{j=1}^{k} l(F|_{S_j}) \qquad \text{and} \qquad (F/G)! := \prod_{j=1}^{k} l(F|_{S_j})!\,.$$

We define the set species Π, L, and Σ by

$$\Pi[I] := \{P : P \text{ is a partition of } I\}, \qquad \mathrm{L}[I] := \{F : F \text{ is a linear ordering of } I\},$$

$$\Sigma[I] := \{F : F \text{ is a composition of } I\}.$$

These are naturally species, not cospecies, if we model partitions as surjections of I into a cardinal, and compositions as surjections of I into an ordinal (which we should). Their elements correspond to regions of space in the root space of type A [AM10, Section 10.2]. If we think of Σ as the torus orbits of tropical permutohedral space $\mathbb{T}\mathfrak{p}$, then the region of a composition F consists of direction vectors of rays with limit points in the torus orbit of F.

1.3. **Preposets.** Let $\mathbf{2} := \{+1, -1\}$ denote the ordinal with two elements (interval category), where the order is '+1 is greater than -1'. Let

$$[\mathbf{2}, I] := \{\text{injective functions } \mathbf{2} \hookrightarrow I\}.$$

For $i_1, i_2 \in I$ with $i_1 \neq i_2$, define $(i_1, i_2) \in [\mathbf{2}, I]$ by

$$(i_1, i_2)(+1) := i_1 \qquad \text{and} \qquad (i_1, i_2)(-1) := i_2.$$

A *preposet* (thin category) p of I is a reflexive and transitive relation $\geq_p$ on I. We make the identification between p and structure preserving probes by $\mathbf{2}$,

$$p := \{(i_1, i_2) \in [\mathbf{2}, I] : i_1 \geq_p i_2\}.$$

This put preposets in one-to-one correspondence with subsets of $[\mathbf{2}, I]$ which are closed under the following partially defined product [2],

$$(i_1, i_2) \circ (i_3, i_4) := \begin{cases} (i_1, i_4) & \text{if } i_2 = i_3 \text{ and } i_1 \neq i_4 \\ (i_3, i_2) & \text{if } i_1 = i_4 \text{ and } i_2 \neq i_3 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

An element $(i_1, i_2) \in p$ is called *symmetric* if $(i_2, i_1) \in p$. A *generator* of p is either a symmetric element, or an element which cannot be written as a product of elements in p. We let $p_<$ denote the set nonsymmetric elements of p. The *opposite* $\bar{p}$ of p is defined by $(i_1, i_2) \in \bar{p}$ if and only if $(i_2, i_1) \in p$. The intersection of two preposets is a preposet. For $p' \subseteq [\mathbf{2}, I]$ any subset, the *transitive closure* $\mathrm{TC}(p')$ is the intersection of all preposets which contain p'.

The *blocks* of p are the equivalence classes of the transitive and symmetric closure of the relation

$$i_1 \sim i_2 \qquad \Longleftrightarrow \qquad (i_1, i_2) \in p \quad \text{or} \quad (i_2, i_1) \in p.$$

The *lumps* of p are the equivalence classes of the equivalence relation

$$i_1 \sim i_2 \qquad \Longleftrightarrow \qquad (i_1, i_2) \in p \quad \text{and} \quad (i_2, i_1) \in p.$$

Let $l(p)$ denote the number of lumps of p. For p and q preposets of I, we write $q \leq p$ if $p \subseteq q$. We write $q \preceq p$ if both $q \leq p$ and $p_< \subseteq q_<$, and we write $q \preceq_l p$ if both $q \preceq p$ and $l(q) = l(p)$. For p a preposet of I and $S \subseteq I$, the *restriction* $p|_S$ is the preposet of S given by

$$(i_1, i_2) \in p|_S \iff (i_1, i_2) \in p, \qquad \text{for all } (i_1, i_2) \in [\mathbf{2}, S].$$

The cospecies of preposets O is defined by

$$\mathrm{O}[I] := \{p : p \text{ is a preposet of } I\}.$$

We say a preposet is *total* if for all $(i_1, i_2) \in [\mathbf{2}, I]$, at least one of $(i_1, i_2) \in p$ and $(i_2, i_1) \in p$ is true. Let Σ^* denote the cospecies of total preposets,

$$\Sigma^*[I] := \{p \in \mathrm{O}[I] : p \text{ is total}\}.$$

We say a preposet is *totally-nonsymmetric* if for all $(i_1, i_2) \in [\mathbf{2}, I]$, exactly one of $(i_1, i_2) \in p$ and $(i_2, i_1) \in p$ is true. Let L^* denote the cospecies of totally-nonsymmetric preposets,

$$\mathrm{L}^*[I] := \{p \in \mathrm{O}[I] : p \text{ is totally-nonsymmetric}\}.$$

Let Π^* denote the cospecies of preposets without nonsymmetric elements,

$$\Pi^*[I] := \{p \in \mathrm{O}[I] : p_< = \emptyset\}.$$

These final three cospecies are in one-to-one correspondence with compositions, linear orders, and partitions respectively. Explicitly, given a partition $P = (S_1| \dots |S_k)$ of I, we let $P \in \Pi^*[I]$ denote the encoding of P as the collection of $(i_1, i_2) \in [\mathbf{2}, I]$ such that $\{i_1, i_2\} \subseteq S_j$ for some $S_j \in P$. Given a composition $F = (S_1, \dots, S_k)$ of I, we let $F \in \Sigma^*[I]$ denote the encoding of F as the collection of $(i_1, i_2) \in [\mathbf{2}, I]$ such that the lump containing i_1 precedes or is equal to the lump containing i_2.

[2] which encodes the addition of coroots of type A, see Section 3.1

1.4. **Transitive Families.** We now define the adjoint analogs of preposets. The adjoint analogs of linear orders have appeared before, as maximal unbalanced families [BMM+12], positive sum set systems [Bjo15], and cells [Eps16]. Let

$$[I, \mathbf{2}] := \{\text{surjective functions } I \twoheadrightarrow \mathbf{2}\}.$$

If we let compositions of I be surjections of I into an ordinal, then $[I, \mathbf{2}]$ *is* the set of compositions of I with two lumps. Explicitly, for $S \sqcup T = I$ with $S, T \neq \emptyset$, define $(S, T) \in [I, \mathbf{2}]$ by

$$(S, T)(i) = +1 \quad \text{if } \ i \in S \qquad \text{and} \qquad (S, T)(i) = -1 \quad \text{if } \ i \in T.$$

Define the following partial product [3],

$$(S, T) \circ (U, V) := \begin{cases} (S \cup U, T \cap V) & \text{if } T \supset U \\ (S \cap U, T \cup V) & \text{if } S \supset V \\ \text{undefined} & \text{otherwise.} \end{cases}$$

A *transitive family* τ of I is a subset of $[I, \mathbf{2}]$ which is closed under the partial product, i.e. for $(S, T), (U, V) \in \tau$, we have either $(S, T) \circ (U, V) \in \tau$ or $(S, T) \circ (U, V)$ is undefined. For $p \in \mathrm{O}[I]$, we obtain a transitive family by taking the structure preserving coprobes by $\mathbf{2}$,

$$\tau_p := \{(S, T) \in [I, \mathbf{2}] : \text{if } i_1 \geq i_2 \text{ and } i_2 \in S, \ \text{then } i_1 \in S\} = \{(S, T) \in [I, \mathbf{2}] : (S, T) \leq p\}.$$

This unforgetfully encodes p in terms of its upward closed subsets of I (equivalently downward closed subsets). Thus, $p \mapsto \tau_p$ is an embedding of preposets into transitive families.

An element $(S, T) \in \tau$ is called *symmetric* if $(T, S) \in \tau$. A *generator* of τ is either a symmetric element, or an element which cannot be written as a product of elements in τ. We let $\tau_<$ denote the set of nonsymmetric elements of τ. The *opposite* $\bar{\tau}$ of τ is defined by $(S, T) \in \bar{\tau}$ if and only if $(T, S) \in \tau$. Then $\bar{\tau}_p = \tau_{\bar{p}}$. The intersection of two transitive families is a transitive family. For $\tau' \subseteq [I, \mathbf{2}]$ any subset, the *transitive closure* $\mathrm{TC}(\tau')$ is the intersection of all transitive families which contain τ'. For τ and υ transitive families of I, we write $\upsilon \leq \tau$ if $\tau \subseteq \upsilon$. We have $q \leq p \iff \tau_p \leq \tau_q$. We write $\upsilon \preceq \tau$ if both $\upsilon \leq \tau$ and $\tau_< \subseteq \upsilon_<$. The species of transitive families $\mathrm{O}^\vee$ is defined by

$$\mathrm{O}^\vee[I] := \{\tau : \tau \text{ is a transitive family of } I\}.$$

A transitive family τ is *total* if for all $(S, T) \in [I, \mathbf{2}]$, at least one of $(S, T) \in \tau$ and $(T, S) \in \tau$ is true. Let $\Sigma^\vee$ denote the species of total transitive families,

$$\Sigma^\vee[I] := \{\tau \in \mathrm{O}^\vee[I] : \tau \text{ is total}\}.$$

A transitive family τ is *totally-nonsymmetric* if for all $(S, T) \in [I, \mathbf{2}]$, exactly one of $(S, T) \in \tau$ and $(T, S) \in \tau$ is true. Let $\mathrm{L}^\vee$ denote the species of totally-symmetric transitive families,

$$\mathrm{L}^\vee[I] := \{\tau \in \mathrm{O}^\vee[I] : \tau \text{ is totally-nonsymmetric}\}.$$

Totally-nonsymmetric transitive families are most commonly known as maximal unbalanced families, although the equivalence of the definitions is not immediately clear. Since maximal unbalanced families index chambers of the adjoint braid arrangement [BMM+12, p.2], the equivalence follows from Corollary 3.3.1. The number of maximal unbalanced families is sequence A034997 in the OEIS. Let $\Pi^\vee$ denote the species of transitive families without nonsymmetric elements,

$$\Pi^\vee[I] := \{\tau \in \mathrm{O}^\vee[I] : \tau_< = \emptyset\}.$$

[3] which encodes the addition of fundamental weights of type A, see Section 3.1

As we shall see, $\mathrm{O}^\vee$ indexes cones of the adjoint braid arrangement, $\Sigma^\vee$ indexes faces, $\mathrm{L}^\vee$ indexes chambers, and $\Pi^\vee$ indexes flats.

2. Algebraic Structures

In this section, we define the various algebras in (co)species which will feature in this paper. Our main references are [AM10], [AM13]. We also prove some additional results which we shall need.

