Big Ramsey spectra of countable chains

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Abstract

A *big Ramsey spectrum* of a countable chain C is a sequence of big Ramsey degrees of finite chains computed in C. In this paper we consider big Ramsey spectra of countable chains. We prove that a countable scattered chain has finite big Ramsey spectrum if and only if its Hausdorff degree is finite. Since big Ramsey spectra of all non-scattered countable chains are infinite, this completes the characterization of countable chains with finite big Ramsey spectra (or degrees).

Key Words: countable chains, big Ramsey degrees, big Ramsey spectrum
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1 Introduction

Ramsey's famous theorem:

Theorem 1.1 (Ramsey's Theorem [18]). *For any $n \geqslant 1$ and an arbitrary coloring $\chi : \binom{\omega}{n} \to k$ of n-element subsets of ω with $k \geqslant 2$ colors there exists a copy $M \subseteq \omega$ of ω which is* monochromatic *in the following sense: $\chi(X) = \chi(Y)$ for all $X, Y \in \binom{M}{n}$.*

was published in 1930, but already in 1933 it was generalized to cardinals by Sierpiński. This marked the beginning of combinatorial set theory which is nowdays a deep and influential part of set theory (see [22]). In contrast to Ramsey theory which abounds with positive results of the form "for any coloring there is a monochromatic copy", the generalization of Ramsey's Theorem to cardinals brought a plethora of negative or conditionally positive results of the form "there is a complicated coloring such that no monochromatic copy exists" or "for any coloring there is a monochromatic copy provided we exclude a certain type of behaviour".

Scaling down to countable chains does not take us back to the realm where monochromatic copies dwell. It is easy to construct a Sierpiński-style coloring of $\binom{\mathbb{Q}}{2}$ with two colors and with no monochromatic subchain isomorphic to $\mathbb{Q}$. However, Galvin showed in [7, 8] that for every coloring $\chi : \binom{\mathbb{Q}}{2} \to k$, $k \geqslant 2$, there is an *oligochromatic* copy of $\mathbb{Q}$ in the following sense: there is a $U \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that χ takes at most two colors on $\binom{U}{2}$. This observation was later generalized by Devlin in his thesis [1]. For each $n \geqslant 1$ Devlin found a positive integer $\mathbb{T}_n$ so that for every coloring $\chi : \binom{\mathbb{Q}}{n} \to k$ where $k \geqslant 2$ there is a $U \subseteq \mathbb{Q}$ order-isomorphic to $\mathbb{Q}$ such that χ takes at most $\mathbb{T}_n$ colors on $\binom{U}{n}$. Devlin actually managed to compute the numbers $\mathbb{T}_n$ and it turns out that $\mathbb{T}_n = \tan^{(2n-1)}(0)$.

The integer $\mathbb{T}_n$ is referred to as the *big Ramsey degree of n in $\mathbb{Q}$* following Kechris, Pestov and Todorčević [12] where big Ramsey degrees were explicitely introduced under this name in the context of structural Ramsey theory of Fraïssé limits. In the concluding remarks the paper [12] points to deep implications Big Ramsey degrees have on topological dynamics. Many of the ideas promoted there were later operationalized by Zucker in [23].

The chain of the rationals, $\mathbb{Q}$, is not only a countable chain but also a Fraïssé limit of the class of all the finite chains. Not surprisingly, $\mathbb{Q}$ is not the only Fraïssé limit whose every finite substructure has finite big Ramsey degree in it. Sauer proved in [20] that several classes of finite structures have finite big Ramsey degrees in the corresponding Fraïssé limits. Most notably, every finite graph has finite big Ramsey degree in the Rado graph — the Fraïssé limit of the class of all the finite graphs. Nugyen Van Thé proved in [17] that for every nonempty finite set S of non-negative reals, every finite S-ultrametric space has finite big Ramsey degree in the Fraïssé limit of the class of all the finite S-ultrametric spaces. Another class of metric spaces was shown to have finite big Ramsey degrees in [15]. Laflamme, Nugyen Van Thé and Sauer proved in [13] that every finite local order has finite big Ramsey degree in the dense local order $\mathcal{S}(2)$ — the Fraïssé limit of the class of all the finite local orders. Finally, a remarkable result of Dobrinen [2, 3] shows that every finite K_n-free graph has finite big Ramsey degree in the Henson graph $\mathcal{H}_n$ — the Fraïssé limit of the class of all the finite K_n-free graphs [3].

In particular, an integer $T \geqslant 1$ is a *big Ramsey degree of a finite chain n in a chain A* if it is the smallest positive integer such that for every coloring $\chi : \binom{A}{n} \to k$ where $k \geqslant 2$ there is a $U \subseteq A$ order-isomorphic to A such that χ takes at most T colors on $\binom{U}{n}$. If no such T exists we say that *n does not have big Ramsey degree in A*. We denote the big Ramsey degree of n in A by $T(n, A)$, and write $T(n, A) = \infty$ if n does not have the big Ramsey degree in A.

Clearly, for every $n \in \mathbb{N}$ there is, up to isomorphism, only one chain of length n. Hence, for any chain A it makes sense to consider the *big Ramsey spectrum of A*:

$$\mathrm{spec}(A) = (T(1, A), T(2, A), T(3, A), \dots) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}.$$

A chain A *has finite big Ramsey spectrum* if $T(n, A) < \infty$ for all $n \geqslant 1$, that is, if $\mathrm{spec}(A) \in \mathbb{N}^{\mathbb{N}}$. In this parlance the Ramsey's theorem and the results of Galvin and Devlin take the following form:

Theorem 1.2. (*a*) *(Ramsey [18])* $\mathrm{spec}(\omega) = (1, 1, 1, \dots)$.

(*b*) *(Galvin [7, 8])* $T(2, \mathbb{Q}) = 2$.

(*c*) *(Devlin [1])* $\mathrm{spec}(\mathbb{Q}) = (\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \dots)$, *which coincides with the OEIS sequence A000182.*

For a chain A let A^* denote A with the order reversed. It is obvious that $\mathrm{spec}(A) = \mathrm{spec}(A^*)$ for all chains A. In particular, $\mathrm{spec}(\omega) = \mathrm{spec}(\omega^*)$. Interestingly, ω and ω^* are the only countable chains whose spectrum is $(1, 1, 1, \dots)$ [19]. We thus get the following strengthening of Ramsey's Theorem:

Theorem 1.3 ([19, Corollary 11.4])**.** *Let A be a countable chain.*

(*a*) $T(2, A) = 1$ *if and only if* $A \cong \omega$ *or* $A \cong \omega^*$.

(*b*) *Consequently,* $\mathrm{spec}(A) = (1, 1, 1, \dots)$ *if and only if* $A \cong \omega$ *or* $A \cong \omega^*$.

It is very easy to show that if A and B are chains such that A embeds into B and B embeds into A then $\mathrm{spec}(A) = \mathrm{spec}(B)$ for all $n \geqslant 1$. Devlin's result, therefore, immediately applies to any non-scattered countable chain (recall that a countable chain is *scattered* if it does *not* embed $\mathbb{Q}$, otherwise it is *non-scattered*):

Theorem 1.4 (Devlin [1])**.** *If A is a non-scattered countable chain then* $\mathrm{spec}(A) = \mathrm{spec}(\mathbb{Q}) = (\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \dots)$.

Not much is known about big Ramsey spectra of scattered chains. One of the most notable results in this direction was proved by R. Laver:

Theorem 1.5 (Laver [14])**.** *$T(1, S) < \infty$ for every scattered chain S.*

In case S is an ordinal a direct (and much simpler) proof can be found in [6, p. 189]:

Theorem 1.6 (Fraïssé [6, p. 189])**.** (*a*) *$T(1, \omega^\alpha) = 1$ for every ordinal α.*
(*b*) *$T(1, \alpha) < \infty$ for every ordinal α.*

There is a bit more clarity in case of finite powers of ω:

Theorem 1.7. (*a*) *$T(n, \omega^m) < \infty$ for all $1 \leqslant n, m < \omega$.*
(*b*) *(Galvin [9]) $\operatorname{spec}(\omega^2)$ coincides with the OEIS sequence A000311.*

A proof of (*a*) in case $n = 2$ (which easily generalizes to other values of n) can be found in [22, Theorem 7.2.7]. Another proof of (*a*) can be found in [16].

