The Euler Polynomial Prime Values Problem

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Abstract: This note provides an effective lower bound for the number of primes in the quadratic progression $p = n^2 + 1 \le x$ as $x \to \infty$.

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1 Introduction

As early as 1760, Euler was developing the theory of prime values of polynomials. In fact, Euler computed a very large table of the primes $p = n^2 + 1$, see [16, p. 123]. Likely, the prime values of polynomials was studied by other researchers before Euler. Later, circa 1910, Landau posed an updated question of the same problem about the primes values of the polynomial $n^2 + 1$. A fully developed conjecture, based on circle methods, was demonstrated about two decades later. A survey of the subsequent developments appears in [30, p. 342], [31, Section 19], and similar references.

Conjecture 1.1. ([23]) Let $x \gg 1$ be a large number. Let $\Lambda(n)$ be the vonMangoldt function, and let $\chi(n) = (n \mid p)$ be the quadratic symbol modulo p. Then

$$\sum_{n \le x} \Lambda \left(n^2 + 1 \right) = a_2 x + O\left(\frac{x}{\log x} \right), \tag{1}$$

where the density constant

$$a_2 = \prod_{p \ge 3} \left(1 - \frac{\chi(-1)}{p-1} \right) = 1.37281346\dots$$
 (2)

This conjecture, also known as one of the Landau primes counting problems, and other related problems are discussed in [8], [10], [19], [30, p. 343], [35, p. 405], [31]. Some partial results are proved in [20], [27], [10], [14], [24], [26], et alii. The results for the associated least common multiple problem $\log \operatorname{lcm}[f(1)f(2)\cdots f(n)]$ appears in [13], and the recent literature. Assuming the Elliott-Halberstam conjecture, see Conjecture 5.1, there is a discussion in [10, p. 5] concerning the existence of infinitely many primes of the form $p = an^2 + 1$, with $a = O(p^{\varepsilon})$, and $\varepsilon > 0$. This note proposes the following partial result.

Theorem 1.1. Let $x \ge 1$ be a large number. Then,

$$\sum_{n \le x^{1/2}} \Lambda(n^2 + 1) \gg x^{1/2} \left(1 + O\left(\frac{(\log \log x)^2}{\log x}\right) \right).$$
(3)

The proof appears in Section 2. The supporting materials are developed in Section 3 to Section 7. The remaining sections are optional. This result seems to resolve the asymptotic part of the polynomial prime values problem for $f(x) = x^2 + 1$. The determination of the true constant remains as an open problem. Some heuristics, and discussions on the difficulty and complexity of the constant appear in [8], [5, Section 3.3], [34], and the significant literature on the Bateman-Horn Conjecture.

2 Main Result

The total number of prime power divisors is $\Omega(n)$, and the Liouville function is $\lambda(n) = (-1)^{\Omega(n)}$. The vonMangoldt function is denoted by $\Lambda(n)$. The characteristic function of square integers $n \in \mathbb{N}$ is precisely

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 & \text{if } n = m^2, \\ 0 & \text{if } n \neq m^2, \end{cases}$$
(4)

for some integer $m \geq 1$. The quadratic to linear identity is the product

$$\Lambda(n+1)\left(\sum_{d|n}\lambda(d)\right)^2 = \begin{cases} \Lambda(m^2+1) & \text{if } n = m^2 \text{ is a square,} \\ 0 & \text{if } n \neq m^2 \text{ is not a square.} \end{cases}$$
(5)

The proof of Theorem 1.1, which is based on the quadratic to linear identity, is broken up into several Lemmas proved in Section 3 to Section 7.

Proof. (Theorem 1.1) Summing the quadratic to linear identity (5) over the integers $n \le x$ leads to

$$\sum_{m \le x^{1/2}} \Lambda(m^2 + 1) = \sum_{n \le x} \Lambda(n+1) \left(\sum_{d|n} \lambda(d) \right) \quad . \tag{6}$$

The lower bound

$$\sum_{m \le x^{1/2}} \Lambda(m^2 + 1) \ge \sum_{n \le x^{1/2}} \Lambda(n+1) \left(\sum_{d|n} \lambda(d)\right)^2$$

$$\ge \log 2$$
(7)

is a nonnegative value for all numbers $x \ge 1$. To derive an asymptotic expression for the lower bound, replace the identity in Lemma 6.4, reverse the order of summation:

$$\sum_{n \le x^{1/2}} \Lambda(n+1) \left(\sum_{d|n} \lambda(d) \right)^2$$

$$= \sum_{n \le x^{1/2}} \Lambda(n+1) \left(2 \sum_{\substack{d,e|n \\ d,e \le \sqrt{n}}} \lambda(d)\lambda(e) + 2\lambda(n) \sum_{\substack{d,e|n \\ d,e \le \sqrt{n}}} \lambda(n/d)\lambda(n/e) \right)$$

$$= 2 \sum_{\substack{d,e \le x^{1/4}}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \Lambda(n+1) + 2 \sum_{\substack{d,e \le x^{1/4}}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \lambda(n)\Lambda(n+1)$$

$$= M(x) + E(x).$$

$$(8)$$

Applying Lemma 3.1 to the main term M(x), and Lemma 4.1 to the error term E(x), return

$$\sum_{n \le x^{1/2}} \Lambda(n+1) \left(\sum_{d|n} \lambda(d) \right)^2 = M(x) + E(x)$$

$$\gg x^{1/2} \left(1 + O\left(\frac{(\log \log x)^2}{\log x}\right) \right) + O\left(\frac{x^{1/2}}{\log x}\right)$$

$$\gg x^{1/2} \left(1 + O\left(\frac{(\log \log x)^2}{\log x}\right) \right),$$
(9)

as $x \to \infty$. Quod erat inveniendum.