2.1. **Operations.** Let $\mathbf{\Sigma}$ denote the linearization of the species Σ. For $(S,T) \in [I,\mathbf{2}]$ and $F \in \Sigma[I]$, we have *deshuffling*,

$$F \parallel_S := \begin{cases} F|_S & \text{if } (S|T) \leq Q_F \\ 0 \in \mathbf{\Sigma}[S] & \text{otherwise.} \end{cases} \qquad \text{and} \qquad F \parallel_T := \begin{cases} F|_T & \text{if } (S|T) \leq Q_F \\ 0 \in \mathbf{\Sigma}[T] & \text{otherwise.} \end{cases}$$

For $p, q \in \mathrm{O}[I]$, let

$$p \cup q \in \mathrm{O}[I]$$

denote the transitive closure of the set union of p and q. For $(S,T) \in [I,\mathbf{2}]$, $p \in \mathrm{O}[S]$ and $q \in \mathrm{O}[T]$, let

$$(p \,|\, q) \in \mathrm{O}[I]$$

denote the (disjoint) set union of p and q. Let $\mathbf{O}$ denote the linearization of the cospecies O. For $(S,T) \in [I,\mathbf{2}]$ and $p \in \mathrm{O}[I]$, we have *deconcatenation*,

$$p\llbracket_S := \begin{cases} p|_S & \text{if } (S,T) \leq p \\ 0 \in \mathbf{O}[S] & \text{otherwise,} \end{cases} \qquad \text{and} \qquad p \mathbin{/\!\!/}_T := \begin{cases} p|_T & \text{if } (S,T) \leq p \\ 0 \in \mathbf{O}[T] & \text{otherwise.} \end{cases}$$

2.2. **The Commutative Hopf Algebras of Preposets.** Given $p \in \mathrm{O}[I]$, we also denote by p the corresponding basis element of $\mathbf{O}[I]$. The vector cospecies $\mathbf{O}$ has the structure of a commutative Hopf algebra, with multiplication union and comultiplication deconcatenation,

$$\mu_{(S|T)}(p \otimes q) := (p \,|\, q) \qquad \text{and} \qquad \Delta_{(S,T)}(p) := p \llbracket_S \otimes p \mathbin{/\!\!/}_T .$$

See [AM17, Section 2] for a quick introduction to species algebra.

2.3. **The Hopf Algebras of Set Compositions.** We now define the Hopf algebras $\mathbf{\Sigma}$ and $\mathbf{\Sigma}^*$, following [AM13, Section 11]. The species of compositions Σ has the structure of a cocommutative Hopf monoid, with multiplication concatenation and comultiplication restriction,

$$\mu_{(S,T)}(F,G) := FG \qquad \text{and} \qquad \Delta_{(S|T)}(F) := (F|_S, F|_T).$$

For $F \in \Sigma[I]$, let $\mathtt{H}_F$ denote the corresponding basis element of $\mathbf{\Sigma}[I]$. The set $\{\mathtt{H}_F : F \in \Sigma[I]\}$ is called the $\mathtt{H}$-*basis* of $\mathbf{\Sigma}[I]$. The linearization of the monoid structure of Σ makes $\mathbf{\Sigma}$ a cocommutative Hopf algebra, with multiplication and comultiplication given by

$$\mu_{(S,T)}(\mathtt{H}_F \otimes \mathtt{H}_G) := \mathtt{H}_{FG} \qquad \text{and} \qquad \Delta_{(S|T)}(\mathtt{H}_F) := \mathtt{H}_{F|_S} \otimes \mathtt{H}_{F|_T}.$$

The antipode is then determined, and is given by

$$s_I(\mathtt{H}_F) := \sum_{G \geq \bar{F}} (-1)^{l(G)} \, \mathtt{H}_G.$$

The Hopf algebra $\mathbf{\Sigma}$ is equivalently the free cocommutative Hopf algebra on the positive coalgebra $\mathbf{E}^*_+$ [AM10, Section 11.2.5]. This construction naturally equips $\mathbf{\Sigma}$ with the $\mathtt{H}$-basis. A second basis of $\mathbf{\Sigma}$, called the $\mathtt{Q}$-*basis*, is given by

$$\mathtt{Q}_F := \sum_{G \geq F} (-1)^{l(G)-l(F)} \frac{1}{l(G/F)} \mathtt{H}_G \qquad \text{or equivalently} \qquad \mathtt{H}_F =: \sum_{G \geq F} \frac{1}{(G/F)!} \mathtt{Q}_G.$$

The $\mathtt{Q}$-basis appears naturally if one constructs $\mathbf{\Sigma}$ according to [AM10, Section 11.2.2], i.e. as the free cocommutative Hopf algebra on the raw species $\mathbf{E}^*_+$, ignoring the coalgebra structure. The algebraic structure of $\mathbf{\Sigma}$ in terms of the $\mathtt{Q}$-basis is

$$\mu_{(S,T)}(\mathtt{Q}_F \otimes \mathtt{Q}_G) = \mathtt{Q}_{FG} \qquad \text{and} \qquad \Delta_{(S,T)}(\mathtt{Q}_F) = \mathtt{Q}_{F|_S} \otimes \mathtt{Q}_{F|_T}.$$

We collect the combinatorial descriptions of the algebraic structure of $\mathbf{\Sigma}$ in the following table.

$\mathbf{\Sigma}$	$\mathtt{H}$-basis	$\mathtt{Q}$-basis
multiplication	concatenation	concatenation
comultiplication	restriction	deshuffling

Let $\mathbf{\Sigma}^*$ denote the linearization of Σ^*. For $F \in \Sigma^*[I]$, let $\mathtt{M}_F$ denote the corresponding basis element of $\mathbf{\Sigma}^*[I]$. The set $\{\mathtt{M}_F : F \in \Sigma^*[I]\}$ is called the $\mathtt{M}$-*basis*, or *monomial basis*, of $\mathbf{\Sigma}^*[I]$. Define the dual pairing

$$\mathbf{\Sigma}^*[I] \otimes \mathbf{\Sigma}[I] \to \mathbb{R}, \qquad \mathtt{M}_F \otimes \mathtt{H}_G = \delta_{FG}.$$

Then the cospecies $\mathbf{\Sigma}^*$ automatically carries the dual Hopf structure of $\mathbf{\Sigma}$, which is the commutative Hopf algebra with multiplication and comultiplication given by

$$\mu_{(S|T)}(\mathtt{M}_F \otimes \mathtt{M}_G) := \sum_{H \preceq (F|G)} \mathtt{M}_H \qquad \text{and} \qquad \Delta_{(S,T)}(\mathtt{M}_F) := \mathtt{M}_{F\llbracket_S} \otimes \mathtt{M}_{F \,/\!\!/_T}.$$

The antipode is then determined, and is given by

$$s_I(\mathtt{M}_F) := (-1)^{l(F)} \sum_{G \leq \bar{F}} \mathtt{M}_G.$$

Consider the map (natural transformation) of cospecies given by

$$\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*, \qquad p \mapsto \mathtt{C}_p := \sum_{F \leq p} \mathtt{M}_F.$$

In particular, the set $\{\mathtt{C}_F : F \in \Sigma^*[I]\}$ is a second basis of $\mathbf{\Sigma}^*[I]$. We call this the $\mathtt{C}$-*basis*, or *cone basis*.

Proposition 2.1. The algebraic structure of $\mathbf{\Sigma}^*$ is given in terms of the $\mathtt{C}$-basis by

$$\mu_{(S|T)}(\mathtt{C}_F \otimes \mathtt{C}_G) = \sum_{H \preceq (F|G)} (-1)^{l(F|G)-l(H)}\, \mathtt{C}_H \qquad \text{and} \qquad \Delta_{(S,T)}(\mathtt{C}_F) = \mathtt{C}_{F\llbracket_S} \otimes \mathtt{C}_{F \,/\!\!/_T}.$$

Proof. Notice that

$$\mathtt{C}_F = (-1)^{l(F)} s_I(\mathtt{M}_{\bar{F}}).$$

The antipode s_I reverses multiplication and comultiplication [AM10, Proposition 1.22]. Therefore, for the multiplication, we have

$$(-1)^{l(F)+l(G)} \mu_{(S|T)}(\mathtt{C}_F \otimes \mathtt{C}_G) = \mu_{(S|T)}(s_I(\mathtt{M}_{\bar{F}}) \otimes s_I(\mathtt{M}_{\bar{G}})) = \sum_{H \preceq (F|G)} s_I(\mathtt{M}_{\bar{H}}) = \sum_{H \preceq (F|G)} (-1)^{l(H)} \mathtt{C}_H.$$

Then

$$\mu_{(S|T)}(\mathtt{C}_F \otimes \mathtt{C}_G) = \sum_{H \preceq (F|G)} (-1)^{l(F)+l(G)+l(H)} \mathtt{C}_H = \sum_{H \preceq (F|G)} (-1)^{l(F|G)-l(H)}\, \mathtt{C}_H.$$

For the comultiplication, we have

$$(-1)^{l(F)} \Delta_{(S,T)}(\mathtt{C}_F) = \Delta_{(S,T)}\big(s_I(\mathtt{M}_{\bar{F}})\big) = s_I(\mathtt{M}_{\bar{F}\rrbracket_T}) \otimes s_I(\mathtt{M}_{\bar{F}\,/\!\!/_S}) = s_I(\mathtt{M}_{\overline{F\,/\!\!/_T}}) \otimes s_I(\mathtt{M}_{\overline{F\rrbracket_S}})$$

$$= (-1)^{l(F\,/\!\!/_T)+l(F\rrbracket_S)}\, \mathtt{C}_{F\,/\!\!/_T} \otimes \mathtt{C}_{F\rrbracket_S} = (-1)^{l(F)}\, \mathtt{C}_{F\,/\!\!/_T} \otimes \mathtt{C}_{F\rrbracket_S}. \qquad \square$$

Using the geometric realization of $\mathbf{\Sigma}^*$ over the braid arrangement, and an inclusion-exclusion argument, we shall see that

$$\mathtt{C}_p = \sum_{G \preceq p} (-1)^{l(p)-l(G)} \mathtt{C}_G.$$

The special case $p = (F|G)$, which gives

$$\mathtt{C}_{(F|G)} = \sum_{H \preceq (F|G)} (-1)^{l(F|G)-l(H)}\, \mathtt{C}_H \quad \Big(= \mu_{(S,T)}(\mathtt{C}_p \otimes \mathtt{C}_q) \Big),$$

is the statement that $\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*$ preserves the multiplication, and is proved explicitly in [Ear17]. The fact that $\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*$ preserves the comultiplication is clear. Thus, $\mathbf{O} \twoheadrightarrow \mathbf{\Sigma}^*$ is a homomorphism of Hopf algebras. The restriction of this homomorphism to posets is considered in [AM10, Section 13.1.2].

