# Riordan arrays, the $A$-matrix, and Somos 4 sequences 

Paul Barry<br>School of Science<br>Waterford Institute of Technology<br>Ireland<br>pbarry@wit.ie


#### Abstract

We characterize certain Riordan arrays by their $A$-matrices and $\rho$ sequences. We conjecture the form of a generic $A$-matrix which leads to Somos 4 sequences. We find an $A$-matrix that produces a Riordan quasi-involution, and we study the $A$-matrices and $\rho$ sequences of the moment matrices of certain perturbed orthogonal polynomials.


## 1 Introduction

An important feature of Riordan arrays $[3,10,11]$ is that they have a sequence characterization. Specifically, a lower-triangular array $\left(t_{n, k}\right)_{0 \leq n, k \leq \infty}$ is a Riordan array if and only if there exists a sequence $a_{n}, n \geq 0$, such that

$$
t_{n, k}=\sum_{i=0}^{\infty} a_{i} t_{n-1, k-1+i} .
$$

This sum is actually a finite sum, since the matrix is lower-triangular. For such a matrix $M$, we let

$$
P=M^{-1} \bar{M}
$$

where $\bar{M}$ is the matrix $M$ with its top row removed. Then the matrix $P$ has the form that begins

$$
\left(\begin{array}{ccccccc}
z_{0} & a_{0} & 0 & 0 & 0 & 0 & 0 \\
z_{1} & a_{1} & a_{0} & 0 & 0 & 0 & 0 \\
z_{2} & a_{2} & a_{1} & a_{0} & 0 & 0 & 0 \\
z_{3} & a_{3} & a_{2} & a_{1} & a_{0} & 0 & 0 \\
z_{4} & a_{4} & a_{3} & a_{2} & a_{1} & a_{0} & 0 \\
z_{5} & a_{5} & a_{4} & a_{3} & a_{2} & a_{1} & a_{0} \\
z_{6} & a_{6} & a_{5} & a_{4} & a_{3} & a_{2} & a_{1}
\end{array}\right) .
$$

Here, $z_{0}, z_{1}, z_{2}, \ldots$ is an ancillary sequence which exists for any lower-triangular matrix. The matrix $P$ is called the production matrix of the Riordan matrix $M$. The sequence characterization of renewal arrays was first described by Rogers $[6,9]$.

An alternative approach to the sequence characterization of a Riordan array is to use a matrix characterization $[7,8]$. One form that such a matrix characterization may take is the following $[7,8]$.

Theorem 1. A lower-triangular array $\left(t_{n, k}\right)_{0 \leq n, k \leq \infty}$ is a Riordan array if and only if there exists another array $A=\left(a_{i, j}\right)_{i, j \in \mathbb{N}_{0}}$ with $a_{0,0} \neq 0$, and a sequence $\left(\rho_{j}\right)_{j \in \mathbb{N}_{0}}$ such that

$$
t_{n+1, k+1}=\sum_{i \geq 0} \sum_{j \geq 0} a_{i, j} t_{n-i, k+j}+\sum_{j \geq 0} \rho_{j} t_{n+1, k+j+2}
$$

The power series definition of a Riordan array is as follows. A Riordan array is defined by a pair of power series, $g(x)$ and $f(x)$, where

$$
g(x)=g_{0}+g_{1} x+g_{2} x^{2}+\cdots, \quad g_{0} \neq 0,
$$

and

$$
f(x)=f_{1} x+f_{2} x^{2}+f_{3} x^{3}+\cdots, \quad f_{0}=0 \text { and } f_{1} \neq 0
$$

We then have

$$
t_{n, k}=\left[x^{n}\right] g(x) f(x)^{k},
$$

where $\left[x^{n}\right]$ is the functional that extracts the coefficient of $x^{n}$. The relationship between $f(x)$ and the pair $(A, \rho)$ is the following.

$$
\frac{f(x)}{x}=\sum_{i \geq 0} x^{i} R^{(i)}(f(x))+\frac{f(x)^{2}}{x} \rho(f(x)),
$$

where $R^{(i)}$ is the generating series of the $i$-th row of $A$, and $\rho(x)$ is the generating series of the sequence $\rho_{n}$.

We shall call the matrix $A=\left(a_{i, j}\right)_{i, j \in \mathbb{N}_{0}}$ the $A$-matrix, while we call the sequence $a_{n}$ the $A$-sequence of the Riordan array. The generating function of $a_{n}$ is denoted by $A(x)$. We have

$$
A(x)=\frac{x}{\bar{f}(x)}
$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$ (thus we have $f(\bar{f}(x))=x$ and $\bar{f}(f(x))=x)$. We sometimes write $\bar{f}(x)=\operatorname{Rev}(f)(x)$.

The generating function of the sequence $z_{n}$ is denoted by $Z(x)$. We have

$$
(g(x), f(x))=\left(\frac{A(x)-x Z(x)}{A(x)}, \frac{x}{A(x)}\right)^{-1}
$$

Note that while the production matrix $P$ of a Riordan array is unique, the pair $(A, \rho)$ is not necessarily unique.

The set of Riordan arrays $(g(x), f(x))$ is a group under the multiplication law

$$
(g(x), f(x)) \cdot(u(x), v(x))=(g(x) u(f(x)), v(f(x))
$$

and the inverse of $(g(x), f(x))$ in the group is given by

$$
(g(x), f(x))^{-1}=\left(\frac{1}{g(\bar{f}(x))}, \bar{f}(x)\right)
$$

where $\bar{f}(x)$ is the compositional inverse of $f(x)$.
The set of Riordan arrays of the form $\left(\frac{f(x)}{x}, f(x)\right)$ is a subgroup of the Riordan group, called the subgroup of Bell matrices.

We shall refer to sequences, where known, by their Annnnnn numbers in the Online Encyclopedia of Integer Sequences [12,13]. The sequence $0^{n}$ is the sequence that begins $1,0,0,0, \ldots$, with generating function $x^{0}=1$. Note that we only show suitable truncations of Riordan arrays and of production arrays.

Example 2. Pascal's triangle (also called the binomial matrix) $\left.\binom{n}{k}\right)$ is the Riordan array

$$
\left(\frac{1}{1-x}, \frac{x}{1-x}\right) .
$$

This begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 0 & 0 & 0 \\
1 & 4 & 6 & 4 & 1 & 0 & 0 \\
1 & 5 & 10 & 10 & 5 & 1 & 0 \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}\right) .
$$

The $A$-sequence for this triangle is given by $A(x)=1+x$. It can also be defined by the $A$-matrix

$$
A=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & -1
\end{array}\right),
$$

and the $\rho$ sequence $1,0,0, \ldots$.
This follow since the solution to the equation

$$
\frac{u}{x}=1-x\left(u+u^{2}\right)+\frac{u^{2}}{x}
$$

is

$$
u=f(x)=\frac{x}{1-x}
$$

We shall call the result of multiplying a Riordan array (on the left) by the binomial matrix as its binomial transform.

