Some Properties and Combinatorial Implications of Weighted Small Schröder Numbers

Yu Hin (Gary) Au
Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon, SK
Canada
au@math.usask.ca

Abstract

The n^{th} small Schröder number is $s(n) = \sum_{k \geq 0} s(n,k)$, where $s(n,k)$ denotes the number of plane rooted trees with n leaves and k internal nodes that each has at least two children. In this manuscript, we focus on the weighted small Schröder numbers $s_d(n) = \sum_{k \geq 0} s(n,k) d^k$, where d is an arbitrary fixed real number. We provide recursive and asymptotic formulas for $s_d(n)$, as well as some identities and combinatorial interpretations for these numbers. We also establish connections between $s_d(n)$ and several families of Dyck paths.

1 Introduction

1.1 Small Schröder numbers

The small Schröder numbers, denoted $s(n)$ for every $n \geq 1$ gives the sequence

n	1	2	3	4	5	6	7	8	$\cdots$
$s(n)$	1	1	3	11	45	197	903	4279	$\cdots$

(A001003 in the On-line Encyclopedia of Integer Sequences (OEIS) [13]). This sequence has been studied extensively and has many combinatorial interpretations (see, for instance, [16, Exercise 6.39]). We list two below:

- Let $\mathcal{T}_n$ denote the set of *Schröder trees* with n leaves, which are plane rooted trees where each internal node has at least 2 children. Then $s(n) = |\mathcal{T}_n|$ for every $n \geq 1$. Figure 1 shows the $s(4) = 11$ Schröder trees with 4 leaves.

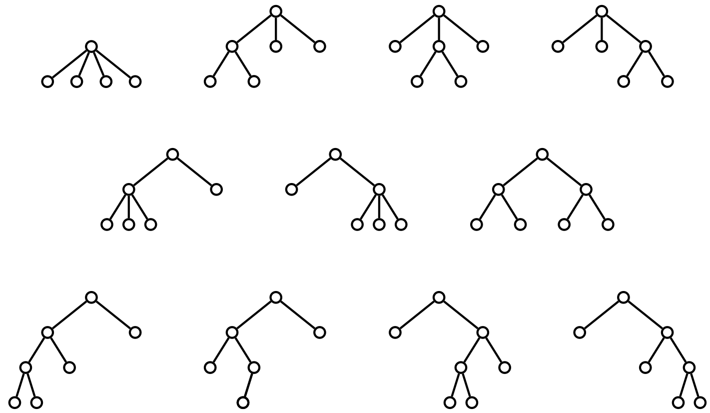

Figure 1: The $s(4) = 11$ Schröder trees with 4 leaves

- Let $\mathcal{P}_n$ be the set of *small Schröder paths* from $(0, 0)$ to $(2n - 2, 0)$, which are lattice paths that

  (P1) use only up steps $U = (1, 1)$, down steps $D = (1, -1)$, and flat steps $F = (2, 0)$;

  (P2) remain on or above the x-axis;

  (P3) doe not contain an F step on the x-axis.

  Then $s(n) = |\mathcal{P}_n|$ for every $n \geq 1$. Figure 2 lists the $s(4) = 11$ small Schröder paths from $(0, 0)$ to $(6, 0)$.

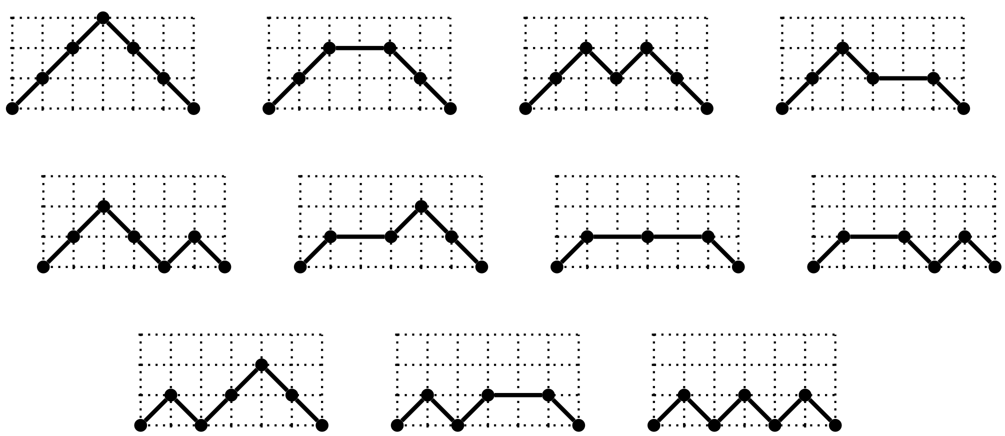

Figure 2: The $s(4) = 11$ small Schröder paths from $(0, 0)$ to $(6, 0)$

To see that $|\mathcal{T}_n| = |\mathcal{P}_n|$ for every $n \geq 1$, we describe the well-known "walk around the tree" procedure that maps plane rooted trees to lattice paths.

Definition 1. Given a Schröder tree $T \in \mathcal{T}_n$, construct the path $\Psi(T) \in \mathcal{P}_n$ as follows:

1. Suppose T has q nodes. We perform a preorder traversal of T, and label the nodes $a_1, a_2, \ldots, a_q$ in that order.

2. Notice that a_1 must be the root of T. For every $i \geq 2$, define the function

$$\psi(a_i) = \begin{cases} U & \text{if } a_i \text{ is the leftmost child of its parent;} \\ D & \text{if } a_i \text{ is the rightmost child of its parent;} \\ F & \text{otherwise.} \end{cases}$$

3. Define

$$\Psi(T) = \psi(a_2)\psi(a_3)\cdots\psi(a_q).$$

Figure 3 illustrates the mapping Ψ for a particular tree. It is not hard to check that $\Psi : \mathcal{T}_n \to \mathcal{P}_n$ is indeed a bijection.

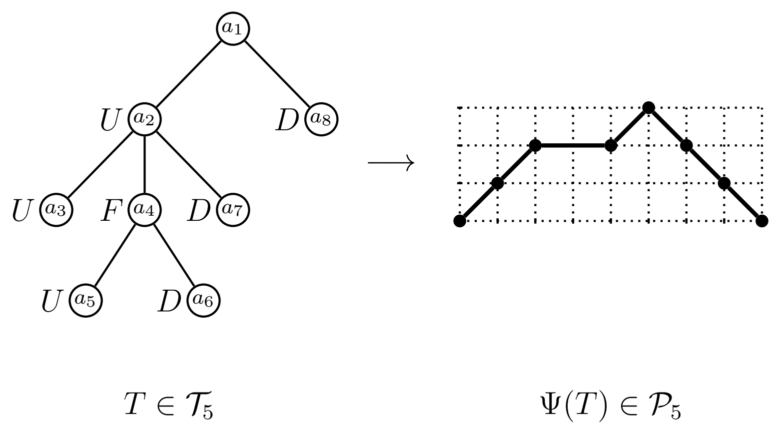

Figure 3: Illustrating the tree-to-path mapping Ψ (Definition 1)

For more historical context, properties, and combinatorial interpretations of the small Schröder numbers, the reader may refer to [15, 16, 12, 7].

1.2 Weighted small Schröder numbers

Before we describe the weighted small Schröder numbers, let us take a closer look at the Schröder trees $\mathcal{T}_n$. Given integers $n \geq 1$ and $0 \leq k < n$, let $\mathcal{T}_{n,k} \subseteq \mathcal{T}_n$ be the set of Schröder trees with n leaves and k internal nodes. Furthermore, let $s(n,k) = |T_{n,k}|$. For instance, Figure 1 shows that $s(4,0) = 0$, $s(4,1) = 1$, and $s(4,2) = s(4,3) = 5$. More generally, the numbers $s(n,k)$ produce the following triangle (A086810 in the OEIS).