The $\mathtt{M}$-basis is naturally extended to preposets $p \in O[I]$ as follows,

$$\mathtt{M}_p := \sum_{F \preceq p} \mathtt{M}_F.$$

Then the multiplication of $\mathbf{\Sigma}^*$ is simply

$$\mu_{(S|T)}(\mathtt{M}_F \otimes \mathtt{M}_G) = \mathtt{M}_{(F|G)} \qquad \text{or} \qquad \mu_{(S|T)}(\mathtt{C}_F \otimes \mathtt{C}_G) = \mathtt{C}_{(F|G)}.$$

Let the $\mathtt{P}$-*basis* be the basis of $\mathbf{\Sigma}^*$ which is dual to the $\mathtt{Q}$-basis, thus

$$\mathtt{P}_F := \sum_{G \leq F} \frac{1}{(F/G)!} \mathtt{M}_G.$$

The algebraic structure of $\mathbf{\Sigma}^*$ in terms of the $\mathtt{P}$-basis is

$$\mu_{(S|T)}(\mathtt{P}_F \otimes \mathtt{P}_G) = \sum_{H \preceq_l (F|G)} \mathtt{P}_H \qquad \text{and} \qquad \Delta_{(S,T)}(\mathtt{P}_F) = \mathtt{P}_{F\rrbracket_S} \otimes \mathtt{P}_{F\,/\!\!/_T}.$$

We collect the combinatorial descriptions of the algebraic structure of $\mathbf{\Sigma}^*$ in the following table.

$\mathbf{\Sigma}^*$	$\mathtt{M}$-basis	$\mathtt{P}$-basis	$\mathtt{C}$-basis
multiplication	quasishuffling	shuffling	signed-quasishuffling
comultiplication	deconcatenation	deconcatenation	deconcatenation

Over the braid arrangement, the $\mathtt{P}$-basis is 'between' the $\mathtt{M}$-basis and the $\mathtt{C}$-basis; $\mathtt{P}$-basis elements are an extension of $\mathtt{M}$-basis elements to fractional values on the higher codimensions, so that one no longer has to add or subtract lumped terms when multiplying, and pure shuffling is obtained.

2.4. **The Primitive Part and the Indecomposable Quotient.** We now describe the Lie algebra which forms the primitive part of $\mathbf{\Sigma}$, and its dual Lie coalgebra, which is the indecomposable quotient of $\mathbf{\Sigma}^*$. The Lie algebra will be denoted by $\mathbf{\Gamma}$, and the Lie coalgebra will be denoted by $\mathbf{\Gamma}^*$. Therefore, we shall have a pair of dual maps,

$$\mathbf{\Gamma} \hookrightarrow \mathbf{\Sigma} \qquad \text{and} \qquad \mathbf{\Sigma}^* \twoheadrightarrow \mathbf{\Gamma}^*.$$

A general result of Hopf theory in species says that $\mathbf{\Gamma}$ is the kernel of the comultiplication of $\mathbf{\Sigma}$, and so dually, $\mathbf{\Gamma}^*$ is the quotient of $\mathbf{\Sigma}^*$ by the image of its multiplication [AM13, Section 5.5-5.6].

Let a *tree* Λ over I be a full binary tree whose leaves are labeled bijectively with the blocks of a partition Q_Λ of I. The blocks of Q_Λ are called the *lumps* of Λ. They form a composition F_Λ by listing in order of appearance from left to right, called the *canopy* of Λ. We may denote trees by nested products '$[\,\cdot\,,\,\cdot\,]$' of subsets of I. Given a tree Λ, let $\mathcal{A}(\Lambda)$ denote the set of trees which are obtained by switching left and right branches at nodes of Λ. For $\Lambda' \in \mathcal{A}(\Lambda)$, let $(\Lambda, \Lambda') \in \mathbb{Z}/2\mathbb{Z}$ denote the number of node switches modulo 2 required to bring Λ to Λ'.

Let $\mathbf{\Gamma}[I]$ be the vector space of formal linear combinations of trees over I, modulo antisymmetry and the Jacobi identity as interpreted on trees in the usual way. This defines the vector species $\mathbf{\Gamma}$. Alternatively, if $\mathbf{Lie}$ is the species of the Lie operad, $\mathbf{E}^*_+$ is the positive exponential species, and '$\circ$' denotes the plethystic monoidal product of species, let

$$\mathbf{\Gamma}[I] := \mathbf{Lie} \circ \mathbf{E}^*_+[I] = \bigoplus_{P \in \Pi[I]} \mathbf{Lie}[P].$$

Define the following map, which is the primitive part of $\mathbf{\Sigma}$,

$$\mathcal{U} : \mathbf{\Gamma} \hookrightarrow \mathbf{\Sigma}, \qquad \Lambda \mapsto \mathtt{Q}_\Lambda := \sum_{\Lambda' \in \mathcal{A}(\Lambda)} (-1)^{(\Lambda,\Lambda')} \mathtt{Q}_{F_{\Lambda'}}.$$

The Lie bracket d^* of $\mathbf{\Gamma}$ connects a pair of trees Λ_1, Λ_2 over disjoint sets by adding a new root whose children are the roots of Λ_1 and Λ_2,

$$d^* : \mathbf{\Gamma} \otimes_{\text{Day}} \mathbf{\Gamma} \to \mathbf{\Gamma}, \qquad d^*_{[S,T]}(\Lambda_1 \otimes \Lambda_2) := [\Lambda_1, \Lambda_2].$$

For $F \in \Sigma[I]$, let $[F] \in \mathbf{\Gamma}[I]$ denote the right comb-tree with canopy F, i.e. if $F = (S_1, \dots, S_k)$, then

$$[F] := [\dots[[S_1, S_2], S_3], \dots, S_k]$$

Given a specified coordinate $i_0 \in I$, let

$$\Sigma_{i_0}[I] := \{F = (S_1, \dots, S_k) \in \Sigma[I] : i_0 \in S_1\}.$$

If $F \in \Sigma_{i_0}[I]$, then $[F]$ is called a *standard right comb-tree* over I (with respect to i_0).

Proposition 2.2. The set of standard right-comb trees

$$\{[F] : F \in \Sigma_{i_0}[I]\}$$

is a basis of $\mathbf{\Gamma}[I]$.

Proof. To show that any tree $\Lambda \in \mathbf{\Gamma}[I]$ is a linear combination of standard right comb-trees, first move i_0 to the left most branch of Λ by using antisymmetry. Then comb branches to the right using the following consequence of antisymmetry and the Jacobi identity,

$$[S, [T, U]] = [[S, T], U] - [[S, U], T].$$

This shows that standard right-comb trees span $\mathbf{\Gamma}[I]$. The result then follows since the dimension of $\mathbf{Lie}[I]$ is $(n-1)!$, see e.g. [AM17, Theorem 10.38]. □

Let $\mathbf{\Gamma}^*$ be the vector cospecies whose component $\mathbf{\Gamma}^*[I]$ is the dual vector space of $\mathbf{\Gamma}[I]$. For $F \in \Sigma^*[I]$, consider the set of trees given by

$$\mathbf{\Lambda}(F) := \bigsqcup_{\Lambda \in \mathbf{\Gamma}[I] : F_\Lambda = F} \mathcal{A}(\Lambda).$$

This is indeed a disjoint union because for each $\Lambda' \in \mathbf{\Lambda}(F)$, there is a unique tree with canopy F which is obtained from Λ' by switching branches at nodes. For $F \in \Sigma^*[I]$, let $\mathtt{p}_F$ be the function on trees over I given by

$$\mathtt{p}_F(\Lambda') := \begin{cases} (-1)^{(\Lambda, \Lambda')} & \text{if } \Lambda' \in \mathcal{A}(\Lambda) \subset \mathbf{\Lambda}(F) \\ 0 & \Lambda' \notin \mathbf{\Lambda}(F). \end{cases}$$

Proposition 2.3. For $F \in \Sigma^*[I]$, we have

$$\mathtt{p}_F \in \mathbf{\Gamma}^*[I].$$

Proof. The definition of $\mathtt{p}_F$ ensures that it satisfies antisymmetry. For the Jacobi identity, suppose that a tree $\Lambda'_{STU} \in \mathbf{\Lambda}(F)$ has a branch $[[S,T],U]$. We may assume that $\mathtt{p}_F(\Lambda'_{STU}) = 1$ by antisymmetry. Then the tree Λ''_{STU} obtained from Λ'_{STU} by switching $[[S,T],U]$ to $[S,[T,U]]$ has $\mathtt{p}_F(\Lambda''_{STU}) = 1$. Therefore the tree Λ'_{TUS} obtained from Λ'_{STU} by switching $[[S,T],U]$ to $[[T,U],S]$ has $\mathtt{p}_F(\Lambda'_{STU}) = -1$. However, the tree Λ'_{UST} obtained from Λ'_{STU} by switching $[[S,T],U]$ to $[[U,S],T]$ has $\mathtt{p}_F(\Lambda'_{UST}) = 0$, because no switching of the nodes of $[[U,S],T]$ can produce the canopy (S,T,U). Then

$$\mathtt{p}_F(\Lambda'_{STU}) + \mathtt{p}_F(\Lambda'_{TUS}) + \mathtt{p}_F(\Lambda'_{UST}) = 1 - 1 + 0 = 0. \qquad \square$$

Given a specified coordinate $i_0 \in I$, let

$$\Sigma^*_{i_0}[I] := \big\{ F = (S_1, \dots, S_k) \in \Sigma^*[I] : i_0 \in S_1 \big\}.$$

Proposition 2.4. The set of functions on trees

$$\big\{ \mathtt{p}_F : F \in \Sigma^*_{i_0}[I] \big\}$$

is the basis of $\mathbf{\Gamma}^*[I]$ which is dual to the basis of standard right-comb trees from Proposition 2.2.

Proof. We have

$$\mathtt{p}_F\big([F]\big) = 1$$

because $\Lambda \in \mathbf{\Lambda}(F)$ with $(-1)^{(\Lambda,\Lambda)} = (-1)^0 = 1$. Let $F \in \Sigma^*_{i_0}[I]$ and $G \in \Sigma_{i_0}[I]$ with

$$[G] \in \mathbf{\Lambda}(F).$$

This means that there exists a tree Λ with canopy F such that $[G] \in \mathcal{A}(\Lambda)$. This implies that $\Lambda \in \mathcal{A}([G])$. But because $[G]$ is a right comb-tree, the only tree in $\mathcal{A}([G])$ which contains i_0 in its left most lump is $[G]$. Therefore we must have $\Lambda = [G]$, and so $F = G$. The contrapositive of this is that if $F \neq G$, then $[G] \notin \mathcal{A}(\Lambda)$, and so

$$\mathtt{p}_F\big([G]\big) = 0 \qquad \square.$$

We call the functions $\mathtt{p}_F$, for $F \in \Sigma^*_{i_0}[I]$, the $\mathtt{p}$-*basis* of $\mathbf{\Gamma}^*$. Of course, it depends on the choice $i_0 \in I$. Let

$$\mathcal{U}^* : \mathbf{\Sigma}^* \twoheadrightarrow \mathbf{\Gamma}^*$$

denote the dual of the map $\mathcal{U}$. Equivalently, $\mathcal{U}^*$ is the indecomposable quotient map of $\mathbf{\Sigma}^*$, i.e. $\mathcal{U}^*$ quotients out the image of the multiplication of $\mathbf{\Sigma}^*$.