Example 3. The identity matrix $(1, x)$ can be defined by the $A$-matrix

$$
A=\left(\begin{array}{ccc}
1 & 0 & 1 \\
-1 & -1 & 0
\end{array}\right)
$$

and the $\rho$ sequence $1,0,0, \ldots$. This follows since the equation

$$
\frac{u}{x}=1+u^{2}-x(1+u)+\frac{u^{2}}{x}
$$

has the solution $u=f(x)=x$.
Another $A$-matrix and $\rho$ sequence pair that define the identity matrix is given by

$$
A=\left(\begin{array}{lll}
1 & -1 & 1 \\
0 & -1 & 0
\end{array}\right)
$$

and the $\rho$ sequence $1,0,0, \ldots$.
It can also be defined by the simpler $A$-matrix $A=(1)$ and $\rho_{n}=0$.
Example 4. We consider the Motzkin triangle defined by the Riordan array

$$
\left(\frac{1-x-\sqrt{1-2 x-3 x^{2}}}{2 x^{2}}, \frac{1-x-\sqrt{1-2 x-3 x^{2}}}{2 x}\right) .
$$

This can be expressed as

$$
\left(\frac{1}{1-x} c\left(\frac{x^{2}}{(1-x)^{2}}\right), \frac{x}{1-x} c\left(\frac{x^{2}}{(1-x)^{2}}\right)\right),
$$

where $c(x)=\frac{1-\sqrt{1-4 x}}{2 x}$ is the generating function of the Catalan numbers $C_{n}=\frac{1}{n+1}\binom{2 n}{n}$ A000108.

This array begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 & 0 & 0 \\
9 & 12 & 9 & 4 & 1 & 0 & 0 \\
21 & 30 & 25 & 14 & 5 & 1 & 0 \\
51 & 76 & 69 & 44 & 20 & 6 & 1
\end{array}\right) .
$$

The production matrix takes the simple form

$$
\left(\begin{array}{lllllll}
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
$$

and hence the $Z$-sequence is the sequence $1,1,0,0,0, \ldots$ and the $A$-sequence is $1,1,1,0,0,0, \ldots$. Then $Z(x)=1+x$ and $A(x)=1+x+x^{2}$.

This Riordan array is also defined by the $A$-matrix

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 1
\end{array}\right)
$$

and the $\rho$ sequence $0,0,0, \ldots$.
In this example, the $A$-sequence is of a simple form, as is the $A$-matrix. In many cases, however, the $A$-matrix may be of a simple form, while the $A$-sequence is of a more complex nature.

Example 5. We now take the example where

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right)
$$

and $\rho_{n}=0$ for all $n \geq 0$. We find that $u=f(x)$ satisfies the equation

$$
\frac{u}{x}=1+u^{2}+x(1+u),
$$

or

$$
f(x)=\frac{1-x^{2}-\sqrt{1-6 x^{2}-4 x^{3}+x^{4}}}{2 x}=\frac{x}{1-x} c\left(\frac{x^{2}(1+x)}{\left(1-x^{2}\right)^{2}}\right) .
$$

The resulting Riordan array is then given by the Bell subgroup element $\left(\frac{f(x)}{x}, f(x)\right)$. This begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 & 0 & 0 \\
3 & 5 & 3 & 1 & 0 & 0 & 0 \\
7 & 10 & 9 & 4 & 1 & 0 & 0 \\
13 & 24 & 22 & 14 & 5 & 1 & 0 \\
31 & 52 & 57 & 40 & 20 & 6 & 1
\end{array}\right) .
$$

We have

$$
\bar{f}(x)=\frac{\sqrt{1+4 x+6 x^{2}+x^{4}}-x^{2}-1}{2(1+x)}
$$

and so $A(x)$ is given by

$$
A(x)=\frac{1+x^{2}+\sqrt{1+4 x+6 x^{2}+x^{4}}}{2}
$$

Thus $a_{n}$ is the sequence that begins

$$
1,1,1,-1,2,-3,3,1,-15,47,-98, \ldots
$$

The expansion of $f(x)$ in this case begins

$$
0,1,1,2,3,7,13,31,65,156,351,849, \ldots
$$

The expansion of $\frac{f(x)}{x}$ is given by A171416. The Hankel transform of this sequence begins

$$
1,1,2,3,7,23,59,314,1529,8209,83313, \ldots
$$

or $\mathrm{A} 006720(n+2)$. This is essentially the Somos 4 sequence.
In this example, we see that while the $A$-sequence is reasonably complicated, the $A$ matrix is simple. In addition, this $A$-matrix leads to a power series $f(x)$ such that the Hankel transform of the expansion of $\frac{f(x)}{x}$ is the Somos 4 sequence.

In the sequel, we shall be interested in determining a form of $A$-matrix that will always lead to a Somos 4 type sequence.

## 2 General Somos 4 sequences

We call a sequence $s_{n}$ a $(\alpha, \beta)$ Somos 4 sequence if the terms $s_{n}$ satisfy

$$
s_{n}=\frac{\alpha s_{n-1} s_{n-3}+\beta s_{n-2}^{2}}{s_{n-4}}, \quad n \geq 4
$$

In exceptional cases, where a divide by zero may occur, the more general condition

$$
s_{n} s_{n-4}=\alpha s_{n-1} s_{n-3}+\beta s_{n-2}^{2}
$$

is appropriate. There are many proven and conjectured results concerning ( $\alpha, \beta$ ) Somos 4 sequences and Riordan arrays [2, 4]. These sequences are closely related to elliptic curves.

## 3 -matrices and Somos conjectures

We consider the case

$$
A=\left(\begin{array}{lll}
1 & a & b \\
1 & c & d
\end{array}\right)
$$

with $\rho_{n}=0$ for all $n$. To find $u=f(x)$, we solve the equation

$$
\frac{u}{x}=1+a u+b u^{2}+x\left(1+c u+d u^{2}\right) .
$$

This gives us

$$
f(x)=\frac{1-a x-c x^{2}-\sqrt{1-2 a x-\left(a^{2}-2(2 b+c)\right) x^{2}+2(a c-2(b+d)) x^{3}+\left(c^{2}-4 d\right) x^{4}}}{2 x(d x+b)} .
$$

It follows that

$$
\frac{f(x)}{x}=\frac{1+x}{1-a x-c x^{2}} c\left(\frac{x^{2}(1+x)(b+d x)}{\left(1-a x-c x^{2}\right)^{2}}\right) .
$$

This now allows us to make the following conjecture.

Conjecture 6. The Hankel transform of the expansion of $\frac{f(x)}{x}$ is a $\left((b+a b+d)^{2}, b^{4}-b^{3}(2+\right.$ $\left.\left.3 a+a^{2}-2 c\right)+b\left(a+a^{2}-a c-2 d\right) d+(1+a-c) d^{2}-b^{2}\left(c+a c-c^{2}+2 d+3 a d\right)\right)$ Somos 4 sequence.

