# INTEGER PARTITIONS PROBABILITY DISTRIBUTIONS 

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#### Abstract

Two closely related discrete probability distributions are introduced. In each case the support is a set of vectors in $\mathbb{R}^{n}$ obtained from the partitions of the fixed positive integer $n$. These distributions arise naturally when considering equally-likely random permutations on the set of $n$ letters. For one of the distributions, the expectation vector and covariance matrix is derived. For the other distribution, conjectures for several elements of the expectation vector are provided.


KEY WORDS: Integer partitions, random partitions, symmetric group, discrete probability distribution.

## 1. Background

The study of random integer partitions is not new, but past work (see Fristedt (1993); Canfield, Corteel, and Hitczenko (2001); Mutafchiev (2002); Mutafchiev (2005)) has focused on probability models where partitions are chosen with equal probability. In contrast, here we assign probabilities to a given partition based on the probability that a given permutation has cycle type indexed by that partition, as explained in detail below.

## 2. Introduction and Notations

Let $n$ and $l$ denote a nonnegative integers. A partition $\lambda$ of size $n$ and length $l$ is an $l$-tuple $\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}\right)$ of integers where

$$
\lambda_{1} \geq \lambda_{2} \geq \ldots \lambda_{l} \geq 1
$$

and

$$
\lambda_{1}+\lambda_{2}+\cdots+\lambda_{l}=n
$$

Each $\lambda_{j}$ is a part of $\lambda$. Notice that the unique partition of 0 is the empty partition $\emptyset$, which has length 0 .

The multiplicity $m_{j}=m_{j}(\lambda)$ of part $j$ in $\lambda$ is the number of times $j$ appears as a part in $\lambda$. Let the multiplicity vector $\mathbf{m}(\lambda)$ of $\lambda$ be given by

$$
\mathbf{m}(\lambda):=\left(m_{1}, m_{2}, \ldots, m_{n}\right)^{\prime}
$$

(For typesetting convenience, we indicate a column vector as the transpose of a row vector, with transpose indicated by the prime ( ${ }^{\prime}$ ) symbol.)

For a partition $\lambda$ of size $n$ and length $l$, define the corresponding partition vector $\boldsymbol{\Lambda}:=\boldsymbol{\Lambda}(\lambda)=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l}, \lambda_{l+1}, \ldots, \lambda_{n}\right)^{\prime}$ where $\lambda_{j}:=0$ for all $l<j \leq n$. In other words, we take a partition $\lambda$ of $n$, pad it on the right

Table 1. The seven partitions of 5 , and their corresponding vectors

| $\lambda$ | $\mathbf{m}(\lambda)$ | $\boldsymbol{\Lambda}(\lambda)$ |
| :---: | :---: | :---: |
| $(5)$ | $(0,0,0,0,1)^{\prime}$ | $(5,0,0,0,0)^{\prime}$ |
| $(4,1)$ | $(1,0,0,1,0)^{\prime}$ | $(4,1,0,0,0)^{\prime}$ |
| $(3,2)$ | $(0,1,1,0,0)^{\prime}$ | $(3,2,0,0,0)^{\prime}$ |
| $(3,1,1)$ | $(2,0,1,0,0)^{\prime}$ | $(3,1,1,0,0)^{\prime}$ |
| $(2,2,1)$ | $(1,2,0,0,0)^{\prime}$ | $(2,2,1,0,0)^{\prime}$ |
| $(2,1,1,1)$ | $(3,1,0,0,0)^{\prime}$ | $(2,1,1,1,0)^{\prime}$ |
| $(1,1,1,1,1)$ | $(5,0,0,0,0)^{\prime}$ | $(1,1,1,1,1)^{\prime}$ |

with 0 's until its length is $n$, convert it to a column vector, and call this vector $\boldsymbol{\Lambda}$.

For a full introduction to integer partitions the standard reference is Andrews (1976). For a gentler introduction to partitions suitable for undergraduates, see Andrews and Eriksson (2004).

