

Generalized continued fraction expansions for π and e

Shirali Kadyrov^a, Farukh Mashurov^{a,b}

^a*Department of Mathematics and Natural Sciences, Suleyman Demirel University,
Kaskelen, Kazakhstan*

^b*Kazakh-British Technical University, Almaty, Kazakhstan*

Abstract

Recently Raayoni et al. announced various conjectures on continued fractions of fundamental constants automatically generated with machine learning techniques. In this paper we prove some of their stated conjectures for Euler number e and show the equivalence of some of the listed conjectures. Moreover, we propose a simple method that can be used to generate other continued fractions using their series representations.

Keywords: Generalized Continued Fraction, π , Euler number e , Ramanujan Machine.

2010 MSC: 11Y65, 30B70

1. Introduction

Continued fraction theory is one of the old area of number theory that dates back to Bombelli from 16th Century [1]. The infinite continued fractions were first studied by Lord Brouncker who stated the well-known continued fraction for $4/\pi$. All giants of number theory including Euler, Gauss, Lagrange, Legendre, and Galois contributed to the field. For any given sequences (a_n) and (b_n)

Email addresses: shirali.kadyrov@sdu.edu.kz (Shirali Kadyrov),
farukh.mashurov@sdu.edu.kz (Farukh Mashurov)

continued fraction is written as

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}}, \text{ or } b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots \quad (1)$$

For any n , a natural number, n 'th convergent of (1) is given by

$$\frac{A_n}{B_n} := b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots + \frac{a_n}{b_n}.$$

We say that a given number x has *continued fraction expansion* (1) provided

$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = x.$$

Inductively it is easy to verify that for $n \geq 1$ the sequences (A_n) and (B_n) satisfy the following difference equations

$$A_n = b_n A_{n-1} + a_n A_{n-2} \quad (2)$$

$$B_n = b_n B_{n-1} + a_n B_{n-2} \quad (3)$$

with initial conditions

$$A_{-1} = 1, A_0 = b_0, B_{-1} = 0, \text{ and } B_0 = 1. \quad (4)$$

When $a_n = 1$ and $b_n \in \mathbb{N}$ for all $n \geq 1$, (1) is called the simple continued fraction and it is a generalized continued fraction otherwise. Computing closed form solutions to difference equations as in (2) and (3) with variable coefficients a_n, b_n is difficult in general [2]. An elementary approach is to guess the closed form of A_n and B_n from the first few terms and prove it using mathematical induction. Recently, this approach was used in [3] to prove one of the many conjectures on generalized continued fraction expansion for Euler number e listed in [4]. Machine learning techniques, namely Meet-In-The-Middle and Gradient Descent, were implemented in [4] to obtain various conjectures for mathematical constants π and e listed in www.RamanujanMachine.com.

In this note our aim is twofold: First, we prove some of the conjectures from www.RamanujanMachine.com using similar approach to [3] or show the equivalence of certain conjectures. Next, we describe a twisted approach to find
 15 and prove new generalized continued fractions for mathematical constants.

For the first objective we have the following, Throughout $e = 2.7182\dots$ stands for the Euler number.

Theorem 1. *We have*

$$\frac{e}{2} = 1 + \frac{1}{2} + \frac{3}{3} + \frac{4}{4} + \frac{5}{5} + \dots$$

Theorem 2. *We have*

$$e - 2 = 1 + \frac{-1}{1} + \frac{2}{1} + \frac{-1}{1} + \frac{3}{1} + \frac{-1}{1} + \frac{4}{1} + \dots,$$

Both were conjectured in www.RamanujanMachine.com. The following simple observations can be used to prove the equivalence of two continued fractions.

Theorem 3. *We have*

$$\begin{aligned} -\left(b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots\right) &= -b_0 + \frac{a_1}{-b_1} + \frac{a_2}{-b_2} + \frac{a_3}{-b_3} + \frac{a_4}{-b_4} + \dots \\ b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots &= b_0 + \frac{-a_1}{-b_1} + \frac{-a_2}{b_2} + \frac{-a_3}{-b_3} + \frac{-a_4}{b_4} + \dots \end{aligned}$$

20 As a corollary, we prove another conjecture from www.RamanujanMachine.com

Corollary 1. *We have*

$$-e = -3 + \frac{-1}{-4} + \frac{-2}{-5} + \frac{-3}{-6} + \dots$$

Theorem 3 applied to the second formula in [3] immediately gives the above corollary. Using Theorem 1, Theorem 3, and a trivial manipulation gives proof for another conjecture in www.RamanujanMachine.com.