Example 7. The matrix

$$
A=\left(\begin{array}{ccc}
1 & -2 & -1 \\
1 & 1 & 0
\end{array}\right)
$$

leads to the Riordan array with

$$
\frac{f(x)}{x}=\frac{1+x}{1+2 x-x^{2}} c\left(-\frac{x^{2}(1+x)}{\left(1+2 x-x^{2}\right)^{2}}\right) .
$$

This expands to give the sequence that begins

$$
1,-1,2,-3,3,1,-15,47,-98,133,-17, \ldots
$$

The Hankel transform of this sequence begins

$$
1,1,-2,-1,3,-5,-7,-4,23,29,-59, \ldots
$$

This is a $(1,1)$ Somos 4 sequence (compare with $\underline{\text { A006769 }}$ ). The expansion $v_{n}$ of $\frac{f(x)}{x}$ thus satisfies the convolution recurrence [1]

$$
v_{n}=-2 v_{n-1}-v_{n-2}-\sum_{i=0}^{n-4} v_{i+1} v_{n-i-3}
$$

with $v_{1}=-1$ and $v_{2}=2$.
Proposition 8. The Bell matrix $\left(t_{n, k}\right)$ which has

$$
A=\left(\begin{array}{lll}
1 & a & b \\
1 & c & d
\end{array}\right)
$$

as its $A$-matrix satisfies

$$
t_{n, k}=t_{n-1, k-1}+a t_{n-1, k}+b t_{n-1, k+1}+t_{n-2, k-1}+c t_{n-2, k}+d t_{n-2, k+1}
$$

with $t_{n, k}=0$ if $k<0$ or $k>n, t_{0,0}=1$ and $t_{1,0}=a+1$. This is the Riordan array

$$
\left(\frac{1+x}{1-a x-c x^{2}} c\left(\frac{x^{2}(1+x)(b+d x)}{\left(1-a x-c x^{2}\right)^{2}}\right), \frac{x(1+x)}{1-a s-c x^{2}} c\left(\frac{s^{2}(1+x)(b+d x)}{\left(1-a x-c x^{2}\right)^{2}}\right)\right) .
$$

Proof. The recurrence follows from the form of the $A$-matrix. We have calculated $f(x)$ above. The Bell element corresponding to this is then $\left(\frac{f(x)}{x}, f(x)\right)$. The value for $t_{1,0}$ follows from this.

We note that the Riordan array

$$
\left(\frac{1-(a-1) x}{1-a x-c x^{2}} c\left(\frac{x^{2}(1+x)(b+d x)}{\left(1-a x-c x^{2}\right)^{2}}\right), \frac{x(1+x)}{1-a x-c x^{2}} c\left(\frac{x^{2}(1+x)(b+d x)}{\left(1-a x-c x^{2}\right)^{2}}\right)\right)
$$

satisfies the same recurrence with $t_{1,0}=1$. Note that we shall always assume that $t_{n, k}=0$ if $k<0$ or $k>n, t_{0,0}=1$ in the following.

## 4 Further examples with $\rho_{n}=0$

In the following examples, we choose $\rho_{n}=0$ for all $n \geq 0$.
Example 9. We let

$$
A=\left(\begin{array}{lll}
1 & 1 & 0 \\
1 & 1 & 1
\end{array}\right)
$$

Then the triangle $\left(t_{n, k}\right)$ with

$$
t_{n, k}=t_{n-1, k-1}+t_{n-1, k}+t_{n-2, k-1}+t_{n-2, k}+t_{n-2, k+1}
$$

and $t_{1,0}=2$ begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 1 & 0 & 0 & 0 & 0 \\
6 & 10 & 6 & 1 & 0 & 0 & 0 \\
13 & 24 & 21 & 8 & 1 & 0 & 0 \\
29 & 59 & 62 & 36 & 10 & 1 & 0 \\
66 & 146 & 174 & 128 & 55 & 12 & 1
\end{array}\right) .
$$

This is the Bell matrix

$$
\left(\frac{1+x}{1-x-x^{2}} c\left(\frac{x^{3}(1+x)}{\left(1-x-x^{2}\right)^{2}}\right), \frac{x(1+x)}{1-x-x^{2}} c\left(\frac{x^{3}(1+x)}{\left(1-x-x^{2}\right)^{2}}\right)\right)
$$

The Hankel transform of the sequence $t_{n, 0}$ begins

$$
1,-1,-4,-3,19,67, \ldots,
$$

which is the $(1,1)$ Somos 4 sequence $\underline{\text { A178628 }}(n+1)$. In fact, the generating function for $t_{n, 0}$ may be expressed as the continued fraction [14]

$$
\frac{1}{1-2 x+\frac{x^{2}}{1+2 x+\frac{4 x^{2}}{1-\frac{11}{4} x-\frac{\frac{3}{16} x^{2}}{1-\frac{43}{12} x-\cdots}}}}
$$

where $1,4,-\frac{3}{16}, \ldots$ are the $x$-coordinates of the multiples of $(0,0)$ on the elliptic curve

$$
y^{2}-x y-y=x^{3}+x^{3}+x .
$$

Example 10. We let

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 0 & 1
\end{array}\right)
$$

Then the triangle $\left(t_{n, k}\right)$ with

$$
t_{n, k}=t_{n-1, k-1}+t_{n-1, k+1}+t_{n-2, k-1}+t_{n-2, k+1},
$$

and $t_{1,0}=1$ begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 1 & 0 & 0 & 0 & 0 \\
3 & 3 & 3 & 1 & 0 & 0 & 0 \\
5 & 8 & 6 & 4 & 1 & 0 & 0 \\
11 & 17 & 16 & 10 & 5 & 1 & 0 \\
25 & 38 & 39 & 28 & 15 & 6 & 1
\end{array}\right) .
$$

This is the Bell matrix

$$
\left((1+x) c\left(x^{2}\left(1+x+x^{2}\right)\right), x(1+x) c\left(x^{2}\left(1+x+x^{2}\right)\right)\right)
$$

The sequence $t_{n, 0}$ begins

$$
1,1,1,3,5,11,25,55,129,303,721,1743, \ldots
$$

This is A104545, which counts the number of Motzkin paths of length $n$ having no consecutive $(1,0)$ steps (Emeric Deutsch). Its Hankel transform begins

$$
1,0,-4,-16,-64,0,4096,65536,1048576,0,-1073741824, \ldots
$$

This is A162547, the Somos 4 variant

$$
\left.s_{n}=\left(4 s_{n-1} s_{n-3}-4 s_{n-2}\right)^{2}\right) / s_{n-4}, \quad \text { if } n \neq 4 k+1,
$$

otherwise $s_{n}=0$, with $s_{-2}=s_{-1}=s_{0}=1$.
Example 11. We let

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 4 & 1
\end{array}\right)
$$

