INTEGER PARTITIONS PROBABILITY DISTRIBUTIONS

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ABSTRACT. Two closely related discrete probability distributions are introduced. In each case the support is a set of vectors in \mathbb{R}^n obtained from the partitions of the fixed positive integer n. These distributions arise naturally when considering equally-likely random permutations on the set of n letters. For one of the distributions, the expectation vector and covariance matrix is derived. For the other distribution, conjectures for several elements of the expectation vector are provided.

KEY WORDS: Integer partitions, random partitions, symmetric group, discrete probability distribution.

1. Background

The study of random integer partitions is not new, but past work (see Fristedt (1993); Canfield, Corteel, and Hitczenko (2001); Mutafchiev (2002); Mutafchiev (2005)) has focused on probability models where partitions are chosen with equal probability. In contrast, here we assign probabilities to a given partition based on the probability that a given permutation has cycle type indexed by that partition, as explained in detail below.

2. INTRODUCTION AND NOTATIONS

Let *n* and *l* denote a nonnegative integers. A partition λ of size *n* and length *l* is an *l*-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_l)$ of integers where

$$\lambda_1 \ge \lambda_2 \ge \dots \lambda_l \ge 1$$

and

$$\lambda_1 + \lambda_2 + \dots + \lambda_l = n.$$

Each λ_j is a *part* of λ . Notice that the unique partition of 0 is the *empty* partition \emptyset , which has length 0.

The multiplicity $m_j = m_j(\lambda)$ of part j in λ is the number of times j appears as a part in λ . Let the multiplicity vector $\mathbf{m}(\lambda)$ of λ be given by

$$\mathbf{m}(\lambda) := (m_1, m_2, \dots, m_n)'.$$

(For typesetting convenience, we indicate a column vector as the transpose of a row vector, with transpose indicated by the prime (') symbol.)

For a partition λ of size n and length l, define the corresponding *partition* vector $\mathbf{\Lambda} := \mathbf{\Lambda}(\lambda) = (\lambda_1, \lambda_2, \dots, \lambda_l, \lambda_{l+1}, \dots, \lambda_n)'$ where $\lambda_j := 0$ for all $l < j \leq n$. In other words, we take a partition λ of n, pad it on the right

λ	$\mathbf{m}(\lambda)$	$oldsymbol{\Lambda}(\lambda)$
(5)	(0, 0, 0, 0, 1)'	(5, 0, 0, 0, 0)'
(4, 1)	(1, 0, 0, 1, 0)'	(4, 1, 0, 0, 0)'
(3,2)	(0, 1, 1, 0, 0)'	(3, 2, 0, 0, 0)'
(3,1,1)	(2, 0, 1, 0, 0)'	(3, 1, 1, 0, 0)'
(2, 2, 1)	(1, 2, 0, 0, 0)'	(2, 2, 1, 0, 0)'
(2, 1, 1, 1)	(3, 1, 0, 0, 0)'	(2, 1, 1, 1, 0)'
(1, 1, 1, 1, 1)	(5, 0, 0, 0, 0)'	(1, 1, 1, 1, 1)'

TABLE 1. The seven partitions of 5, and their corresponding vectors

with 0's until its length is n, convert it to a column vector, and call this vector $\mathbf{\Lambda}$.

For a full introduction to integer partitions the standard reference is Andrews (1976). For a gentler introduction to partitions suitable for undergraduates, see Andrews and Eriksson (2004).

Let \mathfrak{S}_n denote the symmetric group of degree n. Each permutation in \mathfrak{S}_n has a cycle type corresponding to a partition of n. For example, in \mathfrak{S}_3 , permutations (written here in disjoint cycle notation) (1,2,3) and (1,3,2) have cycle type (3); permutations (1,3)(2), (1,2)(3), and (1)(2,3) have cycle type (2,1); and the identity permutation (1)(2)(3) has cycle type (1,1,1).