Corollary 2. *We have*

$$-\frac{e}{2} = -1 + \frac{2}{-2} + \frac{3}{-3} + \frac{4}{-4} + \dots$$

We note that using similar arguments as in the proof of Theorem 1, Zhaniya
 25 Zhanabay provided another proof for Corollary 2. Theorem 2 together with
 Theorem 3 immediately give

Corollary 3. *We have*

$$2 - e = -1 + \frac{-1}{-1} + \frac{2}{-1} + \frac{-1}{-1} + \frac{3}{-1} + \frac{-1}{-1} + \frac{4}{-1} + \dots$$

$$e - 2 = 1 + \frac{1}{-1} + \frac{-2}{1} + \frac{1}{-1} + \frac{-3}{1} + \frac{1}{-1} + \frac{-4}{1} + \dots$$

Both statements were conjectured in www.RamanujanMachine.com.

Corollary 4. *The following conjectures are equivalent:*

$$-\frac{1}{2}(\pi + 2) = -3 \cdot 1 + \frac{-2 \cdot 1}{-3 \cdot 2} + \frac{-3 \cdot 3}{-3 \cdot 3} + \frac{-4 \cdot 5}{-3 \cdot 4} + \frac{-5 \cdot 7}{-3 \cdot 5} + \dots$$

$$\frac{1}{2}(\pi + 2) = 3 \cdot 1 + \frac{-2 \cdot 1}{3 \cdot 2} + \frac{3 \cdot 3}{3 \cdot 3} + \frac{-4 \cdot 5}{3 \cdot 4} + \frac{-5 \cdot 7}{3 \cdot 5} + \dots$$

$$-\frac{1}{2}(\pi + 2) = -3 \cdot 1 + \frac{2 \cdot 1}{3 \cdot 2} + \frac{3 \cdot 3}{-3 \cdot 3} + \frac{4 \cdot 5}{3 \cdot 4} + \frac{5 \cdot 7}{-3 \cdot 5} + \dots$$

$$\frac{1}{2}(\pi + 2) = 3 \cdot 1 + \frac{2 \cdot 1}{-3 \cdot 2} + \frac{-3 \cdot 3}{3 \cdot 3} + \frac{4 \cdot 5}{-3 \cdot 4} + \frac{5 \cdot 7}{3 \cdot 5} + \dots$$

The first two conjectures appear in www.RamanujanMachine.com and the
 remaining two are new. The equivalence of the four conjectures are trivial from
 30 Theorem 3. Attempt to prove the conjectures related to continued fractions for
 π using similar ideas as in the above theorems are not yet successful.

For the second objective we obtain the following.

Theorem 4. *Let $\sum_{k=1}^{\infty} c_k$ be a convergent series with nonzero terms. Define
 $b_1 = 2$, $a_1 = 2c_1$, and*

$$b_n = \frac{2(c_n + c_{n-1})}{c_{n-1}} \text{ and } a_n = \frac{-4c_n}{c_{n-1}}, \text{ for } n \geq 2.$$

Then, we have

$$\sum_{k=1}^{\infty} c_k = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots$$

If we one considers the well-known series representation for $\pi/4$, namely,
 $\pi/4 = \sum_{k=1}^{\infty} (-1)^{k+1}/(2k-1)$, then Theorem 4 gives

$$\frac{\pi}{4} = \frac{1}{1} + \frac{1^2}{2} + \frac{3^2}{2} + \frac{5^2}{2} + \frac{7^2}{2} + \dots,$$

which is well-known (goes back to Lord Brouncker, 17th century), except the first three terms needs to be simplified.

For $(\pi - 3)/4 = \sum_{k=1}^{\infty} (-1)^{k+1}/[2k(2k+1)(2k+2)]$, another series representation of π , we obtain

$$\frac{\pi - 3}{4} = \frac{1}{24} + \frac{36}{6} + \frac{5^2}{6} + \frac{7^2}{6} + \frac{9^2}{6} + \dots,$$

35 yet another well-known representation [5]. We have the following new continued fraction for $\pi/3$:

Corollary 5. *We have*

$$\frac{\pi}{3} = \frac{-6}{-5} + \frac{-75}{63} + \frac{29645}{278} + \frac{a_4}{b_4} + \frac{a_5}{b_5} + \dots,$$

where a_n, b_n are given by

$$b_n = 2(36n^2 - 72n + 31) \text{ and } a_n = (2n - 1)(2n - 5)(6n - 11)^2(6n - 7)^2,$$

for any $n \geq 2$.

In the next section we prove Theorem 1. Then, in § 3 we introduce the twisted method to prove Theorem 4.