A systematic treatment and some explicit calculations of $T(n, \alpha)$ for a countable ordinal α and $n \geqslant 2$ can be found in [16]. The main result of [16] characterizes countable ordinals with finite big Ramsey spectra:

Theorem 1.8 ([16])**.** *Let α be a countable ordinal.*
(*a*) *If $\alpha < \omega^\omega$ then α has finite big Ramsey spectrum.*
(*b*) *If $\alpha \geqslant \omega^\omega$ then $\operatorname{spec}(\alpha) = (n, \infty, \infty, \infty, \ldots)$ for some $n \in \mathbb{N}$.*

In [16] we also managed to compute the big Ramsey spectra in some simple cases.

Theorem 1.9 ([16])**.** *Let $m \in \mathbb{N}$ be arbitrary.*
(*a*) *$T(n, \omega + m) = \sum_{j=0}^{n} \binom{m}{j}$ for all $n \geqslant 1$ (where $\binom{m}{j} = 0$ when $m < j$);*
(*b*) *$\operatorname{spec}(\omega^m) = (1, m^2, m^3, m^4, \ldots)$;*
(*c*) *$\operatorname{spec}(\mathbb{Z}) = (2, 2^2, 2^3, 2^4, \ldots)$.*

In this paper we consider big Ramsey spectra of countable scattered chains. In Section 2 we recall some standard notions and notation. In Section 3 we prove that big Ramsey spectra of countable chains are non-decreasing (where, as usual, we take $n < \infty$ for all $n \in \mathbb{N}$). In Section 4 we prove that countable scattered chains of finite Hausdorff rank (to be defined below) have finite big Ramsey spectra (Theorem 4.3). In Section 5, on the other hand, we prove that big Ramsey spectra of countable scattered chains of infinite Hausdorff rank take the form $(n, \infty, \infty, \infty, \ldots)$ for some $n \in \mathbb{N}$ (Theorem 5.4). In both cases we rely on a Ramsey-type result and an appropriate representation of countable scattered chains. Whereas in Section 4 we use an infinite version of the Product Ramsey Theorem from [16] and work top-down using a syntactical representation of scattered chains based on "chain terms", in Section 5 we use Galvin's result about square bracket partition relation and work bottom-up using a "semantical representation" of scattered chains based on condensations.

Together with the fact that all countable non-scattered chains have finite big Ramsey spectra (Theorem 1.4) the final result of the paper takes the following form:

Main result (Corollary 5.5)**.** *Let A be a countable chain.*

(*a*) *If A is a scattered chain of infinite Hausdorff rank then* $\operatorname{spec}(A) = (n, \infty, \infty, \ldots)$ *for some* $n \in \mathbb{N}$.

(*b*) *In all other cases* $\operatorname{spec}(A)$ *is a non-decreasing chain of integers.*

2 Preliminaries

A *chain* is a pair $(A, <)$ where $<$ is a linear order on A. For $a, b \in A$ let $[a, b]_A = \{x \in A : a \leqslant x \leqslant b\}$. Note that in case $a > b$ we have that $[a, b]_A = \varnothing$. An *interval of* A is a subset $I \subseteq A$ such that $[x, y]_A \subseteq I$ for all $x, y \in I$. If we wish to stress that A and B are isomorphic as ordered sets we shall say that they are *order-isomorphic* and write $A \cong B$. For a well-ordered set A let $\operatorname{tp}(A)$ denote the *order type* of A.

As usual, $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the chain of all the positive integers with the usual ordering, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is the chain of all the integers with the usual ordering, and $\mathbb{Q}$ is the chain of all the rationals with the usual ordering. The order type of $\mathbb{Z}$ will be denoted by ζ.

A chain A is *scattered* if $\mathbb{Q} \not\hookrightarrow A$. Otherwise it is *non-scattered*. In 1908 Hausdorff published a structural characterization of scattered chains [11], which was rediscovered by Erdős and Hajnal in their 1962 paper [4]. We shall now present Hausdorff's characterization of countable scattered chains. Define a sequence Σ_α of chains indexed by ordinals as follows:

- $\Sigma_0 = \{0, 1\}$;
- for an ordinal $\alpha > 0$ let $\Sigma_\alpha = \{\sum_{i \in \mathbb{Z}} S_i : S_i \in \bigcup_{\beta < \alpha} \Sigma_\beta \text{ for all } i \in \mathbb{Z}\}$.

Theorem 2.1 (Hausdorff [11])**.** *For each ordinal α the elements of Σ_α are countable scattered chains. Conversely, for every countable scattered chain S there is an ordinal α such that $S \in \Sigma_\alpha$.*

The least ordinal α such that Σ_α contains a countable scattered chain S is referred to as the *Hausdroff rank of* S and denoted by $r_H(S)$. A countable scattered chain S has *finite Hausdorff rank* if $r_H(S) < \omega$. Otherwise it has *infinite Hausdorff rank*.

Let C be a chain and n a finite chain. Then the set $\binom{C}{n}$ of all the n-element subsets of C clearly corresponds to the set $\mathrm{Emb}(n, C)$ of all the embeddings $n \hookrightarrow C$. We sometimes find it more convenient to formally introduce big Ramsey degrees as follows. For chains A, B, C and integers $k \geqslant 2$ and $t \geqslant 1$ we write $C \longrightarrow (B)^A_{k,t}$ to denote that for every k-coloring $\chi : \mathrm{Emb}(A, C) \to k$ there is an embedding $w \in \mathrm{Emb}(B, C)$ such that $|\chi(w \circ \mathrm{Emb}(A, B))| \leqslant t$. For a chain C and a finite chain n we say that n has *finite big Ramsey degree in* C if there exists a positive integer t such that for each $k \geqslant 2$ we have that $C \longrightarrow (C)^n_{k,t}$. Equivalently, a finite chain n has finite big Ramsey degree in a chain C if there exists a positive integer t such that for every $k \geqslant 2$ and every k-coloring $\chi : \mathrm{Emb}(n, C) \to k$ there is a $U \subseteq C$ order-isomorphic to C such that $|\chi(\mathrm{Emb}(n, U))| \leqslant t$. The least such t is then denoted by $T(n, C)$. If such a t does not exist we say that A *does not have finite big Ramsey degree in* C and write $T(A, C) = \infty$. The sequence

$$\mathrm{spec}(C) = (T(1, C), T(2, C), T(3, C), \dots) \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$$

is referred to as the *big Ramsey spectrum of* C. We say that C *has finite big Ramsey spectrum* if $\mathrm{spec}(C) \in \mathbb{N}^{\mathbb{N}}$. For the sake of convenience, for any chain C we let $T(0, C) = 1$ by definition.

3 Monotonicity

Big Ramsey degrees in Fraïssé limits are monotonous in the following sense (see [23]). Let F be a countable Fraïssé limit in a relational language and let A and B be finite substructures of F. Then $A \hookrightarrow B$ implies that $T(A, F) \leqslant T(B, F)$. In this section we prove that the same holds for countable chains. Consequently, the big Ramsey spectrum of any countable chain is a nondecreasing sequence of elements of $\mathbb{N} \cup \{\infty\}$ where, of course, we take ∞ to be larger than any integer. Since in the context of arbitrary chains we cannot rely on ultrahomogeneity, we shall proceed by discussing the structure of countable chains.

