Riordan arrays, the A-matrix, and Somos 4 sequences

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Abstract

We characterize certain Riordan arrays by their A-matrices and ρ sequences. We conjecture the form of a generic A-matrix which leads to Somos 4 sequences. We find an A-matrix that produces a Riordan quasi-involution, and we study the A-matrices and ρ sequences of the moment matrices of certain perturbed orthogonal polynomials.

1 Introduction

An important feature of Riordan arrays [3, 10, 11] is that they have a sequence characterization. Specifically, a lower-triangular array $(t_{n,k})_{0 \le n,k \le \infty}$ is a Riordan array if and only if there exists a sequence $a_n, n \ge 0$, such that

$$t_{n,k} = \sum_{i=0}^{\infty} a_i t_{n-1,k-1+i}$$

This sum is actually a finite sum, since the matrix is lower-triangular. For such a matrix M, we let

$$P = M^{-1}\overline{M}.$$

where \overline{M} is the matrix M with its top row removed. Then the matrix P has the form that begins

| 1 | z_0 | a_0 | 0 | 0 | 0 | 0 | 0 | ١ |
|---|-------|-------|-------|-------|-------|-------|---------|---|
| | z_1 | a_1 | a_0 | 0 | 0 | 0 | 0 | |
| | z_2 | a_2 | a_1 | a_0 | 0 | 0 | 0 | |
| | z_3 | a_3 | a_2 | a_1 | a_0 | 0 | 0 | |
| | z_4 | a_4 | a_3 | a_2 | a_1 | a_0 | 0 | |
| | z_5 | a_5 | a_4 | a_3 | a_2 | a_1 | a_0 | |
| / | z_6 | a_6 | a_5 | a_4 | a_3 | a_2 | a_1 / | ļ |

Here, z_0, z_1, z_2, \ldots is an ancillary sequence which exists for any lower-triangular matrix. The matrix P is called the *production matrix* of the Riordan matrix M. The sequence characterization of renewal arrays was first described by Rogers [6, 9]. An alternative approach to the sequence characterization of a Riordan array is to use a matrix characterization [7, 8]. One form that such a matrix characterization may take is the following [7, 8].

Theorem 1. A lower-triangular array $(t_{n,k})_{0 \le n,k \le \infty}$ is a Riordan array if and only if there exists another array $A = (a_{i,j})_{i,j \in \mathbb{N}_0}$ with $a_{0,0} \ne 0$, and a sequence $(\rho_j)_{j \in \mathbb{N}_0}$ such that

$$t_{n+1,k+1} = \sum_{i \ge 0} \sum_{j \ge 0} a_{i,j} t_{n-i,k+j} + \sum_{j \ge 0} \rho_j t_{n+1,k+j+2}$$

The power series definition of a Riordan array is as follows. A Riordan array is defined by a pair of power series, g(x) and f(x), where

$$g(x) = g_0 + g_1 x + g_2 x^2 + \cdots, \quad g_0 \neq 0,$$

and

$$f(x) = f_1 x + f_2 x^2 + f_3 x^3 + \cdots, \quad f_0 = 0 \text{ and } f_1 \neq 0.$$

We then have

$$t_{n,k} = [x^n]g(x)f(x)^k,$$

where $[x^n]$ is the functional that extracts the coefficient of x^n . The relationship between f(x) and the pair (A, ρ) is the following.

$$\frac{f(x)}{x} = \sum_{i \ge 0} x^i R^{(i)}(f(x)) + \frac{f(x)^2}{x} \rho(f(x)),$$

where $R^{(i)}$ is the generating series of the *i*-th row of A, and $\rho(x)$ is the generating series of the sequence ρ_n .

We shall call the matrix $A = (a_{i,j})_{i,j \in \mathbb{N}_0}$ the A-matrix, while we call the sequence a_n the A-sequence of the Riordan array. The generating function of a_n is denoted by A(x). We have

$$A(x) = \frac{x}{\bar{f}(x)}$$

where $\bar{f}(x)$ is the compositional inverse of f(x) (thus we have $f(\bar{f}(x)) = x$ and $\bar{f}(f(x)) = x$). We sometimes write $\bar{f}(x) = \text{Rev}(f)(x)$.

The generating function of the sequence z_n is denoted by Z(x). We have

$$(g(x), f(x)) = \left(\frac{A(x) - xZ(x)}{A(x)}, \frac{x}{A(x)}\right)^{-1}.$$

Note that while the production matrix P of a Riordan array is unique, the pair (A, ρ) is not necessarily unique.

The set of Riordan arrays (g(x), f(x)) is a group under the multiplication law

$$(g(x), f(x)) \cdot (u(x), v(x)) = (g(x)u(f(x)), v(f(x)))$$

and the inverse of (g(x), f(x)) in the group is given by

$$(g(x), f(x))^{-1} = \left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right)$$

where $\bar{f}(x)$ is the compositional inverse of f(x).

The set of Riordan arrays of the form $\left(\frac{f(x)}{x}, f(x)\right)$ is a subgroup of the Riordan group, called the subgroup of Bell matrices.

We shall refer to sequences, where known, by their Annnnn numbers in the Online Encyclopedia of Integer Sequences [12, 13]. The sequence 0^n is the sequence that begins $1, 0, 0, 0, \ldots$, with generating function $x^0 = 1$. Note that we only show suitable truncations of Riordan arrays and of production arrays.

Example 2. Pascal's triangle (also called the binomial matrix) $\binom{n}{k}$ is the Riordan array

$$\left(\frac{1}{1-x},\frac{x}{1-x}\right).$$

This begins

The A-sequence for this triangle is given by A(x) = 1 + x. It can also be defined by the A-matrix

$$A = \left(\begin{array}{rrr} 1 & 0 & 0 \\ 0 & -1 & -1 \end{array}\right),$$

and the ρ sequence $1, 0, 0, \ldots$

This follow since the solution to the equation

$$\frac{u}{x} = 1 - x(u + u^2) + \frac{u^2}{x}$$

is

$$u = f(x) = \frac{x}{1 - x}.$$

We shall call the result of multiplying a Riordan array (on the left) by the binomial matrix as its *binomial transform*.