Fix a positive integer n. Select a permutation at random (each permutation is equally likely and thus is chosen with probability 1/n!). Let the random variable \mathbf{X} equal the partition vector $\mathbf{\Lambda}(\lambda)$ for the partition λ corresponding to the cycle type of the random permutation. Then let $\mathbf{Y} := \mathbf{Y}^{(n)} = \mathbf{m}(\lambda)$. The support of the distribution of \mathbf{Y} is those *n*-vectors $(y_1, y_2, \ldots, y_n)'$ of nonnegative integers such that $y_1 + 2y_2 + 3y_3 + \cdots ny_n = n$, i.e. those vectors that are multiplicity vectors for partitions of n. For any given \mathbf{Y} , the *j*th component Y_j is equal to the multiplicity $m_j(\lambda)$ of j in the partition λ , or, equivalently, the number of *j*-cycles in the randomly selected permutation from \mathfrak{S}_n . (See Table 1 for the seven partitions of 5 and their corresponding vectors that comprise the support of \mathbf{X} and \mathbf{Y} in the n = 5case.)

From the theory of the symmetric group, we know that the number of permutations in \mathfrak{S}_n of cycle type λ is

$$\frac{n!}{1^{m_1}2^{m_2}\cdots n^{m_n}m_1!m_2!\cdots m_n!},$$

see, e.g., Sagan (2001, p. 3, Eq. (1.2)). Thus, we may deduce that for a given permutation with cycle type described by the partition λ ,

$$P(\mathbf{X} = \mathbf{\Lambda}(\lambda)) = P(\mathbf{Y} = \mathbf{m}(\lambda)) = \frac{1}{1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n!}$$

That ${\bf X}$ and ${\bf Y}$ are in fact probability distributions follows from the fact that

(2.1)
$$\sum \frac{1}{1^{m_1} 2^{m_2} \cdots n^{m_n} m_1! m_2! \cdots m_n!} = 1,$$

where the sum is extended over all partitions λ of n. Eq. (2.1) was proved by N. J. Fine (1988, p. 38, Eq. (22.2)).

In Section 2, we will study the distribution of \mathbf{Y} , and give an explicit formula for its expectation vector and covariance matrix. In Section 3, we find that the expectation of $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is more subtle than that of \mathbf{Y} , and so we content ourselves with some partial results and conjectures about $E(X_j)$ for various j.

3. The distribution of the random variable ${f Y}$

Before considering arbitrary n, let us record the pmf, expectation, and covariance of **Y** for n = 3 and n = 5.

Example 3.1 (n = 3 case). In the case arising from \mathfrak{S}_3 , we have the following pmf for **Y**:

$$P(\mathbf{Y} = \mathbf{y}) = \begin{cases} 1/6, & \text{if } \mathbf{y} = (3, 0, 0)', \\ 1/2, & \text{if } \mathbf{y} = (1, 1, 0)', \\ 1/3, & \text{if } \mathbf{y} = (0, 0, 1)', \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $E(\mathbf{Y}) = (1, \frac{1}{2}, \frac{1}{3})'$ and the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1/4 & -1/6 \\ -1/3 & -1/6 & 2/9 \end{bmatrix}.$$

Example 3.2 (n = 5 case). In the case arising from \mathfrak{S}_5 , we have the following pmf for **Y**:

$$P(\mathbf{Y} = \mathbf{y}) = \begin{cases} 1/120, & \text{if } \mathbf{y} = (5, 0, 0, 0, 0)', \\ 1/12, & \text{if } \mathbf{y} = (3, 1, 0, 0, 0)', \\ 1/6, & \text{if } \mathbf{y} = (2, 0, 1, 0, 0)', \\ 1/8, & \text{if } \mathbf{y} = (1, 2, 0, 0, 0)' \\ 1/4, & \text{if } \mathbf{y} = (1, 0, 0, 1, 0)', \\ 1/6, & \text{if } \mathbf{y} = (0, 1, 1, 0, 0)', \\ 1/5, & \text{if } \mathbf{y} = (0, 0, 0, 0, 1)', \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $E(\mathbf{Y}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5})'$ and the covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 1 & 0 & 0 & 0 & -1/5 \\ 0 & 1/2 & 0 & -1/8 & -1/10 \\ 0 & 0 & 2/9 & -1/12 & -1/15 \\ 0 & -1/8 & -1/12 & 3/16 & -1/20 \\ -1/5 & -1/10 & -1/15 & -1/20 & 4/25 \end{bmatrix}.$$

The preceding examples suggest that for general n,

$$E(\mathbf{Y}) = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right)'.$$

Let us prove this and some other results.

