On the number of possible resonant algebras

Remigiusz Durka *1 and Kamil Grela †1

1 Institute for Theoretical Physics, University of Wrocław, pl. M. Borna 9, 50-204 Wrocław, Poland

December 2, 2019

Abstract

This article explores the question concerning the number of distinct resonant algebras depending on the generator content, which consists of the Lorentz generator, translation, and new additional Lorentz-like and translation-like generators. Such algebra enlargements originate directly from the so-called Maxwell algebra and implement the S-expansion framework. They find use not only in the construction of gravity and supergravity models but also attract interest in some other applications. The undertaken task is closely related to the subject of finding commutative monoids (semigroups with the identity element) of the particular order, were we additionally enforce the parity condition.

1 Introduction

In the past years we have observed the growing number of works exploring the framework of the so-called semigroup expansion (S-expansion) [1, 2, 3]. The resulting enlargements of the Lorentz/(A)dS/Poincaré algebras correspond to the so-called *Maxwell algebras* containing the original Maxwell algebra introduced in the 70's [4, 5], the Soroka-Soroka algebra [6, 7], as well as their generalizations to $\mathfrak{B}_m$ and $\mathfrak{C}_m$ families [3, 8], along with extension to the new family $\mathfrak{D}_m$ [9], and a recipe for further families [10]. Ultimately, one can generalize it even further to the wider class of the so-called *resonant algebras* [10, 11].

Presented enlargements deliver a very rich structure offering some non-trivial and interesting features. The applications mostly correspond to the construction of the gravity and supergravity actions. Notable examples concern [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. It is worth to mention a recent contribution to the subject of $\mathfrak{bms}$ symmetry [28, 29, 30] and topological insulators [11, 31].

In the indicated constructions of the gauge theories, the particular features and outcomes depend on the chosen Lie algebra, which has a direct impact on the field content, structure constants, symmetries, existence of the particular terms in action and field equations. Therefore, determining the range of possible algebra constructions for a given generator content is of great importance. We are going to tackle this problem by employing computer-assisted computations. After a brief introduction to the semigroup expansion, leading to the discussed algebra enlargements, we will present the outcomes and provide some systematization.

*remigiusz.durka@uwr.edu.pl

†kamilgrela97@gmail.com

2 Semigroup expansion

In this section, we will shortly summarize key elements of the semigroup expansion. First let us recall the nn-Wigner contraction [32] that allows us to obtain the Poincaré algebra as the limit of the Anti-de Sitter (AdS) algebra. This is established by the re-scaling of the translation generator $P_a \to \ell P_a$ with $\Lambda = -\frac{3}{\ell^2}$. The commutator $[P_a, P_b] = J_{ab}$ then becomes $[\ell P_a, \ell P_b] = J_{ab}$ and $[P_a, P_b] = \frac{1}{\ell^2} J_{ab}$. Enforcing the limit corresponding to the vanishing of the cosmological constant makes the right-hand side equal to zero

$$[P_a, P_b] \overset{\ell\to\infty}{=} 0\,. \tag{1}$$

This way we connect two different algebras by the means of a limit. The semigroup expansion framework can be seen as some generalization, which assigns to generators the semigroup elements instead of a scalar parameter like ℓ (see [1, 2, 3] and [12]). Naturally, this new "semigroup scaling" now concerns all the generators. Note, however, that various algebraic outcomes on the right-hand sides are not the consequence of a limit, but the generator redefinition after the evaluation of the original AdS commutators. Choice of the AdS and not dS is due to the possible supersymmetric applications in the future.

The S-expanded algebra is understood as the product $S \times \mathfrak{g}$, where the new generators are given by:

$$J_{ab,(i)} = s_{2i}\tilde{J}_{ab} \qquad \text{and} \qquad P_{a,(i)} = s_{2i+1}\tilde{P}_a, \qquad \text{for } i = \{0, 1, 2, ...\} \tag{2}$$

with the original algebra $\mathfrak{g} = AdS$:

$$\left[\tilde{J}_{ab}, \tilde{J}_{cd}\right] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}\,, \tag{3}$$

$$\left[\tilde{J}_{ab}, \tilde{P}_c\right] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b\,, \tag{4}$$

$$\left[\tilde{P}_a, \tilde{P}_b\right] = \tilde{J}_{ab}\,, \tag{5}$$

and semigroup elements $s_i \in S$. All the commutation relations between the generators belonging to the enlarged algebra are established from the AdS algebra of $\tilde{J}_{ab}$ and $\tilde{P}_a$ along particular semigroup $S = \{s_0, s_1, s_2, ...\}$ multiplication (Cayley) table

$$\begin{array}{c|cccc} & s_0 & s_1 & s_2 & . \\ \hline s_0 & . & . & . & . \\ s_1 & . & . & . & . \\ s_2 & . & . & . & . \\ . & . & . & . & . \end{array} \tag{6}$$