Then the triangle $\left(t_{n, k}\right)$ with

$$
t_{n, k}=t_{n-1, k-1}+t_{n-1, k+1}+t_{n-2, k-1}+4 t_{n-2, k}+t_{n-2, k+1},
$$

and $t_{1,0}=1$ begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
5 & 2 & 1 & 0 & 0 & 0 & 0 \\
7 & 11 & 3 & 1 & 0 & 0 & 0 \\
33 & 24 & 18 & 4 & 1 & 0 & 0 \\
63 & 105 & 52 & 26 & 5 & 1 & 0 \\
261 & 262 & 231 & 92 & 35 & 6 & 1
\end{array}\right) .
$$

The sequence $t_{n, 0}$ which begins

$$
1,1,5,7,33,63,261,619,2333,6355,22669, \ldots
$$

has a Hankel transform that begins

$$
1,4,28,304,14272,676864, \ldots
$$

This is a $(4,12)$ Somos 4 sequence. In general, the $A$-matrix

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & r & 1
\end{array}\right)
$$

leads to a $\left(4, r^{2}-4\right)$ Somos 4 sequence.

## 5 A quasi-involution

We consider the case of the $A$-matrix

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

We solve the equation

$$
\frac{u}{x}=1+u^{2}+x u,
$$

to obtain

$$
u=f(x)=\frac{1-x^{2}-\sqrt{1-6 x^{2}+x^{4}}}{2 x}
$$

Thus the Bell matrix ( $t_{n, k}$ ) defined by the above $A$-matrix is the Riordan array

$$
\left(\frac{1-x^{2}-\sqrt{1-6 x^{2}+x^{4}}}{2 x^{2}}, \frac{1-x^{2}-\sqrt{1-6 x^{2}+x^{4}}}{2 x}\right) .
$$

This triangle begins

$$
\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 6 & 0 & 1 & 0 & 0 & 0 \\
0 & 16 & 0 & 8 & 0 & 1 & 0 & 0 \\
22 & 0 & 30 & 0 & 10 & 0 & 1 & 0 \\
0 & 68 & 0 & 48 & 0 & 12 & 0 & 1
\end{array}\right) .
$$

The inverse of this matrix begins

$$
\left(\begin{array}{cccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 0 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & -6 & 0 & 1 & 0 & 0 & 0 \\
0 & 16 & 0 & -8 & 0 & 1 & 0 & 0 \\
-22 & 0 & 30 & 0 & -10 & 0 & 1 & 0 \\
0 & -68 & 0 & 48 & 0 & -12 & 0 & 1
\end{array}\right) .
$$

We recall that a Riordan array $\left(g\left(x^{2}\right), x g\left(x^{2}\right)\right)$ is called a quasi-involution if we have

$$
\left(g\left(x^{2}\right), x g\left(x^{2}\right)\right)^{-1}=\left(g\left(-x^{2}\right), x g\left(-x^{2}\right)\right)
$$

This is the case for this example [5].

## 6 The case of $r_{n}=0^{n}$

In this section, we take the case of

$$
A=\left(\begin{array}{lll}
1 & a & b \\
1 & c & d
\end{array}\right)
$$

and $\rho_{0}=1$, and $\rho_{n}=0$ for $n>0$. Thus we must solve the equation

$$
\frac{u}{x}=1+a u+b u^{2}+x\left(1+c u+d u^{2}\right)+\rho_{0} \frac{u^{2}}{x}
$$

to get $f(x)=u$. We find that

$$
\frac{f(x)}{x}=\frac{1+x}{1-a x-c x^{2}} c\left(\frac{x(1+x)\left(\rho_{0}+b x+d x^{2}\right)}{\left(1-a x-c x^{2}\right)^{2}}\right) .
$$

In this case, $\frac{f(x)}{x}$ expands to begin

$$
1, a+\rho_{0}+1, a^{2}+a\left(3 \rho_{0}+1\right)+b+c+2 \rho_{0}^{2}+2 \rho_{0}, \ldots,
$$

and the corresponding Bell triangle begins

$$
\left(\begin{array}{ccc}
1 & 0 & 0 \\
a+\rho_{0}+1 & 1 & 0 \\
a^{2}+a\left(3 \rho_{0}+1\right)+b+c+2 \rho_{0}\left(\rho_{0}+1\right) & 2 a+2 \rho_{0}+2 & 1
\end{array}\right) .
$$

In the case that $\rho_{0}=1$, we obtain

$$
\frac{f(x)}{x}=\frac{1+x}{1-a x-c x^{2}} c\left(\frac{x(1+x)\left(1+b x+d x^{2}\right)}{\left(1-a x-c x^{2}\right)^{2}}\right) .
$$

Example 12. We consider the case of

$$
A=\left(\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right)
$$

and $\rho_{n}=0^{n}$. The resulting Bell triangle begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
6 & 4 & 1 & 0 & 0 & 0 & 0 \\
22 & 16 & 6 & 1 & 0 & 0 & 0 \\
90 & 68 & 30 & 8 & 1 & 0 & 0 \\
394 & 304 & 146 & 48 & 10 & 1 & 0 \\
1806 & 1412 & 714 & 264 & 70 & 12 & 1
\end{array}\right) .
$$

This is the Bell matrix defined by the large Schroeder numbers A006318,

$$
\left(t_{n, k}\right)=\left(\frac{1-x-\sqrt{1-6 x+x^{2}}}{2 x}, \frac{1-x-\sqrt{1-6 x+x^{2}}}{2}\right),
$$

and its $A$-sequence is the sequence

$$
1,2,2,2,2,2,2,2, \ldots .
$$

This follows since the solution $u(x)$ with $u(0)=0$ of the equation

$$
\frac{u}{x}=1+u^{2}+x(1+u)+\frac{u^{2}}{x}
$$

is given by

$$
u(x)=\frac{1-x-\sqrt{1-6 x+x^{2}}}{2}
$$

This is also the solution to the equation

$$
\frac{u}{x}=1+u+\frac{u^{2}}{x} .
$$

Thus a simpler $A$-matrix in this case is given by

$$
A=\left(\begin{array}{ll}
1 & 1
\end{array}\right),
$$

along with $\rho_{n}=0^{n}$.
In general, taking

$$
A=\left(\begin{array}{ll}
1 & r
\end{array}\right),
$$

along with $\rho_{n}=0^{n}$, leads to

$$
\frac{f(x)}{x}=\frac{1-(r-1) x-\sqrt{1-2(r+1) x+(r-1)^{2} x^{2}}}{2 x}=\frac{1}{1-(r-1) x} c\left(\frac{x}{(1-(r-1) x)^{2}}\right),
$$

which is the generating function of the Narayana polynomials

$$
1, r, r(r+1), r\left(r^{2}+3 r+1\right), r\left(r^{3}+6 r^{2}+6 r+1\right), \ldots
$$