Example 3. The identity matrix (1, x) can be defined by the A-matrix

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ -1 & -1 & 0 \end{array}\right),$$

and the ρ sequence 1, 0, 0, This follows since the equation

$$\frac{u}{x} = 1 + u^2 - x(1+u) + \frac{u^2}{x}$$

has the solution u = f(x) = x.

Another A-matrix and ρ sequence pair that define the identity matrix is given by

$$A = \left(\begin{array}{rrr} 1 & -1 & 1 \\ 0 & -1 & 0 \end{array}\right),$$

and the ρ sequence $1, 0, 0, \ldots$

It can also be defined by the simpler A-matrix A = (1) and $\rho_n = 0$.

Example 4. We consider the Motzkin triangle defined by the Riordan array

$$\left(\frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}, \frac{1-x-\sqrt{1-2x-3x^2}}{2x}\right).$$

This can be expressed as

$$\left(\frac{1}{1-x}c\left(\frac{x^2}{(1-x)^2}\right), \frac{x}{1-x}c\left(\frac{x^2}{(1-x)^2}\right)\right),$$

where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers $C_n = \frac{1}{n+1} {\binom{2n}{n}}$ <u>A000108</u>.

This array begins

$$\left(\begin{array}{ccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 & 0 \\ 9 & 12 & 9 & 4 & 1 & 0 & 0 \\ 21 & 30 & 25 & 14 & 5 & 1 & 0 \\ 51 & 76 & 69 & 44 & 20 & 6 & 1 \end{array}\right)$$

The production matrix takes the simple form

and hence the Z-sequence is the sequence $1, 1, 0, 0, 0, \ldots$ and the A-sequence is $1, 1, 1, 0, 0, 0, \ldots$. Then Z(x) = 1 + x and $A(x) = 1 + x + x^2$. This Riordan array is also defined by the A-matrix

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 1 & 1 \end{array}\right),$$

and the ρ sequence $0, 0, 0, \ldots$

In this example, the A-sequence is of a simple form, as is the A-matrix. In many cases, however, the A-matrix may be of a simple form, while the A-sequence is of a more complex nature.

Example 5. We now take the example where

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right),$$

and $\rho_n = 0$ for all $n \ge 0$. We find that u = f(x) satisfies the equation

$$\frac{u}{x} = 1 + u^2 + x(1+u),$$

or

$$f(x) = \frac{1 - x^2 - \sqrt{1 - 6x^2 - 4x^3 + x^4}}{2x} = \frac{x}{1 - x} c\left(\frac{x^2(1 + x)}{(1 - x^2)^2}\right).$$

The resulting Riordan array is then given by the Bell subgroup element $\left(\frac{f(x)}{x}, f(x)\right)$. This begins

| (| 1 | 0 | 0 | 0 | 0 | 0 | 0 \ |
|---|----|----|----|----|----|---|-----|
| | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| | 2 | 2 | 1 | 0 | 0 | 0 | 0 |
| | 3 | 5 | 3 | 1 | 0 | 0 | 0 |
| | 7 | 10 | 9 | 4 | 1 | 0 | 0 |
| | | 24 | | 14 | 5 | 1 | 0 |
| (| 31 | 52 | 57 | 40 | 20 | 6 | 1 / |

We have

$$\bar{f}(x) = \frac{\sqrt{1 + 4x + 6x^2 + x^4} - x^2 - 1}{2(1 + x)},$$

and so A(x) is given by

$$A(x) = \frac{1 + x^2 + \sqrt{1 + 4x + 6x^2 + x^4}}{2}.$$

Thus a_n is the sequence that begins

$$1, 1, 1, -1, 2, -3, 3, 1, -15, 47, -98, \ldots$$

The expansion of f(x) in this case begins

 $0, 1, 1, 2, 3, 7, 13, 31, 65, 156, 351, 849, \ldots$

The expansion of $\frac{f(x)}{x}$ is given by <u>A171416</u>. The Hankel transform of this sequence begins

 $1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots$

or <u>A006720</u>(n + 2). This is essentially the Somos 4 sequence.

In this example, we see that while the A-sequence is reasonably complicated, the Amatrix is simple. In addition, this A-matrix leads to a power series f(x) such that the Hankel transform of the expansion of $\frac{f(x)}{x}$ is the Somos 4 sequence. In the sequel, we shall be interested in determining a form of A-matrix that will always

In the sequel, we shall be interested in determining a form of A-matrix that will always lead to a Somos 4 type sequence.

2 General Somos 4 sequences

We call a sequence s_n a (α, β) Somos 4 sequence if the terms s_n satisfy

$$s_n = \frac{\alpha s_{n-1} s_{n-3} + \beta s_{n-2}^2}{s_{n-4}}, \quad n \ge 4.$$

In exceptional cases, where a divide by zero may occur, the more general condition

$$s_n s_{n-4} = \alpha s_{n-1} s_{n-3} + \beta s_{n-2}^2$$

is appropriate. There are many proven and conjectured results concerning (α, β) Somos 4 sequences and Riordan arrays [2, 4]. These sequences are closely related to elliptic curves.

3 A-matrices and Somos conjectures

We consider the case

$$A = \left(\begin{array}{rrr} 1 & a & b \\ 1 & c & d \end{array}\right),$$

with $\rho_n = 0$ for all n. To find u = f(x), we solve the equation

$$\frac{u}{x} = 1 + au + bu^2 + x(1 + cu + du^2).$$

This gives us

$$f(x) = \frac{1 - ax - cx^2 - \sqrt{1 - 2ax - (a^2 - 2(2b + c))x^2 + 2(ac - 2(b + d))x^3 + (c^2 - 4d)x^4}}{2x(dx + b)}$$

It follows that

$$\frac{f(x)}{x} = \frac{1+x}{1-ax-cx^2}c\left(\frac{x^2(1+x)(b+dx)}{(1-ax-cx^2)^2}\right)$$

This now allows us to make the following conjecture.