Elements in these tables must admit the commutative $s_\alpha \cdot s_\beta = s_\beta \cdot s_\alpha$ and associativity conditions $(s_\alpha \cdot s_\beta) \cdot s_\gamma = s_\alpha \cdot (s_\beta \cdot s_\gamma)$. Naturally, the associativity directly translates to the Jacobi identity at the level of the generators $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. For the convenience in the further part of the article, we will call generators using the distinct letters:

$$J_{ab} = s_0\tilde{J}_{ab}\,, \quad P_a = s_1\tilde{P}_a\,, \quad Z_{ab} = s_2\tilde{J}_{ab}\,, \quad R_a = s_3\tilde{P}_a\,, \quad W_{ab} = s_4\tilde{J}_{ab}\,, \quad ... \tag{7}$$

It was pointed out [9, 10] that useful algebras, from the point of the view gauge actions, require the Lorentz generator obeying $[J, J] \sim J$ and $[J, P] \sim P$, so we do not break the definitions of the curvature $R^{ab}(\omega) = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}$ and torsion $T^a = de^a + \omega^a{}_c \wedge e^c$ forms. This should be extended further to all the other generators. Therefore, we start by introducing a semigroup element s_0 playing the role of the identity element $s_0 \cdot s_\alpha = s_\alpha \cdot s_0 = s_\alpha$, which needs to be associated with the Lorentz generator J_{ab} preserving all the generators $[J, X] \sim X$. Similarly, we associate s_1 element with the translation generator. In addition, we require the so-called resonant condition (see [1, 2, 3]), which mathematically is just a parity requirement:

$$s_{even} \cdot s_{even} = s_{even}\,, \qquad s_{even} \cdot s_{odd} = s_{odd}\,, \qquad s_{odd} \cdot s_{odd} = s_{even}, \tag{8}$$

which reflects the AdS algebra structure

$$[\tilde{J}_{..}, \tilde{J}_{..}] \sim \tilde{J}_{..} \qquad [\tilde{J}_{..}, \tilde{P}_{.}] \sim \tilde{P}_{.} \qquad [\tilde{P}_{.}, \tilde{P}_{.}] \sim \tilde{J}_{..}\,. \tag{9}$$

To fully complete the picture, we also separately include the absorbing element 0_S, defined as $0_S \cdot s_\alpha = s_\alpha \cdot 0_S = 0_S$. It is not related to any generator. On the contrary, by acting on any generator T it gives zero $0_S T = 0$, which is needed for example as the output of the Poincaré algebra. One can find more on that subject under the name 0_S-reduction within [1, 2, 3]. As we are going to provide a particular algebra systematization to make semigroup order n and amount of the generators on the same footing, we will not include the separate row and column for 0_S in semigroup/monoid Cayley tables, as they anyway contain only 0_S.

To better understand the presented framework, let us look at one particular example. Focusing on the case of the original Maxwell algebra [4, 5] with the generators J_{ab}, P_a, Z_{ab} we can read off all the commutation relations directly from the schematic "commutation" table

$$\begin{array}{c|ccc} [\,.\,,\,.\,] & J_{..} & P_{.} & Z_{..} \\ \hline J_{..} & J_{..} & P_{.} & Z_{..} \\ P_{.} & P_{.} & Z_{..} & 0 \\ Z_{..} & Z_{..} & 0 & 0 \end{array} \tag{10}$$

where one needs to keep in mind a specific form of the structure constants. Naturally, this is a consequence of the corresponding semigroup *multiplication* table:

$$\begin{array}{c|ccc} \boxed{\mathfrak{B}_4} & s_0 & s_1 & s_2 \\ \hline s_0 & s_0 & s_1 & s_2 \\ s_1 & s_1 & s_2 & 0_S \\ s_2 & s_2 & 0_S & 0_S \end{array} \tag{11}$$

applied to

$$\begin{aligned} [J_{ab}, J_{cd}] &= s_0 \cdot s_0(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} + \eta_{ad}J_{bc} - \eta_{bd}J_{ac}\,, \\ [J_{ab}, Z_{cd}] &= s_0 \cdot s_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} + \eta_{ad}Z_{bc} - \eta_{bd}Z_{ac}\,, \\ [Z_{ab}, Z_{cd}] &= s_2 \cdot s_2(\eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} + \eta_{ad}\tilde{J}_{bc} - \eta_{bd}\tilde{J}_{ac}) = 0\,, \\ [J_{ab}, P_c] &= s_0 \cdot s_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = \eta_{bc}P_a - \eta_{ac}P_b\,, \\ [Z_{ab}, P_c] &= s_2 \cdot s_1(\eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b) = 0\,, \\ [P_a, P_b] &= s_1 \cdot s_1\tilde{J}_{ab} = Z_{ab}\,. \end{aligned} \tag{12}$$