In this case the triangle $\left(t_{n, k}\right)$ begins

$$
\left(\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
r & 1 & 0 & 0 & 0 \\
r(r+1) & 2 r & 1 & 0 & 0 \\
r\left(r^{2}+3 r+1\right) & r(3 r+2) & 3 r & 1 & 0 \\
r\left(r^{3}+6 r^{2}+6 r+1\right) & 2 r\left(2 r^{2}+4 r+1\right) & 3 r(2 r+1) & 4 r & 1
\end{array}\right),
$$

with $A$-sequence

$$
1, r, r, r, \ldots .
$$

Example 13. We take

$$
A=\left(\begin{array}{lll}
1 & -2 & 2 \\
1 & -1 & 1
\end{array}\right)
$$

and $\rho_{n}=0^{n}$. Then the Bell matrix that we obtain is A097609, which begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 1 & 0 & 0 & 0 \\
3 & 2 & 3 & 0 & 1 & 0 & 0 \\
6 & 7 & 3 & 4 & 0 & 1 & 0 \\
15 & 14 & 12 & 4 & 5 & 0 & 1
\end{array}\right)
$$

This is the Bell matrix

$$
\left(\frac{1+x-\sqrt{1-2 x-3 x^{2}}}{2 x(1-x)}, \frac{1+x-\sqrt{1-2 x-3 x^{2}}}{2(1-x)}\right)
$$

defined by the so-called Motzkin sums A005043. This triangle counts the number of Motzkin paths of length $n$ having $k$ horizontal steps at level 0 (Emeric Deutsch). The $A$-sequence of this triangle is

$$
1,0,1,1,1,1, \ldots
$$

We now turn to the issue of the Somos 4 nature of the Hankel transform of the expansion of $\frac{f(x)}{x}$. We have the following conjecture.

Conjecture 14. The Hankel transform of the expansion of $\frac{f(x)}{x}$ in the case that

$$
A=\left(\begin{array}{lll}
1 & a & b \\
1 & c & d
\end{array}\right)
$$

and $\rho_{n}=0^{n}$ is an $(\alpha, \beta)$ Somos 4 sequence, where

$$
\alpha=\left(4+a^{2}+3 b+a(4+b)+c+d\right)^{2}
$$

and

$$
\begin{aligned}
\beta & =-16-a^{5}+b^{4}-3 a^{4}(3+b)+2 b^{3}(-2+c)-8 c-4 c^{2}-c^{3} \\
& +b^{2}\left(-28-c+c^{2}-8 d\right)-8 d-6 c d-2 c^{2} d-d^{2}-c d^{2} \\
& -a^{3}\left(32+23 b+3 b^{2}+c+2 d\right)-2 b\left(20+c^{2}+9 d+d^{2}+3 c(2+d)\right) \\
& -a^{2}\left(56+18 b^{2}+b^{3}+10 d+c(6+d)+b(66+c+5 d)\right) \\
& -a\left(48+3 b^{3}+2 c^{2}+16 d+d^{2}+b^{2}(38-2 c+3 d)+c(12+5 d)+b\left(84-c^{2}+19 d+c(8+d)\right)\right) .
\end{aligned}
$$

Example 15. For

$$
A=\left(\begin{array}{lll}
1 & 0 & -1 \\
1 & 0 & -2
\end{array}\right)
$$

and $\rho_{n}=0^{n}$, we obtain the Bell triangle that begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 4 & 1 & 0 & 0 & 0 & 0 \\
4 & 10 & 6 & 1 & 0 & 0 & 0 \\
2 & 20 & 21 & 8 & 1 & 0 & 0 \\
-11 & 29 & 56 & 36 & 10 & 1 & 0 \\
-59 & 10 & 117 & 120 & 55 & 12 & 1
\end{array}\right) .
$$

For this, we have

$$
\frac{f(x)}{x}=(1+x) c\left(x(1+x)\left(1-x-2 x^{2}\right)\right)
$$

The Hankel transform of the expansion of this generating function begins

$$
1,-1,3,4,5,-31, \ldots
$$

This is a $(1,1)$ Somos 4 sequence.
Example 16. For

$$
A=\left(\begin{array}{ccc}
1 & 0 & -1 \\
1 & -2 & 2
\end{array}\right)
$$

and $\rho_{n}=0^{n}$, we obtain the Bell matrix that begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & 1 & 0 & 0 & 0 & 0 \\
0 & 6 & 6 & 1 & 0 & 0 & 0 \\
4 & 4 & 15 & 8 & 1 & 0 & 0 \\
17 & 9 & 20 & 28 & 10 & 1 & 0 \\
41 & 50 & 27 & 56 & 45 & 12 & 1
\end{array}\right) .
$$

For this, we have

$$
\frac{f(x)}{x}=\frac{1+x}{1+2 x^{2}} c\left(\frac{x(1+x)\left(1-x+2 x^{2}\right.}{\left(1+2 x^{2}\right)^{2}}\right) .
$$

The Hankel transform of the expansion of this generating function begins

$$
1,-3,-13,14,465,1819, \ldots
$$

This is a $(1,3)$ Somos 4 sequence.

## 7 The case $\rho(x)=1+r x$

We briefly consider the case

$$
A=\left(\begin{array}{ll}
1 & 1
\end{array}\right),
$$

along with $\rho(x)=1+x$. We are thus led to the equation

$$
\frac{u}{x}=1+u+\frac{u^{2}(1+u)}{x},
$$

which has solution

$$
u(x)=f(x)=\frac{1}{3}\left(\sqrt{4-3 x} \sin \left(\frac{1}{3} \sin ^{-1}\left(\frac{18 x+11}{2(4-3 x)^{\frac{3}{2}}}\right)\right)-1\right) .
$$

Equivalently, we have

$$
f(x)=\operatorname{Rev} \frac{x\left(1-x-x^{2}\right)}{1+x}
$$

We have that $\frac{f(x)}{x}$ expands to give the sequence that begins

$$
1,2,7,31,154,820,4575,26398,156233, \ldots
$$

or A007863, the number of "hybrid binary trees" with $n$ internal nodes. The corresponding Bell matrix begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
7 & 4 & 1 & 0 & 0 & 0 & 0 \\
31 & 18 & 6 & 1 & 0 & 0 & 0 \\
154 & 90 & 33 & 8 & 1 & 0 & 0 \\
820 & 481 & 185 & 52 & 10 & 1 & 0 \\
4575 & 2690 & 1065 & 324 & 75 & 12 & 1
\end{array}\right) .
$$