Proposition 2.5. The map $\mathcal{U}^*$ is given by

$$\mathcal{U}^* : \mathbf{\Sigma}^* \twoheadrightarrow \mathbf{\Gamma}^*, \qquad \mathtt{P}_F \mapsto \mathtt{p}_F.$$

Proof. Since $\mathcal{U}^*$ is the dual of $\mathcal{U}$, for $F \in \Sigma^*[I]$ and $G \in \Sigma_{i_0}[I]$, we have

$$\mathcal{U}^*(\mathtt{P}_F)([G]) = \mathtt{P}_F(\mathtt{Q}_{[G]}).$$

But, directly from the definitions of $\mathtt{Q}_\Lambda$ and $\mathtt{p}_F$, we see that

$$\mathtt{P}_F(\mathtt{Q}_{[G]}) = \mathtt{p}_F([G]).$$

Thus

$$\mathcal{U}^*(\mathtt{P}_F)([G]) = \mathtt{p}_F([G]).$$

Because the right-comb trees $[G]$, for $G \in \Sigma_{i_0}[I]$, form a basis of $\mathbf{\Gamma}[I]$, we must have $\mathcal{U}^*(\mathtt{P}_F) = \mathtt{p}_F$. $\square$

Let

$$\mathtt{c}_p := \mathcal{U}^*(\mathtt{C}_p) \qquad \text{and} \qquad \mathtt{m}_F := \mathcal{U}^*(\mathtt{M}_F).$$

Then $\{\mathtt{c}_F : F \in \Sigma^*_{i_0}[I]\}$ and $\{\mathtt{m}_F : F \in \Sigma^*_{i_0}[I]\}$ are two more bases of $\mathbf{\Gamma}^*[I]$. We call them the $\mathtt{c}$-*basis* and the $\mathtt{m}$-*basis* respectively. Since $\mathbf{\Gamma}^*$ is the quotient of $\mathbf{\Sigma}^*$ by the image of its multiplication, we have the following three choices of generating relations for $\mathbf{\Gamma}^*$, shuffling, quasishuffling, and signed-quasishuffling,

$$\sum_{H \preceq_l (F|G)} \mathtt{p}_H = 0, \qquad \sum_{H \preceq (F|G)} \mathtt{m}_H = 0, \qquad \sum_{H \preceq (F|G)} (-1)^{l(H)} \mathtt{c}_H = 0.$$

We get a relation for each $(S,T) \in [I, \mathbf{2}]$, $F \in \Sigma^*[S]$, and $G \in \Sigma^*[T]$. The quotient of $\mathbf{\Gamma}^*$ by the relations $\mathtt{p}_F = 0$, for $F \notin \mathrm{L}^*[I]$, is the Lie cooperad $\mathbf{Lie}^*$, whose shuffle relations are well-known.

The cobracket d of $\mathbf{\Gamma}^*$, which is the dual of the bracket d^* of $\mathbf{\Gamma}$, is given in terms of the $\mathtt{p}$-basis (and also the $\mathtt{c}$-basis and $\mathtt{m}$-basis) by the cocommutator of deconcatenation,

$$d : \mathbf{\Gamma}^* \to \mathbf{\Gamma}^* \otimes_{\mathrm{Day}} \mathbf{\Gamma}^*, \qquad d_{[S,T]}(\mathtt{p}_F) := \mathtt{p}_{F|_S} \otimes \mathtt{p}_{F/\!\!/_T} - \mathtt{p}_{F|_T} \otimes \mathtt{p}_{F/\!\!/_S}.$$

3. Geometric Realizations

In this section, we give two geometric realizations of the Hopf algebra $\mathbf{\Sigma}^*$. First, we realize $\mathbf{\Sigma}^*$ as piecewise constant functions on the braid arrangement. Second, we realize $\mathbf{\Sigma}^*$ as certain functionals of piecewise constant functions on the adjoint braid arrangement. The quotients obtained by restricting these realizations to chambers are the commutative Hopf algebra of linear orders $\mathbf{L}^* \cong \mathbf{Com} \circ \mathbf{Lie}^*$ for the braid arrangement (this will be clear), and the Lie coalgebra $\mathbf{\Gamma}^* \cong \mathbf{Lie}^* \circ \mathbf{Com}$ for the adjoint braid arrangement (see Section 4).

3.1. Root Datum and Hyperplane Arrangements.

We begin by describing the braid arrangement and its adjoint, which we call the adjoint braid arrangement. These hyperplane arrangements are naturally associated to the root datum of $\mathrm{SL}_n(\mathbb{C})$, or dually $\mathrm{PGL}_n(\mathbb{C})$.

Let $\mathbb{R}^I$ be the real vector space of functions $\lambda : I \to \mathbb{R}$, and let $\mathbb{Z}^I \subset \mathbb{R}^I$ be the lattice of functions $\lambda : I \to \mathbb{Z}$. Let $\mathbb{R}I$ be the dual space of linear functionals $h : \mathbb{R}^I \to \mathbb{R}$, and let $\mathbb{Z}I \subset \mathbb{R}I$ be the dual lattice of linear functionals $h : \mathbb{Z}^I \to \mathbb{Z}$. For $S \subseteq I$, let $\lambda_S \in \mathbb{Z}^I$ be given by $\lambda_S(i) = 1$ if $i \in S$ and $\lambda_S(i) = 0$ if $i \notin S$. The dual pairing is then

$$\langle h, \lambda \rangle := \sum_{i \in I} h(\lambda_i)\, \lambda(i).$$

Let

$$\mathbf{T}I := \{h \in \mathbb{R}I : \langle h, \lambda_I \rangle = 0\} \qquad \text{and} \qquad Q^\vee I := \{h \in \mathbb{Z}I : \langle h, \lambda_I \rangle = 0\}.$$

The lattice $Q^\vee I$ is the *coweight lattice*. For $(i_1, i_2) \in [\mathbf{2}, I]$, the *coroot* $h_{i_1 i_2} \in Q^\vee I$ is given by $h_{i_1 i_2}(\lambda) := \lambda(i_1) - \lambda(i_2)$. The partial product on $[\mathbf{2}, I]$ encodes coroot addition, restricted to the case where the sum is again a coroot,

$$h_{i_1 i_2} + h_{i_3 i_4} = h_{(i_1,i_2) \circ (i_3,i_4)}.$$

Let $\mathbf{T}^I$ be the dual vector space of $\mathbf{T}I$, and let P^I be the dual lattice of $Q^\vee I$, thus

$$\mathbf{T}^I := \mathbb{R}^I / \mathbb{R}\lambda_I \qquad \text{and} \qquad P^I := \mathbb{Z}^I / \mathbb{Z}\lambda_I.$$

The lattice P^I is the *weight lattice*. For $(S, T) \in [I, \mathbf{2}]$, the *fundamental weight* $\lambda_{ST} \in P^I$ is the image of $\lambda_S \in \mathbb{R}^I$ in $\mathbf{T}^I$. The partial product on $[I, \mathbf{2}]$ encodes fundamental weight addition, restricted to the case where the sum is again a fundamental weight,

$$\lambda_{ST} + \lambda_{UV} = \lambda_{(S,T) \circ (U,V)}.$$

To see this, we have for example,

$T \supset U$	$S \supset V$	otherwise
$[1:1:1:0:0:0:0]$	$[1:1:1:1:1:0:0]$	$[1:1:1:0:0:0:0]$
$+[0:0:0:0:0:1:1]$	$+[0:1:1:1:1:1:1]$	$+[1:0:0:0:0:1:1]$
$=[1:1:1:0:0:1:1]$	$=[1:2:2:2:2:1:1]$	$=[2:1:1:0:0:1:1]$
	$=[0:1:1:1:1:0:0]$	$=[1:0:0:-1:-1:0:0]$

These are indeed homogeneous coordinates from the perspective of tropical geometry. A *reflection hyperplane* is a subspace of $\mathbf{T}^I$ which is the kernel of a coroot, and the collection of all reflection hyperplanes is called the *braid arrangement*. A *special hyperplane* is a subspace of $\mathbf{T}I$ which is the kernel of a fundamental weight, and the collection of all special hyperplanes is called the *adjoint braid arrangement*.

Notice that special hyperplanes are equivalently hyperplanes which are spanned by coroots, whereas fundamental weights can span hyperplanes which are not necessarily reflection hyperplanes.

3.2. **Realization over Braid Arrangement.**

The vector space of functions $\mathbb{R}^{\mathbf{T}^I}$ is an algebra with multiplication the pointwise product. For $(i_1, i_2) \in [\mathbf{2}, I]$, define the *halfspace* $\hat{\mathtt{C}}_{i_1 i_2} \in \mathbb{R}^{\mathbf{T}^I}$ by

$$\hat{\mathtt{C}}_{i_1 i_2} : \mathbf{T}^I \to \mathbb{R}, \qquad \hat{\mathtt{C}}_{i_1 i_2}(\lambda) := \begin{cases} 1 & \text{if } \langle h_{i_1 i_2}, \lambda \rangle \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{\mathbf{\Sigma}}^*[I]$ denote the subalgebra of $\mathbb{R}^{\mathbf{T}^I}$ which is generated by halfspaces, which defines the vector cospecies $\hat{\mathbf{\Sigma}}^*$. This makes each component $\hat{\mathbf{\Sigma}}^*[I]$ a polyhedral algebra in the sense of [BP99]. Monomials in the halfspaces are called *braid cones*. Define the function

$$\mathrm{O}[I] \to \hat{\mathbf{\Sigma}}^*[I], \qquad p \mapsto \hat{\mathtt{C}}_p := \prod^{\text{pointwise}}_{(i_1,i_2) \in p} \hat{\mathtt{C}}_{i_1 i_2}.$$

This is the celebrated one-to-one correspondence between preposets and braid cones [PRW08, Section 3]. Let the *braid signature* be the function

$$\mathbf{T}^I \to \mathrm{O}[I], \qquad \lambda \mapsto F_\lambda := \{(i_1, i_2) \in [\mathbf{2}, I] : \hat{\mathtt{C}}_{i_1 i_2}(\lambda) = 1\}.$$

For $F \in \Sigma^*[I]$, define the (relatively open) *face* of F to be the function given by

$$\hat{\mathtt{M}}_F : \mathbf{T}^I \to \mathbb{R}, \qquad \hat{\mathtt{M}}_F(\lambda) := \begin{cases} 1 & \text{if } F_\lambda = F \\ 0 & \text{otherwise.} \end{cases}$$

For all $F \in \Sigma^*[I]$, we have $\hat{\mathtt{M}}_F \neq 0$. Then the image of the braid signature is $\Sigma^*[I]$, and the image of the complement of the reflection hyperplanes is $\mathrm{L}^*[I]$. Thus, $F \mapsto \hat{\mathtt{M}}_F$ puts L^* in one-to-one correspondence with characteristic functions of connected components of the compliment of the reflection hyperplanes.