The internal product/invariant tensor $\langle\,.\,,\,.\,\rangle$, necessary for the construction of actions, can be similarly read off, according to the same scheme with the particular constants σ_γ depending on the outcome $s_\alpha \cdot s_\beta = s_\gamma$. For example, in 3D theory with S-expanded algebra with $s_2 \cdot s_1 = s_3$ we will have $\langle Z_{ab}, P_c\rangle = \sigma_3\epsilon_{abc}$. Naturally, when one considers $3D$ theory (like in [11]) it is also quite convenient to make a transition to the dual definitions of the generators $X_a = \frac{1}{2}\epsilon_a{}^{bc}X_{bc}$ (and corresponding fields), which simplifies and uniforms the form of the commutation relations

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c\,, \qquad [J_a, P_b] = \epsilon_{ab}{}^c P_c\,, \qquad [P_a, P_b] = \epsilon_{ab}{}^c J_c\,. \tag{13}$$

Further details concerning derivation, notation, and gauging these type of algebras can be found in the literature about the S-expansion and [8, 10, 11].

3 How many resonant algebras is there

Some general considerations about finding all (unrestricted) semigroup expanded algebras were brought in the past [33, 34, 35]. Unfortunately, the overwhelming vastness of algebraic examples does not seem to translate into physically relevant results. We argue that the key to assure the consistent grounds lies in demanding specific properties concerning the commutative semigroups. This includes the resonant/parity condition and requirement of the identity element set to be s_0. Obtained in this way the class of *resonant algebras* [10], being a sub-class of the S-expanded algebras, requires then a specific construction of the semigroups. To this end, we notice that the *parity even* entries can have only $s_{2i} \cup 0_S$ elements related respectively to $\{0, J, Z, W, ...\}$, whereas *parity odd* entries can be filled by odd $s_{2i+1} \cup 0_S$ related to $\{0, P, R, U, ...\}$. By brute-force we can generate all possible *multiplication* tables obeying all given requirements with the exception of associativity. The associativity check has to be employed afterwards upon obtained candidates, which mathematically speaking represent the "unital magma". The number of potential tables can be expressed by the combinatorial formula

$$\left(\frac{2k+5-(-1)^k}{4}\right)^{Floor\left[\frac{k^2}{4}\right]} \cdot \left(\frac{2k+3+(-1)^k}{4}\right)^{Floor\left[\frac{(k-1)^2}{4}\right]} \tag{14}$$

where k represents the number of used generators. Starting point, $k = 2$, corresponds to the J, P setup. For $k = \{2, 3, ..., 8\}$ this gives us, respectively,

$$2, 18, 729, 331776, 1073741824, 64000000000000, 37252902984619140625\,.$$

The associativity filtering significantly reduces the final number of tables, ultimately assuring required commutative semigroup structure with identity and resonant condition. Based on them we generate the resonant algebras. For the available generators $\{J_{ab}, P_a, Z_{ab}, R_a, W_{ab}, U_a\}$, therefore up to three copies of the Lorentz/translation-like generators, we can summarize our explicit findings as

	$\{J,P\}$	$\{J,P,Z\}$	$\{J,P,Z,R\}$	$\{J,P,Z,R,W\}$	$\{J,P,Z,R,W,U\}$
Possible algebras	2	6	30	347	3786
Algebras without 0	1	1	6	28	222

Obviously, the pair $\{J_{ab}, P_a\}$ results in the Poincaré and AdS algebra. By enlarging algebra to the set of $\{J_{ab}, P_a, Z_{ab}\}$ we end up with the Maxwell algebra [4, 5] along with the Soroka-Soroka algebra [6, 7] and four more examples showed in [10] and [11]. Due to the plethora of examples, we suggest that more emphasis should be placed on the fully non-abelian algebras, i.e. not having any zeros in the commutation outcomes, thus offering the most general content and the widest scope of the corresponding phenomena.

The explicit form of algebras, up to the case of $\{J_{ab}, P_a, Z_{ab}, R_a\}$, will be presented in the next section, whereas rest will be attached to the publication.