Let $\mathbf{t} = (t_1, t_2, \dots, t_n)'$. The joint moment generating function $M_{\mathbf{Y}}^{(n)}(\mathbf{t})$ of Y is

$$M_{\mathbf{Y}}^{(n)}(\mathbf{t}) = \sum_{\substack{(y_1, y_2, \dots, y_n)\\y_1+2y_2+\dots ny_n = n}} \frac{e^{y_1 t_1 + y_2 t_2 + \dots + y_n t_n}}{1^{y_1} 2^{y_2} \dots n^{y_n} y_1! y_2! \dots y_n!}$$

Notice that $M_{\mathbf{Y}}^{(n)}(\mathbf{t}) = 0$ if n < 0, since it is the empty sum, and $M_{\mathbf{Y}}^{(0)} = \sum_{\emptyset \text{ empty product}} \frac{e^0}{1} = 1$. In order to justify one of the steps in Theorem 3.4 below, we will require

the following simple lemma.

Lemma 3.3. There exists a bijection between the set of partitions of n in which part i appears at least once and the set of unrestricted partitions of n-i.

Proof. Let λ be a partition of n in which $m_i(\lambda) > 0$. Map λ to the partition μ for which

$$m_j(\mu) = \begin{cases} m_j(\lambda) & \text{if } j \neq i, \\ m_j(\lambda) - 1 & \text{if } j = i \end{cases}$$

•

The partition μ is clearly a partition of n - i, and the map is reversible and is therefore a bijection.

Theorem 3.4.

$$\frac{\partial}{\partial t_i} M_{\mathbf{Y}}^{(n)}(\mathbf{t}) = \frac{e^{t_i}}{i} M_{\mathbf{Y}}^{(n-i)}(\mathbf{t}).$$

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$$\begin{split} Proof. \\ \frac{\partial}{\partial t_i} M_{\mathbf{Y}}^{(n)}(\mathbf{t}) &= \frac{\partial}{\partial t_i} \sum_{\substack{(y_1, y_2, \dots, y_n) \\ y_1 + 2y_2 + \dots ny_n = n}} \frac{e^{y_1 t_1 + y_2 t_2 + \dots + y_n t_n}}{1^{y_1 2y_2} \dots n^{y_n} y_1 ! y_2 ! \dots y_n !} \\ &= \frac{\partial}{\partial t_i} \sum_{\substack{(y_1, y_2, \dots, y_n) \\ y_1 + 2y_2 + \dots ny_n = n \\ y_i > 0}} \frac{e^{y_1 t_1 + y_2 t_2 + \dots + y_n t_n}}{1^{y_1 2y_2} \dots n^{y_n} y_1 ! y_2 ! \dots y_n !} \\ &= \sum_{\substack{(y_1, y_2, \dots, y_n) \\ y_1 + 2y_2 + \dots ny_n = n \\ y_i > 0}} \frac{y_i e^{y_1 t_1 + y_2 t_2 + \dots + y_n t_n}}{1^{y_1 2y_2} \dots n^{y_n} y_1 ! y_2 ! \dots y_n !} \\ &= \frac{e^{t_i}}{i} \sum_{\substack{(y_1, y_2, \dots, y_n) \\ y_1 + 2y_2 + \dots ny_n = n \\ y_1 + 2y_2 + \dots ny_n = n}} \frac{i y_i e^{y_1 t_1 + y_2 t_2 + \dots + y_{i-1} t_{i-1} + (y_i - 1) t_i + y_{i+1} t_{i+1} + \dots + y_n t_n}}{1^{y_1 2y_2} \dots n^{y_n} y_1 ! y_2 ! \dots y_n !} \\ &= \frac{e^{t_i}}{i} \sum_{\substack{(y_1, y_2, \dots, y_n) \\ y_1 + 2y_2 + \dots (n-i) y_{n-i} = n-i}} \frac{e^{y_1 t_1 + y_2 t_2 + \dots + y_{i-1} t_{i-1} + (y_i - 1) t_i + y_{i+1} t_{i+1} + \dots + y_n t_n}}{1^{y_1 2y_2} \dots (n-i)^{y_{n-i}} y_1 ! y_2 ! \dots y_{n-i} !} \\ &= \frac{e^{t_i}}{i} M_{\mathbf{Y}}^{(n-i)}(\mathbf{t}), \end{split}$$

where the penultimate equality follows from Lemma 3.3.