Conjecture 6. The Hankel transform of the expansion of $\frac{f(x)}{x}$ is a $((b+ab+d)^2, b^4-b^3(2+3a+a^2-2c)+b(a+a^2-ac-2d)d+(1+a-c)d^2-b^2(c+ac-c^2+2d+3ad))$ Somos 4 sequence.

Example 7. The matrix

$$A = \left(\begin{array}{rrr} 1 & -2 & -1 \\ 1 & 1 & 0 \end{array}\right),$$

leads to the Riordan array with

$$\frac{f(x)}{x} = \frac{1+x}{1+2x-x^2}c\left(-\frac{x^2(1+x)}{(1+2x-x^2)^2}\right)$$

This expands to give the sequence that begins

$$1, -1, 2, -3, 3, 1, -15, 47, -98, 133, -17, \ldots$$

The Hankel transform of this sequence begins

$$1, 1, -2, -1, 3, -5, -7, -4, 23, 29, -59, \ldots$$

This is a (1,1) Somos 4 sequence (compare with <u>A006769</u>). The expansion v_n of $\frac{f(x)}{x}$ thus satisfies the convolution recurrence [1]

$$v_n = -2v_{n-1} - v_{n-2} - \sum_{i=0}^{n-4} v_{i+1}v_{n-i-3},$$

with $v_1 = -1$ and $v_2 = 2$.

Proposition 8. The Bell matrix $(t_{n,k})$ which has

$$A = \left(\begin{array}{rrr} 1 & a & b \\ 1 & c & d \end{array}\right)$$

as its A-matrix satisfies

$$t_{n,k} = t_{n-1,k-1} + at_{n-1,k} + bt_{n-1,k+1} + t_{n-2,k-1} + ct_{n-2,k} + dt_{n-2,k+1}$$

with $t_{n,k} = 0$ if k < 0 or k > n, $t_{0,0} = 1$ and $t_{1,0} = a + 1$. This is the Riordan array

$$\left(\frac{1+x}{1-ax-cx^2}c\left(\frac{x^2(1+x)(b+dx)}{(1-ax-cx^2)^2}\right), \frac{x(1+x)}{1-as-cx^2}c\left(\frac{s^2(1+x)(b+dx)}{(1-ax-cx^2)^2}\right)\right)$$

Proof. The recurrence follows from the form of the A-matrix. We have calculated f(x) above. The Bell element corresponding to this is then $\left(\frac{f(x)}{x}, f(x)\right)$. The value for $t_{1,0}$ follows from this.

We note that the Riordan array

$$\left(\frac{1-(a-1)x}{1-ax-cx^2}c\left(\frac{x^2(1+x)(b+dx)}{(1-ax-cx^2)^2}\right),\frac{x(1+x)}{1-ax-cx^2}c\left(\frac{x^2(1+x)(b+dx)}{(1-ax-cx^2)^2}\right)\right)$$

satisfies the same recurrence with $t_{1,0} = 1$. Note that we shall always assume that $t_{n,k} = 0$ if k < 0 or k > n, $t_{0,0} = 1$ in the following.

4 Further examples with $\rho_n = 0$

In the following examples, we choose $\rho_n = 0$ for all $n \ge 0$.

Example 9. We let

$$A = \left(\begin{array}{rrr} 1 & 1 & 0 \\ 1 & 1 & 1 \end{array}\right).$$

Then the triangle $(t_{n,k})$ with

$$t_{n,k} = t_{n-1,k-1} + t_{n-1,k} + t_{n-2,k-1} + t_{n-2,k} + t_{n-2,k+1},$$

and $t_{1,0} = 2$ begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
|---|----|-----|-----|-----|----|----|-----|--|
| | 2 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 3 | 4 | 1 | 0 | 0 | 0 | 0 | |
| | 6 | 10 | 6 | 1 | 0 | 0 | 0 | |
| | 13 | 24 | 21 | 8 | 1 | 0 | 0 | |
| | 29 | 59 | 62 | 36 | 10 | 1 | 0 | |
| ĺ | 66 | 146 | 174 | 128 | 55 | 12 | 1 / | |

This is the Bell matrix

$$\left(\frac{1+x}{1-x-x^2}c\left(\frac{x^3(1+x)}{(1-x-x^2)^2}\right), \frac{x(1+x)}{1-x-x^2}c\left(\frac{x^3(1+x)}{(1-x-x^2)^2}\right)\right)$$

The Hankel transform of the sequence $t_{n,0}$ begins

 $1, -1, -4, -3, 19, 67, \ldots,$

which is the (1,1) Somos 4 sequence <u>A178628</u>(n + 1). In fact, the generating function for $t_{n,0}$ may be expressed as the continued fraction [14]

$$\frac{1}{1 - 2x + \frac{x^2}{1 + 2x + \frac{4x^2}{1 - \frac{11}{4}x - \frac{\frac{3}{16}x^2}{1 - \frac{43}{12}x - \dots}}}$$

where $1, 4, -\frac{3}{16}, \ldots$ are the x-coordinates of the multiples of (0, 0) on the elliptic curve

$$y^2 - xy - y = x^3 + x^3 + x.$$

Example 10. We let

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 0 & 1 \end{array}\right).$$

Then the triangle $(t_{n,k})$ with

$$t_{n,k} = t_{n-1,k-1} + t_{n-1,k+1} + t_{n-2,k-1} + t_{n-2,k+1},$$

and $t_{1,0} = 1$ begins

| (| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
|---|----|----|----|----|----|---|-----|
| | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| | 1 | 2 | 1 | 0 | 0 | 0 | 0 |
| | 3 | 3 | 3 | 1 | 0 | 0 | 0 |
| | 5 | 8 | 6 | 4 | 1 | 0 | 0 |
| | 11 | 17 | 16 | 10 | 5 | 1 | 0 |
| ĺ | 25 | 38 | 39 | 28 | 15 | 6 | 1 / |

This is the Bell matrix

$$((1+x)c(x^2(1+x+x^2)), x(1+x)c(x^2(1+x+x^2))).$$

The sequence $t_{n,0}$ begins

 $1, 1, 1, 3, 5, 11, 25, 55, 129, 303, 721, 1743, \ldots$

This is <u>A104545</u>, which counts the number of Motzkin paths of length n having no consecutive (1,0) steps (Emeric Deutsch). Its Hankel transform begins

 $1, 0, -4, -16, -64, 0, 4096, 65536, 1048576, 0, -1073741824, \ldots.$

This is A162547, the Somos 4 variant

$$s_n = (4s_{n-1}s_{n-3} - 4s_{n-2})^2)/s_{n-4}, \quad \text{if } n \neq 4k+1,$$

otherwise $s_n = 0$, with $s_{-2} = s_{-1} = s_0 = 1$.