$s(n,k)$	$k=0$	1	2	3	4	5	$\cdots$
$n=1$	1						
2	0	1					
3	0	1	2				
4	0	1	5	5			
5	0	1	9	21	14		
6	0	1	14	56	84	42	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

In general,

$$s(n,k) = \begin{cases} 1 & \text{if } n = 1 \text{ and } k = 0; \\ \frac{1}{n-1}\binom{n-1}{k}\binom{n+k-1}{n} & \text{if } n \geq 2. \end{cases}$$

(See, for instance, [6, Lemma 1] for a proof.) Notice that $s(n) = \sum_{k=0}^{n-1} s(n,k)$. Next, we likewise define $\mathcal{P}_{n,k} \subseteq \mathcal{P}_n$ to be the set of small Schröder paths in $\mathcal{P}_n$ with exactly k up steps. Now notice that given $T \in \mathcal{T}_{n,k}$, there must be exactly k nodes in T that is each the leftmost child of its parent. Thus, $\Psi(T)$ would have exactly k up steps, and so Ψ is in fact a bijection between $\mathcal{T}_{n,k}$ and $\mathcal{P}_{n,k}$ for every n and k, and it follows that $s(n,k) = |\mathcal{P}_{n,k}|$. Moreover, observe that $s(n,n-1)$ counts the number of Schröder paths from $(0,0)$ to $(2n-2,0)$ with $n-1$ up steps (and hence $n-1$ down steps). These paths must then have no flat steps, and therefore are in fact Dyck paths. Hence, $s(n,n-1)$ gives the n^{th} Catalan number (A000108 in the OEIS).

We are now ready to define the sequences that are of our main focus in this manuscript. Given a real number d, we define

$$s_d(n) = \sum_{k=0}^{n-1} s(n,k) d^k \tag{1}$$

for every integer $n \geq 1$. Intuitively, one can interpret assigning a weight of d^k to each tree in $\mathcal{T}_{n,k}$, and let $s_d(n)$ be the total weight of all trees in $\mathcal{T}_n$.

Clearly, $s_1(n) = s(n)$, so the above is indeed a generalization of the small Schröder numbers. For $d \geq 2$, we get the following sequences:

n	1	2	3	4	5	6	7	8	$\cdots$
$s_2(n)$	1	2	10	62	430	3194	24850	199910	$\cdots$
$s_3(n)$	1	3	21	183	1785	18651	204141	2310447	$\cdots$
$s_4(n)$	1	4	36	404	5076	68324	963396	14046964	$\cdots$

(See A107841, A131763, and A131765 for the cases $d = 2, 3, 4$ respectively.) While we could not find any literature in which these sequences were the main focus of study, they (especially $s_2(n)$) have made appearances in many areas, such as queuing theory [1], embedded Riordan arrays [2], operads from posets [8], and vorticity equations in fluid dynamics [10]. Chan and Pan [3] also came across these sequences when enumerating a certain family of valley-type weighted Dyck paths, and related these quantities to a variant of generalized large Schröder numbers that is somewhat similar to $s_d(n)$. We will make this connection more explicit in Section 4. For other generalizations of small Schröder numbers, see (among others) [18, 11, 9].

1.3 A roadmap of this paper

In Section 2, we will discuss some basic properties of $s_d(n)$ by studying its generating function. We will see that the tools for establishing known formulas for $s(n)$ readily extend to proving analogous results for $s_d(n)$. After that, we prove a few identities for $s_d(n)$ (Section 3), and describe several families of Dyck paths that are counted by $s_d(n)$ (Section 4). Our analysis of the weighted Schröder numbers $s_d(n)$ leads to several results regarding Schröder paths and Dyck paths, such as:

- For every $n \geq 1$, the number of small Schröder paths in $\mathcal{P}_n$ with an odd number of up steps and that with an even number of up steps differ by exactly one (Proposition 6);
- For every $n \geq 1$, $s_{kl-1}(n)$ gives the number of Dyck paths from $(0, 0)$ to $(2n - 2, 0)$ with k possible colors for each up step $(U_1, \ldots, U_k)$, ℓ possible colors for each down step $(D_1, \ldots, D_\ell)$, and avoid peaks of type U_1D_1 (Proposition 15).

We close in Section 5 by mentioning some possible future research directions.

2 Basic properties and formulas

Let d be a fixed real number, and consider the generating function $y = \sum_{n\geq1} s_d(n)x^n$. We first establish a functional equation for y.

Proposition 2. *The generating function $y = \sum_{n\geq1} s_d(n)x^n$ satisfies the functional equation*

$$(d + 1)y^2 - (x + 1)y + x = 0. \quad (2)$$

Proof. Let $\mathcal{T} = \bigcup_{n\geq 1} \mathcal{T}_n$ be the set of all Schröder trees, and for each tree $T \in \mathcal{T}$ we let $k(T), n(T)$ be the number of internal nodes and leaves of T, respectively. Then notice that for each $T \in \mathcal{T}$ where $n(T) \geq 2$, the root of T must be an internal node with $\ell \geq 2$ subtrees $T_1, \ldots, T_\ell$, in which case $\sum_{j=1}^{\ell} k(T_j) + 1 = k(T)$ and $\sum_{j=1}^{\ell} n(T_j) = n(T)$. Thus,

$$\begin{aligned} y &= \sum_{n\geq 1} s_d(n) x^n \\ &= \sum_{T\in\mathcal{T}} d^{k(T)} x^{n(T)} \\ &= \sum_{T\in\mathcal{T}_1} d^{k(T)} x^{n(T)} + sum_{T\in\mathcal{T}\setminus\mathcal{T}_1} d^{k(T)} x^{n(T)} \\ &= x + \sum_{\ell\geq 2} \sum_{T_1,\ldots,T_\ell\in\mathcal{T}} d^{1+\sum_{j=1}^{\ell} k(T_j)} x^{\sum_{j=1}^{\ell} n(T_j)} \\ &= x + d\sum_{\ell\geq 2} y^\ell \\ &= x + \frac{dy^2}{1-y}, \end{aligned}$$

which easily rearranges to give (2). $\square$

When $d \neq -1$, solving for y in (2) (and noting $[x^0]y = 0$) yields

$$y = \frac{1}{2(d+1)}\left(1 + x - \sqrt{1-(4d+2)x+x^2}\right). \tag{3}$$

We next use (2) and (3) to prove a recurrence relation for $s_d(n)$ that would allow for very efficient computation of these numbers. For the case $d = 1$, Stanley [15] proved the recurrence

$$ns(n) = 3(2n-3)s(n-1) - (n-3)s(n-2), \quad n \geq 3 \tag{4}$$

using generating functions. Subsequently, a combinatorial proof using weighted binary trees is given [5]. Shortly after, Sulanke [17] also showed that the above recurrence applies for the closely-related large Schröder numbers using Schröder paths.

Herein, we adapt Stanley's [15] argument to obtain a similar recurrence relation for $s_d(n)$ for all d.

Proposition 3. *For every real number d, $s_d(1) = 1, s_d(2) = d$, and*

$$ns_d(n) = (2d+1)(2n-3)s_d(n-1) - (n-3)s_d(n-2) \tag{5}$$

for all $n \geq 3$.