Lemma 3.1. *Let A be an infinite chain and $m, n \in \mathbb{N}$ such that $2 \leqslant m \leqslant n$. If $T(n, A) < \infty$ then $T(m, A) < \infty$.*

Proof. Let $T(n, A) = t \in \mathbb{N}$ and let us show that $T(m, A) \leqslant t \cdot \binom{n}{m}$. Take any $k \geqslant 2$ and any $\chi : \mathrm{Emb}(m, A) \to k$. Define

$$\chi' : \mathrm{Emb}(n, A) \to \mathcal{P}(k)$$

by

$$\chi'(f) = \chi(f \circ \mathrm{Emb}(m, n)) \subseteq k.$$

Since $T(n, A) = t$, there is an $A' \subseteq A$ order-isomorphic to A such that

$$|\chi'(\mathrm{Emb}(n, A'))| \leqslant t.$$

Therefore,

$$\begin{aligned}\chi'(\mathrm{Emb}(n, A')) &= \{\chi'(f) : f \in \mathrm{Emb}(n, A')\} \\ &= \{\chi(f \circ \mathrm{Emb}(m, n)) : f \in \mathrm{Emb}(n, A')\}\end{aligned}$$

has at most t elements, so there exist not necessarily distinct $g_0, g_1, \ldots, g_{t-1} \in \mathrm{Emb}(n, A')$ such that

$$\{\chi(f \circ \mathrm{Emb}(m, n)) : f \in \mathrm{Emb}(n, A')\} = \{\chi(g_i \circ \mathrm{Emb}(m, n)) : i < t\}. \quad (3.1)$$

On the other hand, it is easy to see that $\mathrm{Emb}(m, A') = \mathrm{Emb}(n, A') \circ \mathrm{Emb}(m, n)$ because A', as a chain isomorphic to A, is infinite. So,

$$\begin{aligned}\chi(\mathrm{Emb}(m, A')) &= \chi(\mathrm{Emb}(n, A') \circ \mathrm{Emb}(m, n)) \\ &= \chi(\textstyle\bigcup_{f \in \mathrm{Emb}(n, A')} f \circ \mathrm{Emb}(m, n)) \\ &= \textstyle\bigcup_{f \in \mathrm{Emb}(n, A')} \chi(f \circ \mathrm{Emb}(m, n)) \\ &= \textstyle\bigcup_{i<t} \chi(g_i \circ \mathrm{Emb}(m, n)) \qquad \text{because of (3.1).}\end{aligned}$$

Therefore,

$$|\chi(\mathrm{Emb}(m, A'))| \leqslant \sum_{i<t} |\chi(g_i \circ \mathrm{Emb}(m, n))| \leqslant t \cdot \binom{n}{m},$$

because $|\chi(g_i \circ \mathrm{Emb}(m, n))| \leqslant \binom{n}{m}$. □

Lemma 3.2. *Let A be a chain with no maximal element. Then $m \leqslant n$ implies $T(m, A) \leqslant T(n, A)$ for all $m, n \in \mathbb{N}$.*

Proof. Let $T(n, A) = t \in \mathbb{N}$. Take any $k \geqslant 2$ and let $\chi : \mathrm{Emb}(m, A) \to k$ be a coloring. Define $\chi' : \mathrm{Emb}(n, A) \to k$ by $\chi'(h) = \chi(h\restriction_m)$. Then there is an $A' \subseteq A$ order-isomorphic to A such that $|\chi'(\mathrm{Emb}(n, A'))| \leqslant t$. Since A' is a chain with no maximal element, every m-element subchain of A' can be extended to an n-element subchain of A', whence $\chi(\mathrm{Emb}(m, A')) \subseteq \chi'(\mathrm{Emb}(n, A'))$. Therefore, $|\chi(\mathrm{Emb}(m, A'))| \leqslant t$. □

Lemma 3.3. *Let A be a chain such that $A = B + \omega^*$ for some chain B. Then $m \leqslant n$ implies $T(m, A) \leqslant T(n, A)$ for all $m, n \in \mathbb{N}$.*

Proof. Without loss of generality we may assume that $B \cap \omega^* = \varnothing$. Fix $m, n \in \mathbb{N}$ such that $m \leqslant n$. Let $f : m \hookrightarrow n$ be the inclusion $f(i) = i$, and let $g : A \hookrightarrow A$ be the self-embedding of A where $g(b) = b$ for all $b \in B$ and $g(i) = i + n$ for all $i \in \omega^*$. Because g "leaves enough room towards the end of the chain" it is easy to show that $g \circ \mathrm{Emb}(m, A) \subseteq \mathrm{Emb}(n, A) \circ f$.

Let $T(n, A) = t \in \mathbb{N}$. Take any $k \geqslant 2$ and let $\chi : \mathrm{Emb}(m, A) \to k$ be a coloring. Define $\chi' : \mathrm{Emb}(n, A) \to k$ by $\chi'(h) = \chi(h \circ f)$. Then there is a $w : A \hookrightarrow A$ such that $|\chi'(w \circ \mathrm{Emb}(n, A))| \leqslant t$. The definition of χ' then yields that $|\chi(w \circ \mathrm{Emb}(n, A) \circ f)| \leqslant t$. Therefore, $|\chi(w \circ g \circ \mathrm{Emb}(m, A))| \leqslant t$ because $g \circ \mathrm{Emb}(m, A) \subseteq \mathrm{Emb}(n, A) \circ f$. □

Lemma 3.4. *Let A be a chain such that $A = B + r$ for some $r \in \mathbb{N}$ and some chain B with no maximal element. Assume that $B \cap r = \varnothing$ where $r = \{0, 1, \ldots, r - 1\}$, and that there exists an embedding $\hat{g} : A \hookrightarrow A$ such that $\hat{g}(0) \notin r$. Then $m \leqslant n$ implies $T(m, A) \leqslant T(n, A)$ for all $m, n \in \mathbb{N}$.*

Proof. Let $\hat{g}(0) = b_0 \in B$. Since B does not have the maximal element there exist $b_1, \ldots, b_{r-1} \in B$ such that $b_0 < b_1 < \ldots < b_{r-1}$. Let $g : A \hookrightarrow A$ be the self-embedding of A where $g(b) = \hat{g}(b)$ for all $b \in B$ and $g(i) = b_i$ for all $i < r$. Then g is clearly a self embedding of A. Let $f : m \hookrightarrow n$ be the inclusion $f(i) = i$. As in the proof of Lemma 3.3 it is easy to show that $g \circ \mathrm{Emb}(m, A) \subseteq \mathrm{Emb}(n, A) \circ f$ because g "leaves enough room towards the end of the chain". We can now simply repeat the argument of Lemma 3.3 to conclude the proof. □

Let $f : n \hookrightarrow B + r$ be an embedding, where $n, r \in \mathbb{N}$ and B is a chain such that $B \cap r = \varnothing$. Then $\mathrm{tp}(f) = \mathrm{im}(f) \cap r$ will be referred to as the *type of f*. (For a set map $f : A \to B$ by $\mathrm{im}(f)$ we denote the image of f, that is, the set $\{f(a) : a \in A\}$.) Given a type $\tau \subseteq r$, let

$$\mathrm{Emb}_\tau(n, B + r) = \{f \in \mathrm{Emb}(n, B + r) : \mathrm{tp}(f) = \tau\}.$$

Lemma 3.5. *Let $n, r \in \mathbb{N}$ and let B be a chain such that $B \cap r = \varnothing$. For every type $\tau \subseteq r$ with $|\tau| \leqslant n$, every $k \geqslant 2$ and every coloring $\chi : \mathrm{Emb}_\tau(n, B + r) \to k$ there is a $U \subseteq B$ order-isomorphic to B such that $|\chi(\mathrm{Emb}_\tau(n, U + r))| \leqslant T(n - |\tau|, B)$.*

Proof. Take an type $\tau \subseteq r$ such that $|\tau| \leqslant n$ and assume that $T(n - |\tau|, B) < \infty$. If $|\tau| = n$ then for every $U \subseteq B$ we have that $|\mathrm{Emb}_\tau(n, U + r)| = 1$, whence $|\chi(\mathrm{Emb}_\tau(n, U + r))| = 1 = T(0, B)$. So, let $s = |\tau| < n$ and let $\Phi : \mathrm{Emb}_\tau(n, B + r) \to \mathrm{Emb}(n - s, B)$ be the bijection that takes $f \in \mathrm{Emb}_\tau(n, B + r)$ to $f\restriction_{n-s} \in \mathrm{Emb}(n - s, B)$. Fix a $k \geqslant 2$ and a coloring $\chi : \mathrm{Emb}_\tau(n, B+r) \to k$. Let $\chi' : \mathrm{Emb}(n-s, B) \to k$ be the coloring defined by $\chi'(f) = \chi(\Phi^{-1}(f))$. Then there is a $U \subseteq B$ order-isomorphic to B such that $|\chi'(\mathrm{Emb}(n - s, U))| \leqslant T(n - s, B)$. But then it easily follows that $|\chi(\mathrm{Emb}_\tau(n, U + r))| \leqslant T(n - s, B)$. □

Lemma 3.6. *Let B be a chain with no maximal element and let $r \in \mathbb{N}$. Assume that $B \cap r = \varnothing$ and that $g\restriction_r = \mathrm{id}_r$ for every embedding $g : B + r \hookrightarrow B + r$.*

(a) If $T(n, B + r) < \infty$ then $T(n - j, B) < \infty$ for all $n \in \mathbb{N}$ and $0 \leqslant j \leqslant \min\{n, r\}$.