Example 11. We let

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 4 & 1 \end{array}\right).$$

Then the triangle $(t_{n,k})$ with

$$t_{n,k} = t_{n-1,k-1} + t_{n-1,k+1} + t_{n-2,k-1} + 4t_{n-2,k} + t_{n-2,k+1},$$

and $t_{1,0} = 1$ begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 \ | |
|---|-----|-----|-----|----|----|---|-----|--|
| | 1 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 5 | 2 | 1 | 0 | 0 | 0 | 0 | |
| | 7 | 11 | 3 | 1 | 0 | 0 | 0 | |
| | 33 | 24 | 18 | 4 | 1 | 0 | 0 | |
| | 63 | 105 | 52 | 26 | 5 | 1 | 0 | |
| | 261 | 262 | 231 | 92 | 35 | 6 | 1 / | |

The sequence $t_{n,0}$ which begins

 $1, 1, 5, 7, 33, 63, 261, 619, 2333, 6355, 22669, \ldots$

has a Hankel transform that begins

 $1, 4, 28, 304, 14272, 676864, \ldots$

This is a (4, 12) Somos 4 sequence. In general, the A-matrix

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & r & 1 \end{array}\right)$$

leads to a $(4, r^2 - 4)$ Somos 4 sequence.

5 A quasi-involution

We consider the case of the A-matrix

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right).$$

We solve the equation

$$\frac{u}{x} = 1 + u^2 + xu,$$

to obtain

$$u = f(x) = \frac{1 - x^2 - \sqrt{1 - 6x^2 + x^4}}{2x}.$$

Thus the Bell matrix $(t_{n,k})$ defined by the above A-matrix is the Riordan array

$$\left(\frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x^2}, \frac{1-x^2-\sqrt{1-6x^2+x^4}}{2x}\right)$$

This triangle begins

The inverse of this matrix begins

We recall that a Riordan array $(g(x^2), xg(x^2))$ is called a quasi-involution if we have

$$(g(x^2), xg(x^2))^{-1} = (g(-x^2), xg(-x^2)).$$

This is the case for this example [5].

6 The case of $r_n = 0^n$

In this section, we take the case of

$$A = \left(\begin{array}{rrr} 1 & a & b \\ 1 & c & d \end{array}\right)$$

and $\rho_0 = 1$, and $\rho_n = 0$ for n > 0. Thus we must solve the equation

$$\frac{u}{x} = 1 + au + bu^{2} + x(1 + cu + du^{2}) + \rho_{0}\frac{u^{2}}{x}$$

to get f(x) = u. We find that

$$\frac{f(x)}{x} = \frac{1+x}{1-ax-cx^2} c\left(\frac{x(1+x)(\rho_0+bx+dx^2)}{(1-ax-cx^2)^2}\right).$$

In this case, $\frac{f(x)}{x}$ expands to begin

$$1, a + \rho_0 + 1, a^2 + a(3\rho_0 + 1) + b + c + 2\rho_0^2 + 2\rho_0, \dots,$$

and the corresponding Bell triangle begins

$$\left(\begin{array}{cccc} 1 & 0 & 0\\ a+\rho_0+1 & 1 & 0\\ a^2+a(3\rho_0+1)+b+c+2\rho_0(\rho_0+1) & 2a+2\rho_0+2 & 1 \end{array}\right).$$

In the case that $\rho_0 = 1$, we obtain

$$\frac{f(x)}{x} = \frac{1+x}{1-ax-cx^2}c\left(\frac{x(1+x)(1+bx+dx^2)}{(1-ax-cx^2)^2}\right).$$

Example 12. We consider the case of

$$A = \left(\begin{array}{rrr} 1 & 0 & 1 \\ 1 & 1 & 0 \end{array}\right)$$

and $\rho_n = 0^n$. The resulting Bell triangle begins

| 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
|------|------|---|--|--|--|--|--|
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | |
| 6 | 4 | 1 | 0 | 0 | 0 | 0 | |
| 22 | 16 | 6 | 1 | 0 | 0 | 0 | |
| 90 | 68 | 30 | 8 | 1 | 0 | 0 | |
| 394 | 304 | 146 | 48 | 10 | 1 | 0 | |
| 1806 | 1412 | 714 | 264 | 70 | 12 | 1 / | |
| | | $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ |

This is the Bell matrix defined by the large Schroeder numbers $\underline{A006318}$,

$$(t_{n,k}) = \left(\frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}, \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}\right),$$

and its A-sequence is the sequence

$$1, 2, 2, 2, 2, 2, 2, 2, \ldots$$

This follows since the solution u(x) with u(0) = 0 of the equation

$$\frac{u}{x} = 1 + u^2 + x(1+u) + \frac{u^2}{x}$$

is given by

$$u(x) = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2}$$

This is also the solution to the equation

$$\frac{u}{x} = 1 + u + \frac{u^2}{x}.$$

Thus a simpler A-matrix in this case is given by

$$A = \left(\begin{array}{cc} 1 & 1 \end{array} \right),$$

along with $\rho_n = 0^n$.