Proof. First, $s_d(1) = 1$ and $s_d(2) = d$ follows readily from the definition of $s_d(n)$. For the recurrence, we start with (2) and differentiate both sides with respect to x to obtain

$$2(d+1)yy' - y - (x+1)y' + 1 = 0.$$

Rearranging gives

$$y' = \frac{y-1}{2(d+1)y - x - 1}.$$

We also have

$$2(d+1)y - x - 1 = -\sqrt{x^2 - (4d+2)x + 1}.$$

This obviously holds when $d = -1$, and can be derived from (3) for all other values of d. Thus,

$$\begin{aligned} y' &= \frac{y-1}{2(d+1)y - x - 1} = \frac{y-1}{-\sqrt{x^2 - (4d+2)x + 1}} = \frac{(y-1)(2(d+1)y - x - 1)}{x^2 - (4d+2)x + 1} \\ &= \frac{2(d+1)y^2 - (2d + x + 3)y + x + 1}{x^2 - (4d+2)x + 1} = \frac{(x - 2d - 1)y - x + 1}{x^2 - (4d+2)x + 1}, \end{aligned}$$

where the last equality made use of the fact that $(d+1)y^2 = (x+1)y - x$ from (2). From the above, we obtain the equation

$$(x^2 - (4d+2)x + 1)y' - (x - 2d - 1)y - x + 1 = 0. \tag{6}$$

Since $y = \sum_{n \geq 0} s_d(n)x^n$ and $y' = \sum_{n \geq 0}(n+1)s_d(n+1)x^n$, taking the coefficient of x^{n-1} of (6) (for any $n \geq 3$) yields

$$(n-2)s_d(n-2) - (4d+2)(n-1)s_d(n-1) + ns_d(n) - s_d(n-2) + (2d+1)s_d(n-1) = 0,$$

which can be rearranged to give the desired recurrence (5). □

Next, we look into the asymptotic behavior of $s_d(n)$. The asymptotic formula for $s_1(n)$ is well known — see, for instance, [4, p. 474]. Here, we extend that formula to one that applies for arbitrary, positive d.

Proposition 4. *For every real number $d > 0$, as $n \to \infty$,*

$$s_d(n) \sim \left(\frac{(\sqrt{d+1} - \sqrt{d}) \cdot d^{1/4}}{2(d+1)^{3/4}\pi^{1/2}} \right) \cdot n^{-3/2} \cdot \left(2d + 1 + 2\sqrt{d^2 + d}\right)^n.$$

Proof. We follow the template given in [4, Theorem VI.6, p. 420] to prove our claim. First, recall that $y = \sum_{n \geq 1} s_d(n)x^n$ satisfies the functional equation $y = x\phi(y)$ where $\phi(y) = \frac{1-y}{1-(d+1)y}$. We need to verify the following analytic conditions for ϕ:

H_1: ϕ is a nonlinear function that is analytic at 0 with $\phi(0) \neq 0$ and $[z^n]\phi(z) \geq 0$ for all $n \geq 0$.

H_2: Within the open disc of convergence of ϕ at 0, $|z| < R$, there exists a (then necessarily unique) positive solution s to the characteristic equation $\phi(s) = s\phi'(s)$.

Notice that, expanding $\phi(z)$, we obtain

$$\phi(z) = \frac{1-z}{1-(d+1)z} = 1 + \sum_{n\geq 1} \left((d+1)^n - (d+1)^{n-1}\right) z^n = 1 + d\sum_{n\geq 1}(d+1)^{n-1}z^n,$$

and so H_1 holds for all $d > 0$. For H_2, the radius of convergence of $\phi(z)$ (which is a geometric series) is obviously $R = \frac{1}{d+1}$. Now since $\phi'(z) = \frac{d}{(1+(d+1)z)^2}$, solving $\phi(s) = s\phi'(s)$ yields one solution $s = 1 - \sqrt{\frac{d}{d+1}}$, which lies in $(0, R)$ given $d > 0$. Thus, H_2 holds as well.

Hence, the aforementioned result in [4] applies, and

$$s_d(n) \sim \sqrt{\frac{\phi(s)}{2\phi''(s)}} \cdot \frac{\rho^n}{\sqrt{\pi n^3}}, \tag{7}$$

where $\rho = \frac{\phi(s)}{s}$. It is not hard to check that

$$\begin{aligned}
\phi(s) &= \frac{1}{\sqrt{d+1}(\sqrt{d+1}-\sqrt{d})}, \\
\phi''(s) &= \frac{4(d+1)}{\sqrt{d}\left(\sqrt{d+1}-\sqrt{d}\right)^3}, \\
\rho &= 2d+1+2\sqrt{d^2+d}.
\end{aligned}$$

Substituting these expressions into (7) and simplifying gives the desired result. $\square$

In particular, from Proposition 4 we obtain that the sequence $s_d(n)$ has growth rate

$$\lim_{n\to\infty} \frac{s_d(n+1)}{s_d(n)} = \rho = 2d+1+2\sqrt{d^2+d} \in (4d+1, 4d+2)$$

for all $d > 0$.

3 Some identities and implications

In this section, we will prove several identities related to $s_d(n)$, and describe their combinatorial implications. We will first focus on two special sequences $s_{-1/2}(n)$ and $s_{-1}(n)$, then prove an identity that seems to have a natural connection with large Schröder numbers.

3.1 The case $d = -1/2$

Notice that in the recurrence (5), the coefficient of $s_d(n-1)$ vanishes when $d = \frac{-1}{2}$. In this case, we obtain the sequence:

n	1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$
$s_{-1/2}(n)$	1	$\frac{-1}{2}$	0	$\frac{1}{4}$	0	$\frac{-2}{8}$	0	$\frac{5}{16}$	0	$\frac{-14}{32}$	0	$\frac{42}{64}$	$\cdots$

For every $n \geq 1$, let $c(n) = \frac{1}{n}\binom{2n-2}{n-1}$ denote the n^{th} Catalan number. It is easy to check that $c(1) = 1$, and that

$$c(n) = \frac{2(2n-3)}{n} c(n-1) \tag{8}$$

for all $n \geq 2$. Using this recurrence of $c(n)$ and Proposition 3, we prove the following:

Proposition 5. *For every integer $m \geq 1$, $s_{-1/2}(2m+1) = 0$ and $s_{-1/2}(2m) = \frac{(-1)^m}{2^{2m-1}} c(m)$.*

Proof. We prove our claim by induction on m. First, in the case of $d = \frac{-1}{2}$, (5) reduces to

$$s_{-1/2}(n) = \frac{-(n-3)}{n} s_{-1/2}(n-2). \tag{9}$$

Thus, $s_{-1/2}(3) = 0 s_{-1/2}(1) = 0$. From there on, since $s_{-1/2}(2m+1)$ is a multiple of $s_{-1/2}(2m-1)$ for all $m \geq 1$, we obtain that $s_{-1/2}(2m+1) = 0$ for all $m \geq 1$.

Next, we establish the claim for $s_{-1/2}(2m)$. When $m = 1$,

$$s_{-1/2}(2) = \frac{-1}{2} = \frac{(-1)^1}{2^{2-1}} c(1),$$

so the base case holds. For the inductive step, notice that

$$\begin{aligned} s_{-1/2}(2m) &= \frac{-(2m-3)}{2m} s_{-1/2}(2m-2) \\ &= \frac{-(2m-3)}{2m} \left(\frac{(-1)^{n-1}}{2^{2m-3}} c(m-1) \right) \\ &= \frac{(-1)^n}{2^{2m-1}} \left(\frac{2(2m-3)}{m} c(m-1) \right) \\ &= \frac{(-1)^n}{2^{2m-1}} c(m), \end{aligned}$$

where the last equality follows from (8). This finishes our proof. $\square$

We will return to the quantity $s_{-1/2}(n)$ subsequently in Section 4.