(b) If $T(n, B + r) < \infty$ then $T(n, B + r) = \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} \cdot T(n - j, B)$.

Proof. (a) Assume that $T(n - j, B) = \infty$ for some $0 \leqslant j \leqslant \min\{n, r\}$ and let us show that $T(n, B + r) = \infty$ by showing that $T(n, B + r) \geqslant t$ for every $t \in \mathbb{N}$. Fix a $t \in \mathbb{N}$. Because $T(n - j, B) = \infty$ there is a coloring $\chi : \mathrm{Emb}(n - j, B) \to k$ for some $k \geqslant t$ such that $|\chi(w \circ \mathrm{Emb}(n - j, B))| \geqslant t$ for every $w : B \hookrightarrow B$. Define $\chi' : \mathrm{Emb}(n, B + r) \to k$ as follows:

$$\chi'(f) = \begin{cases} \chi(f\restriction_{n-j}), & |\mathrm{tp}(f)| = j, \\ 0, & \text{otherwise.} \end{cases}$$

Take any $g : B + r \hookrightarrow B + r$. Clearly, $g\restriction_B : B \hookrightarrow B$. Let us show that

$$\chi'(g \circ \mathrm{Emb}(n, B + r)) \supseteq \chi(g\restriction_B \circ \mathrm{Emb}(n - j, B)).$$

Take any $f \in \mathrm{Emb}(n - j, B)$ and let $h : j \to r$ be the inclusion $i \mapsto i$. Put $f' = f + h : n \hookrightarrow B + r$. Since $|\mathrm{tp}(f')| = j$ we have that

$$\chi'(g \circ f') = \chi((g \circ f')\restriction_{n-j}) = \chi(g\restriction_B \circ f'\restriction_{n-j}) = \chi(g\restriction_B \circ f).$$

Therefore, $|\chi'(g \circ \mathrm{Emb}(n, B + r))| \geqslant |\chi(g\restriction_B \circ \mathrm{Emb}(n - j, B))| \geqslant t$.

(b) Fix an $n \in \mathbb{N}$ and assume that $T(n, B+r) < \infty$. Then $T(n-j, B) < \infty$ for all $0 \leqslant j \leqslant \min\{n, r\}$ (by (a)). Let $Q = \{\tau \subseteq m : |\tau| \leqslant n\}$ be the set of all the types realized by members of $\mathrm{Emb}(n, B + r)$. Let $Q = \{\tau_0, \tau_1, \ldots, \tau_{t-1}\}$ so that $|Q| = t$. Note that $t = \sum_{j=0}^{\min\{n,r\}} \binom{r}{j}$.

Fix a $k \geqslant 2$ and a coloring $\chi : \mathrm{Emb}(n, B + r) \to k$. By Lemma 3.5 there is a $U_0 \subseteq B$ order-isomorphic to B such that

$$|\chi(\mathrm{Emb}_{\tau_0}(n, U_0 + r))| \leqslant T(n - |\tau_0|, B).$$

By the same lemma for each $j \in \{1, \dots, t-1\}$ we then inductively obtain a $U_j \subseteq U_{j-1}$ order-isomorphic to U_{j-1} (and hence to B) such that

$$|\chi(\mathrm{Emb}_{\tau_j}(n, U_j + r))| \leqslant T(n - |\tau_j|, B).$$

Then, using the fact that $U_{t-1} \subseteq U_j$ we have that

$$\begin{aligned} |\chi(\mathrm{Emb}(n, U_{t-1} + r))| &= \sum_{j<t} |\chi(\mathrm{Emb}_{\tau_j}(n, U_{t-1} + r))| \\ &\leqslant \sum_{j<t} |\chi(\mathrm{Emb}_{\tau_j}(n, U_j + r))| \\ &\leqslant \sum_{j<t} T(n - |\tau_j|, B) \leqslant \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} \cdot T(n-j, B). \end{aligned}$$

In order to conclude the proof we have to show that there exists a coloring $\chi : \mathrm{Emb}(n, B + r) \to k$ where $k \geqslant \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} \cdot T(n-j, B)$ such that $|\chi(w \circ \mathrm{Emb}(n, B + r))| \geqslant \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} \cdot T(n-j, B)$ for every embedding $w : B + r \hookrightarrow B + r$.

Since $T(n-j, B)$ is the big Ramsey degree of $n-j$ in B, for every $0 \leqslant j \leqslant \min\{n, r\}$ there is a coloring $\chi_j : \mathrm{Emb}(n-j, B) \to k_j$ where $k_j \geqslant T(n-j, B)$ such that $|\chi_j(v \circ \mathrm{Emb}(n-j, B))| \geqslant T(n-j, B)$ for every embedding $v : B \hookrightarrow B$. Define

$$\chi : \mathrm{Emb}(n, B + r) \to \bigcup_{\tau \in Q} \{\tau\} \times k_{|\tau|}$$

as follows: for an $f \in \mathrm{Emb}(n, B+r)$ let $\tau = \mathrm{tp}(f)$ and $j = |\tau|$, and then put

$$\chi(f) = (\tau, \chi_j(f\restriction_{n-j})).$$

Take any embedding $w : B + r \hookrightarrow B + r$. By the assumption we know that $w\restriction_r = \mathrm{id}_r$. Clearly, $\chi(w \circ \mathrm{Emb}(n, B+r)) = \bigcup_{\tau \in Q} \chi(w \circ \mathrm{Emb}_\tau(n, B+r))$. Let us show that this is a disjoint union.

Take any $f \in \mathrm{Emb}(n, B + r)$. Note first that $\mathrm{tp}(w \circ f) = \mathrm{tp}(f)$ because $w\restriction_r = \mathrm{id}_r$. For $j = |\mathrm{tp}(f)|$ we then have

$$\chi(w \circ f) = (\mathrm{tp}(f), \chi_j((w \circ f)\restriction_{n-j})) = (\mathrm{tp}(f), \chi_j(w\restriction_B \circ f\restriction_{n-j})).$$

The claim now follows immediately, because the first component of $\chi(w \circ f)$ is $\mathrm{tp}(f)$.

Consequently, $|\chi(w \circ \mathrm{Emb}(n, B + r))| = \sum_{\tau\in Q} |\chi(w \circ \mathrm{Emb}_\tau(n, B + r))|$. Now, take any $\tau \in Q$ and let $j = |\tau|$. Then

$$\begin{aligned}\chi(w \circ \mathrm{Emb}_\tau(n, B + r)) &= \{(\tau, \chi_j(w\restriction_B \circ f\restriction_{n-j})) : f \in \mathrm{Emb}_\tau(n, B + r)\}\\ &= \{(\tau, \chi_j(w\restriction_B \circ f')) : f' \in \mathrm{Emb}(n - j, B)\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|\chi(w \circ \mathrm{Emb}(n, B + r))| &= \sum_{\tau\in Q} |\chi(w \circ \mathrm{Emb}_\tau(n, B + r))|\\ &= \sum_{\tau\in Q} |\chi_{|\tau|}(w\restriction_B \circ \mathrm{Emb}(n - |\tau|, B)))|\\ &\geqslant \sum_{\tau\in Q} T(n - |\tau|, B) \quad [\text{by the choice of } \chi_{|\tau|}]\\ &= \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} T(n - j, B).\end{aligned}$$

□

Theorem 3.7. *Let A be a countable chain and $m, n \in \mathbb{N}$. If $m \leqslant n$ then $T(m, A) \leqslant T(n, A)$.*

Proof. Case 1. If A is a non-scattered chain then $\mathrm{spec}(A) = (\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \ldots)$ by Theorem 1.4, and it is a well known fact that $\mathbb{T}_1 < \mathbb{T}_2 < \mathbb{T}_3 < \ldots$.

Case 2. If A has no maximal element then Lemma 3.2 applies.

Case 3. If $A = B + \omega^*$ for some chain B then Lemma 3.3 applies.

Case 4. Assume that A is a scattered chain with the maximal element, but $A = B + \omega^*$ for *no* chain B.

Then $A = B + r$ for some $r \in \mathbb{N}$ and some chain B with no maximal element. Without loss of generality we can assume that $B \cap r = \varnothing$. If there is an embedding $g : A \hookrightarrow A$ such that $g(0) \in B$ Lemma 3.4 applies. Therefore, for the rest of the proof assume that for every embedding $g : A \hookrightarrow A$ we have that $g\restriction_r = \mathrm{id}_r$.