In general, taking

$$A = \left(\begin{array}{cc} 1 & r \end{array} \right),$$

along with $\rho_n = 0^n$, leads to

$$\frac{f(x)}{x} = \frac{1 - (r-1)x - \sqrt{1 - 2(r+1)x + (r-1)^2 x^2}}{2x} = \frac{1}{1 - (r-1)x} c\left(\frac{x}{(1 - (r-1)x)^2}\right),$$

which is the generating function of the Narayana polynomials

$$1, r, r(r+1), r(r^2 + 3r + 1), r(r^3 + 6r^2 + 6r + 1), \dots$$

In this case the triangle $(t_{n,k})$ begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ r & 1 & 0 & 0 & 0 \\ r(r+1) & 2r & 1 & 0 & 0 \\ r(r^2+3r+1) & r(3r+2) & 3r & 1 & 0 \\ r(r^3+6r^2+6r+1) & 2r(2r^2+4r+1) & 3r(2r+1) & 4r & 1 \end{pmatrix},$$

with A-sequence

 $1, r, r, r, \ldots$

Example 13. We take

$$A = \left(\begin{array}{rrr} 1 & -2 & 2\\ 1 & -1 & 1 \end{array}\right)$$

and $\rho_n = 0^n$. Then the Bell matrix that we obtain is <u>A097609</u>, which begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 \ | |
|---|----|----|----|---|---|---|-----|--|
| | 0 | 1 | 0 | 0 | 0 | 0 | 0 | |
| I | 1 | 0 | 1 | 0 | 0 | 0 | 0 | |
| | 1 | 2 | 0 | 1 | 0 | 0 | 0 | |
| | 3 | 2 | 3 | 0 | 1 | 0 | 0 | |
| | 6 | 7 | 3 | 4 | 0 | 1 | 0 | |
| | 15 | 14 | 12 | 4 | 5 | 0 | 1 / | |

This is the Bell matrix

$$\left(\frac{1+x-\sqrt{1-2x-3x^2}}{2x(1-x)}, \frac{1+x-\sqrt{1-2x-3x^2}}{2(1-x)}\right)$$

defined by the so-called Motzkin sums <u>A005043</u>. This triangle counts the number of Motzkin paths of length n having k horizontal steps at level 0 (Emeric Deutsch). The A-sequence of this triangle is

 $1, 0, 1, 1, 1, 1, \dots$

We now turn to the issue of the Somos 4 nature of the Hankel transform of the expansion of $\frac{f(x)}{x}$. We have the following conjecture.

Conjecture 14. The Hankel transform of the expansion of $\frac{f(x)}{x}$ in the case that

$$A = \left(\begin{array}{rrr} 1 & a & b \\ 1 & c & d \end{array}\right)$$

and $\rho_n = 0^n$ is an (α, β) Somos 4 sequence, where

$$\alpha = (4 + a^2 + 3b + a(4 + b) + c + d)^2,$$

and

$$\begin{split} \beta &= -16 - a^5 + b^4 - 3a^4(3+b) + 2b^3(-2+c) - 8c - 4c^2 - c^3 \\ &+ b^2(-28 - c + c^2 - 8d) - 8d - 6cd - 2c^2d - d^2 - cd^2 \\ &- a^3(32+23b+3b^2+c+2d) - 2b(20+c^2+9d+d^2+3c(2+d)) \\ &- a^2(56+18b^2+b^3+10d+c(6+d)+b(66+c+5d)) \\ &- a(48+3b^3+2c^2+16d+d^2+b^2(38-2c+3d)+c(12+5d)+b(84-c^2+19d+c(8+d))). \end{split}$$

Example 15. For

$$A = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 1 & 0 & -2 \end{array}\right)$$

and $\rho_n = 0^n$, we obtain the Bell triangle that begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | $0 \rangle$ | |
|---|-----|----|-----|-----|----|----|-------------|--|
| | 2 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 3 | 4 | 1 | 0 | 0 | 0 | 0 | |
| | 4 | 10 | 6 | 1 | 0 | 0 | 0 | |
| | 2 | 20 | 21 | 8 | | 0 | 0 | |
| | -11 | 29 | 56 | 36 | 10 | 1 | 0 | |
| | -59 | 10 | 117 | 120 | 55 | 12 | 1 / | |

For this, we have

$$\frac{f(x)}{x} = (1+x)c(x(1+x)(1-x-2x^2)).$$

The Hankel transform of the expansion of this generating function begins

$$1, -1, 3, 4, 5, -31, \ldots$$

This is a (1, 1) Somos 4 sequence.

Example 16. For

$$A = \left(\begin{array}{rrr} 1 & 0 & -1 \\ 1 & -2 & 2 \end{array}\right)$$

and $\rho_n = 0^n$, we obtain the Bell matrix that begins

| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
|----|----|---|---|--|--|--|
| 2 | 1 | 0 | 0 | 0 | 0 | 0 |
| 1 | 4 | 1 | 0 | 0 | 0 | 0 |
| 0 | 6 | 6 | 1 | 0 | 0 | 0 |
| 4 | 4 | 15 | 8 | 1 | 0 | 0 |
| 17 | 9 | 20 | 28 | 10 | 1 | 0 |
| 41 | 50 | 27 | 56 | 45 | 12 | 1 / |
| | | $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ | $\begin{array}{cccccccccccccccccccccccccccccccccccc$ |

•

For this, we have

$$\frac{f(x)}{x} = \frac{1+x}{1+2x^2} c\left(\frac{x(1+x)(1-x+2x^2)}{(1+2x^2)^2}\right)$$

The Hankel transform of the expansion of this generating function begins

 $1, -3, -13, 14, 465, 1819, \ldots$

This is a (1,3) Somos 4 sequence.