Let $\mathfrak{S}_{n}$ denote the symmetric group of degree $n$. Each permutation in $\mathfrak{S}_{n}$ has a cycle type corresponding to a partition of $n$. For example, in $\mathfrak{S}_{3}$, permutations (written here in disjoint cycle notation) $(1,2,3)$ and $(1,3,2)$ have cycle type (3); permutations $(1,3)(2),(1,2)(3)$, and $(1)(2,3)$ have cycle type $(2,1)$; and the identity permutation $(1)(2)(3)$ has cycle type $(1,1,1)$.

Fix a positive integer $n$. Select a permutation at random (each permutation is equally likely and thus is chosen with probability $1 / n!$ ). Let the random variable $\mathbf{X}$ equal the partition vector $\boldsymbol{\Lambda}(\lambda)$ for the partition $\lambda$ corresponding to the cycle type of the random permutation. Then let $\mathbf{Y}:=\mathbf{Y}^{(n)}=\mathbf{m}(\lambda)$. The support of the distribution of $\mathbf{Y}$ is those $n$-vectors $\left(y_{1}, y_{2}, \ldots, y_{n}\right)^{\prime}$ of nonnegative integers such that $y_{1}+2 y_{2}+3 y_{3}+\cdots n y_{n}=n$, i.e. those vectors that are multiplicity vectors for partitions of $n$. For any given $\mathbf{Y}$, the $j$ th component $Y_{j}$ is equal to the multiplicity $m_{j}(\lambda)$ of $j$ in the partition $\lambda$, or, equivalently, the number of $j$-cycles in the randomly selected permutation from $\mathfrak{S}_{n}$. (See Table 1 for the seven partitions of 5 and their corresponding vectors that comprise the support of $\mathbf{X}$ and $\mathbf{Y}$ in the $n=5$ case.)

From the theory of the symmetric group, we know that the number of permutations in $\mathfrak{S}_{n}$ of cycle type $\lambda$ is

$$
\frac{n!}{1^{m_{1}} 2^{m_{2}} \cdots n^{m_{n}} m_{1}!m_{2}!\cdots m_{n}!},
$$

see, e.g., Sagan (2001, p. 3, Eq. (1.2)). Thus, we may deduce that for a given permutation with cycle type described by the partition $\lambda$,

$$
P(\mathbf{X}=\boldsymbol{\Lambda}(\lambda))=P(\mathbf{Y}=\mathbf{m}(\lambda))=\frac{1}{1^{m_{1}} 2^{m_{2}} \cdots n^{m_{n}} m_{1}!m_{2}!\cdots m_{n}!} .
$$

That $\mathbf{X}$ and $\mathbf{Y}$ are in fact probability distributions follows from the fact that

$$
\begin{equation*}
\sum \frac{1}{1^{m_{1}} 2^{m_{2}} \cdots n^{m_{n}} m_{1}!m_{2}!\cdots m_{n}!}=1 \tag{2.1}
\end{equation*}
$$

where the sum is extended over all partitions $\lambda$ of $n$. Eq. (2.1) was proved by N. J. Fine (1988, p. 38, Eq. (22.2)).

In Section 2, we will study the distribution of $\mathbf{Y}$, and give an explicit formula for its expectation vector and covariance matrix. In Section 3, we find that the expectation of $\mathbf{X}=\left(X_{1}, X_{2}, \ldots, X_{n}\right)$ is more subtle than that of $\mathbf{Y}$, and so we content ourselves with some partial results and conjectures about $E\left(X_{j}\right)$ for various $j$.

## 3. The distribution of the random variable $\mathbf{Y}$

Before considering arbitrary $n$, let us record the pmf, expectation, and covariance of $\mathbf{Y}$ for $n=3$ and $n=5$.