40 2. Proofs of conjectures

Proof of Theorem 1. We note that $b_n = n + 1$ for $n \geq 0$ and $a_1 = 1$, $a_n = n + 1$ for $n \geq 1$. Substituting into (2) and (3) we see that $(A_n)_{n \geq -1} = (1, 1, 3, 12, 60, 360, 2520, \dots)$ and $(B_n)_{n \geq -1} = (0, 1, 2, 9, 44, 265, 1854, \dots)$. From [6], <https://oeis.org/> we see that A_n is eventually the sequence A001710, the number of even permutations of n letters, which is given by $n!/2$. So, we derive that $A_n = (n + 2)!/2$ for $n \geq 0$. Similarly, we see that B_n is the shifted sequence A182386 from [6]. This gives $B_n = (n + 2)! \sum_{k=0}^{n+2} (-1)^k/k!$ for $n \geq 1$. It is straightforward to verify by induction these formulas, then

$$\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = \lim_{n \rightarrow \infty} \frac{1/2}{\sum_{k=0}^{n+2} (-1)^k/k!} = \frac{2^{-1}}{e^{-1}} = e/2.$$

□

Proof of Theorem 3. Note that

$$\begin{aligned} -\left(b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots\right) &= -b_0 + \frac{-a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots \\ &= -b_0 + \frac{a_1}{-\left(b_1 + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots\right)}. \end{aligned}$$

Hence, the assertion follows by induction. For the second claim, we notice that

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots = b_0 + \frac{-a_1}{-b_1 + \left(\frac{-a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots\right)}.$$

Again inductively we deduce the claim. \square

Proof of Theorem 2. For $b_n = 1, \geq 0$ and $(a_n)_{\geq 1} = (-1, 2, -1, 3, -1, 4, -1, \dots)$ we get

$$(A_n)_{\geq -1} = (1, 1, 0, 2, 2, 8, 6, 38, 32, 222, 190, \dots),$$

$$(B_n)_{\geq -1} = (0, 1, 1, 3, 2, 11, 9, 53, 44, 309, 265, \dots).$$

These sequences are not recognized in [6]. However, if we consider subsequences with odd and even indices separately then we see for $n \geq 1$ that

$$A_{2n-1} = -n! \left(1 + 2 \sum_{k=1}^n (-1)^k \frac{1}{k!}\right) = -n! \left(2 \sum_{k=0}^n (-1)^k \frac{1}{k!} - 1\right)$$

$$A_{2n} = (n+1)! + (n+2)! - 2(A000166(n+1) + A000166(n+2))$$

$$B_{2n-1} = n! \sum_{k=0}^n (-1)^k \frac{1}{k!}$$

$$B_{2n} = A000166(n+1) + A000166(n+2)$$

where $A000166(n) = n! \sum_{k=0}^n (-1)^k \frac{1}{k!}$. We see that

$$\lim_{n \rightarrow \infty} \frac{A_{2n-1}}{B_{2n-1}} = -2 + \lim_{n \rightarrow \infty} \frac{1}{\sum_{k=0}^n (-1)^k \frac{1}{k!}} = e - 2.$$

Similarly,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{A_{2n}}{B_{2n}} &= \lim_{n \rightarrow \infty} \frac{(n+1)! + (n+2)!}{(n+1)! \sum_{k=0}^{n+1} (-1)^k \frac{1}{k!} + (n+2)! \sum_{k=0}^{n+2} (-1)^k \frac{1}{k!}} - 2 \\ &= \lim_{n \rightarrow \infty} \frac{(n+1)! + (n+2)!}{[(n+1)! + (n+2)!] \sum_{k=0}^{n+1} (-1)^k \frac{1}{k!} + (-1)^{n+2}/(n+2)!} - 2 \\ &= \frac{1}{\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}} - 2 = e - 2. \end{aligned}$$

\square

3. Generating new continued fractions

45 In this section we explain a general method to obtain various generalized continued fraction formulas for a given number. It is based on the following very simple idea:

1. For a given number x , find sequences A_n, B_n such that $\lim_{n \rightarrow \infty} \frac{A_n}{B_n} = x$
2. Find a_n, b_n satisfying the difference equations (2) and (3) with initial
- 50 conditions (4).