Proposition 3.1. We have

$$\hat{\mathtt{C}}_p = \sum_{F \leq p} \hat{\mathtt{M}}_F.$$

Proof. Let $\lambda \in \mathbf{T}^I$. We have

$$\hat{\mathtt{C}}_p(\lambda) = \prod_{(i_1,i_2)\in p}^{\text{pointwise}} \hat{\mathtt{C}}_{i_1 i_2}(\lambda) = 1 \quad \iff \quad \hat{\mathtt{C}}_{i_1 i_2}(\lambda) = 1, \text{ for all } (i_1, i_2) \in p \quad \iff \quad F_\lambda \leq p.$$

The support of $\hat{\mathtt{M}}_F$ is the preimage of F under the braid signature. Therefore λ is in the support of exactly one face, and so

$$F_\lambda \leq p \quad \iff \quad \sum_{F \leq p} \hat{\mathtt{M}}_F(\lambda) = 1.$$

Since $\hat{\mathtt{C}}_p$ and $\sum_{F \leq p} \hat{\mathtt{M}}_F$ take values 0 and 1 only, the result follows. □

The set $\{\hat{\mathtt{M}}_F : F \in \Sigma^*[I]\}$ spans $\hat{\mathbf{\Sigma}}^*[I]$ by Proposition 3.1, and is linearly independent because the faces $\hat{\mathtt{M}}_F$ are supported by disjoint sets. Therefore we have an isomorphism of cospecies, given by

$$\mathbf{\Sigma}^* \to \hat{\mathbf{\Sigma}}^*, \qquad \mathtt{M}_F \mapsto \hat{\mathtt{M}}_F \quad \text{or} \quad \mathtt{C}_p \mapsto \hat{\mathtt{C}}_p.$$

We let this isomorphism induce the structure of the commutative Hopf algebra of set compositions on $\hat{\mathbf{\Sigma}}^*$.

Remark 3.1. In the introduction, we mentioned that the realization $\hat{\mathbf{\Sigma}}^*$ can be extended to functions on the permutohedral compactification of $\mathbf{T}^I$. Then, the comultiplication of $\hat{\mathbf{\Sigma}}^*$ is induced by embedding facets of the permutohedron, and the multiplication of $\hat{\mathbf{\Sigma}}^*$ is induced by quotienting the permutohedron in the direction of fundamental weights.

A *conical space* is a module of the rig $(\mathbb{R}_{\geq 0}, +, \times)$, and an *open conical space* is a module of the rig $(\mathbb{R}_{>0}, +, \times)$. Let $\sigma_{i_1 i_2} \subset \mathbf{T}^I$ denote the support of the halfspace $\hat{\mathtt{C}}_{i_1 i_2}$. For $p \in \mathrm{O}[I]$, we have the conical space given by

$$\sigma_p := \{\lambda \in \mathbf{T}^I : \hat{\mathtt{C}}_p(\lambda) = 1\} = \{\lambda \in \mathbf{T}^I : F_\lambda \leq p\} = \bigcap_{(i_1,i_2)\in p} \sigma_{i_1 i_2}.$$

Let $\widetilde{\sigma}_p$ denote the open conical space which is the interior of σ_p relative to its support σ_{Q_p}, where Q_p is the partition with blocks the lumps of p. Then

$$\sigma_q \subseteq \sigma_p \iff q \leq p \qquad \text{and} \qquad \widetilde{\sigma}_q \subseteq \widetilde{\sigma}_p \iff q \preceq p.$$

Thus, the images of $\mathtt{M}_p$ and $\mathtt{C}_p$ in $\hat{\boldsymbol{\Sigma}}^*$ are the characteristic functions of $\widetilde{\sigma}_p$ and σ_p respectively. Notice also that $\sigma_p \cap \sigma_q = \sigma_{p\cup q}$. Therefore the polyhedral algebraic structure of $\hat{\boldsymbol{\Sigma}}^*$ may be given by

$$\hat{\boldsymbol{\Sigma}}^* \times \hat{\boldsymbol{\Sigma}}^* \to \hat{\boldsymbol{\Sigma}}^*, \qquad \hat{\mathtt{C}}_p \otimes \hat{\mathtt{C}}_q \mapsto \hat{\mathtt{C}}_p \cdot \hat{\mathtt{C}}_q := \hat{\mathtt{C}}_{p\cup q}.$$

Here, $\times$ denotes the Hadamard tensor product of cospecies.

Proposition 3.2. The conical space σ_p is generated by the fundamental weights $\{\lambda_{ST} : (S,T) \leq p\}$. Equivalently, σ_p is given by the following Minkowski sum,

$$\sigma_p = \bigcup_{(S,T)\leq p}^{\text{Mink}} \sigma_{(S,T)}.$$

Proof. Recall that $\lambda \in \sigma_p \iff F_\lambda \leq p$. If $(S,T) \leq p$, then S is an upward closed subset of I with respect to $\leq_p$. Therefore, if λ is in the conical span of $\{\lambda_{ST} : (S,T) \leq p\}$, we have $F_\lambda \leq p$. Conversely, suppose that $F_\lambda \leq p$. In particular, $\lambda \in \sigma_{F_\lambda}$. But σ_{F_λ} is easily seen to be the conical space generated by $\{\lambda_{ST} : (S,T) \leq F_\lambda\}$. Since $F_\lambda \leq p$, we have $(S,T) \leq F_\lambda \implies (S,T) \leq p$, and so λ is also in the conical span of $\{\lambda_{ST} : (S,T) \leq p\}$. □

3.3. **Adjoint Braid Arrangement.** The vector space of functions $\mathbb{R}^{\mathbf{T}I}$ is an algebra with multiplication the pointwise product. For $(S,T) \in [I,\mathbf{2}]$, define the *halfspace* $\mathtt{Y}_{ST} \in \mathbb{R}^{\mathbf{T}I}$ by

$$\mathtt{Y}_{ST} : \mathbf{T}I \to \mathbb{R}, \qquad \mathtt{Y}_{ST}(h) := \begin{cases} 1 & \text{if } \langle h, \lambda_{ST}\rangle \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\check{\boldsymbol{\Sigma}}^\vee[I]$ denote the subalgebra of $\mathbb{R}^{\mathbf{T}I}$ which is generated by halfspaces, which defines the vector species $\check{\boldsymbol{\Sigma}}^\vee$. This makes each component $\check{\boldsymbol{\Sigma}}^\vee[I]$ a polyhedral algebra in the sense of [BP99]. Let an *adjoint cone* be a monomial in the halfspaces. Define the function

$$\mathrm{O}^\vee[I] \to \check{\boldsymbol{\Sigma}}^\vee[I], \qquad \tau \mapsto \mathtt{Y}_\tau := \prod_{(S,T)\in\tau}^{\text{pointwise}} \mathtt{Y}_{ST}.$$

Let the *adjoint signature* be the function

$$\mathbf{T}I \to \mathrm{O}^\vee[I], \qquad h \mapsto \mathcal{S}_h := \{(S,T) \in [I,\mathbf{2}] : \mathtt{Y}_{ST}(h) = 1\}.$$

Notice that the image of the adjoint signature is contained in $\Sigma^\vee[I]$, and a point $h \in \mathbf{T}I$ does not lie on a special hyperplane if and only if $\mathcal{S}_h \in \mathrm{L}^\vee[I]$. For $\mathcal{S} \in \Sigma^\vee[I]$, defined the *shard* (adjoint face) of $\mathcal{S}$ to be the function given by

$$\mathtt{X}_{\mathcal{S}} : \mathbf{T}I \to \mathbb{R}, \qquad \mathtt{X}_{\mathcal{S}}(h) := \begin{cases} 1 & \text{if } \mathcal{S}_h = \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma_{ST} \subset \mathbf{T}I$ denote the support of the halfspace $\mathtt{Y}_{ST}$. For $\tau \in \mathrm{O}^\vee[I]$, we have the conical space given by

$$\sigma_\tau := \{h \in \mathbf{T}I : \mathtt{Y}_\tau(h) = 1\} = \{h \in \mathbf{T}I : \mathcal{S}_h \leq \tau\} = \bigcap_{(S,T)\in\tau} \sigma_{ST}.$$

Let $\widetilde{\sigma}_\tau$ denote the open conical space which is the interior of σ_τ relative to its support. Then

$$\sigma_\upsilon \subseteq \sigma_\tau \iff \upsilon \leq \tau \qquad \text{and} \qquad \widetilde{\sigma}_\upsilon \subseteq \widetilde{\sigma}_\tau \iff \upsilon \preceq \tau.$$

For $\tau \in \mathrm{O}^\vee[I]$, let $\sigma^\vee_\tau$ denote the dual conical space of σ_τ, given by

$$\sigma^\vee_\tau := \big\{\lambda \in \mathbf{T}^I : \langle h, \lambda\rangle \geq 0,\ \text{for all } h \in \sigma_\tau\big\}.$$

For $p \in \mathrm{O}[I]$, let $\sigma^\vee_p$ denote the dual conical space of σ_p, given by

$$\sigma^\vee_p := \big\{h \in \mathbf{T}I : \langle h, \lambda\rangle \geq 0,\ \text{for all } \lambda \in \sigma_p\big\}.$$

The conical spaces $\sigma^\vee_F$ are tangent cones to permutohedra, and the conical spaces $\sigma^\vee_p$ are tangent cones to generalized permutohedra. We have

$$\sigma^\vee_{\tau_p} = \sigma_p \qquad \text{or equivalently} \qquad \sigma^\vee_p = \sigma_{\tau_p}$$

by Proposition 3.2, and the fact that conical space duality interchanges intersections with Minkowski sums. Then the conical space σ_{τ_p} is generated by the coroots $\{h_{i_1 i_2} : (i_1, i_2) \in p\}$, or equivalently

$$\sigma_{\tau_p} = \bigcup^{\text{Mink}}_{(i_1,i_2)\in p} \sigma^\vee_{i_1 i_2}.$$

Proposition 3.3. The function $\tau \mapsto \mathtt{Y}_\tau$ is a one-to-one correspondence between adjoint cones on $\mathbf{T}I$ and transitive families of I.

Proof. Let $\mathtt{Y} \in \check{\boldsymbol{\Sigma}}^\vee[I]$ be an adjoint cone. By definition, there exists a subset $\tau' \subseteq [I, \mathbf{2}]$ such that

$$\mathtt{Y} = \prod^{\text{pointwise}}_{(S,T)\in\tau'} \mathtt{Y}_{ST}.$$

Let

$$\tau(\mathtt{Y}) = \big\{(S,T) \in [I, \mathbf{2}] : \mathtt{Y}_{ST} \cdot \mathtt{Y} = \mathtt{Y}\big\}.$$

Then $\tau(\mathtt{Y}) \supseteq \mathrm{TC}(\tau')$, and

$$\mathtt{Y}_{\tau(\mathtt{Y})} = \mathtt{Y}.$$

Then $\sigma^\vee_{\tau(\mathtt{Y})}$ is the conical space which is generated by $\{\lambda_{ST} : (S,T) \in \tau(\mathtt{Y})\}$ on the one hand, and $\{\lambda_{ST} : (S,T) \in \tau'\}$ on the other. Therefore $(S,T) \in \tau(\mathtt{Y}) \implies (S,T) \in \mathrm{TC}(\tau')$, and so $\tau(\mathtt{Y}) = \mathrm{TC}(\tau')$. For $\tau \in \mathrm{O}^\vee[I]$, we have

$$\mathtt{Y} = \prod^{\text{pointwise}}_{(S,T)\in\tau} \mathtt{Y}_{ST} \qquad \implies \qquad \tau(\mathtt{Y}) = \mathrm{TC}(\tau) = \tau.$$

Therefore $\mathtt{Y} \mapsto \tau(\mathtt{Y})$ is the inverse of $\tau \mapsto \mathtt{Y}_\tau$. $\square$

Corollary 3.3.1. For all $\mathcal{S} \in \Sigma^\vee[I]$, we have $\mathtt{X}_\mathcal{S} \neq 0$.