Example 3.1 ( $n=3$ case). In the case arising from $\mathfrak{S}_{3}$, we have the following pmf for $\mathbf{Y}$ :

$$
P(\mathbf{Y}=\mathbf{y})= \begin{cases}1 / 6, & \text { if } \mathbf{y}=(3,0,0)^{\prime} \\ 1 / 2, & \text { if } \mathbf{y}=(1,1,0)^{\prime} \\ 1 / 3, & \text { if } \mathbf{y}=(0,0,1)^{\prime} \\ 0, & \text { otherwise }\end{cases}
$$

Observe that $E(\mathbf{Y})=\left(1, \frac{1}{2}, \frac{1}{3}\right)^{\prime}$ and the covariance matrix

$$
\boldsymbol{\Sigma}=\left[\begin{array}{ccc}
1 & 0 & -1 / 3 \\
0 & 1 / 4 & -1 / 6 \\
-1 / 3 & -1 / 6 & 2 / 9
\end{array}\right]
$$

Example 3.2 ( $n=5$ case). In the case arising from $\mathfrak{S}_{5}$, we have the following pmf for $\mathbf{Y}$ :

$$
P(\mathbf{Y}=\mathbf{y})= \begin{cases}1 / 120, & \text { if } \mathbf{y}=(5,0,0,0,0)^{\prime} \\ 1 / 12, & \text { if } \mathbf{y}=(3,1,0,0,0)^{\prime} \\ 1 / 6, & \text { if } \mathbf{y}=(2,0,1,0,0)^{\prime} \\ 1 / 8, & \text { if } \mathbf{y}=(1,2,0,0,0)^{\prime} \\ 1 / 4, & \text { if } \mathbf{y}=(1,0,0,1,0)^{\prime} \\ 1 / 6, & \text { if } \mathbf{y}=(0,1,1,0,0)^{\prime} \\ 1 / 5, & \text { if } \mathbf{y}=(0,0,0,0,1)^{\prime} \\ 0, & \text { otherwise }\end{cases}
$$

Observe that $E(\mathbf{Y})=\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right)^{\prime}$ and the covariance matrix

$$
\boldsymbol{\Sigma}=\left[\begin{array}{ccccc}
1 & 0 & 0 & 0 & -1 / 5 \\
0 & 1 / 2 & 0 & -1 / 8 & -1 / 10 \\
0 & 0 & 2 / 9 & -1 / 12 & -1 / 15 \\
0 & -1 / 8 & -1 / 12 & 3 / 16 & -1 / 20 \\
-1 / 5 & -1 / 10 & -1 / 15 & -1 / 20 & 4 / 25
\end{array}\right]
$$

The preceding examples suggest that for general $n$,

$$
E(\mathbf{Y})=\left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right)^{\prime}
$$

Let us prove this and some other results.
Let $\mathbf{t}=\left(t_{1}, t_{2}, \ldots, t_{n}\right)^{\prime}$. The joint moment generating function $M_{\mathbf{Y}}^{(n)}(\mathbf{t})$ of $\mathbf{Y}$ is

$$
M_{\mathbf{Y}}^{(n)}(\mathbf{t})=\sum_{\substack{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \\ y_{1}+2 y_{2}+\cdots y_{n}=n}} \frac{e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{n} t_{n}}}{1^{y_{1}} 2^{y_{2} \cdots n^{y_{n}} y_{1}!y_{2}!\cdots y_{n}!}}
$$

Notice that $M_{\mathbf{Y}}^{(n)}(\mathbf{t})=0$ if $n<0$, since it is the empty sum, and $M_{\mathbf{Y}}^{(0)}=$ $\sum_{\emptyset} \frac{e^{0}}{\text { empty product }}=1$.

In order to justify one of the steps in Theorem 3.4 below, we will require the following simple lemma.

Lemma 3.3. There exists a bijection between the set of partitions of $n$ in which part $i$ appears at least once and the set of unrestricted partitions of $n-i$.

Proof. Let $\lambda$ be a partition of $n$ in which $m_{i}(\lambda)>0$. Map $\lambda$ to the partition $\mu$ for which

$$
m_{j}(\mu)=\left\{\begin{array}{ll}
m_{j}(\lambda) & \text { if } j \neq i, \\
m_{j}(\lambda)-1 & \text { if } j=i
\end{array} .\right.
$$

The partition $\mu$ is clearly a partition of $n-i$, and the map is reversible and is therefore a bijection.