If $T(n, A) = \infty$ the statement is trivially true.

If $T(n, A) < \infty$ then $T(m, A) < \infty$ either because $m = 1$ in which case Theorem 1.5 applies, or $m \geqslant 2$ in which case Lemma 3.1 applies. Anyhow, both $T(m, A)$ and $T(n, A)$ are finite, so by Lemma 3.6 (recall that $A = B+r$),

$$T(m, B + r) = \sum_{j=0}^{\min\{m,r\}} \binom{r}{j} \cdot T(m - j, B) \text{ and}$$
$$T(n, B + r) = \sum_{j=0}^{\min\{n,r\}} \binom{r}{j} \cdot T(n - j, B).$$

Since $m \leqslant n$ and B is a chain with no maximal element, Lemma 3.2 ensures that $T(m - j, B) \leqslant T(n - j, B)$ for all $0 \leqslant j \leqslant \min\{m, r\}$. Therefore, $T(m, B + r) \leqslant T(n, B + r)$. This completes the proof. □

4 Countable scattered chains of finite rank

In this section we prove that countable scattered chains of finite rank have finite big Ramsey spectra. The tool we rely on is a result from [16] that we see as the infinite version of the Product Ramsey Theorem. The proof presented in [16] is rather involved because one of our aims was to compute the upper bound on the number of colors along the way and prove that the bound is tight. For the sake of completeness we shall now give a less ambitious, but much shorter proof.

Theorem 4.1 (cf. [16, Corollary 4.9]). *For every choice of integers $s \geqslant 1$ and $m_0, m_1, \ldots, m_{s-1} \geqslant 1$ there is an integer $D = D(s; m_0, m_1, \ldots, m_{s-1})$ such that for every $k \geqslant 2$ and every coloring $\chi : \binom{\omega}{m_0} \times \ldots \times \binom{\omega}{m_{s-1}} \to k$ there is an infinite $U \subseteq \omega$ satisfying $\left|\chi\left(\binom{U}{m_0} \times \ldots \times \binom{U}{m_{s-1}}\right)\right| \leqslant D$.*

Proof. For notational convenience let $\overline{m} = (m_0, m_1, \ldots, m_{s-1})$, let $\|\overline{m}\| = m_0 + m_1 + \ldots + m_{s-1}$, and for a set X let

$$E(\overline{m}, X) = \binom{X}{m_0} \times \binom{X}{m_1} \times \ldots \times \binom{X}{m_{s-1}}.$$

For a tuple $\overline{A} = (A_0, A_1, \ldots, A_{s-1}) \in E(\overline{m}, \omega)$ let

$$V(\overline{A}) = A_0 \cup A_1 \cup \ldots \cup A_{s-1} = \{v_0 < v_1 < \ldots < v_{q-1}\} \subseteq \omega,$$

and for $i < q$ let $S_i = \{j < s : v_i \in A_j\}$. Then we refer to $(S_0, S_1, \ldots, S_{q-1}) \in \mathcal{P}(s)^q$ as the *type of* $\overline{A}$ and denote it by $\mathrm{tp}(\overline{A})$. Clearly, $\overline{A}$ is uniquely determined by $\mathrm{tp}(\overline{A})$ and $V(\overline{A})$. We say that a tuple $\sigma \in \mathcal{P}(s)^q$ is a *type* if there exists a tuple $\overline{A} \in E(\overline{m}, \omega)$ such that $\sigma = \mathrm{tp}(\overline{A})$. Note that $q \leqslant \|\overline{m}\|$, so there are at most $\sum_{q=1}^{\|\overline{m}\|} 2^{s \cdot q}$ types.

For $X \subseteq \omega$ and a type σ let

$$E_\sigma(\overline{m}, X) = \{\overline{A} \in E(\overline{m}, X) : \mathrm{tp}(\overline{A}) = \sigma\}.$$

Claim. For any infinite $X \subseteq \omega$, any type σ, any $k \geqslant 2$ and any coloring $\chi : E_\sigma(\overline{m}, X) \to k$ there is an infinite $Y \subseteq X$ such that $|\chi(E_\sigma(\overline{m}, Y))| = 1$.

Proof. Let q be the length of the tuple σ. As we have already seen, any $\overline{A} \in E_\sigma(\overline{m}, X)$ is uniquely determined by $V(\overline{A})$, which is a q-element subset of X. Conversely, every q-element subset of X is $V(\overline{A})$ for some $\overline{A} \in E_\sigma(\overline{m}, X)$. So, $V : E_\sigma(\overline{m}, X) \to \binom{X}{q}$ is a bijection. Define $\chi' : \binom{X}{q} \to k$ by $\chi'(B) = \chi(V^{-1}(B))$. By Ramsey's Theorem there is an infinite $Y \subseteq X$ such that χ' takes only one color on $\binom{Y}{q}$. Therefore, χ takes only one color on $E_\sigma(\overline{m}, Y)$. This concludes the proof of the claim.

Let us now resume with the proof of the theorem. Let σ_0, σ_1, ..., σ_{D-1} be all the types of tuples from $E(\overline{m}, \omega)$. Take any $k \geqslant 2$ and any coloring $\chi : E(\overline{m}, \omega) \to k$. The Claim implies there is an infinite $U_0 \subseteq \omega$ such that

$$|\chi(E_{\sigma_0}(\overline{m}, \omega))| = 1.$$

By the same argument for each $j \in \{1, \dots, D-1\}$ we can inductively construct an infinite $U_j \subseteq U_{j-1}$ such that

$$|\chi(E_{\sigma_j}(\overline{m}, U_j))| = 1.$$

Then, having in mind that $U_{D-1} \subseteq U_j$,

$$\begin{aligned} |\chi(E(\overline{m}, U_{D-1}))| &\leqslant \sum_{j<D} |\chi(E_{\sigma_j}(\overline{m}, U_{D-1}))| \\ &\leqslant \sum_{j<D} |\chi(E_{\tau_j}(\overline{m}, U_j))| \leqslant D. \end{aligned}$$

□

The discussion that follows is based on the syntactic representation of chains in terms of *chain terms*, which are trees that capture the way the chain can be constructed from simpler chains. Let us, therefore, fix some basic notions.

A *rooted tree* is a pair $\tau = (T, v_0)$ where T is partially ordered, $v_0 \in T$ is the *root* of T and $[v_0, x]_T$ is well-ordered for every $x \in T$. Maximal chains in T are called the *branches* of τ. The *height* of a rooted tree is the supremum of order-types of branches in T:

$$\mathrm{ht}(\tau) = \sup\{\mathrm{tp}(b) : b \text{ is a branch in } \tau\}.$$

For a vertex $x \in T$ let $\mathrm{succ}_\tau(x)$ be the set of all the immediate successors of x and let $\mathrm{out}_\tau(x) = \{(x, y) : y \in \mathrm{succ}_\tau(x)\}$ be the set of the *outgoing edges*. A vertex $x \in T$ is a *leaf of* τ if $\mathrm{succ}_\tau(x) = \varnothing$.

A rooted tree $\tau = (T, v_0)$ is *ordered* if $\mathrm{out}_\tau(x)$ is a chain for every $x \in T$. If $\mathrm{ht}(\tau) \leqslant \omega$ then the linear orders on $\mathrm{out}_\tau(x)$, $x \in T$, uniquely determine a linear ordering on the vertices of T: just traverse the tree using the breadth-first-search strategy. This means that we start with the root v_0, then list the immediate successors of v_0 according to the ordering of $\mathrm{out}_\tau(v_0)$, and so on. We refer to this ordering as the *BFS-ordering of* τ.

A *labelled ordered rooted tree* is an ordered rooted tree where some (but not necessarily all) of the vertices are labelled by the elements of some set L_1, and some (but not necessarily all) of the edges are labelled by the elements of some set L_2.