3.2 The case $d = -1$

Next, we turn to the case when $d = -1$. In this case, (2) gives

$$y = \frac{x}{1-x} = \sum_{n \geq 1} (-1)^{n-1} x^i,$$

and so $s_{-1}(n) = (-1)^{n-1}$ for every $n \geq 1$. Since $s_{-1}(n) = \sum_{k \geq 0} s(n,k)(-1)^k$ by definition, we have shown the following:

Proposition 6. *For every integer $n \geq 1$,*

$$\sum_{k \text{ odd}} s(n,k) = \sum_{k \text{ even}} s(n,k) + (-1)^n.$$

Proposition 6 implies that, for every n, the number of trees in $\mathcal{T}_n$ with an odd number of internal nodes and that with an even number of internal nodes differ by exactly 1. Likewise, in terms of small Schröder paths, we now know that the number of paths in $\mathcal{P}_n$ with an odd number of up steps is exactly 1 away from that with an even number of up steps. This leads to the following consequence, which might be folklore but we could not find a mention of in the literature:

Corollary 7. *The small Schröder numbers $s(n)$ is odd for all $n \geq 1$.*

Corollary 7 further implies that, for all odd, positive integers d and for all $n \geq 1$, $s_d(n)$ is odd (since then d^k is odd for all $k \geq 0$, and so $s_d(n)$ would be a sum of an odd number of odd quantities). Likewise, we obtain that for all $n \geq 2$, the n^{th} large Schröder number (which is twice the n^{th} small Schröder number) is congruent to 2 (mod 4).

While we have already described a very short algebraic proof to Proposition 6, we will also provide a simple combinatorial proof.

Proof of Proposition 6. Given a fixed n, consider the set of paths $\mathcal{P}_n$. Let $Q = U^{n-1}D^{n-1}$ (i.e., the path consisting of $n-1$ up steps followed by $n-1$ down steps), and define

$$\mathcal{P}_n^O = \left(\bigcup_{k \text{ odd}} \mathcal{P}_{n,k} \right) \setminus \{Q\}, \quad \mathcal{P}_n^E = \left(\bigcup_{k \text{ even}} \mathcal{P}_{n,k} \right) \setminus \{Q\}.$$

To prove our claim, it then suffices to find a bijection between $\mathcal{P}_n^O$ and $\mathcal{P}_n^E$ for every n. Given a path $P \in \mathcal{P}_n$ where $P \neq Q$, P must then contain either a flat step or a valley (or both). We define $\alpha(P)$ as follows:

(1) If the first valley in P occurs before the first flat step, we write $P = P_1 DU P_2$ where P_1 does not contain a flat step, and define $\alpha(P) = P_1 F P_2$.

(2) Otherwise, the first flat step occurs before the first valley. In this case, we write P as $P_1 F P_2$ where P_1 does not contain an instance of DU, and define $\alpha(P) = P_1 DU P_2$.

Figure 4 illustrates the mapping α on paths in $\mathcal{P}_4$. Intuitively, given a path, α finds the first appearance of a valley DU or a flat step F, whichever comes first. If it is a valley, then α replaces it by a flat step, and vice versa.

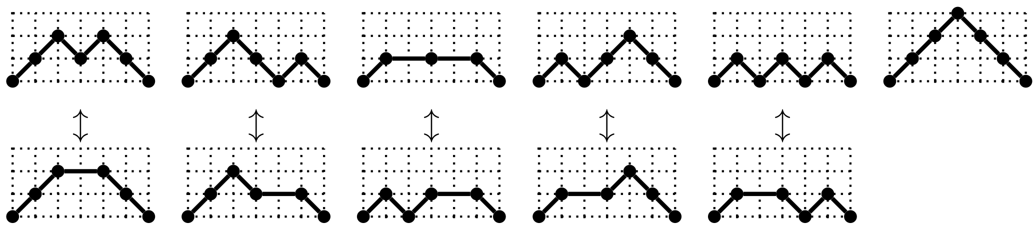

Figure 4: Illustrating the mapping α on paths in $\mathcal{P}_4$ (Proof of Proposition 6). Paths on the first (resp., second) row have an odd (resp., even) number of up steps.

Now observe that in any case, the number of up steps in P and in $\alpha(P)$ differ by exactly one, so α can be considered as a mapping from $\mathcal{P}_n^O$ to $\mathcal{P}_n^E$. Moreover, it is easy to see that $\alpha(\alpha(P)) = P$ for all paths $P \in \mathcal{P}_n \setminus \{Q\}$, and so α is in fact a bijection between P_n^O and P_n^E. This finishes our proof. $\square$

3.3 Connection to large Schröder numbers

Here, we prove another identity for $s_d(n)$, and briefly discuss a version of weighted large Schröder numbers that is analogous to $s_d(n)$. We first have the following:

Proposition 8. *For every real number $d \neq -1$ and every integer $n \geq 2$,*

$$s_d(n) = \frac{(-1)^{n-1} d}{d+1} s_{-d-1}(n).$$

Proof. Recall the generating function $y = \sum_{n \geq 1} s_d(n) x^n$. We can rewrite (2) as the functional equation $y = x \left(\frac{1-y}{1-(d+1)y} \right)$, apply Lagrange inversion, and obtain that

$$\begin{aligned} s_d(n) = [x^n]y &= \frac{1}{n}[y^{n-1}] \left(\frac{1-y}{1-(d+1)y} \right)^n \\ &= \frac{1}{n}[y^{n-1}] \left(\sum_{i \geq 0} \binom{n}{i} (-1)^i \right) \left(\sum_{j \geq 0} \binom{n+j-1}{n-1} (d+1)^j \right) \\ &= \frac{1}{n} \left(\sum_{i \geq 0} (-1)^i \binom{n}{i} \binom{2n-i-2}{n-1} (d+1)^{n-i-1} \right). \end{aligned}$$

Since $\binom{2n-i-2}{n-1} = 0$ when $i > n-1$, the sum could simply run from $i = 0$ to $i = n-1$. Moreover, by re-indexing the sum using $j = n-i-1$, we have

$$\begin{aligned} s_d(n) &= \frac{1}{n}\left(\sum_{j=0}^{n-1}(-1)^{n-j-1}\binom{n}{n-j-1}\binom{2n-(n-j-1)-2}{n-1}(d+1)^j\right) \\ &= (-1)^{n-1}\left(\sum_{j=0}^{n-1}\frac{1}{n}\binom{n}{j+1}\binom{n+j-1}{n-1}(-d-1)^j\right) \end{aligned}$$

Now notice that

$$\begin{aligned} \frac{1}{n}\binom{n}{j+1}\binom{n+j-1}{n-1} &= \frac{1}{n-1}\left(\binom{n-1}{j}\binom{n+j-1}{n} + \binom{n-1}{j+1}\binom{n+j}{n}\right) \\ &= s(n,j) + s(n,j+1). \end{aligned}$$