Before we go any further let us classify obtained resonant algebras. Depending on the closing commutator $[P_a, P_b]$ we can organize algebras according to the resulting sub-algebras:

	$\{J,P\}$	$\{J,P,Z\}$	$\{J,P,Z,R\}$	$\{J,P,Z,R,W\}$	$\{J,P,Z,R,W,U\}$
Possible algebras	2	6	30	347	3786
Poincaré-like	1	4	17	211	2062
AdS-like	1	0	3	0	38
Maxwell-like	-	2	10	68	843
Extra-Maxwell-like	-	-	-	68	843

Obviously Poincaré-like algebras are the ones possessing $[P_a, P_b] = 0$, AdS-like $[P_a, P_b] = J_{ab}$, Maxwell-like $[P_a, P_b] = Z_{ab}$, and so on.

As we can see, the task of generating the resonant algebras is inextricably related to the subject of finding the semigroups (and monoids) of a particular order n. Unfortunately, there is little known about the inclusion of the parity condition. As for now, the amounts of the particular resonant algebras seem not to follow any known general sequence, just like for the semigroups and monoids. In literature [33, 36, 37, 38, 39, 40] and through oeis.org we find:

- A001423 Number of semigroups of order n for $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$:
  $1, 4, 18, 126, 1160, 15973, 836021, 1843120128, 52989400714478, 12418001077381302684$
- A001426 Number of commutative semigroups of order n:
  $1, 3, 12, 58, 325, 2143, 17291, 221805, 11545843, 3518930337$
- A058129 Number of monoids (semigroups with identity) of order n:
  $1, 2, 7, 35, 228, 2237, 31559, 1668997, \ldots$
- A058131 Number of commutative monoids of order n:
  $1, 2, 5, 19, 78, 421, 2637, \ldots$

One might wonder, why not start with the given (commutative) semigroups available even up to order $n = 10$, and then just filter them due to the identity and parity. That path naturally has much higher efficiency (paid by someone's else prior work). Unfortunately, all the available tables are given up to isomorphism. For instance, without the resonant condition, for the commutative monoids of order $n = 3$ with an identity fixed as the first element, we can find 9 tables:

$$
\begin{array}{c|ccc}
M_1 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_2 & s_0 \\
s_2 & s_2 & s_0 & s_1
\end{array}
\quad
\begin{array}{c|ccc}
M_2 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_0 & s_2 \\
s_2 & s_2 & s_2 & s_2
\end{array}
\quad
\begin{array}{c|ccc}
M_3 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_1 & s_1 \\
s_2 & s_2 & s_1 & s_2
\end{array}
\quad
\begin{array}{c|ccc}
M_4 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_1 & s_1 \\
s_2 & s_2 & s_1 & s_1
\end{array}
\quad
\begin{array}{c|ccc}
M_5 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_1 & s_2 \\
s_2 & s_2 & s_2 & s_1
\end{array}
$$

$$
\begin{array}{c|ccc}
M_6 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_1 & s_1 \\
s_2 & s_2 & s_1 & s_0
\end{array}
\quad
\begin{array}{c|ccc}
M_7 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_1 & s_2 \\
s_2 & s_2 & s_2 & s_2
\end{array}
\quad
\begin{array}{c|ccc}
M_8 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_2 & s_2 \\
s_2 & s_2 & s_2 & s_2
\end{array}
\quad
\begin{array}{c|ccc}
M_9 & s_0 & s_1 & s_2 \\ \hline
s_0 & s_0 & s_1 & s_2 \\
s_1 & s_1 & s_2 & s_1 \\
s_2 & s_2 & s_1 & s_2
\end{array}
$$

The four bottom examples will be rejected, because with a redefinition of elements $s_1 \leftrightarrow s_2$, and changing the order of rows and columns, we reproduce exactly four examples lying above them. We could even miss the Soroka-Soroka $\mathfrak{C}_4$ algebra, represented by the last M_9 table (incidentally, the only one obeying resonant condition!) as it might not be directly visible in an isomorphic output. The presence of the absorbing element introduce yet another complication. Including 0_S means searching among monoids of a higher order. Notice, that here M_8 table essentially gives the Poincaré algebra with s_2 not associated with Z_{ab} generator but playing the role of 0_S.

The setup we have presented in the previous section is much more comfortable. The physical properties attached to s_0 and s_1, along with the resonant/parity condition, remove the isomorphic ambiguities because similar re-definitions as above are just impossible. The lack of physical restrictions concerning Z_{ab} and W_{ab}, thus some interchangeability, might lead to some issues, but for now, we will just assume that the role of all generators is somehow unique.

4 Panorama of algebras

Below we present an overview of the resonant algebras obtained by subsequent including further generators.