The production matrix of this array starts

$$
\left(\begin{array}{cccccc}
2 & 1 & 0 & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 \\
5 & 3 & 2 & 1 & 0 & 0 \\
8 & 5 & 3 & 2 & 1 & 0 \\
13 & 8 & 5 & 3 & 2 & 1 \\
21 & 13 & 8 & 5 & 3 & 2
\end{array}\right),
$$

indicating that

$$
A(x)=\frac{1+x}{1-x-x^{2}}, \quad \text { and } Z(x)=\frac{2+x}{1-x-x^{2}}
$$

Here, we recognize the shifted Fibonacci numbers $\underline{\text { A000045. The Bell matrix }\left(t_{n, k}\right) \text { in this }}$ case is given by

$$
\left(\frac{1-x-x^{2}}{1+x}, \frac{x\left(1-x-x^{2}\right)}{1+x}\right)^{-1}
$$

Similarly, the $A$-matrix

$$
A=\left(\begin{array}{ll}
1 & 2
\end{array}\right),
$$

along with $\rho(x)=1+x$, leads to the Bell matrix

$$
\left(\frac{1-x-x^{2}}{1+2 x}, \frac{x\left(1-x-x^{2}\right)}{1+2 x}\right)^{-1}
$$

which begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
13 & 6 & 1 & 0 & 0 & 0 & 0 \\
70 & 35 & 9 & 1 & 0 & 0 & 0 \\
424 & 218 & 66 & 12 & 1 & 0 & 0 \\
2756 & 1437 & 471 & 106 & 15 & 1 & 0 \\
18778 & 9876 & 3390 & 856 & 155 & 18 & 1
\end{array}\right) .
$$

The case of

$$
A=\left(\begin{array}{ll}
1 & 1
\end{array}\right),
$$

along with $\rho(x)=1+2 x$ is especially interesting. We obtain the Bell matrix that begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 1 & 0 & 0 & 0 & 0 \\
40 & 20 & 6 & 1 & 0 & 0 & 0 \\
224 & 112 & 36 & 8 & 1 & 0 & 0 \\
1344 & 672 & 224 & 56 & 10 & 1 & 0 \\
8448 & 4224 & 1440 & 384 & 80 & 12 & 1
\end{array}\right),
$$

which has a production matrix that begins

$$
\left(\begin{array}{cccccc}
2 & 1 & 0 & 0 & 0 & 0 \\
4 & 2 & 1 & 0 & 0 & 0 \\
8 & 4 & 2 & 1 & 0 & 0 \\
16 & 8 & 4 & 2 & 1 & 0 \\
32 & 16 & 8 & 4 & 2 & 1 \\
64 & 32 & 16 & 8 & 4 & 2
\end{array}\right) .
$$

Thus we have

$$
A(x)=\frac{1}{1-2 x}, \quad \text { and } Z(x)=\frac{2}{1-2 x}
$$

The corresponding Bell matrix $\left(t_{n, k}\right)$ is given by

$$
(1-2 x, x(1-2 x))^{-1}
$$

In effect, the solution to the equation

$$
\frac{u}{x}=1+u+\frac{u^{2}(1+2 u)}{x}
$$

is given by

$$
u(x)=f(x)=\frac{1-\sqrt{1-8 x}}{4}
$$

This is the generating function of $2^{n} C_{n} \underline{\text { A151374 }}$. Extending the two previous cases, we look at the case

$$
A=\left(\begin{array}{ll}
1 & 2
\end{array}\right),
$$

along with $\rho(x)=1+2 x$. The equation to be solved is now

$$
\frac{u}{x}=1+2 u+\frac{u^{2}(1+2 u)}{x}
$$

with solution

$$
u(x)=f(x)=\frac{1}{3}\left(\sqrt{7-12 x} \sin \left(\frac{1}{3} \sin ^{-1}\left(\frac{2(18 x+5)}{(7-12 x)^{\frac{3}{2}}}\right)\right)-\frac{1}{2}\right) .
$$

Equivalently, we have

$$
f(x)=\operatorname{Rev} \frac{x\left(1-x-2 x^{2}\right)}{1+2 x}
$$

The Bell matrix $\left(t_{n, k}\right)$ in this case begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 0 \\
14 & 6 & 1 & 0 & 0 & 0 & 0 \\
83 & 37 & 9 & 1 & 0 & 0 & 0 \\
554 & 250 & 69 & 12 & 1 & 0 & 0 \\
3966 & 1802 & 528 & 110 & 15 & 1 & 0 \\
29756 & 13580 & 4122 & 944 & 160 & 18 & 1
\end{array}\right),
$$

with a production matrix that begins

$$
\left(\begin{array}{cccccc}
3 & 1 & 0 & 0 & 0 & 0 \\
5 & 3 & 1 & 0 & 0 & 0 \\
11 & 5 & 3 & 1 & 0 & 0 \\
21 & 11 & 5 & 3 & 1 & 0 \\
43 & 21 & 11 & 5 & 3 & 1 \\
85 & 43 & 21 & 11 & 5 & 3
\end{array}\right) .
$$

We recognise the shifted Jacobsthal numbers A001045. We deduce that the above Bell matrix is given by

$$
\left(\frac{1-x-2 x^{2}}{1+2 x}, \frac{x\left(1-x-2 x^{2}\right)}{1+2 x}\right)^{-1}
$$

The sequence $t_{n, 0}$ is given by A215661.
In general, the Bell matrix $\left(t_{n, k}\right)$ defined by

$$
A=\left(\begin{array}{ll}
1 & r
\end{array}\right),
$$

along with $\rho(x)=1+r x$ is given by

$$
\left(\frac{1-x-r x^{2}}{1+r x}, \frac{x\left(1-x-r x^{2}\right)}{1+r x}\right)^{-1}
$$

We have

$$
t_{n, k}=t_{n-1, k-1}+r t_{n-1, k}+t_{n, k+1}+r t_{n, k+2}
$$

with $t_{0,0}=1, t_{1,0}=r+1$.
The sequence $t_{n, 0}$ (the first column of the Bell matrix) then begins

$$
1, r+1, r^{2}+4 r+2, r^{3}+10 r^{2}+15 r+5, r^{4}+20 r^{3}+63 r^{2}+56 r+14, \ldots
$$

This polynomial sequence $P_{n}(r)$ can be expressed as

$$
P_{n}(r)=\sum_{k=0^{n}} \frac{1}{k+(2 n+1) 0^{k}}\binom{2 n-k}{k-1+0^{k}}\binom{2 n-k+1}{n-k} r^{k}
$$

where the coefficient array begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 4 & 1 & 0 & 0 & 0 & 0 \\
5 & 15 & 10 & 1 & 0 & 0 & 0 \\
14 & 56 & 63 & 20 & 1 & 0 & 0 \\
42 & 210 & 336 & 196 & 35 & 1 & 0 \\
132 & 792 & 1650 & 1440 & 504 & 56 & 1
\end{array}\right) .
$$