Hence, we have

$$\begin{aligned} s_d(n) &= (-1)^{n-1}\sum_{j=0}^{n-1}(s(n,j)+s(n,j+1))(-d-1)^j \\ &= (-1)^{n-1}\left(\sum_{j=0}^{n-1}s(n,j)(-d-1)^j - \frac{1}{d+1}\sum_{j=0}^{n-1}s(n,j+1)(-d-1)^{j+1}\right) \\ &= (-1)^{n-1}\frac{d}{d+1}s_{-d-1}(n), \end{aligned}$$

finishing the proof of our claim. Note that in the second equality above we used the fact that $s(n,0) = s(n,n) = 0$ for all $n \geq 2$, as well as the assumption that $d \neq -1$. $\square$

Notice that Proposition 8 rearranges to $(-1)^{n-1}s_{-d-1}(n) = \frac{d+1}{d}s_d(n)$ (for $d \neq 0, -1$), and that we encountered the expression $s(n,j) + s(n,j+1)$ in its proof. We note that these quantities have a natural connection with the large Schröder numbers. For every integer $n \geq 1$, we define the set of *large Schröder paths* $\overline{\mathcal{P}}_n$ to be the set of lattice paths from $(0,0)$ to $(2n-2,0)$ that satisfy properties (P1) and (P2) (but not necessarily (P3)) in the definition of small Schröder paths. Thus, $\mathcal{P}_n$ is a subset of $\overline{\mathcal{P}}_n$ for every n. Figure 5 illustrates the large Schröder paths in $\overline{\mathcal{P}}_4$ that do not belong to $\mathcal{P}_4$.

Let $\overline{s}(n) = |\overline{\mathcal{P}}_n|$. Then $\overline{s}(n)$ gives the large Schröder numbers (A006318 in the OEIS):

n	1	2	3	4	5	6	7	8	$\cdots$
$\overline{s}(n)$	1	2	6	22	90	394	1806	8558	$\cdots$

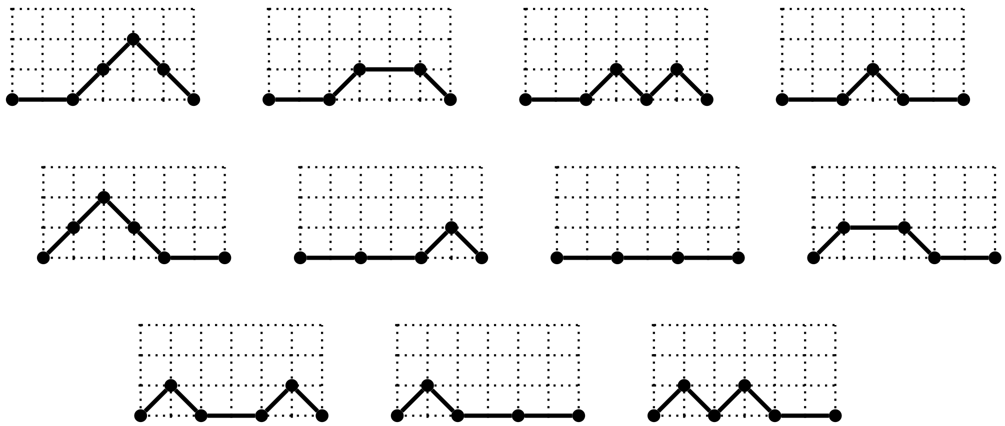

Figure 5: The 11 large Schröder paths in $\overline{\mathcal{P}}_4 \setminus \mathcal{P}_4$

It is well known that $\overline{s}(n) = 2s(n)$ for every $n \geq 2$. To see this, consider a path $P \in \overline{\mathcal{P}_n} \setminus \mathcal{P}_n$, and write $P = P_1FP_2$ where P_1 does not contain a flat step on the x-axis. Then it is easy to show that the mapping $\beta : \overline{\mathcal{P}}_n \setminus \mathcal{P}_n \to \mathcal{P}_n$ where $\beta(P) = P_1UP_2D$ is a bijection, thus showing that $|\overline{\mathcal{P}}_n| = 2|\mathcal{P}_n|$. Moreover, notice that $\beta(P)$ has exactly one more up step than P. Thus, if we let $\overline{\mathcal{P}}_{n,k}$ denote the set of paths in $\overline{\mathcal{P}}_n$ with exactly k up steps, then we have

$$\overline{\mathcal{P}}_{n,k} = \mathcal{P}_{n,k} \cup \{P : \beta(P) \in \mathcal{P}_{n,k+1}\}.$$

The sets $\mathcal{P}_{n,k}$ and $\{P : \beta(P) \in \mathcal{P}_{n,k+1}\}$ are obviously disjoint. Thus, if we let $\overline{s}(n,k) = |\overline{\mathcal{P}}_{n,k}|$, we have

$$\overline{s}(n,k) = s(n,k) + s(n,k+1)$$

for every $n \geq 2$ and every $k \in \{0, \ldots, n-1\}$. Furthermore, given a real number d, if we define the weighted large Schröder number $\overline{s}_d(n) = \sum_{k=0}^{n-1} \overline{s}(n,k)d^k$, then we see that

$$\overline{s}_d(n) = \sum_{k=0}^{n-1} \overline{s}(n,k)d^k = \sum_{k=0}^{n-1} (s(n,k) + s(n,k+1))d^k = \frac{d+1}{d} s_d(n).$$

Thus, Proposition 8 can be written as simply

$$\overline{s}_d(n) = (-1)^{n-1} s_{-d-1}(n)$$

for all $d \neq 0, -1$ and $n \geq 2$. We will revisit Proposition 8 from the perspective of Dyck paths in the next section.

4 $s_d(n)$ in terms of Dyck paths

We next discuss several families of Dyck paths that are counted by $s_d(n)$. Recall that a Dyck path is a small Schröder path with only up and down steps (i.e., no flat steps). Let

$\mathcal{D}_n$ be the set of Dyck paths that starts at $(0,0)$ and ends at $(2n-2,0)$. (Equivalently, $\mathcal{D}_n = \mathcal{P}_{n,n-1}$.) Also, it will be convenient to have the following notation for our subsequent discussion: Given a path $P \in \mathcal{P}_n$, let

- $U(P)$, $F(P)$, and $D(P)$ respectively denote the number of up, flat, and down steps in P;
- $V(P)$ denote the number valleys (i.e., occurrences of DU) in P;
- $K(P)$ denote the number peaks (i.e., occurrences of UD) in P;
- $U_V(P)$ (resp., $D_V(P)$) denote the number of up (resp., down) steps in P that is not contained in a valley;
- $U_K(P)$ (resp., $D_K(P)$) denote the number of up (resp., down) steps in P that is not contained in a peak.

Recently, Chen and Pan [3] studied the following notion of weighted Catalan numbers: Given real numbers a and b, define

$$c^V_{a,b}(n) = \sum_{P \in \mathcal{D}_n} a^{U_V(P)} b^{V(P)}. \tag{10}$$

Then $c^V_{1,1}(n)$ gives the ordinary Catalan numbers $c(n)$. Interestingly, $c^V_{a,b}(n)$ also coincides with $s_d(n)$ for certain choices of a and b.

Proposition 9. *For every real number d and integer $n \geq 1$,*

$$s_d(n) = c^V_{d,d+1}(n).$$

Proof. Given a path $P \in \mathcal{D}_n$, let $S \subseteq \{1, \ldots, V(P)\}$ be an arbitrary subset of the valleys of P. Define the function $f_S : \mathcal{D}_n \to \mathcal{P}_n$ such that $f_S(P)$ is the path obtained from replacing every valley not in S by a flat step F. Observe that by this construction, the resulting path $f_S(P)$ is a small Schröder path with $U_V(P) + |S|$ up steps.