Example 3.5. For clarity, let us observe Theorem 3.4 in the case n = 5, i = 2.

$$\begin{split} \frac{\partial}{\partial t_2} M_{\mathbf{Y}}^{(5)}(\mathbf{t}) &= \frac{\partial}{\partial t_2} \left(\frac{e^{5t_1}}{120} + \frac{e^{3t_1 + t_2}}{12} + \frac{e^{2t_1 + t_3}}{6} + \frac{e^{t_1 + 2t_2}}{8} + \frac{e^{t_1 + t_4}}{4} + \frac{e^{t_2 + t_3}}{6} + \frac{e^{t_5}}{5} \right) \\ &= 1 \frac{e^{3t_1 + t_2}}{12} + 2 \frac{e^{t_1 + 2t_2}}{8} + 1 \frac{e^{t_2 + t_3}}{6} \\ &= \frac{e^{t_2}}{2} \left(\frac{e^{3t_1}}{6} + \frac{e^{t_1 + t_2}}{2} + \frac{e^{t_3}}{3} \right) \\ &= \frac{e^{t_2}}{2} M_{\mathbf{Y}}^{(3)}(\mathbf{t}) \end{split}$$

With Theorem 3.4 in hand, we can use it to derive the moments in the usual way. In particular, we have the following results.

Corollary 3.6.

$$E(\mathbf{Y}) = (1, 1/2, 1/3, \dots, 1/n)'$$

Remark 3.7. Corollary 3.6 tells us that in \mathfrak{S}_n , the expected number of *i*-cycles in a randomly selected permutation is 1/i for $1 \leq i \leq n$.

Remark 3.8. Corollary 3.6 can be deduced from Zhang et al. (2017, Eq. (14)).

Corollary 3.9. $E(\mathbf{Y}\mathbf{Y}')$ can be decomposed into a sum of two matrices

$$E(\mathbf{Y}\mathbf{Y}') = \mathbf{A} + \mathbf{B},$$

where $\mathbf{A} = (a_{ij})_{1 \le i,j \le n}$ with

$$a_{ij} = \begin{cases} 1/(ij), & \text{if } i \leq j \text{ and } j < n-i, \\ 0, & \text{if } i \leq j \text{ and } j \geq n-i, \\ a_{ji}, & \text{otherwise} \end{cases};$$

and $\mathbf{B} = (b_{ij})_{1 \le i,j \le n}$ with

$$b_{ij} = \begin{cases} 1/i, & \text{if } i = j, \\ 0, & \text{otherwise} \end{cases}$$

Of course, it is always possible to write a covariance matrix as

$$\boldsymbol{\Sigma} = E(\mathbf{Y}\mathbf{Y}') - E(\mathbf{Y})E(\mathbf{Y}'),$$

and so we may deduce the following result.

Corollary 3.10. The entries of the covariance matrix Σ are given by

$$\Sigma_{ij} = \begin{cases} 1/i, & \text{if } i = j \text{ and } i \le n/2; \\ (i-1)/i^2, & \text{if } i = j \text{ and } i > n/2; \\ 0, & \text{if } i < j \text{ and } j \le n-i; \\ -1/(ij), & \text{if } i < j \text{ and } j > n-i; \\ \Sigma_{ji}, & \text{if } i > j. \end{cases}$$

4. The distribution of the random variable \mathbf{X}

We now turn our attention to the random variable \mathbf{X} , first observing the pmf in the n = 4 case.