4.1 J algebra

We can consider the Lorentz algebra as the starting point obeying all the requirements.

Lorentz	J
J	J

(15)

4.2 J and P algebras

With the translation generator we see only two possibilities: Poincaré and AdS algebra.

Poincaré	J	P
J	J	P
P	P	0

AdS	J	P
J	J	P
P	P	J

(16)

4.3 J and P and Z algebras

Including another generator, Z_{ab}, brings much richer structure:

- 2 Maxwell-like algebras (i.e. containing $[P,P] \sim Z$): of type $\mathfrak{B}_4$ (original Maxwell algebra introduced in the 70's) and type $\mathfrak{C}_4 \equiv AdS \oplus Lorentz$ (introduced by Soroka-Soroka, which was shown to represent under a change of basis the direct sum of two algebras)

$\boxed{\mathfrak{B}_4}$	J	P	Z
J	J	P	Z
P	P	Z	0
Z	Z	0	0

$\boxed{\mathfrak{C}_4}$	J	P	Z
J	J	P	Z
P	P	Z	P
Z	Z	P	Z

(17)

- 0 of AdS-like algebras, which would contain $[P,P] \sim J$

- 4 Poincaré-like algebras (i.e. having $[P,P] \sim 0$), which we denote as type: B_4, $\tilde{B}_4$, $\tilde{C}_4$, and $C_4 \equiv Poincare \oplus Lorentz$

$\boxed{B_4}$	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	0

$\boxed{\tilde{B}_4}$	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	0

$\boxed{\tilde{C}_4}$	J	P	Z
J	J	P	Z
P	P	0	0
Z	Z	0	Z

$\boxed{C_4}$	J	P	Z
J	J	P	Z
P	P	0	P
Z	Z	P	Z

(18)

For the set of three generators $\{J,P,Z\}$ we have 6 different algebras, which can correspond to six different Lagrangians, thus six different configurations of the field equations, just as it was recently shown in the analysis of topological insulators [11].

4.4 J and P and Z and R algebras

The last presented here explicitly example consists of 30 cases:

- 10 Maxwell-like

$\boxed{\mathfrak{B}_5}$	J	P	Z	R
J	J	P	Z	R
P	P	Z	R	0
Z	Z	R	0	0
R	R	0	0	0

$\boxed{\mathfrak{C}_4}$	J	P	Z	R
J	J	P	Z	R
P	P	Z	R	J
Z	Z	R	J	P
R	R	J	P	Z

$\boxed{\mathfrak{D}_4}$	J	P	Z	R
J	J	P	Z	R
P	P	Z	R	Z
Z	Z	R	Z	R
R	R	Z	R	Z

	J	P	Z	R
J	J	P	Z	R
P	P	Z	P	0
Z	Z	P	Z	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	Z	P	Z
Z	Z	P	Z	P
R	R	Z	P	J

	J	P	Z	R
J	J	P	Z	R
P	P	Z	P	Z
Z	Z	P	Z	P
R	R	Z	P	Z

	J	P	Z	R
J	J	P	Z	R
P	P	Z	0	0
Z	Z	0	0	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	Z	0	0
Z	Z	0	0	0
R	R	0	0	Z

	J	P	Z	R
J	J	P	Z	R
P	P	Z	0	Z
Z	Z	0	0	0
R	R	Z	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	Z	0	Z
Z	Z	0	0	0
R	R	Z	0	Z

(19)

- 3 AdS-like

$\boxed{\mathcal{B}_5}$	J	P	Z	R
J	J	P	Z	R
P	P	J	R	Z
Z	Z	R	0	0
R	R	Z	0	0

$\boxed{\mathcal{C}_5}$	J	P	Z	R
J	J	P	Z	R
P	P	J	R	Z
Z	Z	R	J	P
R	R	Z	P	J

$\boxed{\mathcal{D}_5}$	J	P	Z	R
J	J	P	Z	R
P	P	J	R	Z
Z	Z	R	Z	R
R	R	Z	R	Z

(20)

- 17 Poincaré-like

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	0	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	Z	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	0	0
R	R	0	0	Z

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	Z
Z	Z	0	0	0
R	R	Z	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	Z
Z	Z	0	0	0
R	R	Z	0	Z

	J	P	Z	R
J	J	P	Z	R
P	P	0	R	0
Z	Z	R	0	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	0	P
R	R	0	P	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	P	0
Z	Z	P	Z	0
R	R	0	0	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	Z	R
R	R	0	R	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	Z
Z	Z	0	0	P
R	R	Z	P	J