In this section it will be convenient to ignore the empty chain 0 and to restrict finite sums of chains to sums of pairs. Let X be a variable which ranges over chains. Let us define a hierarchy of *chain terms* as follows. Let $\mathcal{T}_0(X) = \{1\}$ and $\mathcal{S}_0(X) = \{1\}$. For each ordinal $\alpha \geqslant 1$ define $\mathcal{T}_\alpha(X)$ and $\mathcal{S}_\alpha(X)$ by:

$$\begin{aligned} \mathcal{T}_\alpha(X) =& \textstyle\bigcup_{\beta<\alpha} \mathcal{S}_\beta(X) \\ \mathcal{S}_\alpha(X) =& \mathcal{T}_\alpha(X) \cup \{\textstyle\sum_{i\in 2} \varphi_i : \varphi_0, \varphi_1 \in \mathcal{T}_\alpha(X)\} \\ & \cup \{\textstyle\sum_{i\in X} \varphi_i : \varphi_0, \varphi_1, \ldots \in \mathcal{T}_\alpha(X)\} \\ & \cup \{\textstyle\sum_{i\in X^*} \varphi_i : \varphi_0, \varphi_1, \ldots \in \mathcal{T}_\alpha(X)\}. \end{aligned}$$

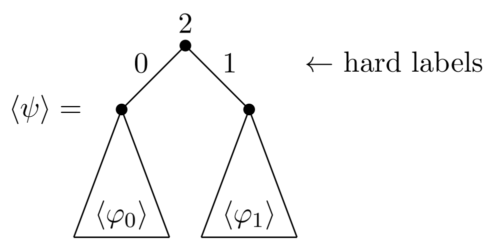
For a chain term $\psi(X) \in \mathcal{S}_\alpha(X)$ and a chain C, by $\psi(C)$ we denote the chain that is obtained by substituting C for the variable X in the term $\psi(X)$, and by $\mathcal{T}_\alpha(C)$ and $\mathcal{S}_\alpha(C)$ we denote the corresponding classes of chains.

Clearly, elements of each $\mathcal{S}_\alpha(\omega)$ are nonempty scattered chains and every nonempty scattered chain appears in some $\mathcal{S}_\alpha(\omega)$. For a nonempty scattered chain S let $r(S)$ denote the *rank of* S in this hierarchy, which is the smallest ordinal α such that $S \in \mathcal{S}_\alpha(\omega)$. A scattered chain S is of *finite rank* if $r(S) < \omega$.

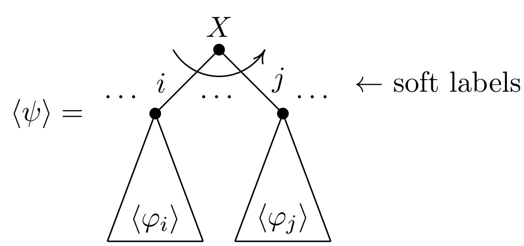
Note that $r(S)$ and $r_H(S)$ need not coincide. For example, for $n \in \mathbb{N}$ we have that $r_H(n) = 1$ while $r(n) = \lceil \log_2 n \rceil$. Nevertheless, for every nonempty scattered chain S we have that $r(S)$ is finite if and only if $r_H(S)$ is finite.

Every chain term $\psi \in \mathcal{S}_\alpha(X)$ can be represented as a labelled ordered rooted tree $\langle\psi\rangle$ of height α. The leaves of $\langle\psi\rangle$ will not be labelled while all other vertices will be labelled by 2, X or X^*. The labels on the edges going out of vertices labelled by 2 will be referred as *hard labels* and labels on all other edges will be referred to as *soft labels*.

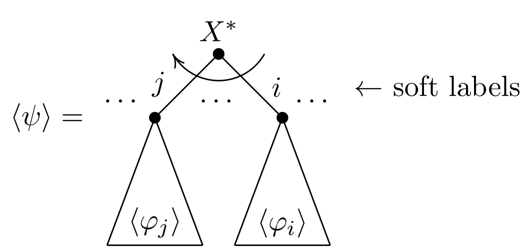
- In case $\psi = 1$ the corresponding tree is an unlabelled one-vertex tree $\langle\psi\rangle = \bullet$.
- If $\psi = \sum_{i\in 2} \varphi_i$ then $\langle\psi\rangle$ has the root labelled by 2, the edges going out of the root are labelled by 0 and 1, respectively, ordered that way, and lead to the subtrees $\langle\varphi_0\rangle$ and $\langle\varphi_1\rangle$:



- If $\psi = \sum_{i\in X} \varphi_i$ then $\langle\psi\rangle$ has the root labelled by X, the edges going out of the root are labelled by elements of X, ordered that way, and lead to the subtrees $\langle\varphi_i\rangle$, $i \in X$:



- If $\psi = \sum_{i\in X^*} \varphi_i$ then $\langle\psi\rangle$ has the root labelled by X^*, the edges going out of the root are labelled by elements of X^*, ordered that way, and lead to the subtrees $\langle\varphi_i\rangle$, $i \in X^*$:



Let $\psi(\omega) \in \mathcal{S}_r(\omega)$ be a nonempty scattered chain of finite rank $r \in \mathbb{N}$. Every embedding $f : n \hookrightarrow \psi(\omega)$ corresponds to a subtree of $\langle \psi \rangle$ induced by n branches. Let us denote this tree by $\langle f \rangle$. Clearly, $\langle f \rangle$ has n leaves and has height $\leqslant r$.

Assume, now, that $\langle f \rangle$ has p vertices. If we replace the vertex set of $\langle f \rangle$ by $\{0, 1, \ldots, p-1\}$ so that the usual ordering of the integers agrees with the BFS-ordering of the new tree, and then erase the soft labels, the resulting labelled ordered rooted tree on the set of vertices $\{0, 1, \ldots, p-1\}$ will be referred to as the *type of* f and will be denoted by $\mathrm{tp}(f)$. A finite labelled ordered rooted tree τ is an $(n, \psi(\omega))$-type if $\tau = \mathrm{tp}(g)$ for some embedding $g : n \hookrightarrow \psi(\omega)$.

Therefore, for all $n, r \in \mathbb{N}$ and all $\psi(\omega) \in \mathcal{S}_r(\omega)$ each $(n, \psi(\omega))$-type is a labelled ordered rooted tree with the following properties:

- its set of vertices is $\{0, 1, \ldots, p-1\}$ for some $p \in \mathbb{N}$ and the BFS-order of the tree coincides with the usual ordering of the integers (hence 0 is the root of the tree);
- it has n leaves and its height is $\leqslant r$;
- its leaves are not labelled while other vertices are labelled by 2, ω or ω^*; and
- its edges going out of vertices labelled by 2 are labelled by 0, 1 or both while other edges are not labelled.

(Note that a labelled ordered rooted tree with the above properties need not be an $(n, \psi(\omega))$-type.) Clearly, given $n, r \in \mathbb{N}$ and a $\psi(\omega) \in \mathcal{S}_r(\omega)$ there are only finitely many $(n, \psi(\omega))$-types. For an $(n, \psi(\omega))$-type τ let

$$\mathrm{Emb}_\tau(n, \psi(\omega)) = \{f \in \mathrm{Emb}(n, \psi(\omega)) : \mathrm{tp}(f) = \tau\}.$$

Lemma 4.2. *Let $S \subseteq \omega$ be an infinite subset of ω, let $n, r \in \mathbb{N}$ and let $\psi(X) \in \mathcal{S}_r(X)$. For every $(n, \psi(S))$-type τ there is a $D_\tau \in \mathbb{N}$ such that for every $k \geqslant 2$ and every coloring $\chi : \mathrm{Emb}_\tau(n, \psi(S)) \to k$ there is an infinite $U \subseteq S$ satisfying*

$$|\chi(\mathrm{Emb}_\tau(n, \psi(U)))| \leqslant D_\tau.$$

Proof. Without loss of generality we can take $S = \omega$. Let $\ell_0 < \ell_1 < \ldots < \ell_{s-1}$ be all the vertices of τ labelled by ω or ω^*. Let $m_i = |\mathrm{out}_\tau(\ell_i)|$, $i < s$, and let $D_\tau = D(s; m_0, m_1, \ldots, m_{s-1})$ be the number provided by Theorem 4.1.

Take any $f \in \mathrm{Emb}_\tau(n, \psi(\omega))$ and let $(v_0, v_1, \dots, v_{p-1})$ be the vertex set of $\langle f \rangle$ ordered by the BFS-order of $\langle f \rangle$. Since $\mathrm{tp}(f) = \tau$, the only vertices in $\langle f \rangle$ labelled by ω or ω^* are v_{ℓ_0}, v_{ℓ_1}, ..., $v_{\ell_{s-1}}$. Let $L_f(i) \subseteq \omega$ be the set of all the labels used to label the edges in $\mathrm{out}(v_{\ell_i})$, $i < s$. Clearly, $|L_f(i)| = m_i$, $i < s$.