Proof. The support of $\mathtt{X}_\mathcal{S}$ is $\widetilde{\sigma}_\mathcal{S}$, where $\sigma_\mathcal{S}$ is a top dimensional conical space because $\mathcal{S}$ has no symmetric elements. Therefore $\widetilde{\sigma}_\mathcal{S}$ is nonempty. $\square$

It follows that the image of the adjoint signature map is $\Sigma^\vee[I]$, and the image of the complement of the special hyperplanes is $\mathrm{L}^\vee[I]$. Therefore $\mathcal{S} \mapsto \mathtt{X}_\mathcal{S}$ puts $\mathrm{L}^\vee[I]$ in one-to-one correspondence with characteristic functions of connected components of the compliment of the special hyperplanes.

Proposition 3.4. We have

$$\mathtt{Y}_\tau = \sum_{\mathcal{S}\leq\tau} \mathcal{S}_h.$$

Proof. Let $h \in \mathbf{T}I$. We have

$$\mathtt{Y}_\tau(h) = \overset{\text{pointwise}}{\prod_{(S,T)\in\tau}} \mathtt{Y}_{ST}(h) = 1 \quad \iff \quad \mathtt{Y}_{ST}(h) = 1, \text{ for all } (S,T) \in \tau \quad \iff \quad \mathcal{S}_h \leq \tau.$$

The support of $\mathtt{X}_\mathcal{S}$ is the preimage of $\mathcal{S}$ under the adjoint signature. Therefore h is in the support of exactly one shard, and so

$$\mathcal{S}_h \leq \tau \quad \iff \quad \sum_{\mathcal{S}\leq\tau} \mathtt{X}_\mathcal{S}(h) = 1.$$

Since $\mathtt{Y}_\tau$ and $\sum_{\mathcal{S}\leq\tau} \mathtt{X}_\mathcal{S}$ take values 0 and 1, the result follows. □

The set $\{\mathtt{X}_\mathcal{S} : \mathcal{S} \in \Sigma^\vee[I]\}$ spans $\check{\boldsymbol{\Sigma}}^\vee[I]$ by Proposition 3.4, and is linearly independent because the shards $\mathtt{X}_\mathcal{S}$ are supported by disjoint sets. We give each component $\check{\boldsymbol{\Sigma}}^\vee[I]$ the structure of a real Hilbert space by letting shards be an orthonormal basis. If $\mathtt{Y} \in \check{\boldsymbol{\Sigma}}^\vee[I]$ is the characteristic function of a region $X \subset \mathbf{T}I$, let the *characteristic functional* of X be the Riesz representation of $\mathtt{Y}$.

3.4. **Realization over Adjoint Braid Arrangement.** For $p \in \mathrm{O}[I]$, let the *coroot cone* $\check{\mathtt{C}}_p$ be the characteristic functional of $\sigma_p^\vee$, thus

$$\check{\mathtt{C}}_p : \check{\boldsymbol{\Sigma}}^\vee[I] \to \mathbb{R}, \qquad \check{\mathtt{C}}_p(\mathtt{X}_\mathcal{S}) := \begin{cases} 1 & \text{if } \mathcal{S} \leq \tau_p \\ 0 & \text{otherwise.} \end{cases}$$

Let $\check{\boldsymbol{\Sigma}}^*[I]$ denote the span of $\{\check{\mathtt{C}}_p : p \in O[I]\}$ in linear functionals on shards $\mathrm{Hom}(\check{\boldsymbol{\Sigma}}^\vee[I], \mathbb{R})$. This defines the cospecies $\check{\boldsymbol{\Sigma}}^*$.

Proposition 3.5. We have a cospecies isomorphism, given by

$$\mathcal{D} : \hat{\boldsymbol{\Sigma}}^* \to \check{\boldsymbol{\Sigma}}^*, \qquad \mathcal{D}(\hat{\mathtt{C}}_p) := \check{\mathtt{C}}_p.$$

Proof. Since $\hat{\mathtt{C}}_p$ is the characteristic function of σ_p, and the Riesz representation of $\check{\mathtt{C}}_p$ is the characteristic function of $\sigma_p^\vee$, each $\check{\boldsymbol{\Sigma}}^*[I]$ is the dual of the polyhedral algebra $\hat{\boldsymbol{\Sigma}}^*[I]$ in the sense of [BP99, Theorem 2.7], and $\mathcal{D}$ is the duality map. □

We let $\mathcal{D}$ induce the structure of the commutative Hopf algebra of compositions on $\check{\boldsymbol{\Sigma}}^*$. The image of the pointwise product in $\check{\boldsymbol{\Sigma}}^*$ is called *convolution*,

$$\check{\boldsymbol{\Sigma}}^* \times \check{\boldsymbol{\Sigma}}^* \to \check{\boldsymbol{\Sigma}}^*, \qquad \check{\mathtt{C}}_p \otimes \check{\mathtt{C}}_q \mapsto \check{\mathtt{C}}_p \star \check{\mathtt{C}}_q := \check{\mathtt{C}}_{p\cup q}.$$

The restriction of convolution to coroot cones is Minkowski sum.

For $F \in \Sigma^*[I]$, let $\check{\mathtt{M}}_F \in \mathrm{Hom}(\check{\boldsymbol{\Sigma}}^\vee[I], \mathbb{R})$ be given by

$$\check{\mathtt{M}}_F(\mathtt{X}_\mathcal{S}) := \begin{cases} (-1)^{l(F)-1} & \text{if } \mathcal{S} \preceq \bar{\tau}_F \\ 0 & \text{otherwise.} \end{cases}$$

Equivalently, $\check{\mathtt{M}}_F$ is the characteristic functional of the relative interior of the permutohedral cone $\sigma_{\bar{F}}^\vee$, with sign $(-1)^{l(F)-1}$.

Theorem 3.6. For $F \in \Sigma^*[I]$, we have

$$\mathcal{D}(\hat{\mathtt{M}}_F) = \check{\mathtt{M}}_F.$$

Proof. For $F \in \Sigma^*[I]$, we have

$$\mathcal{D}(\hat{\mathtt{M}}_F) = \mathcal{D}\Bigg(\overset{\text{pointwise}}{\prod_{(i_1,i_2)\in F}} (\hat{\mathtt{C}}_I - \hat{\mathtt{C}}_{i_2 i_1}) \Bigg) = \overset{\text{convol}}{\prod_{(i_1,i_2)\in F}} \mathcal{D}(\hat{\mathtt{C}}_I - \hat{\mathtt{C}}_{i_2 i_1}) = \overset{\text{convol}}{\prod_{(i_1,i_2)\in \bar{F}}} (\check{\mathtt{C}}_I - \check{\mathtt{C}}_{i_1 i_2}) = \check{\mathtt{M}}_F.$$

The final equality follows by multiplying out the convolution product, and then inclusion-exclusion of faces of the permutohedral cone $\sigma^\vee_F$. □

Therefore, the isomorphism $\mathcal{D} : \hat{\mathbf{\Sigma}}^* \to \check{\mathbf{\Sigma}}^*$ is also given by $\hat{\mathtt{M}}_F \mapsto \check{\mathtt{M}}_F$. We let $\check{\mathtt{P}}_F$ denote the image of $\mathtt{P}_F$ in $\check{\mathbf{\Sigma}}^*$.

4. The Indecomposable Quotient

In this section, we show that the indecomposable quotient of the adjoint realization of $\mathbf{\Sigma}^*$ is the restriction to chambers. Moreover, we show that the resulting Lie coalgebra consists of functionals which satisfy the Steinmann relations, with cobracket the discrete differentiation of functionals across hyperplanes.

4.1. Permutohedral Cones and the Steinmann Relations.

Let $\check{\mathbf{\Gamma}}^*$ be the quotient cospecies of $\check{\mathbf{\Sigma}}^*$ which is obtained by restricting functionals to top dimensional shards, thus

$$\check{\mathbf{\Gamma}}^*[I] := \check{\mathbf{\Sigma}}^*[I] \Big/ \big\langle f \in \check{\mathbf{\Sigma}}^*[I] : f(\mathtt{X}_{\mathcal{S}}) = 0 \text{ for all } \mathcal{S} \in \mathrm{L}^\vee[I] \big\rangle .$$

We denote the corresponding quotient map by

$$\check{\mathcal{U}}^* : \check{\mathbf{\Sigma}}^* \twoheadrightarrow \check{\mathbf{\Gamma}}^*.$$

Define the following functionals on top dimensional shards,

$$\check{\mathtt{p}}_F := \check{\mathcal{U}}^*(\check{\mathtt{P}}_F), \qquad \check{\mathtt{c}}_p := \check{\mathcal{U}}^*(\check{\mathtt{C}}_p), \qquad \check{\mathtt{m}}_F := \check{\mathcal{U}}^*(\check{\mathtt{M}}_F).$$

In particular, the functionals $\check{\mathtt{c}}_p$ are characteristic functionals of generalized permutohedral tangent cones, taken modulo higher codimensions. Therefore we may characterize the subspace $\check{\mathbf{\Gamma}}^*[I] \subset \mathrm{Hom}(\mathbf{L}^\vee[I], \mathbb{R})$ as the span of characteristic functionals of (generalized) permutohedral tangent cones.

Remark 4.1. The adjoint analog of $\check{\mathbf{\Gamma}}^*$, i.e. the quotient of $\check{\mathbf{\Sigma}}^*$ obtained by restricting functions to Weyl chambers, is the cospecies $\hat{\mathbf{L}}^*$ defined by

$$\hat{\mathbf{L}}^*[I] := \hat{\mathbf{\Sigma}}^*[I] \Big/ \big\langle \mathtt{M}_F : F \notin \mathrm{L}^*[I] \big\rangle .$$

This is a geometric realization of the commutative Hopf algebra of linear orders $\mathbf{L}^*$. Since $\mathbf{L}^*$ is the coenveloping algebra of the Lie cooperad $\mathbf{Lie}^*$, we have a natural isomorphism $\mathbf{L}^* \cong \mathbf{E} \circ \mathbf{Lie}^*$. Therefore $\mathbf{L}^*$ is a right $\mathbf{Lie}^*$-comodule. If we look at what the $\mathbf{Lie}^*$-coaction should be for the realization $\hat{\mathbf{L}}^*$, we see that it is discrete differentiation of functions across reflection hyperplanes. This is geometrically the same as the Lie structure we shall give $\check{\mathbf{\Gamma}}^*$.