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	0	P
R	R	0	P	Z

	J	P	Z	R
J	J	P	Z	R
P	P	0	0	0
Z	Z	0	Z	R
R	R	0	R	Z

	J	P	Z	R
J	J	P	Z	R
P	P	0	P	0
Z	Z	P	J	R
R	R	0	R	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	R	0
Z	Z	R	J	P
R	R	0	P	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	P	0
Z	Z	P	Z	R
R	R	0	R	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	P	0
Z	Z	P	Z	P
R	R	0	P	0

	J	P	Z	R
J	J	P	Z	R
P	P	0	R	0
Z	Z	R	Z	R
R	R	0	R	0

(21)

4.5 Algebras with more generators

We will not present explicitly all the 347 $\{J, P, Z, R, W\}$ algebras, nor 3876 examples of $\{J, P, Z, R, W, U\}$ but they will available through a separately supplemented file.

Values given in (14) show that incorporating further generators, beyond the last pair of $\{W_{ab}, U_a\}$, becomes quite a time-consuming exercise, requiring in the next steps the associativity analysis of $6,4 \times 10^{13}$ and $3,725 \times 10^{19}$ candidates. The last result of 3876 resonant algebras was established after 31 hours of medium-class PC computations concerning "only" 10^9 candidates.

5 Other observations

Besides the systematization over the appearing sub-algebras, one could look for other things. More emphasis should be placed on the non-abelian algebras (without zeros in the outcome). They are particularly interesting because they give the most general actions. Intriguingly, we notice quite a small number of AdS-like algebras. It seems even that we are unable to construct such an algebraic type for the odd number of generators.

Another thing is matching some numbers concerning the Maxwell-like and algebras of a further type. Generators Z_{ab} and W_{ab} seem to be interchangeable from a certain point. This will be always the case, until we deliver some interpretation or enforce special relations for Z_{ab} generator, similarly as we did it for Lorentz and translation. One would expect more in the future on the meaning of these additional bosonic fields.

For the few resonant tables, it happens that their elements form the *cyclic group* $\mathbb{Z}_k$. There is one such table corresponding to the $\{J, P\}$ content, one for $\{J, P, Z, R\}$, and six for $\{J, P, Z, R, W, U\}$. When we look for the tables forming *groups* (monoids with the inverse) this requires adding to a list above one more table for $\{J, P, Z, R\}$. While studying the gauged actions, this has no extra meaning or impact. As far as we know, there is nothing meaningful about the generators and fields related to elements being inverses.

The extended set of possible algebras asks for further analysis corresponding to gauge models (Chern-Simons in odd dimensions, Born-Infeld in even dimensions [8, 18] or BF $\to$ BFCG construction [24, 27]). This also includes the supergravity models, where the supersymmetric extensions should be now explored for all the resonant algebras.

Finally, the setup with the two sets of Lorentz- and translation-like generators: $\{J, P, Z, R\}$ might be particularly interesting from the perspective of the bi-metric theories [41]. The spin connection ω^{ab} and vierbein e^a (associated with J_{ab} and P_a) would correspond to the background metric, whereas the other pair k^{ab} and h^a (associated with the Z_{ab} and R_a) could be related to other metric field. Closing these algebras in thirty different ways means many various non-trivial interaction terms, being usually put there by hand. With more generators in play, this opens doors to the multi-metric formulation.

6 Summary

Within this paper, we have provided the complete overview of the *resonant algebras*, of which only a limited portion [11] has been analyzed and incorporated in various applications. Provided scope and classification helps us better understand all the relations between algebras, which in the future might turn essential in realizing various formal and physical goals.

7 Acknowledgment

Authors would like to thank B. Koszałka for the past contribution in generating the resonant algebras. Work was supported by the Institute Grant for Young Researchers 0420/2716/18.

References

[1] F. Izaurieta, E. Rodríguez and P. Salgado, "Expanding Lie (super)algebras through Abelian semi-groups," J. Math. Phys. **47**, 123512 (2006) [hep-th/0606215].

[2] J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodríguez, P. Salgado and O. Valdivia, "A generalized action for $(2+1)$-dimensional Chern-Simons gravity," J. Phys. A **45**, 255207 (2012) [arXiv:1311.2215 [gr-qc]].

[3] P. Salgado and S. Salgado, "$\mathfrak{so}(D-1,1)\otimes\mathfrak{so}(D-1,2)$ algebras and gravity," Phys. Lett. B **728**, 5 (2014).