Now observe that, for every path $P \in \mathcal{D}_n$,

$$d^{U_V(P)}(d+1)^{V(P)} = \sum_{S \subseteq \{1,\ldots,V(P)\}} d^{U_V(P)+|S|} = \sum_{S \subseteq \{1,\ldots,V(P)\}} d^{U(f_S(P))} \tag{11}$$

Also, every path $Q \in \mathcal{P}_n$ can be written as $f_S(P)$ for a unique choice of $P \in \mathcal{D}_n$ and subset S (namely, let P be the path with all flat steps in Q replaced by valleys, and let S be the set of valleys in P that are valleys in Q). Thus, if we sum over all paths in $\mathcal{D}_n$ on both sides of (11), we obtain

$$c^V_{d,d+1}(n) = \sum_{P \in \mathcal{D}_n} d^{U_V(P)}(d+1)^{V(P)} = \sum_{Q \in \mathcal{P}_n} d^{U(Q)} = s_d(n).$$

□

We remark that Proposition 9 also follows from [3, eq. (1.15)] and our discussion in Section 3.3 showing that $\overline{s}_d(n) = \frac{d+1}{d} s_d(n)$. Also, notice that when d is a positive integer, we obtain that $s_d(n)$ counts the number ways to construct a Dyck path from $(0,0)$ to $(2n-2,0)$ where each valley can be painted one of $d+1$ colors, and each up step that does not belong to a valley can be painted one of d colors. Moreover, it is obvious that

$$c^V_{-1,0}(n) = \sum_{P \in \mathcal{D}_n} (-1)^{U_V(P)} (0)^{V(P)} = (-1)^{n-1}$$

for ever $n \geq 1$, since the only Dyck path in $\mathcal{D}_n$ with no valleys is the path $U^{n-1}D^{n-1}$. Thus, in the case of $d = -1$, Proposition 9 implies Proposition 6.

Next, we show that $c^V_{a,b}(n)$ can be related to $s_d(n)$ even when $b \neq a+1$.

Corollary 10. *For every integer $n \geq 1$ and every real number a, b where $a \neq b$,*

$$c^V_{a,b}(n) = (b-a)^{n-1} s_{a/(b-a)}(n).$$

Proof.

$$\begin{aligned}
c^V_{a,b}(n) &= \sum_{P \in \mathcal{D}_n} a^{U_V(P)} b^{V(P)} \\
&= (b-a)^{n-1} \sum_{P \in \mathcal{D}_n} \left(\frac{a}{b-a} \right)^{U_V(P)} \left(\frac{a}{b-a} + 1 \right)^{V(P)} \\
&= (b-a)^{n-1} c^V_{a/(b-a).a/(b-a)+1}(n) \\
&= (b-a)^{n-1} s_{a/(b-a)}(n).
\end{aligned}$$

Note that the second equality follows from the fact that $U_V(P) + V(P) = U(P) = n-1$ for all $P \in \mathcal{D}_n$. ∎

Corollary 10 readily implies that certain families of small Schröder paths and Dyck paths have the same size. For instance:

Corollary 11. *For all integers $m, n \geq 1$, the following quantities are both equal to $m^{n-1} s_{1/m}(n)$:*

(i) the number of small Schröder paths in $\mathcal{P}_n$ where each flat step can be painted one of m colors;

(ii) the number of Dyck paths in $\mathcal{D}_n$ where each valley can be painted one of $m+1$ colors.

Proof. From Corollary 10, we obtain that

$$m^{n-1} s_{1/m}(n) = c^V_{1,m+1}(n),$$

showing (ii). For (i),

$$m^{n-1} s_{1/m}(n) = m^{n-1} \sum_{P \in \mathcal{P}_n} (1/m)^{U(P)} 1^{F(P)} = \sum_{P \in \mathcal{P}_n} m^{F(P)},$$

since $U(P) + F(P) = n-1$ for every $P \in \mathcal{P}_n$. This finishes the proof. ∎

We next study another variant of weighted Catalan numbers that, as we shall see, is equal to $c^V_{a,b}(n)$. Given real numbers a and b, define

$$c^K_{a,b}(n) = \sum_{P \in \mathcal{D}_n} a^{K(P)} b^{U_K(P)}.$$

Then we have the following:

Proposition 12. *For all real numbers a, b and integer $n \geq 1$*

$$c^V_{a,b}(n) = c^K_{a,b}(n).$$

To prove Proposition 12, we need another tree-to-path procedure. Recall the mapping Ψ described in Definition 1, where we turn a tree into a path by walking counterclockwise around the tree starting at the root. The following mapping Ψ' is very similar to Ψ, except we walk clockwise around the tree this time.

Definition 13. Given a Schröder tree $T \in \mathcal{T}_n$, construct the path $\Psi'(T) \in \mathcal{P}_n$ as follows:

1. Suppose T has q nodes. We perform an postorder traversal of T in reverse (i.e., we recursively traverse the tree in the order of root, then right, then left), and label the nodes $a_1, a_2, \ldots, a_q$ in that order.

2. Notice that a_1 must be the root of T. For every $i \geq 2$, define the function

$$\psi'(a_i) = \begin{cases} U & \text{if } a_i \text{ is the rightmost child of its parent;} \\ D & \text{if } a_i \text{ is the leftmost child of its parent;} \\ F & \text{otherwise.} \end{cases}$$

3. Define
$$\Psi'(T) = \psi'(a_2)\psi'(a_3)\cdots\psi'(a_q).$$

Figure 6 illustrates the mapping Ψ' applied to the same tree shown in Figure 3.

For essentially the same reasons why $\Psi : \mathcal{T}_n \to \mathcal{P}_n$ is a bijection, Ψ' is a bijection between these two sets as well. Moreover, both Ψ, Ψ' are in fact bijections between $\mathcal{T}_{n,n-1}$ (which are necessarily full binary trees) and $\mathcal{P}_{n,n-1} = \mathcal{D}_n$. The follow result relates various statistics of the Dyck paths obtained from applying Ψ and Ψ' to the same binary tree.

Lemma 14. *For every tree $T \in \mathcal{T}_{n,n-1}$ where $n \geq 2$, we have*

$$K(\Psi(T)) + K(\Psi'(T)) = n, \tag{12}$$
$$K(\Psi(T)) = U_V(\Psi'(T)), \tag{13}$$
$$U_K(\Psi(T)) = V(\Psi'(T)). \tag{14}$$

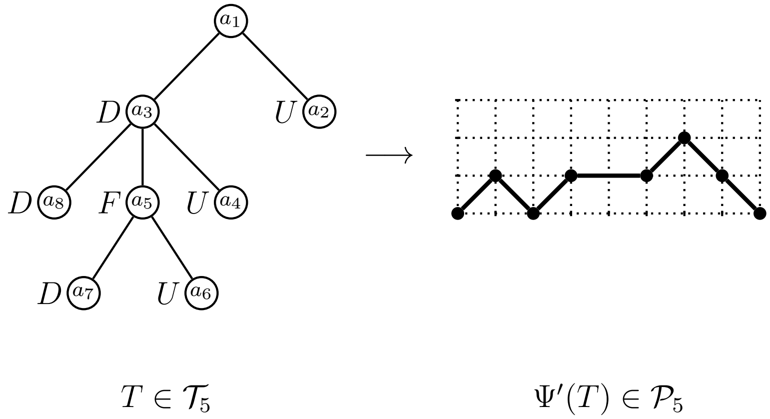

Figure 6: Illustrating the mapping Ψ' (Definition 13)

Proof. We first prove (12) by induction on n. When $n = 2$, there is a unique binary tree T in $\mathcal{T}_{2,1}$ and it is easy to check that $K(\Psi(T)) + K(\Psi'(T)) = 1 + 1 = 2$, so the base case holds. For the inductive step, suppose the root of T has left subtree T_1 and right subtree T_2, where T_1, T_2 have respectively n_1 and n_2 leaves.