An interesting feature of this array is the following. The diagonal sums of this array begin

$$
1,1,3,9,30,108,406,1577,6280,25499,105169, \ldots .
$$

This is A200074. The generating function $g(x)$ of this sequence can be expressed as $g(x)=$ $\frac{u(x)}{x}$, where we have

$$
\frac{u}{x}=1+u^{3}+x u+\frac{u^{2}}{x}
$$

Thus this sequence is the first column of the Bell matrix $\left(t_{n, k}\right)$ defined by the $A$-matrix

$$
\left(\begin{array}{llll}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{array}\right),
$$

with $\rho_{n}=0^{n}$. This triangle $\left(t_{n, k}\right)$ begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 & 0 \\
9 & 7 & 3 & 1 & 0 & 0 & 0 \\
30 & 24 & 12 & 4 & 1 & 0 & 0 \\
108 & 87 & 46 & 18 & 5 & 1 & 0 \\
406 & 330 & 180 & 76 & 25 & 6 & 1
\end{array}\right) .
$$

We have

$$
t_{n, k}=t_{n-1, k-1}+t_{n-1, k+2}+t_{n-2, k}+t_{n, k+1},
$$

with $t_{1,0}=1$.

## 8 Perturbed orthogonal polynomials

We consider the case when

$$
A=\left(\begin{array}{lll}
1 & a & b
\end{array}\right),
$$

and $\rho_{n}=c 0^{n}$.
Thus we must solve the equation

$$
\frac{u}{x}=1+a u+b u^{2}+\frac{c u^{2}}{x} .
$$

The required solution is

$$
u=\frac{1-a x-\sqrt{1-2(a+2 c) x+\left(a^{2}-4 b\right) x^{2}}}{2(b x+c)}=\frac{x}{1-a x} c\left(\frac{x(b x+c)}{(1-a x)^{2}}\right) .
$$

Alternatively, we have

$$
u=\operatorname{Rev} \frac{x(1-c x)}{1+a x+b x^{2}}
$$

Thus the Bell matrix $\left(t_{n, k}\right)$ with

$$
t_{n, k}=t_{n-1, k-1}+a t_{n-1, k}+b t_{n-1, k+1}+c t_{n, k+1}
$$

and $t_{1,0}=a+c$ is given by

$$
\left(\frac{1-c x}{1+a x+b x^{2}}, \frac{x(1-c x)}{1+a x+b x^{2}}\right)^{-1}
$$

or

$$
\left(\frac{1}{x} \operatorname{Rev} \frac{x(1-c x)}{1+a x+b x^{2}}, \operatorname{Rev} \frac{x(1-c x)}{1+a x+b x^{2}}\right) .
$$

In the case that $c=0$, this is the moment matrix of the orthogonal polynomials

$$
P_{n}(x)=(x-a) P_{n-1}(x)-b P_{n-2}(x),
$$

with $P_{0}(x)=1, P_{1}(x)=x-a$.
Similarly, the triangle ( $t_{n, k}$ ) with

$$
t_{n, k}=t_{n-1, k-1}+a t_{n-1, k}+b t_{n-1, k+1}+c t_{n-1, k+2}+d t_{n, k+1}
$$

and $t_{1,0}=a+d$ is given by

$$
\left(\frac{1}{x} \operatorname{Rev} \frac{x(1-d x)}{1+a x+b x^{2}+c x^{3}}, \operatorname{Rev} \frac{x(1-d x)}{1+a x+b x^{2}+c x^{3}}\right) .
$$

The binomial transform of the Riordan array

$$
\left(\frac{1}{1+a x+b x^{2}}, \frac{x}{1+a x+b x^{2}}\right)^{-1}
$$

is the Riordan array

$$
\left(\frac{1}{1+(a+1) x+b x^{2}}, \frac{x}{1+(a+1) x+b x^{2}}\right)^{-1}
$$

In a similar vein, we have the following result.

Proposition 17. The binomial transform of $\frac{u}{x}$ where

$$
\frac{u}{x}=1+a u+b u^{2}+\frac{c u^{2}}{x}
$$

is given by $\frac{v}{x}$ where

$$
\frac{v}{x}=\frac{1}{1-x}\left(1+a v+b v^{2}\right)+\frac{c v^{2}}{x} .
$$

Proof. We have

$$
\frac{u}{x}=\frac{1}{1-a x} c\left(\frac{x(b x+c)}{(1-a x)^{2}}\right) .
$$

We find that the binomial transform $\frac{1}{1-x} u\left(\frac{x}{1-x}\right)$ of $u$ is given by

$$
\frac{1}{1-x} u\left(\frac{x}{1-x}\right)=\frac{1}{1-(a+1) x} c\left(\frac{x(x(b-c)+c)}{(1-(a+1) x)^{2}}\right) .
$$

But this is precisely $\frac{v}{x}$ where

$$
\frac{v}{x}=\frac{1}{1-x}\left(1+a v+b v^{2}\right)+\frac{c v^{2}}{x} .
$$

Thus the Riordan array with $A$-matrix $A=(1, a, b)$ and $\rho_{n}=c 0^{n}$ will have a binomial transform with the infinite $A$-matrix

$$
\left(\begin{array}{ccc}
1 & a & b \\
1 & a & b \\
1 & a & b \\
1 & a & b \\
\vdots & \vdots & \vdots
\end{array}\right)
$$

and $\rho_{n}=c 0^{n}$.
Example 18. The Riordan array

$$
\left(\frac{1-5 x}{1+2 x+3 x^{2}}, \frac{x(1-5 x)}{1+2 x+3 x^{2}}\right)^{-1}
$$

begins

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 1 & 0 & 0 & 0 & 0 & 0 \\
87 & 14 & 1 & 0 & 0 & 0 & 0 \\
1331 & 223 & 21 & 1 & 0 & 0 & 0 \\
22731 & 3880 & 408 & 28 & 1 & 0 & 0 \\
415427 & 71665 & 7990 & 642 & 35 & 1 & 0 \\
7949259 & 1380682 & 159591 & 14004 & 925 & 42 & 1
\end{array}\right) .
$$

Note for instance that

$$
3880=1331+2 \cdot 223+3 \cdot 21+5 \cdot 408
$$

Consider the binomial transform of this matrix, given by

$$
\left(\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 1 & 0 & 0 & 0 & 0 & 0 \\
102 & 16 & 1 & 0 & 0 & 0 & 0 \\
1614 & 268 & 24 & 1 & 0 & 0 & 0 \\
28606 & 4860 & 498 & 32 & 1 & 0 & 0 \\
543298 & 93440 & 10250 & 792 & 40 & 1 & 0 \\
10810754 & 1873548 & 214086 & 18296 & 1150 & 48 & 1
\end{array}\right) .
$$