7 The case $\rho(x) = 1 + rx$

We briefly consider the case

 $A = \left(\begin{array}{cc} 1 & 1 \end{array}\right),$

along with $\rho(x) = 1 + x$. We are thus led to the equation

$$\frac{u}{x} = 1 + u + \frac{u^2(1+u)}{x},$$

which has solution

$$u(x) = f(x) = \frac{1}{3} \left(\sqrt{4 - 3x} \sin\left(\frac{1}{3}\sin^{-1}\left(\frac{18x + 11}{2(4 - 3x)^{\frac{3}{2}}}\right) \right) - 1 \right).$$

Equivalently, we have

$$f(x) = \text{Rev}\frac{x(1-x-x^2)}{1+x}$$

We have that $\frac{f(x)}{x}$ expands to give the sequence that begins

 $1, 2, 7, 31, 154, 820, 4575, 26398, 156233, \ldots,$

or <u>A007863</u>, the number of "hybrid binary trees" with n internal nodes. The corresponding Bell matrix begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
|---|------|------|------|-----|----|----|-----|--|
| | 2 | 1 | 0 | 0 | 0 | 0 | 0 | |
| I | 7 | 4 | 1 | 0 | 0 | 0 | 0 | |
| | 31 | 18 | 6 | 1 | 0 | 0 | 0 | |
| | 154 | 90 | 33 | 8 | 1 | 0 | 0 | |
| | 820 | 481 | 185 | 52 | 10 | 1 | 0 | |
| ĺ | 4575 | 2690 | 1065 | 324 | 75 | 12 | 1 / | |

The production matrix of this array starts

indicating that

$$A(x) = \frac{1+x}{1-x-x^2}$$
, and $Z(x) = \frac{2+x}{1-x-x^2}$.

Here, we recognize the shifted Fibonacci numbers <u>A000045</u>. The Bell matrix $(t_{n,k})$ in this case is given by

$$\left(\frac{1-x-x^2}{1+x}, \frac{x(1-x-x^2)}{1+x}\right)^{-1}$$

Similarly, the A-matrix

 $A=\left(\begin{array}{cc}1&2\end{array}\right),$

along with $\rho(x) = 1 + x$, leads to the Bell matrix

$$\left(\begin{array}{cccc} \frac{1-x-x^2}{1+2x}, \frac{x(1-x-x^2)}{1+2x}\right)^{-1}$$

$$\left(\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 \\ 13 & 6 & 1 & 0 & 0 & 0 & 0 \\ 13 & 6 & 1 & 0 & 0 & 0 & 0 \\ 70 & 35 & 9 & 1 & 0 & 0 & 0 \\ 424 & 218 & 66 & 12 & 1 & 0 & 0 \\ 2756 & 1437 & 471 & 106 & 15 & 1 & 0 \\ 2756 & 1437 & 471 & 106 & 15 & 1 & 0 \\ 18778 & 9876 & 3390 & 856 & 155 & 18 & 1 \end{array}\right).$$

The case of

which begins

$$A = \left(\begin{array}{cc} 1 & 1 \end{array} \right),$$

along with $\rho(x) = 1 + 2x$ is especially interesting. We obtain the Bell matrix that begins

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 8 & 4 & 1 & 0 & 0 & 0 & 0 \\ 40 & 20 & 6 & 1 & 0 & 0 & 0 \\ 224 & 112 & 36 & 8 & 1 & 0 & 0 \\ 1344 & 672 & 224 & 56 & 10 & 1 & 0 \\ 8448 & 4224 & 1440 & 384 & 80 & 12 & 1 \end{pmatrix},$$

which has a production matrix that begins

Thus we have

$$A(x) = \frac{1}{1 - 2x}$$
, and $Z(x) = \frac{2}{1 - 2x}$.

The corresponding Bell matrix $(t_{n,k})$ is given by

$$(1-2x, x(1-2x))^{-1}$$
.

In effect, the solution to the equation

$$\frac{u}{x} = 1 + u + \frac{u^2(1+2u)}{x}$$

is given by

$$u(x) = f(x) = \frac{1 - \sqrt{1 - 8x}}{4}$$

This is the generating function of $2^n C_n \underline{A151374}$. Extending the two previous cases, we look at the case

$$A = \left(\begin{array}{cc} 1 & 2 \end{array} \right),$$

along with $\rho(x) = 1 + 2x$. The equation to be solved is now

$$\frac{u}{x} = 1 + 2u + \frac{u^2(1+2u)}{x}$$

with solution

$$u(x) = f(x) = \frac{1}{3} \left(\sqrt{7 - 12x} \sin\left(\frac{1}{3}\sin^{-1}\left(\frac{2(18x + 5)}{(7 - 12x)^{\frac{3}{2}}}\right) \right) - \frac{1}{2} \right).$$

Equivalently, we have

$$f(x) = \operatorname{Rev} \frac{x(1 - x - 2x^2)}{1 + 2x}$$

The Bell matrix $(t_{n,k})$ in this case begins

$$\left(\begin{array}{cccccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 14 & 6 & 1 & 0 & 0 & 0 & 0 \\ 83 & 37 & 9 & 1 & 0 & 0 & 0 \\ 554 & 250 & 69 & 12 & 1 & 0 & 0 \\ 3966 & 1802 & 528 & 110 & 15 & 1 & 0 \\ 29756 & 13580 & 4122 & 944 & 160 & 18 & 1 \end{array}\right),$$

with a production matrix that begins

We recognise the shifted Jacobsthal numbers <u>A001045</u>. We deduce that the above Bell matrix is given by

$$\left(\frac{1-x-2x^2}{1+2x},\frac{x(1-x-2x^2)}{1+2x}\right)^{-1}$$
.

The sequence $t_{n,0}$ is given by <u>A215661</u>. In general, the Bell matrix $(t_{n,k})$ defined by

$$A = \left(\begin{array}{cc} 1 & r \end{array} \right),$$

along with $\rho(x) = 1 + rx$ is given by

$$\left(\frac{1-x-rx^2}{1+rx}, \frac{x(1-x-rx^2)}{1+rx}\right)^{-1}$$

We have

$$t_{n,k} = t_{n-1,k-1} + rt_{n-1,k} + t_{n,k+1} + rt_{n,k+2},$$

with $t_{0,0} = 1$, $t_{1,0} = r + 1$.