[4] H. Bacry, P. Combe and J. L. Richard, "Group-theoretical analysis of elementary particles in an external electromagnetic field. 1. the relativistic particle in a constant and uniform field," Nuovo Cim. A **67**, 267 (1970). doi:10.1007/BF02725178

[5] R. Schrader, "The maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields," Fortsch. Phys. **20** (1972) 701. doi:10.1002/prop.19720201202

[6] D. V. Soroka and V. A. Soroka, "Tensor extension of the Poincaré algebra," Phys. Lett. B **607**, 302 (2005) [hep-th/0410012].

[7] D. V. Soroka and V. A. Soroka, "Semi-simple extension of the (super)Poincaré algebra," Adv. High Energy Phys. **2009**, 234147 (2009) [hep-th/0605251].

[8] P. K. Concha, R. Durka, C. Inostroza, N. Merino and E. K. Rodrguez, "Pure Lovelock gravity and Chern-Simons theory," Phys. Rev. D **94**, no. 2, 024055 (2016) doi:10.1103/PhysRevD.94.024055 [arXiv:1603.09424 [hep-th]].

[9] P. K. Concha, R. Durka, N. Merino and E. K. Rodrguez, "New family of Maxwell like algebras," Phys. Lett. B **759**, 507 (2016) doi:10.1016/j.physletb.2016.06.016 [arXiv:1601.06443 [hep-th]].

[10] R. Durka, "Resonant algebras and gravity," J. Phys. A **50**, no. 14, 145202 (2017) doi:10.1088/1751-8121/aa5c0b [arXiv:1605.00059 [hep-th]].

[11] R. Durka and J. Kowalski-Glikman, "Resonant algebras in Chern-Simons model of topological insulators," Phys. Lett. B **795**, 516 (2019) doi:10.1016/j.physletb.2019.06.058 [arXiv:1906.02356 [hep-th]].

[12] J. D. Edelstein, M. Hassaine, R. Troncoso and J. Zanelli, "Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian," Phys. Lett. B **640**, 278 (2006) [hep-th/0605174].

[13] J. A. de Azcarraga, K. Kamimura and J. Lukierski, "Generalized cosmological term from Maxwell symmetries," Phys. Rev. D **83**, 124036 (2011) [arXiv:1012.4402 [hep-th]].

[14] D. M. Peafiel and L. Ravera, "Generalized cosmological term in $D=4$ supergravity from a new AdSLorentz superalgebra," Eur. Phys. J. C **78**, no. 11, 945 (2018) doi:10.1140/epjc/s10052-018-6421-9 [arXiv:1807.07673 [hep-th]].

[15] O. Fierro, F. Izaurieta, P. Salgado and O. Valdivia, "Minimal AdS-Lorentz supergravity in three-dimensions," Phys. Lett. B **788**, 198 (2019) doi:10.1016/j.physletb.2018.10.066 [arXiv:1401.3697 [hep-th]].

[16] K. Kamimura and J. Lukierski, "Supersymmetrization Schemes of D=4 Maxwell Algebra," Phys. Lett. B **707**, 292 (2012) [arXiv:1111.3598 [math-ph]].

[17] J. A. de Azcarraga, J. M. Izquierdo, J. Lukierski and M. Woronowicz, "Generalizations of Maxwell (super)algebras by the expansion method," Nucl. Phys. B **869**, 303 (2013) [arXiv:1210.1117 [hep-th]].

[18] P. K. Concha, D. M. Peñafiel, E. K. Rodríguez and P. Salgado, "Chern-Simons and Born-Infeld gravity theories and Maxwell algebras type," Eur. Phys. J. C **74**, 2741 (2014) [arXiv:1402.0023 [hep-th]].

[19] P. K. Concha, O. Fierro and E. K. Rodrguez, "InnWigner contraction and $D=2+1$ supergravity," Eur. Phys. J. C **77**, no. 1, 48 (2017) doi:10.1140/epjc/s10052-017-4615-1 [arXiv:1611.05018 [hep-th]].

[20] P. Concha, N. Merino, O. Miskovic, E. Rodríguez, P. Salgado-Rebolledo and O. Valdivia, "Asymptotic symmetries of three-dimensional Chern-Simons gravity for the Maxwell algebra," JHEP **1810**, 079 (2018) doi:10.1007/JHEP10(2018)079 [arXiv:1805.08834 [hep-th]].

[21] L. Aviles, E. Frodden, J. Gomis, D. Hidalgo and J. Zanelli, "Non-Relativistic Maxwell Chern-Simons Gravity," JHEP **1805**, 047 (2018) doi:10.1007/JHEP05(2018)047 [arXiv:1802.08453 [hep-th]].