Then, obviously (1) converges to x . To this end, we need to find formulas for a_n, b_n in terms of A_n, B_n and pay special attention to initial conditions as we study now. We may write (2) and (3) as a matrix equation:

$$\begin{bmatrix} A_{n-1} & A_{n-2} \\ B_{n-1} & B_{n-2} \end{bmatrix} \begin{bmatrix} b_n \\ a_n \end{bmatrix} = \begin{bmatrix} A_n \\ B_n \end{bmatrix} \quad (5)$$

Assuming invertibility ($A_{n-1}B_{n-2} - A_{n-2}B_{n-1} \neq 0$) we obtain

$$\begin{bmatrix} b_n \\ a_n \end{bmatrix} = \frac{1}{A_{n-1}B_{n-2} - A_{n-2}B_{n-1}} \begin{bmatrix} B_{n-2} & -A_{n-2} \\ -B_{n-1} & A_{n-1} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix},$$

which summarizing gives

Theorem 5. *Let x be a number. Let $(A_n)_{\geq -1}$ and $(B_n)_{\geq -1}$ be two sequences of numbers satisfying*

- 55 1. $\lim_{n \rightarrow \infty} A_n/B_n = x$,
2. $A_{-1} = 1, B_{-1} = 0, B_0 = 1$,
3. $A_{n-1}B_{n-2} - A_{n-2}B_{n-1} \neq 0$ for $n \geq 2$.

Then, for the sequences $(a_n)_{\geq 1}$ and $(b_n)_{\geq 0}$ given by

$$b_n := \frac{A_n B_{n-2} - A_{n-2} B_n}{A_{n-1} B_{n-2} - A_{n-2} B_{n-1}}, a_n := \frac{-A_n B_{n-1} + A_{n-1} B_n}{A_{n-1} B_{n-2} - A_{n-2} B_{n-1}} \text{ for } n \geq 1, \quad (6)$$

and $b_0 = A_0$ we have

$$x = b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \frac{a_4}{b_4} + \dots$$

We now apply Theorem 5 to prove Theorem 4.

Proof of Theorem 4. Let $\sum_{k=1}^{\infty} c_k$ converge to some x . We let

$$A_n = 2^n \sum_{k=1}^n c_k, B_n = 2^n, n \geq 1 \text{ and } A_{-1} = 1, A_0 = 0, B_{-1} = 0, B_0 = 1. \quad (7)$$

Then, $\lim_{n \rightarrow \infty} A_n/B_n = x$ and A_n, B_n satisfy the conditions of Theorem 4. We may rewrite (6) as

$$b_n = \frac{\frac{A_n}{B_n} - \frac{A_{n-2}}{B_{n-2}}}{\frac{A_{n-1}}{B_{n-1}} - \frac{A_{n-2}}{B_{n-2}}} \cdot \frac{B_n}{B_{n-1}} \text{ and } a_n = \frac{-\frac{A_n}{B_n} + \frac{A_{n-1}}{B_{n-1}}}{\frac{A_{n-1}}{B_{n-1}} - \frac{A_{n-2}}{B_{n-2}}} \cdot \frac{B_n}{B_{n-2}}, n \geq 2. \quad (8)$$

Using (7) we see that (8) simplifies to

$$b_n = \frac{2(c_n + c_{n-1})}{c_{n-1}} \text{ and } a_n = \frac{-4c_n}{c_{n-1}}, \text{ for } n \geq 3. \quad (9)$$

The remaining cases $n = 1, 2$ require special attention due to initial conditions.

For $n = 2$, we have

$$b_2 = \frac{2(c_1 + c_2)}{c_1} \text{ and } a_2 = \frac{-4c_2}{c_1}$$

For $n = 1$, we have $B_{n-2} = B_{-1} = 0$, hence we cannot use (8). Instead we use

60 (6) to obtain $b_1 = 2$ and $a_1 = 2c_1$. Applying Theorem 5 finishes the proof. \square

Proof of Corollary 5. Let us apply Theorem 4 to

$$\frac{\pi}{3} = \sum_{k=1}^{\infty} (-1)^{k+1} \left(\frac{1}{6k-5} + \frac{1}{6k-1} \right).$$

This leads to

$$b_n = \frac{4(36n^2 - 72n + 31)}{(6n-5)(6n-1)(2n-3)} \text{ and } a_n = \frac{(2n-1)(6n-11)(6n-7)}{(6n-5)(6n-1)(2n-3)}, n \geq 2,$$

and $b_1 = 2, a_1 = 6/5$. Note that for any nonzero scalar c we have $\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \dots = \frac{ca_1}{cb_1} + \frac{ca_2}{b_2} + \frac{a_3}{b_3} + \dots$. Repeatedly using this fact to clear the denominators we arrive at

$$\pi/3 = \frac{-6}{-5} + \frac{-75}{63} + \frac{29645}{278} + \frac{a'_4}{b'_4} + \frac{a'_5}{b'_5} + \dots,$$

with a'_n and b'_n given by

$$b'_n = 2(36n^2 - 72n + 31) \text{ and } a'_n = (2n-1)(2n-5)(6n-11)^2(6n-7)^2,$$

for any $n \geq 2$. \square

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