## Theorem 3.4.

$$
\frac{\partial}{\partial t_{i}} M_{\mathbf{Y}}^{(n)}(\mathbf{t})=\frac{e^{t_{i}}}{i} M_{\mathbf{Y}}^{(n-i)}(\mathbf{t})
$$

Proof.

$$
\begin{aligned}
& \frac{\partial}{\partial t_{i}} M_{\mathbf{Y}}^{(n)}(\mathbf{t})=\frac{\partial}{\partial t_{i}} \sum_{\substack{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \\
y_{1}+2 y_{2}+\cdots n y_{n}=n}} \frac{e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{n} t_{n}}}{1^{y_{1}} 2^{y_{2}} \cdots n^{y_{n}} y_{1}!y_{2}!\cdots y_{n}!} \\
& =\frac{\partial}{\partial t_{i}} \sum_{\substack{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \\
y_{1}+2 y_{2}+\cdots y_{n}=n \\
y_{i}>0}} \frac{e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{n} t_{n}}}{1^{y_{1}} 2^{y_{2} \cdots n^{y_{n}} y_{1}!y_{2}!\cdots y_{n}!}} \\
& =\sum_{\substack{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \\
y_{1}+2 y_{2}+\cdots, n y_{n}=n \\
y_{i}>0}} \frac{y_{i} e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{n} t_{n}}}{1^{y_{1}} 2^{y_{2} \cdots n^{y_{n}} y_{1}!y_{2}!\cdots y_{n}!}} \\
& =\frac{e^{t_{i}}}{i} \sum_{\substack{\left(y_{1}, y_{2}, \ldots, y_{n}\right) \\
y_{1}+2 y_{2}+\cdots n y_{n}=n}} \frac{i y_{i} e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{i-1} t_{i-1}+\left(y_{i}-1\right) t_{i}+y_{i+1} t_{i+1}+\cdots+y_{n} t_{n}}}{1^{y_{1}} 2^{y_{2}} \cdots n^{y_{n}} y_{1}!y_{2}!\cdots y_{n}!} \\
& =\frac{e^{t_{i}}}{i} \sum_{\left(y_{1}, y_{2}, \ldots, y_{n-i}\right)} \frac{e^{y_{1} t_{1}+y_{2} t_{2}+\cdots+y_{n-i} t_{n-i}}}{1^{y_{1}} 2^{y_{2} \cdots(n-i)^{y_{n-i}} y_{1}!y_{2}!\cdots y_{n-i}!}} \\
& y_{1}+2 y_{2}+\cdots(n-i) y_{n-i}=n-i \\
& =\frac{e^{t_{i}}}{i} M_{\mathbf{Y}}^{(n-i)}(\mathbf{t}),
\end{aligned}
$$

where the penultimate equality follows from Lemma 3.3,
Example 3.5. For clarity, let us observe Theorem 3.4 in the case $n=5$, $i=2$.

$$
\begin{aligned}
\frac{\partial}{\partial t_{2}} M_{\mathbf{Y}}^{(5)}(\mathbf{t}) & =\frac{\partial}{\partial t_{2}}\left(\frac{e^{5 t_{1}}}{120}+\frac{e^{3 t_{1}+t_{2}}}{12}+\frac{e^{2 t_{1}+t_{3}}}{6}+\frac{e^{t_{1}+2 t_{2}}}{8}+\frac{e^{t_{1}+t_{4}}}{4}+\frac{e^{t_{2}+t_{3}}}{6}+\frac{e^{t_{5}}}{5}\right) \\
& =1 \frac{e^{3 t_{1}+t_{2}}}{12}+2 \frac{e^{t_{1}+2 t_{2}}}{8}+1 \frac{e^{t_{2}+t_{3}}}{6} \\
& =\frac{e^{t_{2}}}{2}\left(\frac{e^{3 t_{1}}}{6}+\frac{e^{t_{1}+t_{2}}}{2}+\frac{e^{t_{3}}}{3}\right) \\
& =\frac{e^{t_{2}}}{2} M_{\mathbf{Y}}^{(3)}(\mathbf{t})
\end{aligned}
$$

With Theorem 3.4 in hand, we can use it to derive the moments in the usual way. In particular, we have the following results.