The sequence $t_{n,0}$ (the first column of the Bell matrix) then begins

$$1, r + 1, r^{2} + 4r + 2, r^{3} + 10r^{2} + 15r + 5, r^{4} + 20r^{3} + 63r^{2} + 56r + 14, \dots$$

This polynomial sequence $P_n(r)$ can be expressed as

$$P_n(r) = \sum_{k=0^n} \frac{1}{k + (2n+1)0^k} \binom{2n-k}{k-1+0^k} \binom{2n-k+1}{n-k} r^k,$$

where the coefficient array begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 \ | |
|---|-----|-----|------|------|-----|----|-----|--|
| | 1 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 2 | 4 | 1 | 0 | 0 | 0 | 0 | |
| | 5 | 15 | 10 | 1 | 0 | 0 | 0 | |
| | 14 | 56 | 63 | 20 | 1 | 0 | 0 | |
| | 42 | 210 | 336 | 196 | 35 | 1 | 0 | |
| ſ | 132 | 792 | 1650 | 1440 | 504 | 56 | 1 / | |

An interesting feature of this array is the following. The diagonal sums of this array begin

 $1, 1, 3, 9, 30, 108, 406, 1577, 6280, 25499, 105169, \ldots$

This is <u>A200074</u>. The generating function g(x) of this sequence can be expressed as $g(x) = \frac{u(x)}{x}$, where we have

$$\frac{u}{x} = 1 + u^3 + xu + \frac{u^2}{x}$$

Thus this sequence is the first column of the Bell matrix $(t_{n,k})$ defined by the A-matrix

$$\left(\begin{array}{rrrr} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right),$$

with $\rho_n = 0^n$. This triangle $(t_{n,k})$ begins

$$\left(\begin{array}{cccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 & 0 \\ 9 & 7 & 3 & 1 & 0 & 0 & 0 \\ 30 & 24 & 12 & 4 & 1 & 0 & 0 \\ 108 & 87 & 46 & 18 & 5 & 1 & 0 \\ 406 & 330 & 180 & 76 & 25 & 6 & 1 \end{array}\right).$$

We have

$$t_{n,k} = t_{n-1,k-1} + t_{n-1,k+2} + t_{n-2,k} + t_{n,k+1},$$

with $t_{1,0} = 1$.

8 Perturbed orthogonal polynomials

We consider the case when

$$A = \left(\begin{array}{ccc} 1 & a & b \end{array}\right),$$

and $\rho_n = c0^n$.

Thus we must solve the equation

$$\frac{u}{x} = 1 + au + bu^2 + \frac{cu^2}{x}.$$

The required solution is

$$u = \frac{1 - ax - \sqrt{1 - 2(a + 2c)x + (a^2 - 4b)x^2}}{2(bx + c)} = \frac{x}{1 - ax}c\left(\frac{x(bx + c)}{(1 - ax)^2}\right).$$

Alternatively, we have

$$u = \operatorname{Rev} \frac{x(1-cx)}{1+ax+bx^2}$$

Thus the Bell matrix $(t_{n,k})$ with

$$t_{n,k} = t_{n-1,k-1} + at_{n-1,k} + bt_{n-1,k+1} + ct_{n,k+1}$$

and $t_{1,0} = a + c$ is given by

$$\left(\frac{1-cx}{1+ax+bx^2}, \frac{x(1-cx)}{1+ax+bx^2}\right)^{-1},$$

or

$$\left(\frac{1}{x}\operatorname{Rev}\frac{x(1-cx)}{1+ax+bx^2},\operatorname{Rev}\frac{x(1-cx)}{1+ax+bx^2}\right)$$

In the case that c = 0, this is the moment matrix of the orthogonal polynomials

$$P_n(x) = (x - a)P_{n-1}(x) - bP_{n-2}(x),$$

with $P_0(x) = 1$, $P_1(x) = x - a$.

Similarly, the triangle $(t_{n,k})$ with

$$t_{n,k} = t_{n-1,k-1} + at_{n-1,k} + bt_{n-1,k+1} + ct_{n-1,k+2} + dt_{n,k+1}$$

and $t_{1,0} = a + d$ is given by

$$\left(\frac{1}{x}\operatorname{Rev}\frac{x(1-dx)}{1+ax+bx^2+cx^3},\operatorname{Rev}\frac{x(1-dx)}{1+ax+bx^2+cx^3}\right)$$

The binomial transform of the Riordan array

$$\left(\frac{1}{1+ax+bx^2},\frac{x}{1+ax+bx^2}\right)^{-1}$$

is the Riordan array

$$\left(\frac{1}{1+(a+1)x+bx^2}, \frac{x}{1+(a+1)x+bx^2}\right)^{-1}.$$

In a similar vein, we have the following result.

Proposition 17. The binomial transform of $\frac{u}{x}$ where

$$\frac{u}{x} = 1 + au + bu^2 + \frac{cu^2}{x}$$

is given by $\frac{v}{x}$ where

$$\frac{v}{x} = \frac{1}{1-x}(1+av+bv^2) + \frac{cv^2}{x}$$

Proof. We have

$$\frac{u}{x} = \frac{1}{1-ax}c\left(\frac{x(bx+c)}{(1-ax)^2}\right).$$

We find that the binomial transform $\frac{1}{1-x}u\left(\frac{x}{1-x}\right)$ of u is given by

$$\frac{1}{1-x}u\left(\frac{x}{1-x}\right) = \frac{1}{1-(a+1)x}c\left(\frac{x(x(b-c)+c)}{(1-(a+1)x)^2}\right).$$

But this is precisely $\frac{v}{x}$ where

$$\frac{v}{x} = \frac{1}{1-x}(1+av+bv^2) + \frac{cv^2}{x}.$$

Thus the Riordan array with A-matrix A = (1, a, b) and $\rho_n = c0^n$ will have a binomial transform with the infinite A-matrix

$$\left(\begin{array}{rrrrr} 1 & a & b \\ \vdots & \vdots & \vdots \end{array}\right)$$

and $\rho_n = c0^n$.