Example 4.1 (n = 4 case). In the case arising from \mathfrak{S}_4 , we have the following pmf for **X**:

$$P(\mathbf{X} = \mathbf{x}) = \begin{cases} 1/24, & \text{if } \mathbf{x} = (1, 1, 1, 1)', \\ 1/4, & \text{if } \mathbf{x} = (2, 1, 1, 0)', \\ 1/8, & \text{if } \mathbf{x} = (2, 2, 0, 0)' \\ 1/3, & \text{if } \mathbf{x} = (3, 1, 0, 0)', \\ 1/4, & \text{if } \mathbf{x} = (4, 0, 0, 0)', \\ 0, & \text{otherwise.} \end{cases}$$

The moments of **X** are not as easily described as those of **Y**. When we need to consider more than one *n* at a time, let us write $\mathbf{X} = \mathbf{X}^{(n)} = (X_1^{(n)}, X_2^{(n)}, \dots, X_n^{(n)})'$. We note that by direct calculation, we find that $n!E(X_1^{(n)})$ for $n = 1, 2, 3, \dots, 10$ is the sequence

 $\{1, 3, 13, 67, 411, 2911, 23 563, 213 543, 2 149 927, 23 759 791\},\$

which is in the Online Encyclopedia of Integer Sequences Sloane (2019, sequence A028418). We may interpret that, e.g., in \mathfrak{S}_4 , the expected largest cycle length is $67/24 \approx 2.792$; in \mathfrak{S}_5 , the expected largest cycle length is 411/120 = 3.425, etc.

If we similarly calculate $n!E(X_2^{(n)})$ for $n=2,3,4,\ldots$, we obtain the sequence

$\{1, 4, 21, 131, 950, 7694, 70343, \dots\},\$

which is not directly in the OEIS, although both of the preceding appear as subsequences of A322384 (in exactly the context at hand: the entry gives the "number T(n, k) of entries in the kth cycles of all permutations of [n] when the cycles are ordered by decreasing lengths").

We can easily conclude that $E(X_n^{(n)}) = 1/n!$ since there is only one partition of n with n parts (the partition consisting of n 1s), and this partition occurs with probability 1/n!.

All of the subsequent formulas are conjectural; the author guesses them based on calculations performed with *Mathematica*. For $n \ge 3$,

(4.1)
$$n!E(X_{n-1}^{(n)}) = \frac{n^2 - n + 2}{2} = \binom{n}{2} + 1;$$

for $n \ge 5$, (4.2)

$$n!E(X_{n-2}^{(n)}) = \frac{3n^4 - 10n^3 + 21n^2 - 14n + 24}{24} = 3\binom{n}{4} + 2\binom{n}{3} + \binom{n}{2} + 1;$$

and for $n \geq 7$,

$$n!E(X_{n-3}^{(n)}) = \frac{n^6 - 7n^5 + 23n^4 - 37n^3 + 48n^2 - 28n + 48}{48}$$

$$(4.3) = 15\binom{n}{6} + 20\binom{n}{5} + 9\binom{n}{4} + 2\binom{n}{3} + \binom{n}{2} + 1.$$

The idea for expressing the preceding sequence of polynomials in terms of a "binomial basis" was suggested by the work of Breuer, Eichhorn, and Kronholm (2017).

It appears plausible from the above and for analogous data for larger j, that $n!E(X_{n-j}^{(n)})$ is a polynomial in n of degree 2j for all $n \ge 2j + 1$, and that

$$n!E(X_{n-j}^{(n)}) = \frac{n^{2j}}{j!2^j} + \frac{2j+1}{3\cdot 2^j(j-1)!}n^{2j-1} + O(n^{2j-2}),$$

which would follow from the conjecture that

(4.4)
$$n!E(X_{n-j}^{(n)}) = 1 + \sum_{i=2}^{2j} a_i \binom{n}{i},$$

for some positive integers a_2, a_3, \ldots, a_{2j} ; with $a_{2j} = (2j - 1)!!$,

$$a_{2j-1} = \frac{(2j+1)!!}{3} - (2j-1)!!,$$

 $a_4 = 9$ (when $j \ge 3$), $a_3 = 2$, $a_2 = 1$. Here we are using the double factorial notation; recall that

$$(2n+1)!! = \frac{(2n+1)!}{2^n n!}.$$

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5. Conclusion

As already remarked, previous research on random partitions has focused on both unrestricted and restricted partitions of the integer n, but invariably with such partitions being chosen with equal probability. Here we consider some of the consequences of choosing the random partition with a nonuniform, but nonetheless natural assignment of probabilities, and observe the effect of representing the partition in two different ways: by multiplicities and by parts. It is hoped that this study will help open the door to future studies of random partitions with other probability distributions.

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