## Corollary 3.6.

$$
E(\mathbf{Y})=(1,1 / 2,1 / 3, \ldots, 1 / n)^{\prime} .
$$

Remark 3.7. Corollary 3.6 tells us that in $\mathfrak{S}_{n}$, the expected number of $i$ cycles in a randomly selected permutation is $1 / i$ for $1 \leq i \leq n$.
Remark 3.8. Corollary 3.6 can be deduced from Zhang et al. (2017, Eq. (14)).

Corollary 3.9. $E\left(\mathbf{Y} \mathbf{Y}^{\prime}\right)$ can be decomposed into a sum of two matrices

$$
E\left(\mathbf{Y} \mathbf{Y}^{\prime}\right)=\mathbf{A}+\mathbf{B},
$$

where $\mathbf{A}=\left(a_{i j}\right)_{1 \leq i, j \leq n}$ with

$$
a_{i j}= \begin{cases}1 /(i j), & \text { if } i \leq j \text { and } j<n-i, \\ 0, & \text { if } i \leq j \text { and } j \geq n-i, \quad ; \\ a_{j i}, & \text { otherwise }\end{cases}
$$

and $\mathbf{B}=\left(b_{i j}\right)_{1 \leq i, j \leq n}$ with

$$
b_{i j}=\left\{\begin{array}{ll}
1 / i, & \text { if } i=j, \\
0, & \text { otherwise }
\end{array} .\right.
$$

Of course, it is always possible to write a covariance matrix as

$$
\boldsymbol{\Sigma}=E\left(\mathbf{Y} \mathbf{Y}^{\prime}\right)-E(\mathbf{Y}) E\left(\mathbf{Y}^{\prime}\right),
$$

and so we may deduce the following result.
Corollary 3.10. The entries of the covariance matrix $\boldsymbol{\Sigma}$ are given by

$$
\Sigma_{i j}= \begin{cases}1 / i, & \text { if } i=j \text { and } i \leq n / 2 ; \\ (i-1) / i^{2}, & \text { if } i=j \text { and } i>n / 2 ; \\ 0, & \text { if } i<j \text { and } j \leq n-i ; \\ -1 /(i j), & \text { if } i<j \text { and } j>n-i ; \\ \Sigma_{j i}, & \text { if } i>j .\end{cases}
$$

## 4. The distribution of the random variable $\mathbf{X}$

We now turn our attention to the random variable $\mathbf{X}$, first observing the pmf in the $n=4$ case.
Example 4.1 ( $n=4$ case). In the case arising from $\mathfrak{S}_{4}$, we have the following pmf for $\mathbf{X}$ :

$$
P(\mathbf{X}=\mathbf{x})= \begin{cases}1 / 24, & \text { if } \mathbf{x}=(1,1,1,1)^{\prime} \\ 1 / 4, & \text { if } \mathbf{x}=(2,1,1,0)^{\prime} \\ 1 / 8, & \text { if } \mathbf{x}=(2,2,0,0)^{\prime} \\ 1 / 3, & \text { if } \mathbf{x}=(3,1,0,0)^{\prime} \\ 1 / 4, & \text { if } \mathbf{x}=(4,0,0,0)^{\prime} \\ 0, & \text { otherwise }\end{cases}
$$

The moments of $\mathbf{X}$ are not as easily described as those of $\mathbf{Y}$. When we need to consider more than one $n$ at a time, let us write $\mathbf{X}=\mathbf{X}^{(n)}=$ $\left(X_{1}^{(n)}, X_{2}^{(n)}, \ldots X_{n}^{(n)}\right)^{\prime}$. We note that by direct calculation, we find that $n!E\left(X_{1}^{(n)}\right)$ for $n=1,2,3, \ldots, 10$ is the sequence

$$
\{1,3,13,67,411,2911,23563,213543,2149927,23759791\},
$$

which is in the Online Encyclopedia of Integer Sequences Sloane (2019, sequence A028418). We may interpret that, e.g., in $\mathfrak{S}_{4}$, the expected largest cycle length is $67 / 24 \approx 2.792$; in $\mathfrak{S}_{5}$, the expected largest cycle length is $411 / 120=3.425$, etc.