Let a *Steinmann functional* be a real valued function on top dimensional shards which satisfies the Steinmann relations. See [LNO19, Section 5.2] for the definition of the Steinmann relations. In [LNO19], it was shown that restricting to Steinmann functionals is necessary and sufficient for the discrete differentiation of functionals across hyperplanes [LNO19, Definition 3.3] to be a Lie cobracket. We now show that $\check{\mathbf{\Gamma}}^*$ coincides with Steinmann functionals. In the following, see [LNO19, Definition 2.6] for the definition of $\mu_{(S|T)}$.

Lemma 4.1. For $F \in \Sigma^*[I]$ and $(S,T) \in [I,\mathbf{2}]$, the discrete derivative $\partial_{[S,T]}\, \check{\mathtt{c}}_F$ of the functional $\check{\mathtt{c}}_F$ across the special hyperplane $\sigma^\vee_{(S|T)}$ is given by

$$\partial_{[S,T]}\, \check{\mathtt{c}}_F = \mu_{(S|T)}\big(\check{\mathtt{c}}_{F\rrbracket_S} \otimes \check{\mathtt{c}}_{F\,/\!\!/_T} - \check{\mathtt{c}}_{F\rrbracket_T} \otimes \check{\mathtt{c}}_{F\,/\!\!/_S}\big).$$

Proof. Let $\mathtt{X}$ be a codimension one shard which is supported by $\sigma^\vee_{(S|T)}$. Let $\mathtt{X}^{[S,T]}$, respectively $\mathtt{X}^{[T,S]}$, be the top dimensional shard with facet $\mathtt{X}$ such that $\mathtt{Y}_{ST} \cdot \mathtt{X}^{[S,T]} = \mathtt{X}^{[S,T]}$, respectively $\mathtt{Y}_{ST} \cdot \mathtt{X}^{[T,S]} = 0$. Assume $(S,T) \leq F$. In this case, we need to show that

$$\partial_{[S,T]}\, \check{\mathtt{c}}_F = \mu_{(S|T)}(\check{\mathtt{c}}_{F|_S} \otimes \check{\mathtt{c}}_{F|_T}).$$

By the definition of the derivative, we have

$$\partial_{[S,T]}\check{\mathtt{c}}_F(\mathtt{X}) = \check{\mathtt{c}}_F(\mathtt{X}^{[S,T]}) - \check{\mathtt{c}}_F(\mathtt{X}^{[T,S]}).$$

However, since $(S,T) \leq F$, we have $\check{\mathtt{c}}_F(\mathtt{X}^{[T,S]}) = 0$, so that

$$\partial_{[S,T]}\check{\mathtt{c}}_F(\mathtt{X}) = \check{\mathtt{c}}_F(\mathtt{X}^{[S,T]}).$$

Then, directly from the definitions, we see that

$$\mu_{(S|T)}(\check{\mathtt{c}}_{F|_S} \otimes \check{\mathtt{c}}_{F|_T})(\mathtt{X}) = 1 \qquad \iff \qquad \check{\mathtt{c}}_F(\mathtt{X}^{[S,T]}) = 1.$$

Since these functionals take values 0 or 1, the result follows. The case $(T,S) \leq F$ then follows by antisymmetry of the derivative. Finally, if S is not an initial or terminal interval of F, then

$$\mu_{(S|T)}\big(\check{\mathtt{c}}_{F\rrbracket_S} \otimes \check{\mathtt{c}}_{F\,/\!\!/_T} - \check{\mathtt{c}}_{F\rrbracket_T} \otimes \check{\mathtt{c}}_{F\,/\!\!/_S}\big) = \mu_{(S|T)}(0-0) = 0.$$

Also, in this case, we have

$$\check{\mathtt{c}}_F(\mathtt{X}^{[S,T]}) = \check{\mathtt{c}}_F(\mathtt{X}^{[T,S]}),$$

and so $\partial_{[S,T]}\, \check{\mathtt{c}}_F = 0$ as required. $\square$

Lemma 4.2. The functionals $\{\check{\mathtt{c}}_F : F \in \Sigma^*[I]\}$ satisfy the Steinmann relations. More generally, characteristic functionals of generalized permutohedral tangent cones $\check{\mathtt{c}}_p$ satisfy the Steinmann relations.

Proof. In Lemma 4.1, we showed that the derivative of $\check{\mathtt{c}}_F$ decomposes as a tensor product. The result then follows by [LNO19, Theorem 5.3]. $\square$

Theorem 4.3. The cospecies $\check{\mathbf{\Gamma}}^*$ coincides with the cospecies of Steinmann functionals. Thus, a functional on top dimensional shards satisfies the Steinmann relations if and only if it is a linear combination of characteristic functionals of permutohedral tangent cones.

Proof. Let $n = |I|$. For $m \in \mathbb{N}$, $m \leq n$, let f_m denote a Steinmann functional such that $\partial_{[F]} f_m = 0$ for all $F \in \Sigma^*_{i_0}[I]$ with $l(F) > m$. If $l(F) = m$, then $\partial_{[F]} f_m$ is a constant function. Let us denote its valued by $\upsilon(\partial_{[F]} f_m)$. Define

$$f_{m-1} = f_m - \sum_{F \in \Sigma^*_{i_0}[I]\,:\, l(F)=m} \upsilon(\partial_{[F]} f_m)\, \check{\mathtt{c}}_F$$

Then, for $F \in \Sigma^*_{i_0}[I]$ with $l(F) > m-1$, we have

$$\partial_{[F]} f_{m-1} = \partial_{[F]} f_m - \sum_{G \in \Sigma^*_{i_0}[I]\,:\, l(G)=m} \upsilon(\partial_{[G]} f_m)\, \partial_{[F]}\check{\mathtt{c}}_G = 0.$$

Therefore we can systematically perturb a Steinmann functional by the functionals $\{\check{\mathtt{c}}_F : F \in \Sigma^*[I]\}$ to obtain the zero functional. $\square$

Let us equip the cospecies $\check{\mathbf{\Gamma}}^*$ with the cobracket d of discrete differentiation across hyperplanes, so that it is now exactly the Lie coalgebra in [LNO19],

$$d : \check{\mathbf{\Gamma}}^* \to \check{\mathbf{\Gamma}}^* \otimes_{\mathrm{Day}} \check{\mathbf{\Gamma}}^*, \qquad d_{[S,T]}\check{\mathtt{c}}_F := \check{\mathtt{c}}_{F \rrbracket_S} \otimes \check{\mathtt{c}}_{F\,/\!\!/_T} - \check{\mathtt{c}}_{F\rrbracket_T} \otimes \check{\mathtt{c}}_{F\,/\!\!/_S}.$$

4.2. **An Isomorphism of Lie Coalgebras.** If $p \in O[I]$ has at least two blocks, then $\check{\mathtt{C}}_p(\mathtt{X}_{\mathcal{S}}) = 0$ for all $\mathcal{S} \in \mathrm{L}^\vee[I]$, and so $\check{\mathtt{c}}_p = 0$. Therefore the Lie coalgebra $\check{\mathbf{\Gamma}}^*$ is a quotient of the indecomposable quotient of $\check{\mathbf{\Sigma}}^*$. We now show that this quotient is an isomorphism.

Lemma 4.4. The set of functionals

$$\big\{ \check{\mathtt{c}}_F : F \in \Sigma^*_{i_0}[I] \big\}$$

is linearly independent.

Proof. We prove by induction on $n = |I|$. Suppose we have coefficients $a_F \in \mathbb{R}$ such that

$$\sum_{F \in \Sigma^*_{i_0}[I]} a_F \, \check{\mathtt{c}}_F = 0.$$

Let $(S,T) \in (I, \mathbf{2})$ with $i_0 \in S$. We have

$$\sum_{F \in \Sigma^*_{i_0}[I]} a_F \left(d_{[S,T]} \, \check{\mathtt{c}}_F \right) = d_{[S,T]} \Bigg(\sum_{F \in \Sigma^*_{i_0}[I]} a_F \, \check{\mathtt{c}}_F \Bigg) = d_{[S,T]} \, 0 = 0.$$

Let $m = |S|$. If $m = 1$, then each term $d_{[S,T]} \, \check{\mathtt{c}}_F \neq 0$ will be of the form $\check{\mathtt{c}}_{(i_0)} \otimes \check{\mathtt{c}}_G$, for some $G \in \Sigma^*[T]$. Then, since $|T| = n - 1 < n$, by the induction hypothesis we have $a_F = 0$ for all $F \in \Sigma^*_{i_0}[I]$ with a first lump of size one. We now do induction on m, and assume that $a_F = 0$ for all $F \in \Sigma^*_{i_0}[I]$ with a first lump of size less than m. By the induction hypothesis on m, each $d_{[S,T]} \, \check{\mathtt{c}}_F \neq 0$ such that $a_F \neq 0$ will be of the form $\check{\mathtt{c}}_{(S)} \otimes \check{\mathtt{c}}_G$, for some $G \in \Sigma^*[T]$. Therefore $a_F = 0$ for F with a first lump of size m by the induction hypothesis on n. $\square$

Lemma 4.5. The set of functionals

$$\big\{ \check{\mathtt{c}}_F : F \in \Sigma^*_{i_0}[I] \big\}$$

is a basis of $\check{\mathbf{\Gamma}}^*[I]$.

Proof. This set of functionals is linearly independent by Lemma 4.4. It spans $\check{\mathbf{\Gamma}}^*[I]$ because it is the image of the $\mathtt{c}$-basis under the quotient $\mathbf{\Gamma}^* \twoheadrightarrow \check{\mathbf{\Gamma}}^*$. $\square$

Theorem 4.6. The quotient

$$\mathbf{\Gamma}^* \twoheadrightarrow \check{\mathbf{\Gamma}}^*, \qquad \mathtt{p}_F \mapsto \check{\mathtt{p}}_F \quad \text{or} \quad \mathtt{m}_F \mapsto \check{\mathtt{m}}_F \quad \text{or} \quad \mathtt{c}_p \mapsto \check{\mathtt{c}}_p$$

is an isomorphism of Lie coalgebras.

Proof. This map is an isomorphism at the level of cospecies by Lemma 4.5. It preserves the cobracket by Lemma 4.1. $\square$

Thus, we have shown that Steinmann functionals, equivalently the span of permutohedral cones modulo higher codimensions, equipped with the discrete differentiation of functionals across hyperplanes, is the indecomposable quotient of the adjoint realization of $\mathbf{\Sigma}^*$. A consequence of this is the following. For a partition $P \in \Pi^*[I]$, let $\check{\mathbf{\Sigma}}^*_P[I]$ denote the subspace of $\check{\mathbf{\Sigma}}^*[I]$ which consists of those functionals that are supported by the semisimple flat $\sigma^\vee_P$,

$$\check{\mathbf{\Sigma}}^*_P[I] := \big\{ f \in \check{\mathbf{\Sigma}}^*[I] : f(\mathtt{X}_{\mathcal{S}}) \neq 0 \implies \mathtt{X}_{\mathcal{S}} \subset \sigma^\vee_P \big\}.$$

Since the Hopf structure of $\check{\mathbf{\Sigma}}^*$ was induced by the identification $\mathtt{C}_p \mapsto \check{\mathtt{C}}_p$ with $\mathbf{\Sigma}^*$, the higher multiplication of $\check{\mathbf{\Sigma}}^*$ is given by

$$\Delta_P : \check{\mathbf{\Sigma}}^*(P) \hookrightarrow \check{\mathbf{\Sigma}}^*[I], \qquad \check{\mathtt{C}}_{p_1} \otimes \cdots \otimes \check{\mathtt{C}}_{p_k} \mapsto \check{\mathtt{C}}_{(p_1|\dots|p_k)}.$$

Then, since $\check{\mathbf{\Gamma}}^*$ was the indecomposable quotient of $\check{\mathbf{\Sigma}}^*$, this must be injective with image $\check{\mathbf{\Sigma}}^*_P[I]$,

$$\check{\mathbf{\Sigma}}^*_P[I] \cong \check{\mathbf{\Sigma}}^*(P).$$

In this sense, the higher multiplication of the adjoint realization of $\mathbf{\Sigma}^*$ is simply the embedding of semisimple flats. For the Lie coalgebra $\mathbf{\Gamma}^*$, one then quotients out the images of all these embeddings, leaving just the chambers.