Example 18. The Riordan array

$$\left(\frac{1-5x}{1+2x+3x^2}, \frac{x(1-5x)}{1+2x+3x^2}\right)^{-1}$$

begins

| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 \ |
|---|---------|---------|--------|-------|-----|----|-----|
| | 7 | 1 | 0 | 0 | 0 | 0 | 0 |
| | 87 | 14 | 1 | 0 | 0 | 0 | 0 |
| | 1331 | 223 | 21 | 1 | 0 | 0 | 0 |
| | 22731 | 3880 | 408 | 28 | 1 | 0 | 0 |
| | 415427 | 71665 | 7990 | 642 | 35 | 1 | 0 |
| (| 7949259 | 1380682 | 159591 | 14004 | 925 | 42 | 1 / |

Note for instance that

$$3880 = 1331 + 2 \cdot 223 + 3 \cdot 21 + 5 \cdot 408.$$

Consider the binomial transform of this matrix, given by

| (| 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
|---|----------|---------|--------|-------|------|----|-----|---|
| | 8 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 102 | 16 | 1 | 0 | 0 | 0 | 0 | |
| | 1614 | 268 | 24 | 1 | 0 | 0 | 0 | . |
| | 28606 | 4860 | 498 | 32 | 1 | 0 | 0 | |
| | 543298 | 93440 | 10250 | 792 | 40 | 1 | 0 | |
| | 10810754 | 1873548 | 214086 | 18296 | 1150 | 48 | 1 / | |

Note now that, for instance, we have

$$10250 = 1 \cdot (4860 + 268 + 16 + 1) + 2 \cdot (498 + 24 + 1) + 3 \cdot (32 + 1) + 5 \cdot 792$$

9 Calculating the A-sequence

We give an example of how to calculate the A sequence, given a suitable A-matrix and ρ sequence. Thus we take

$$A = \left(\begin{array}{rrr} 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right)$$

and $\rho_n = 0^n$. This leads us to the equation

$$\frac{u}{x} = 1 + u + u^2 + xu + \frac{u^2}{x},$$

whose solution is given by

$$u = f(x) = \frac{1 - x - x^2 - \sqrt{1 - 6x - 5x^2 + 2x^3 + x^4}}{2(1 + x)} = \frac{x}{1 - x - x^2} c\left(\frac{x(1 + x)}{(1 - x - x^2)^2}\right).$$

From this we can calculate the A-sequence since its generating function A(x) is given by

$$A(x) = \frac{x}{\bar{f}(x)}.$$

An alternative method is to start with the equation

$$\frac{u}{x} = 1 + u + u^2 + xu + \frac{u^2}{x},$$

which we write using u = f(x) as

$$\frac{f(x)}{x} = 1 + f(x) + f(x)^2 + xf(x) + \frac{(f(x))^2}{x}.$$

Substituting $\bar{f}(x)$ for x, and using the fact that $f(\bar{f}(x)) = x$, we then obtain

$$\frac{x}{\bar{f}(x)} = 1 + x + x^2 + x\bar{f}(x) + \frac{x^2}{\bar{f}(x)}.$$

Thus we must solve the equation

$$v = 1 + x + x^2 + \frac{x^2}{v} + xv,$$

for $v = A(x) = \frac{x}{f(x)}$. We obtain that

$$A(x) = v = \frac{1 + x + x^2 + \sqrt{1 + 2x + 7x^2 - 2x^3 + x^4}}{2(1 - x)}$$

This expands to give the sequence that begins

 $1, 2, 4, 2, 2, 8, -2, -10, 52, -26, -202, 576, \ldots$

The Bell matrix $(t_{n,k})$ in this case begins

| / 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
|--------|-------|-------|-------|------|-----|-----|----|-----|---|
| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 8 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 34 | 20 | 6 | 1 | 0 | 0 | 0 | 0 | 0 | |
| 162 | 100 | 36 | 8 | 1 | 0 | 0 | 0 | 0 | , |
| 820 | 524 | 206 | 56 | 10 | 1 | 0 | 0 | 0 | |
| 4338 | 2832 | 1182 | 360 | 80 | 12 | 1 | 0 | 0 | |
| 23694 | 15704 | 6828 | 2248 | 570 | 108 | 14 | 1 | 0 | |
| 132612 | 88876 | 39818 | 13856 | 3850 | 844 | 140 | 16 | 1 / | |

and its production matrix takes the form

| (| 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 \ | |
|---|-----|-----|----|---|---|---|---|-----|---|
| | 4 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | |
| | 2 | 4 | 2 | 1 | 0 | 0 | 0 | 0 | |
| | 2 | 2 | 4 | 2 | 1 | 0 | 0 | 0 | |
| | 8 | 2 | 2 | 4 | 2 | 1 | 0 | 0 | • |
| | -2 | 8 | 2 | 2 | 4 | 2 | 1 | 0 | |
| | -10 | -2 | 8 | 2 | 2 | 4 | 2 | 1 | |
| ĺ | 52 | -10 | -2 | 8 | 2 | 2 | 4 | 2 / | |

10 Conclusions

We have given examples of Riordan arrays defined by simple A-matrices and simple *rho* sequences, and we have shown the form of the calculations required to go from these data to a corresponding Bell matrix. In some cases, we have examined the associated A-sequence.

Some special matrices, including a Riordan quasi-involution and certain "perturbed" moment matrices have been studied in terms of their defining A-matrix and ρ sequence.

We have conjectured that an A-matrix of the form

$$A = \left(\begin{array}{rrr} 1 & a & b \\ 1 & c & d \end{array}\right)$$

leads to Hankel transforms that are (α, β) Somos 4 sequences, in the case that $\rho_n = 0$ for all n and when $\rho_n = 0^n$. Specific examples bear this out, but the form of β in general makes it uncertain at the moment how such a conjecture might be proven, or generalized. Other open questions that remain are the relationship between the parameters (a, b, c, d) and the corresponding coefficients of the related elliptic curves. It is clear that these problems deserve further study.

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(Concerned with sequences <u>A000045</u>, <u>A000108</u>, <u>A001045</u>, <u>A005043</u>, <u>A006318</u>, <u>A006720</u>, <u>A006769</u>, <u>A007863</u>, <u>A097609</u>, <u>A104545</u>, <u>A151374</u>, <u>A162547</u>, <u>A171416</u>, <u>A178628</u>, and <u>A215661</u>.)