If we similarly calculate $n!E\left(X_{2}^{(n)}\right)$ for $n=2,3,4, \ldots$, we obtain the sequence

$$
\{1,4,21,131,950,7694,70343, \ldots\}
$$

which is not directly in the OEIS, although both of the preceding appear as subsequences of A322384 (in exactly the context at hand: the entry gives the "number $T(n, k)$ of entries in the $k$ th cycles of all permutations of $[n]$ when the cycles are ordered by decreasing lengths").

We can easily conclude that $E\left(X_{n}^{(n)}\right)=1 / n$ ! since there is only one partition of $n$ with $n$ parts (the partition consisting of $n 1 \mathrm{~s}$ ), and this partition occurs with probability $1 / n!$.

All of the subsequent formulas are conjectural; the author guesses them based on calculations performed with Mathematica. For $n \geq 3$,

$$
\begin{equation*}
n!E\left(X_{n-1}^{(n)}\right)=\frac{n^{2}-n+2}{2}=\binom{n}{2}+1 \tag{4.1}
\end{equation*}
$$

for $n \geq 5$,

$$
\begin{equation*}
n!E\left(X_{n-2}^{(n)}\right)=\frac{3 n^{4}-10 n^{3}+21 n^{2}-14 n+24}{24}=3\binom{n}{4}+2\binom{n}{3}+\binom{n}{2}+1 \tag{4.2}
\end{equation*}
$$

and for $n \geq 7$,

$$
\begin{align*}
n!E\left(X_{n-3}^{(n)}\right) & =\frac{n^{6}-7 n^{5}+23 n^{4}-37 n^{3}+48 n^{2}-28 n+48}{48} \\
& =15\binom{n}{6}+20\binom{n}{5}+9\binom{n}{4}+2\binom{n}{3}+\binom{n}{2}+1 \tag{4.3}
\end{align*}
$$

The idea for expressing the preceding sequence of polynomials in terms of a "binomial basis" was suggested by the work of Breuer, Eichhorn, and Kronholm (2017).

It appears plausible from the above and for analogous data for larger $j$, that $n!E\left(X_{n-j}^{(n)}\right)$ is a polynomial in $n$ of degree $2 j$ for all $n \geq 2 j+1$, and that

$$
n!E\left(X_{n-j}^{(n)}\right)=\frac{n^{2 j}}{j!2^{j}}+\frac{2 j+1}{3 \cdot 2^{j}(j-1)!} n^{2 j-1}+O\left(n^{2 j-2}\right)
$$

which would follow from the conjecture that

$$
\begin{equation*}
n!E\left(X_{n-j}^{(n)}\right)=1+\sum_{i=2}^{2 j} a_{i}\binom{n}{i} \tag{4.4}
\end{equation*}
$$

for some positive integers $a_{2}, a_{3}, \ldots, a_{2 j} ;$ with $a_{2 j}=(2 j-1)!$ !,

$$
a_{2 j-1}=\frac{(2 j+1)!!}{3}-(2 j-1)!!,
$$

$a_{4}=9($ when $j \geq 3), a_{3}=2, a_{2}=1$. Here we are using the double factorial notation; recall that

$$
(2 n+1)!!=\frac{(2 n+1)!}{2^{n} n!}
$$

## 5. Conclusion

As already remarked, previous research on random partitions has focused on both unrestricted and restricted partitions of the integer $n$, but invariably with such partitions being chosen with equal probability. Here we consider some of the consequences of choosing the random partition with a nonuniform, but nonetheless natural assignment of probabilities, and observe the effect of representing the partition in two different ways: by multiplicities and by parts. It is hoped that this study will help open the door to future studies of random partitions with other probability distributions.

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