Note now that, for instance, we have

$$
10250=1 \cdot(4860+268+16+1)+2 \cdot(498+24+1)+3 \cdot(32+1)+5 \cdot 792
$$

## 9 Calculating the $A$-sequence

We give an example of how to calculate the $A$ sequence, given a suitable $A$-matrix and $\rho$ sequence. Thus we take

$$
A=\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 0
\end{array}\right)
$$

and $\rho_{n}=0^{n}$. This leads us to the equation

$$
\frac{u}{x}=1+u+u^{2}+x u+\frac{u^{2}}{x}
$$

whose solution is given by

$$
u=f(x)=\frac{1-x-x^{2}-\sqrt{1-6 x-5 x^{2}+2 x^{3}+x^{4}}}{2(1+x)}=\frac{x}{1-x-x^{2}} c\left(\frac{x(1+x)}{\left(1-x-x^{2}\right)^{2}}\right)
$$

From this we can calculate the $A$-sequence since its generating function $A(x)$ is given by

$$
A(x)=\frac{x}{\bar{f}(x)}
$$

An alternative method is to start with the equation

$$
\frac{u}{x}=1+u+u^{2}+x u+\frac{u^{2}}{x}
$$

which we write using $u=f(x)$ as

$$
\frac{f(x)}{x}=1+f(x)+f(x)^{2}+x f(x)+\frac{(f(x))^{2}}{x} .
$$

Substituting $\bar{f}(x)$ for $x$, and using the fact that $f(\bar{f}(x))=x$, we then obtain

$$
\frac{x}{\bar{f}(x)}=1+x+x^{2}+x \bar{f}(x)+\frac{x^{2}}{\bar{f}(x)} .
$$

Thus we must solve the equation

$$
v=1+x+x^{2}+\frac{x^{2}}{v}+x v
$$

for $v=A(x)=\frac{x}{f(x)}$. We obtain that

$$
A(x)=v=\frac{1+x+x^{2}+\sqrt{1+2 x+7 x^{2}-2 x^{3}+x^{4}}}{2(1-x)}
$$

This expands to give the sequence that begins

$$
1,2,4,2,2,8,-2,-10,52,-26,-202,576, \ldots
$$

The Bell matrix $\left(t_{n, k}\right)$ in this case begins

$$
\left(\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
8 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
34 & 20 & 6 & 1 & 0 & 0 & 0 & 0 & 0 \\
162 & 100 & 36 & 8 & 1 & 0 & 0 & 0 & 0 \\
820 & 524 & 206 & 56 & 10 & 1 & 0 & 0 & 0 \\
4338 & 2832 & 1182 & 360 & 80 & 12 & 1 & 0 & 0 \\
23694 & 15704 & 6828 & 2248 & 570 & 108 & 14 & 1 & 0 \\
132612 & 88876 & 39818 & 13856 & 3850 & 844 & 140 & 16 & 1
\end{array}\right),
$$

and its production matrix takes the form

$$
\left(\begin{array}{cccccccc}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 4 & 2 & 1 & 0 & 0 & 0 & 0 \\
2 & 2 & 4 & 2 & 1 & 0 & 0 & 0 \\
8 & 2 & 2 & 4 & 2 & 1 & 0 & 0 \\
-2 & 8 & 2 & 2 & 4 & 2 & 1 & 0 \\
-10 & -2 & 8 & 2 & 2 & 4 & 2 & 1 \\
52 & -10 & -2 & 8 & 2 & 2 & 4 & 2
\end{array}\right) .
$$

## 10 Conclusions

We have given examples of Riordan arrays defined by simple $A$-matrices and simple rho sequences, and we have shown the form of the calculations required to go from these data to a corresponding Bell matrix. In some cases, we have examined the associated $A$-sequence.

Some special matrices, including a Riordan quasi-involution and certain "perturbed" moment matrices have been studied in terms of their defining $A$-matrix and $\rho$ sequence.

We have conjectured that an $A$-matrix of the form

$$
A=\left(\begin{array}{lll}
1 & a & b \\
1 & c & d
\end{array}\right)
$$

leads to Hankel transforms that are $(\alpha, \beta)$ Somos 4 sequences, in the case that $\rho_{n}=0$ for all $n$ and when $\rho_{n}=0^{n}$. Specific examples bear this out, but the form of $\beta$ in general makes it uncertain at the moment how such a conjecture might be proven, or generalized. Other open questions that remain are the relationship between the parameters ( $a, b, c, d$ ) and the corresponding coefficients of the related elliptic curves. It is clear that these problems deserve further study.

## References

[1] P. Barry, Generalized Catalan recurrences, Riordan arrays, elliptic curves, and orthogonal polynomials
[2] P. Barry, Riordan Pseudo-Involutions, Continued Fractions and Somos 4 Sequences, arXiv:1807.05794 [math.CO], 2018.
[3] P. Barry, Riordan Arrays: A Primer, Logic Press, 2017.
[4] P. Barry, Invariant number triangles, eigentriangles and Somos-4 sequences, arXiv preprint arXiv:1107.5490 [math.CO], 2011.
[5] G-S. Cheon, S-T. Jin, Structural properties of Riordan matrices and extending the matrices, Linear Algebra Appl., 435 (2011), 2019-2032.
[6] Tian-Xiao He, R. Sprugnoli, Sequence characterization of Riordan arrays, Discrete Maths., 309 (2009), 3962-3974.
[7] Tian-Xiao He, Matrix characterizations of Riordan arrays, Linear Algebra Appl., 465 (2015), 15-42.
[8] D. Merlini, D. G. Rogers, R. Sprugnoli, M. C. Verri, On Some Alternative Characterizations of Riordan Arrays, Canad. J. Math., 49 (1997), 301-320.
[9] D. G. Rogers, Pascal triangles, Catalan numbers and renewal arrays, Discrete Math., 22 (1978), 301-310.
[10] L. Shapiro, A survey of the Riordan group, available electronically at Center for Combinatorics, Nankai University, 2018.
[11] L. W. Shapiro, S. Getu, W.-J. Woan, and L. C. Woodson, The Riordan group, Discr. Appl. Math. 34 (1991), 229-239.
[12] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2018.
[13] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, Notices Amer. Math. Soc., 50 (2003), 912-915.
[14] H. S. Wall, Analytic Theory of Continued Fractions, AMS Chelsea Publishing, 2001.

2010 Mathematics Subject Classification: Primary 15B36; Secondary 05A15, 11C20, 11B37, 11B83, 15B36. Keywords: Number triangle, Riordan array, $A$-matrix, $A$-sequence, Somos 4 sequence, Hankel transform.
(Concerned with sequences $\underline{A 000045}, \underline{A 000108}, \underline{A 001045}, \underline{A 005043}, \underline{A 006318}, \underline{A 006720}, \underline{A 006769}$, A007863, $\mathbf{A 0 9 7 6 0 9}, \underline{A 104545, ~ A 151374}, \underline{A 162547, ~ A 171416, ~ A 178628, ~ a n d ~ A 215661 .) ~}$