Now observe that $\Psi(T) = U\Psi(T_1)D\Psi(T_2)$. Since both $\Psi(T_1)$ and $\Psi(T_2)$ are Dyck paths in their own rights, each must either start with a U and end with a D, or is empty. Hence, we obtain that $K(\Psi(T)) = K(\Psi(T_1)) + K(\Psi(T_2))$.

Similarly, we see that $\Psi'(T) = U\Psi'(T_2)D\Psi'(T_1)$, and $K(\Psi'(T)) = K(\Psi'(T_1)) + K(\Psi'(T_2))$. Thus, using the inductive hypothesis, we obtain

$$K(\Psi(T)) + K(\Psi'(T)) = K(\Psi(T_1)) + K(\Psi(T_2)) + K(\Psi'(T_1)) + K(\Psi'(T_2)) = n_1 + n_2 = n.$$

This proves (12).

We next prove (13) and (14). Notice that $V(P) = K(P) - 1$ for every $P \in \mathcal{D}_n$. Thus, applying (12) gives

$$K(\Psi(T)) + V(\Psi'(T)) = n - 1.$$

Furthermore, since $U(\Psi(T)) = U(\Psi'(T)) = n - 1$, we have

$$U_K(\Psi(T)) + K(\Psi(T)) = n - 1,$$
$$U_V(\Psi'(T)) + V(\Psi'(T)) = n - 1.$$

This implies that $K(\Psi(T)) = U_V(\Psi'(T))$ and $U_K(\Psi(T)) = V(\Psi'(T))$, as desired. $\square$

We are now ready to prove Proposition 12.

Proof of Proposition 12. Define the mapping $\gamma : \mathcal{D}_n \to \mathcal{D}_n$ where $\gamma(P) = \Psi'(\Psi^{-1}(P))$. Since $\Psi, \Psi' : \mathcal{T}_{n,n-1} \to \mathcal{D}_n$ are both bijections, γ is a bijection between $\mathcal{D}_n$ and itself. Hence,

$$c^K_{a,b}(n) = \sum_{P \in \mathcal{D}_n} a^{K(P)} b^{U_K(P)} = \sum_{P \in \mathcal{D}_n} a^{U_V(\gamma(P))} b^{V(\gamma(P))} = c^V_{a,b}(n),$$

as claimed. Note that the second equality follows from Lemma 14. $\square$

We next point out how Propositions 5 and 8 in Section 3 can be alternatively shown using the connections between $s_d(n), c^V_{a,b}(n)$, and $c^K_{a,b}(n)$ established above. First, we revisit Propsition 5 and the sequence $s_{-1/2}(n)$. For every integer $n \geq 2$ and $k \in \{0, \ldots, n-1\}$, define

$$c(n,k) = \frac{1}{n-1}\binom{n-1}{k-1}\binom{n-1}{k},$$

and let $c(1,0) = 1$. These are known as the Narayana numbers (A090181 in the OEIS), and $c(n,k)$ counts the number of Dyck paths in $\mathcal{D}_n$ with exactly k up steps.

$c(n,k)$	$k=0$	1	2	3	4	5	$\cdots$
$n=1$	1						
2	0	1					
3	0	1	1				
4	0	1	3	1			
5	0	1	6	6	1		
6	0	1	10	20	10	1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

If we define the generating function

$$C_d(x) = \sum_{n \geq 1} \sum_{k \geq 0} c(n,k) d^k x^n,$$

then it is not hard to check that $C_d(x)$ satisfies the functional equation

$$C_d(x)^2 + (dx - x - 1)C_d(x) + x = 0.$$

(See, for instance, [16, Exercise 6.36b] for the details of the derivation.) Solving the above gives

$$C_d(x) = \frac{1 + x - dx - \sqrt{(dx-x-1)^2 - 4x}}{2}.$$

When $d = -1, 1$, this specializes to

$$C_1(x) = \frac{1 - \sqrt{1-4x}}{2}, C_{-1}(x) = \frac{1 + 2x - \sqrt{1+4x^2}}{2}.$$

Obviously, $C_1(x)$ is the generating function for the ordinary Catalan numbers, and so $[x^n]C_1(x) = c(n)$ for every $n \geq 1$. Also notice that $C_1(-x^2) = C_{-1}(x) - x$. Putting things together, we see that, for every $n \geq 2$,

$$2^{n-1} s_{-1/2}(n) = c^K_{-1,1}(n) = \sum_{P \in \mathcal{D}_n} (-1)^{K(P)} = \sum_{P \in \mathcal{D}_n} c(n,k)(-1)^k$$

$$= [x^n]C_{-1}(x) = [x^n]C_1(-x^2)$$
$$= \begin{cases} 0 & \text{if } n \text{ is odd;} \\ c(n/2) & \text{if } n \equiv 0 \pmod 4; \\ -c(n/2) & \text{if } n \equiv 2 \pmod 4, \end{cases}$$

which aligns with Proposition 5. Moreover, the fact that $2^{n-1}s_{-1/2}(n) = c^K_{-1,1}(n)$ also implies that

$$2^{n-1}s_{-1/2}(n) = \sum_{k \text{ even}} c(n,k) + \sum_{k \text{ odd}} c(n,k)$$

for all $n \geq 1$. Thus, we see that given a fixed length, the number of Dyck paths with an odd number of peaks and that with an even number of peaks are either identical, or differ by a Catalan number. This produces the following two sequences:

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
$\lvert\{P \in \mathcal{D}_n : K(P) \text{ is even}\}\rvert$	0	0	1	3	7	20	66	217	715	2424	$\cdots$
$\lvert\{P \in \mathcal{D}_n : K(P) \text{ is odd}\}\rvert$	1	1	1	2	7	22	66	212	715	2438	$\cdots$

These two sequences, with the terms for $n = 1$ removed, give A071688 and A071684 in the OEIS, respectively.

Next, we re-consider Proposition 8. Notice that for every real number $m \neq 1$,

$$c^K_{m,1}(n) = \sum_{P \in \mathcal{D}_n} m^{K(P)} = m \sum_{P \in \mathcal{D}_n} m^{V(P)} = mc^V_{1,m}(n).$$

Thus, applying Corollary 10 and substituting $d = \frac{1}{m-1}$, we obtain

$$mc^V_{1,m}(n) = m(m-1)^{n-1}s_{1/(m-1)}(n) = \left(\frac{d+1}{d}\right) d^{-(n-1)}s_d(n)$$

Likewise, we also have

$$c^K_{m,1}(n) = (1-m)^{n-1}s_{m/(1-m)}(n) = (-1)^{n-1}d^{-(n-1)}s_{-d-1}(n),$$

which yields a proof of Proposition 8 for all cases where $d \neq 0$.