[22] C. Inostroza, A. Salazar and P. Salgado, "BransDicke gravity theory from topological gravity," Phys. Lett. B **734**, 377 (2014). doi:10.1016/j.physletb.2014.05.080

[23] S. Hoseinzadeh and A. Rezaei-Aghdam, "(2+1)-dimensional gravity from Maxwell and semisimple extension of the Poincaré gauge symmetric models," Phys. Rev. D **90**, no. 8, 084008 (2014) doi:10.1103/PhysRevD.90.084008 [arXiv:1402.0320 [hep-th]].

[24] R. Durka, J. Kowalski-Glikman and M. Szczachor, "Gauged AdS-Maxwell algebra and gravity," Mod. Phys. Lett. A **26**, 2689 (2011) [arXiv:1107.4728 [hep-th]].

[25] R. Durka, J. Kowalski-Glikman and M. Szczachor, "AdS-Maxwell superalgebra and supergravity," Mod. Phys. Lett. A **27**, 1250023 (2012) doi:10.1142/S021773231250023X [arXiv:1107.5731 [hep-th]].

[26] R. Durka and J. Kowalski-Glikman, "Local Maxwell symmetry and gravity," arXiv:1110.6812 [hep-th].

[27] R. Durka, "Deformed BF theory as a theory of gravity and supergravity," arXiv:1208.5185 [gr-qc].

[28] P. Concha, D. M. Peñafiel and E. Rodríguez, "On the Maxwell supergravity and flat limit in 2+1 dimensions," Phys. Lett. B **785**, 247 (2018) doi:10.1016/j.physletb.2018.08.050 [arXiv:1807.00194 [hep-th]].

[29] P. Concha, N. Merino, E. Rodríguez, P. Salgado-Rebolledo and O. Valdivia, "Semi-simple enlargement of the $\mathfrak{bms}_3$ algebra from a $\mathfrak{so}(2,2) \oplus \mathfrak{so}(2,1)$ Chern-Simons theory," JHEP **1902**, 002 (2019) doi:10.1007/JHEP02(2019)002 [arXiv:1810.12256 [hep-th]].

[30] P. Salgado-Rebolledo, "The Maxwell group in 2+1 dimensions and its infinite-dimensional enhancements," JHEP **1910**, 039 (2019) doi:10.1007/JHEP10(2019)039 [arXiv:1905.09421 [hep-th]].

[31] G. Palumbo, "Geometric Model of Topological Insulators from the Maxwell Algebra," Annals Phys. **386**, 15 (2017) doi:10.1016/j.aop.2017.08.018 [arXiv:1610.04734 [cond-mat.mes-hall]].

[32] E. Inonu and E. P. Wigner, "On the Contraction of groups and their represenations," Proc. Nat. Acad. Sci. **39**, 510 (1953). doi:10.1073/pnas.39.6.510

[33] L. Andrianopoli, N. Merino, F. Nadal and M. Trigiante, "General properties of the expansion methods of Lie algebras," J. Phys. A **46**, 365204 (2013) doi:10.1088/1751-8113/46/36/365204 [arXiv:1308.4832 [gr-qc]].

[34] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, "A Java library to perform S-expansions of Lie algebras," arXiv:1703.04036 [cs.MS].

[35] C. Inostroza, I. Kondrashuk, N. Merino and F. Nadal, "On the algorithm to find S-related Lie algebras," J. Phys. Conf. Ser. **1085**, no. 5, 052011 (2018) doi:10.1088/1742-6596/1085/5/052011 [arXiv:1802.05765 [physics.comp-ph]].

[36] G. E. Forsythe, "SWAC Computes 126 Distinct Semigroups of Order 4," Proceedings of the American Mathematical Society **6**, no. 3, 443-447 (1955) doi:10.2307/2032786.

[37] T. S. Motzkin, J. L. Selfridge, "Semigroups of order five," Bull. Amer. Math. Soc, **62**, 14, 13-23 (1956)

[38] R. J. Plemmons, "There are 15973 semigroups of order 6," Math. Algorithms **2**, 2-17 (1967)

[39] A. Distler, T. Kelseym "The monoids of orders eight, nine & ten," Annals of Mathematics and Artificial Intelligence, **56**, (1) (2009) doi: 10.1007/s10472-009-9140-y

[40] A. Distler, C. Jefferson, T. Kelseymm, L. Kotthoff "The semigroups of order 10," International Conference on Principles and Practice of Constraint Programming (pp. 883-899). Springer, doi:10.1007/978-3-642-33558-7_3

[41] S. Hoseinzadeh and A. Rezaei-Aghdam, "(2+1)-dimensional Chern-Simons bi-gravity with AdS Lie bialgebra as an interacting theory of two massless spin-2 fields," arXiv:1706.02129 [hep-th].