By construction, each embedding $f \in \mathrm{Emb}_\tau(n, \psi(\omega))$ is uniquely determined by the sequence $(L_f(0), L_f(1), \dots, L_f(s-1))$ of subsets of ω of sizes m_0, m_1, ..., m_{s-1}, respectively. Therefore,

$$\Phi : \mathrm{Emb}_\tau(n, \psi(\omega)) \to \binom{\omega}{m_0} \times \binom{\omega}{m_1} \times \ldots \times \binom{\omega}{m_{s-1}}$$

given by

$$\Phi(f) = (L_f(0), L_f(1), \dots, L_f(s-1))$$

is an injective mapping.

Now, take any $k \geqslant 2$ and any coloring $\chi : \mathrm{Emb}_\tau(n, \psi(\omega)) \to k$, and define

$$\chi' : \binom{\omega}{m_1} \times \ldots \times \binom{\omega}{m_{s-1}} \to k$$

by

$$\chi'(A_0, A_1, \dots, A_{s-1}) = \begin{cases} \chi(f), & \Phi(f) = (A_0, A_1, \dots, A_{s-1}), \\ 0, & \text{otherwise.} \end{cases}$$

Then by Theorem 4.1 there exists an infinite $U \subseteq \omega$ such that

$$\left| \chi' \left(\binom{U}{m_0} \times \ldots \times \binom{U}{m_{s-1}} \right) \right| \leqslant D_\tau.$$

The construction of χ' ensures that

$$\chi(\mathrm{Emb}_\tau(n, \psi(U))) \subseteq \chi' \left(\binom{U}{m_0} \times \ldots \times \binom{U}{m_{s-1}} \right),$$

whence $|\chi(\mathrm{Emb}_\tau(n, \psi(U)))| \leqslant D_\tau$. This completes the proof, having in mind that $\psi(U)$ is order-isomorphic to $\psi(\omega)$. ☐

Theorem 4.3. *Let S be a countable scattered chain such that $r_H(S) < \omega$. Then* $\mathrm{spec}(S)$ *is finite.*

Proof. Let $r(S) = r$. Note that $r(S) < \omega$ because $r_H(S) < \omega$. Then S is isomorphic to some $\psi(\omega) \in \mathcal{S}_r(\omega)$. Take any $n \in \mathbb{N}$ and let us show that $T(n, \psi(\omega))$ is finite. Let τ_0, τ_1, ..., τ_{s-1} be all the $(n, \psi(\omega))$-types and let

D_{τ_0}, D_{τ_1}, ..., $D_{\tau_{s-1}}$ be the integers provided by Lemma 4.2. We are going to show that $T(n, \psi(\omega)) \leqslant \sum_{j<s} D_{\tau_j} < \infty$.

Take any $k \geqslant 2$ and any coloring $\chi : \mathrm{Emb}(n, \psi(\omega)) \to k$. By Lemma 4.2 there is an infinite $U_0 \subseteq \omega$ such that

$$|\chi(\mathrm{Emb}_{\tau_0}(n, \psi(U_0)))| \leqslant D_{\tau_0}.$$

By the same lemma for each $j \in \{1, \ldots, s-1\}$ we can inductively construct an infinite $U_j \subseteq U_{j-1}$ such that

$$|\chi(\mathrm{Emb}_{\tau_j}(n, \psi(U_j)))| \leqslant D_{\tau_j}.$$

Then, having in mind that $U_{s-1} \subseteq U_j$,

$$\begin{aligned} |\chi(\mathrm{Emb}(n, \psi(U_{s-1})))| &= \sum_{j<t} |\chi(\mathrm{Emb}_{\tau_j}(n, \psi(U_{s-1})))| \\ &\leqslant \sum_{j<t} |\chi(\mathrm{Emb}_{\tau_j}(n, \psi(U_j)))| \leqslant \sum_{j<t} D_{\tau_j}. \end{aligned}$$ □

5 Countable scattered chains of infinite rank

In this section we prove that countable scattered chains of infinite Hausdorff rank do not have finite big Ramsey spectra. In contrast to the previous section where we worked top-down using a syntactical representation of scattered chains based on chain terms, in this section we work bottom-up using a "semantical representation" of scattered chains based on condensations.

A map $f : A \to B$ between two chains is a *homomorphism* if $x \leqslant y \Rightarrow f(x) \leqslant (y)$ for all $x, y \in A$. A *condensation of* A [19] is a surjective homomorphism $c : A \to B$. Note that any condensation of a scattered chain is scattered.

If $c : A \to B$ is a condensation then $\theta = \ker c$ is an equivalence relation whose classes are intervals of A. Conversely, for every equivalence relation θ whose classes are intervals of A the linear order carries from A to A/θ in the obvious way and the quotient map $c : A \to A/\theta$ given by $c(a) = [a]_\theta$ is a condensation.

A condensation $c : A \to B$ is referred to as *finite* if the following holds: $c(x) = c(y)$ if and only if $[x, y]_A \cup [y, x]_A$ is finite [19]. It is easy to see that if $c_1 : A \to B_1$ and $c_2 : A \to B_2$ are finite condensations of A then $B_1 \cong B_2$. Hence, up to isomorphism of codomains, there is a unique finite condensation of A that we refer to as *the finite condensation of* A and denote by $c_{\rm fin}$. For each finite condensation $c_{\rm fin} : A \to B$ and each $y \in B$ the order-type of $c_{\rm fin}^{-1}(y)$ is either n for some $n \in \mathbb{N}$, or ω, or ω^* or ζ. We say that $y \in B$ is a *finitary point* if $c_{\rm fin}^{-1}(y)$ is finite; otherwise we say that y is an *infinitary point*. Clearly, there do not exist $x, y \in B$ such that $[x, y]_B = \{x, y\}$ and both x and y are finitary points.

Let A be a chain. For each ordinal α let us define $c_{\rm fin}^\alpha : A \to A/\theta_\alpha$ inductively as follows. Let $\theta_0 = \{(a, a) : a \in A\}$ and define $c_{\rm fin}^0 : A \to A/\theta_0$ by $c_{\rm fin}^0(a) = \{a\}$. For a successor ordinal $\alpha = \beta + 1$ let $c_{\rm fin}^\alpha = c_{\rm fin} \circ c_{\rm fin}^\beta$ and $\theta_\alpha = \ker c_{\rm fin}^\alpha$, while for a limit ordinal λ let $\theta_\lambda = \bigcup_{\alpha<\lambda} \theta_\alpha$ and define $c_{\rm fin}^\lambda : A \to A/\theta_\lambda$ by $c_{\rm fin}^\lambda(a) = [a]_{\theta_\lambda}$.

We say that an ordinal α is the *finite condensation rank* of a chain A and write $r_F(A) = \alpha$ if α is the least ordinal such that $c_{\rm fin}^\alpha(A) \cong 1$. For every countable scattered chain S the finite condensation rank $r_F(S)$ exists and $r_F(S) = r_H(S)$ [19].

The proof that we present in this section heavily relies on a powerful result of Galvin about square bracket partition relations which express strong counterexamples to ordinary partition relations. For chains C, B_0, B_1, B_2, ..., and $n < \omega$ write

$$C \longrightarrow [B_0, B_1, B_2, \ldots]^n$$

to denote that for every coloring $\chi : \mathrm{Emb}(n, C) \to \omega$ there is an $i < \omega$ and a subchain $U \subseteq C$ such that $U \cong B_i$ and $i \notin \chi(\mathrm{Emb}(n, U))$. Erdős and Hajnal note in [5, p. 275] that in 1971 Galvin proved the following:

Theorem 5.1 (Galvin 1971)**.** *If S is a scattered chain that contains no uncountable well-ordered subsets then $S \not\longrightarrow [\omega, \omega^2, \omega^2, \omega^3, \omega^3, \ldots]^2$.*

A recent proof of Galvin's result can be found in [21].

Lemma 5.2. *Let S be a countable scattered chain, let $c_{\rm fin} : S \to c_{\rm fin}(S)$ be the finite condensation of S and let $I \subseteq c_{\rm fin}(S)$ be an infinite interval of $c_{\rm fin}(S)$. Then there are infinitely many infinitary points in I.*

Proof. Let us show that for any finitary points $a, b \in c_{\rm fin}(S)$ there is an infinitary point $x \in c_{\rm fin}(S)$ such that $a < x < b$. Suppose this is not the case. Then there exist $a, b \in c(S)$ such that $a < b$ and every $x \in [a, b]_{c_{\rm fin}(S)}$ is finitary. Since $c_{\rm fin}(S)$ is scattered, there exist $a', b' \in [a, b]_{c_{\rm fin}(S)}$ such that $a' < b'$ and $[a', b']_{c_{\rm fin}(S)} = \{a', b'\}$ — contradiction with the fact that $c_{\rm fin}$ is a finite condensation.