Finally, inspired by David Scambler's comments in the OEIS, we show how $s_d(n)$ counts families of Dyck paths with certain forbidden peak types. Scambler claimed (in A107841, slightly paraphrased here) that $s_2(n)$ counts the number of Dyck paths from $(0,0)$ to $(2n-2,0)$ with 3 types of up steps U_1, U_2, U_3, one type of down step D, and avoid U_1D. Similarly, he commented (A131763) that $s_3(n)$ gives the number of Dyck paths from $(0,0)$ to $(2n-2,0)$ with two types of up steps U_1, U_2, two types of down steps D_1, D_2, and avoid U_1D_1. Independently, Geffner and Noy [6, Theorem 3] showed that $s(n)$ counts the number of Dyck paths in $\mathcal{D}_n$ with one type of up step U, two types of down steps D_1, D_2, and avoid UD_1. We show that a generalization of these statements follow readily from our findings above.

Proposition 15. *Given integers $k, \ell \geq 1$, let $D_{k,\ell}(n)$ denote the number of ways to construct Dyck paths from $(0,0)$ to $(2n-2, 0)$ with k types of up steps $U_1, \ldots, U_k$, ℓ types of down steps $D_1, \ldots, D_\ell$, and avoid peaks of the type U_1D_1. Then*

$$D_{k,\ell}(n) = s_{k\ell-1}(n).$$

for every $n \geq 1$.

Proof. Notice that $D_{k,\ell}(n)$ can be reinterpreted as the number of Dyck paths where

- each of the $K(P)$ peaks of P is assigned $k\ell - 1$ colors (since there are a total of $k\ell$ types of possible peaks, 1 of which is forbidden);
- each of the $U_K(P)$ up steps not contained in a peak is assigned one of k colors;
- each of the $D_K(P)$ down steps not contained in a peak is assigned one of ℓ colors.

Hence, we obtain that

$$\begin{aligned} D_{k,\ell}(n) &= \sum_{P \in \mathcal{D}_n} k^{U_K(P)}(k\ell-1)^{K(P)}\ell^{D_K(P)} \\ &= \sum_{P \in \mathcal{D}_n} (k\ell)^{U_K(P)}(k\ell-1)^{K(P)} \\ &= c^K_{k\ell-1,k\ell}(n) \\ &= c^V_{k\ell-1,k\ell}(n) \\ &= s_{k\ell-1}(n). \end{aligned}$$

Note that the second equality above is due to $U_K(P) = D_K(P)$ for every Dyck path P, and the last two equalities are due to Propositions 9 and 12, respectively. □

Note that the above argument can be easily extended to cases where more than 1 of the kl possible peak types are forbidden. More precisely, one could show that the number of Dyck paths from $(0,0)$ to $(2n-2, 0)$ with k types of up steps, l types of down steps, with p of the $k\ell$ peak types forbidden, is $c^K_{k\ell-p,k\ell}(n) = p^{n-1}s_{(k\ell-p)/p}(n)$.

5 Concluding Remarks

In this manuscript, we looked at $s_d(n)$, a natural generalization of the small Schröder numbers that had made cameos in a number of contexts but was never the main focus of study until now. We also saw that the study of $s_d(n)$ led to implications for familiar combinatorial objects such as Schröder paths and Dyck paths.

We remark that there are many natural combinatorial interpretations of $s_d(n)$ that we have yet to mention. For instance, as we have seen with Schröder trees and small Schröder paths, any combinatorial interpretation of $s(n,k)$ readily extends to a corresponding set of objects counted by $s_d(n)$ for positive integers d. For another example, it is known [14] that $s(n,k)$ counts the number of ways to subdivide the regular $(n+1)$-gon into k regions using non-crossing diagonals. Therefore, $s_d(n)$ gives the number of ways to subdivide the regular $(n+1)$-gon and color each of the regions using one of d colors.

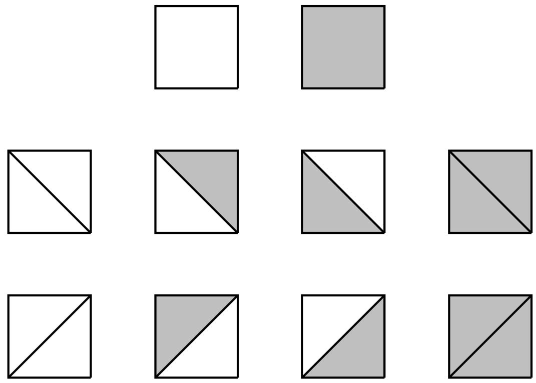

Figure 7: The $s_2(3) = 10$ ways to 2-color subdivisions of a square

Other interpretations of $s(n,k)$ (e.g., in terms of standard Young tableaux [14], and loopless outerplanar maps [6]) can be extended similarly, and it is possible that $s_d(n)$ — or other generalizations of related integer sequences — can lend new perspectives to solving combinatorial problems involving these familiar quantities.

6 Acknowledgments

Finally, we express our gratitude to Neil J. A. Sloane and the numerous contributors to the On-Line Encyclopedia of Integer Sequences for developing and curating such a valuable resource for mathematical discovery.

References

[1] J. Abate and W. Whitt, Integer sequences from queuing theory, *Journal of Integer Sequences*, **13** (2010), Article 10.5.5.

[2] P. Barry, Embedding structures associated with Riordan arrays and moment matrices, *International Journal of Combinatorics*, **2014** (2014).

[3] Z. Chen and H. Pan, Identities involving weighted Catalan, Schröder and Motzkin paths, *Advances in Applied Mathematics*, **86** (2017), 81–98.

[4] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press, 2009.

[5] D. Foata and D. Zeilberger, A classic proof of a recurrence for a very classical sequence, *Journal of Combinatorial Theory, Series A*, **80** (1997), 380–384.

[6] I. Geffner and M. Noy, Counting outerplanar maps, *The Electronic Journal of Combinatorics*, **24(2)** (2017), 1–8.

[7] I. Gessel, Schröder numbers, large and small, 2009. Available at http://people.brandeis.edu/~gessel/homepage/slides/schroder.pdf.

[8] S. Giraudo, Operads from posets and Koszul duality, *European Journal of Combinatorics*, **56** (2016), 1–32.

[9] J. Huh and S. Park, Generalized small Schröder numbers, *The Electronic Journal of Combinatorics*, **22(3)** (2015), 3–14.

[10] F. Lam, Integral invariance and non-linearity reduction for proliferating vorticity scales in fluid dynamics, 2013. Available at https://arxiv.org/abs/1311.6395

[11] J. Schröder, Generalized Schröder numbers and the rotation principle, *Journal of Integer Sequences*, **7** (2007), Article 07.7.7.

[12] L. W. Shapiro and R. A. Sulanke, Bijections for the Schröder numbers, *Mathematics Magazine*, **73(5)** (2000), 369–376.

[13] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2019. Available at https://oeis.org.

[14] R. P. Stanley, Polygon dissections and standard Young tableaux, *Journal of Combinatorial Theory, Series A*, **76(1)** (1996), 175–177.

[15] R. P. Stanley, Hipparchus, Plutarch, Schröder, and Hough, *The American Mathematical Monthly*, **104(4)** (1997), 344–350.

[16] R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge Studies in Advanced Mathematics, 1999.

[17] R. A. Sulanke, Bijective recurrences concerning Schröder paths, *The Electronic Journal of Combinatorics*, **5** (1998), Article R47.

[18] R. A. Sulanke, Generalizing Narayana and Schroder numbers to higher dimensions, *The Electronic Journal of Combinatorics*, **11** (2004), Article R54.

2010 *Mathematics Subject Classification*: Primary 05A15. Secondary 05A16, 05A19.

Keywords: Schröder number, Schröder path, Catalan number, Dyck path, Narayana number, plane rooted tree.

Concerned with sequences A000108, A001003, A006318, A071684, A071688, A086810, A090181, A107841, A131763, A131765.