4.3. **Bring to Basis for Steinmann Functionals.** In this section, we evaluate derivatives of functionals at the first Eulerian idempotent in order to bring Steinmann functionals to the $\mathtt{p}$-basis. This will work in the same way as Taylor series expansions.

Let $\check{\mathbf{\Sigma}}$ be the dual species of $\check{\mathbf{\Sigma}}^*$. This is naturally a quotient of the linearization of shards,

$$\check{\mathbf{\Sigma}} = \mathbf{\Sigma}^\vee /\!\sim\ .$$

Let $\check{\mathbf{\Gamma}}$ denote the Lie algebra which is dual to $\check{\mathbf{\Gamma}}^*$. This is naturally the subspecies of $\check{\mathbf{\Sigma}}$ which is spanned by top dimensional shards. The underlying species of $\check{\mathbf{\Gamma}}$ is given by

$$\check{\mathbf{\Gamma}} = \mathbf{L}^\vee /\mathbf{Stein}\ .$$

See [LNO19, Section 5.2] for the definition of **Stein**. The *first Eulerian idempotent* $\mathtt{E}_I \in \mathbf{\Sigma}[I]$ is defined by $\mathtt{E}_\emptyset = 0$, and for I nonempty,

$$\mathtt{E}_I := \mathtt{Q}_{(I)} = \sum_{F \in \Sigma[I]} (-1)^{l(F)-1} \frac{1}{l(F)} \mathtt{H}_F.$$

See [AM13, Section 14.1]. The first Eulerian idempotent is a primitive series. Its image in the adjoint realization $\check{\mathbf{\Gamma}}$ is then as follows. For I nonempty, let $\check{\mathtt{E}}_I \in \check{\mathbf{\Gamma}}[I]$ such that for all $F \in \Sigma^*[I]$, we have

$$\check{\mathtt{p}}_F(\check{\mathtt{E}}_I) := \begin{cases} 1 & \text{if } F = (I) \\ 0 & \text{otherwise.} \end{cases}$$

To define $\check{\mathtt{E}}_I$, it is enough to consider just the basis elements $\{\check{\mathtt{p}}_F : \Sigma^*_{i_0}[I]\}$. The definition we gave is then satisfied because the $\mathtt{p}$-basis elements satisfy shuffle relations. This defines the series

$$\mathbf{E} \hookrightarrow \check{\mathbf{\Gamma}}, \qquad \{I\} \mapsto \check{\mathtt{E}}_I.$$

In order to obtain the explicit isomorphism between $\mathbf{\Gamma}$ and $\check{\mathbf{\Gamma}}$, we should now act on $\check{\mathtt{E}}_I$ with the antiderivative d^*_Λ of [LNO19], giving

$$\mathbf{\Gamma} \to \check{\mathbf{\Gamma}}, \qquad \mathtt{Q}_\Lambda \mapsto \check{\mathtt{Q}}_\Lambda := d^*_\Lambda(\check{\mathtt{E}}_{S_1} \otimes \cdots \otimes \check{\mathtt{E}}_{S_k}).$$

We have computed $\check{\mathtt{E}}_I$ for $n \leq 4$, see Figure 1. In these cases, $\check{\mathtt{E}}_I$ may be presented as a sum of $n!$ shards with coefficients $1/n!$.

Theorem 4.7. Given a Steinmann functional $f \in \check{\mathbf{\Gamma}}^*[I]$, we have

$$f = \sum_{F \in \Sigma_{i_0}[I]} d_{[F]} f(\check{\mathtt{E}}_{S_1} \otimes \cdots \otimes \check{\mathtt{E}}_{S_k})\ \check{\mathtt{p}}_F.$$

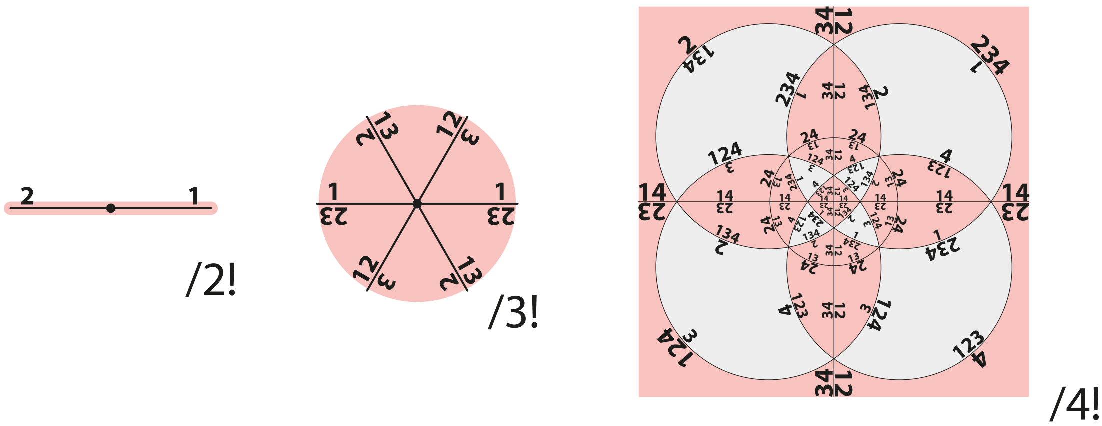

FIGURE 1. The adjoint realization of the first Eulerian idempotent $\mathtt{E}_I$ for $n \leq 4$. The picture for $n=4$ is a stereographic projection of the Steinmann planet, in which case the Steinmann relations are nontrivial, and so the presentation is not unique. One can check in these pictures that the evaluation of the $\check{\mathtt{M}}$-basis on $\check{\mathtt{E}}_I$ is correct.

Proof. Let $a_G \in \mathbb{R}$ be the coefficients in the expansion of f in the $\check{\mathtt{p}}$-basis,

$$f = \sum_{G \in \Sigma^*_{i_0}[I]} a_G \, \check{\mathtt{p}}_G .$$

For $F = (S_1, \dots, S_k) \in \Sigma_{i_0}[I]$, we have

$$d_{[F]} f = \sum_{G \in \Sigma^*_{i_0}[I]} a_G \, d_{[F]} \check{\mathtt{p}}_G .$$

Then

$$d_{[F]} \check{\mathtt{p}}_G = \begin{cases} \check{\mathtt{p}}_{G|_{S_1}} \otimes \dots \otimes \check{\mathtt{p}}_{G|_{S_k}} & \text{if } F \leq G \\ 0 \in \check{\mathbf{\Gamma}}^*(Q_F) & \text{otherwise.} \end{cases}$$

Therefore, since the first Eulerian idempotent is equal to $\mathtt{Q}_{(I)}$ for I nonempty, and the $\mathtt{Q}$-basis is dual to the $\mathtt{P}$-basis, we have

$$d_{[F]} \check{\mathtt{p}}_G (\check{\mathtt{E}}_{S_1} \otimes \dots \otimes \check{\mathtt{E}}_{S_k}) = \delta_{FG} .$$

Thus,

$$d_{[F]} f (\check{\mathtt{E}}_{S_1} \otimes \dots \otimes \check{\mathtt{E}}_{S_k}) = a_F . \qquad \square$$

5. Axiomatic QFT

Let $\check{\mathcal{U}} : \check{\mathbf{\Gamma}} \hookrightarrow \mathbf{\Sigma}$ denote the dual map of the composite

$$\mathbf{\Sigma}^* \xrightarrow{\sim} \check{\mathbf{\Sigma}}^* \overset{\check{\mathcal{U}}^*}{\twoheadrightarrow} \check{\mathbf{\Gamma}}^* .$$

For $\mathcal{S} \in \mathrm{L}^\vee[I]$, define the following element of $\mathbf{\Sigma}[I]$,

$$\mathcal{U}_{\mathcal{S}} := \sum_{F \in \Sigma[I]} \check{\mathtt{m}}_F(\mathtt{X}_{\mathcal{S}}) \, \mathtt{H}_F .$$

Proposition 5.1. We have

$$\check{\mathcal{U}} : \check{\mathbf{\Gamma}} \hookrightarrow \mathbf{\Sigma}, \qquad \mathtt{X}_{\mathcal{S}} \mapsto \mathcal{U}_{\mathcal{S}}.$$

Proof. Recall that $\check{\mathcal{U}}^*$ may be given by $\check{\mathtt{M}}_F \mapsto \check{\mathtt{m}}_F$, and that the $\mathtt{H}$-basis is the dual of the $\mathtt{M}$-basis. □

We have the corresponding map

$$\mathrm{L}^\vee \to \mathbf{\Sigma}, \qquad \mathcal{S} \mapsto \mathcal{U}_{\mathcal{S}}.$$

This map is exactly [EGS75, Equation 1, p.26], or more recently [Eps16, Equation 2.13]. Then the Steinmann relations turn up as the kernel of the linearization $\mathbf{L}^\vee \to \mathbf{\Sigma}$. Thus, the generalized retarded functions (operators) correspond to the image of the primitive part of $\mathbf{\Sigma}$ in distributions (operator valued distributions).

The *Tits product* is the action of Σ on itself by Hopf powers, thus

$$\Sigma \times \Sigma \to \Sigma, \qquad (F, G) \mapsto \mu_F(\Delta_F(G)).$$

We then linearize the Tits product to obtain,

$$\mathbf{\Sigma} \times \mathbf{\Sigma} \to \mathbf{\Sigma}, \qquad \mathtt{H}_F \otimes \mathtt{H}_G \mapsto \mu_F(\Delta_F(\mathtt{H}_G)).$$

For physicists, the Tits product is motivated by considering causal locality [Eps16, p. 2]. In [EGS75], the following expression for the primitive element corresponding to a top dimensional shard, equivalently maximal unbalanced family, is given,

$$\mathcal{U}_{\mathcal{S}} = \prod_{(S,T)\in\mathcal{S}}^{\mathrm{Tits}} \big(\mathtt{H}_{(I)} - \mathtt{H}_{(T,S)}\big),$$

where the right-hand side is well-defined because these elements commute.

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