Now, if I contains only finitely many finitary points we are done because I is infinite. Suppose, therefore, that there are infinitely many finitary points in I. Take any $n \geqslant 1$ and let $b_0 < b_1 < \ldots < b_n$ be some finitary points from I. Then, as we have just seen, there are infinitary points $a_1, \ldots, a_n \in c_{\text{fin}}(S)$ such that $b_0 < a_1 < b_1 < a_2 < \ldots < a_n < b_n$. Note that $a_1, \ldots, a_n \in I$ because I is an interval. This is true for any $n \geqslant 1$, so there are infinitely many infinitary points in I. $\square$

For notational convenience let $\omega^{(+)} = \omega$ and $\omega^{(-)} = \omega^*$. For a finite sequence $\delta = (\delta_0, \delta_1, \ldots, \delta_{n-1}) \in \{+,-\}^n$ let

$$\omega^{(\delta)} = \omega^{(\delta_0)} \cdot \omega^{(\delta_1)} \cdot \ldots \cdot \omega^{(\delta_{n-1})}.$$

Let α be an odrinal. For an α-sequence $\delta = (\delta_i)_{i<\alpha} \in \{+,-\}^\alpha$ and $n < \alpha$ we let $\delta\restriction_n = (\delta_0, \delta_1, \ldots, \delta_{n-1})$.

Lemma 5.3. *Let S be a countable scattered chain such that $r_H(S) \geqslant \omega$. There exists an ω-sequence $\delta \in \{+,-\}^\omega = (\delta_0, \delta_1, \ldots)$ such that $\omega^{(\delta\restriction_k)} \hookrightarrow S$ for all $k \geqslant 1$.*

Proof. Each of the chains S, $c_{\text{fin}}(S)$, $c^2_{\text{fin}}(S)$, …, $c^n_{\text{fin}}(S)$, … is a countably infinite scattered chain. For each $j \in \mathbb{N}$ we shall label infinitary points of $c^j_{\text{fin}}(S)$ by elements of $\{+,-\}^j$ and along the way build a set $\mathcal{T}$ of finite words over $\{+,-\}$ as follows. To start the induction put the empty word ε in $\mathcal{T}$ and label each infinitary point $y \in c_{\text{fin}}(S)$ by $+$ if the order type of $c^{-1}_{\text{fin}}(y)$ is ω or ζ; otherwise label the point by $-$. Add all the labels assigned to infinitary points of $c_{\text{fin}}(S)$ to $\mathcal{T}$. Note that $\ell \in \mathcal{T}$ means that $\omega^{(\ell)} \hookrightarrow S$.

Assume that all the infinitary points of $c^j_{\text{fin}}(S)$ have been labelled by elements of $\{+,-\}^j$. Take any infinitary point $y \in c^{j+1}_{\text{fin}}(S)$. Since $c^{-1}_{\text{fin}}(y)$ is an infinite interval of $c^j_{\text{fin}}(S)$, it contains infinitely many infinitary points (Lemma 5.2). Each of the infinitary points in $c^{-1}_{\text{fin}}(y)$ is labelled by one of the 2^j labels, so there is a label $\ell \in \{+,-\}^j$ which occurs infinitely many times in $c^{-1}_{\text{fin}}(y)$. If the order type of $c^{-1}_{\text{fin}}(y)$ is ω or ζ label y by $\ell+$; otherwise label y by $\ell-$. Add all the labels assigned to infinitary points of $c^{j+1}_{\text{fin}}(S)$ to $\mathcal{T}$. Note again that $\ell \in \mathcal{T}$ means that $\omega^{(\ell)} \hookrightarrow S$.

The prefix ordering turns $\mathcal{T}$ into an infinite (not necessarily full) binary tree, so by the Kőnig's Lemma there is an infinite branch $\delta \in \{+,-\}^\omega = (\delta_0, \delta_1, \ldots)$. Clearly, the construction ensures that $\omega^{(\delta\restriction_k)} \hookrightarrow S$ for all $k \geqslant 1$. $\square$

Theorem 5.4. *Let S be a countable scattered chain such that $r_H(S) \geqslant \omega$. Then $T(n, S) = \infty$ for every $2 \leqslant n < \omega$.*

Proof. Due to Lemma 3.1 it suffices to show that $T(2, S) = \infty$.

According to Lemma 5.3 there exists an ω-sequence $\delta \in \{+,-\}^\omega = (\delta_0, \delta_1, \dots)$ such that $\omega^{(\delta\restriction_k)} \hookrightarrow S$ for all $k \geqslant 1$.

Case 1: The symbol $+$ occurs infinitely many times in δ.

Since S is a countable scattered chain Theorem 5.1 applies, so

$$S \not\longrightarrow [\omega^{n_0}, \omega^{n_1}, \omega^{n_2}, \omega^{n_3}, \omega^{n_4}, \dots]^2,$$

where $n_0 = 1$, $n_1 = n_2 = 2$, $n_3 = n_4 = 3$, and so on. Therefore, there exists a coloring $\gamma : \mathrm{Emb}(2, S) \to \omega$ with the following property: for every $i < \omega$ and every subchain $H \subseteq S$ such that $H \cong \omega^{n_i}$ we have that $i \in \gamma(\mathrm{Emb}(2, H))$.

Note that if $+$ appears m times in $\delta\restriction_k$ then, clearly, $\omega^m \hookrightarrow \omega^{(\delta\restriction_k)}$. Analogously, if $-$ appears m times in $\delta\restriction_k$ then $(\omega^*)^m \hookrightarrow \omega^{(\delta\restriction_k)}$. Since $+$ occurs infinitely many times in δ it follows that $\omega^i \hookrightarrow S$ for all $i \in \mathbb{N}$.

Now, take any $t \geqslant 2$ and consider the coloring $\chi_t : \mathrm{Emb}(2, S) \to t$ given by $\chi_t(f) = \min\{t-1, \gamma(f)\}$. Let S' be an arbitrary subchain of S order-isomorphic to S. Since $\omega^i \hookrightarrow S \cong S'$ for all $i \in \mathbb{N}$, for every $i < t$ there is a subchain $H_i \subseteq S'$ order-isomorphic to ω^{n_i}. By the construction of χ_t it then follows that $i \in \chi_t(\mathrm{Emb}(2, H_i)) \subseteq \chi_t(\mathrm{Emb}(2, S'))$. Therefore, $|\chi_t(\mathrm{Emb}(2, S'))| \geqslant t$. This concludes the proof that $T(2, S) = \infty$ in Case 1.

Case 2: The symbol $+$ occurs only finitely many times in δ.

Then the symbol $-$ occurs infinitely many times in δ, so $(\omega^*)^i \hookrightarrow S$ for all $i \in \mathbb{N}$. Therefore, $\omega^i \hookrightarrow S^*$ for all $i \in \mathbb{N}$. This time we apply Theorem 5.1 to S^* to conclude that $S^* \not\longrightarrow [\omega^{n_0}, \omega^{n_1}, \omega^{n_2}, \omega^{n_3}, \omega^{n_4}, \dots]^2$, and as in Case 1 we conclude that $T(2, S^*) = \infty$. But it is easy to see that $T(2, S) = T(2, S^*)$ for every chain S. Therefore, $T(2, S) = \infty$. □

Corollary 5.5. *Let A be a countable chain.*

(*a*) *If A is a scattered chain of infinite Hausdorff rank then* $\mathrm{spec}(A) = (n, \infty, \infty, \dots)$ *for some $n \in \mathbb{N}$.*

(*b*) *In all other cases* $\mathrm{spec}(A)$ *is a non-decreasing chain of integers.*

Proof. If A is a countable scattered chain of infinite Hausdorff rank then $T(n, A) = \infty$ for all $n \geqslant 2$ (Theorem 5.4). On the other hand, if A is a countable scattered chain of finite Hausdorff rank then $T(n, A) < \infty$ for all $n \geqslant 2$ (Theorem 4.3) and this sequence is non-decreasing by Theorem 3.7. Finally, if A is a non-scattered countable chain then Theorem 1.4 applies. □

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