The earliest numerical data seems to be the Euler table in [16, p. 123]. An instructive numerical experiment for the polynomial $f(x) = x^2 + 1$ is conducted in [5, Section 3.3]. Some other numerical data and experiments are reported in [23, p. 50], [37], and [34]. A list of the prime values $p = n^2 + 1$ is also archived in OEIS A002496.

3 Estimate For The Main Term

Lemma 3.1. If $x \ge 1$ is a large number, then,

$$M(x) = 2 \sum_{d,e \le x^{1/4}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \Lambda(n+1) = 2a_0 x^{1/2} \left(1 + O\left(\frac{(\log\log x)^2}{\log x}\right) \right),$$
(10)

where $a_0 > 0$ is a constant.

Proof. Take a partition of the finite sum into two subsums:

$$M(x) = 2 \sum_{d,e \le x^{1/4}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \Lambda(n+1)$$
(11)
$$= 2 \sum_{d \le x^{1/4}} \lambda(d^2) \sum_{\substack{n \le x^{1/2} \\ d^2|n}} \Lambda(n+1) + 2 \sum_{\substack{d,e \le x^{1/4} \\ d \ne e}} \lambda(d)\lambda(e) \sum_{\substack{n \le x^{1/2} \\ d,e|n}} \Lambda(n+1)$$
$$= S_0 + S_1.$$

The finite sum S_0 is estimated in Lemma 3.2, and the finite sum S_1 is estimated in Lemma 3.5. Summing yields

$$M(x) = S_0 + S_1$$

$$= 2a_0 x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right) \right) + O\left(\frac{x^{1/2} (\log \log x)^2}{\log x}\right)$$

$$= 2a_0 x^{1/2} \left(1 + O\left(\frac{(\log \log x)^2}{\log x}\right) \right).$$
(12)

3.1 The Subsum S_0

Lemma 3.2. If $x \ge 1$ is a large number, then,

$$S_0 = 2 \sum_{d \le x^{1/4}} \lambda(d^2) \sum_{\substack{n \le x^{1/2} \\ d^2 \mid n}} \Lambda(n+1) = 2a_0 x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$
(13)

Proof. Take a partition

$$S_{0} = 2 \sum_{d \leq x^{1/4}} \lambda(d^{2}) \sum_{\substack{n \leq x^{1/2} \\ d^{2}|n}} \Lambda(n+1)$$

$$= 2 \sum_{d \leq x_{0}} \lambda(d^{2}) \sum_{\substack{n \leq x^{1/2} \\ d^{2}|n}} \Lambda(n+1) + 2 \sum_{x_{0} < d \leq x^{1/4}} \lambda(d^{2}) \sum_{\substack{n \leq x^{1/2} \\ d^{2}|n}} \Lambda(n+1)$$

$$= T_{0} + T_{1},$$
(14)

where $x_0 = (\log x)^B$, and B > 2 is a constant. The subsums T_0 and T_1 are evaluated or estimated separately in Lemma 3.3 and Lemma 3.4 respectively. Combining these results yield

$$T_{0} + T_{1} = 2a_{0}x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right)\right) + O\left(\frac{x^{1/2}}{\log x}\right) = 2a_{0}x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right)\right),$$
(15)

where $a_0 = \sum_{n \ge 1} 1/\varphi(n^2) > 0$ is a constant.

Lemma 3.3. Let $x \ge 1$ be a large number, and let $x_0 = (\log x)^B$, with B > 2 an arbitrary constant. Then,

$$T_0 = 2 \sum_{d \le x_0} \lambda(d^2) \sum_{\substack{n \le x^{1/2} \\ d^2|n}} \Lambda(n+1) = 2a_0 x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$
(16)

Proof. The vonMangoldt function sieves the prime values $p = n + 1 \equiv 1 \mod d^2$. Hence, applying Corollary 5.1 yields

$$\sum_{d \le x_0} \lambda(d^2) \sum_{\substack{n \le x^{1/2} \\ d^2 \mid n}} \Lambda(n+1) = 2 \sum_{d \le x_0} \frac{x^{1/2}}{\varphi(d^2)} \left(1 + O\left(\frac{1}{\log x}\right) \right)$$

$$= 2x^{1/2} \left(1 + O\left(\frac{1}{\log x}\right) \right) \sum_{d \le x_0} \frac{1}{\varphi(d^2)}.$$
(17)

The finite sum in (17) converges to a constant

$$2\sum_{d \le x_0} \frac{1}{\varphi(d^2)} = 2\sum_{d \ge 1} \frac{1}{\varphi(d^2)} - \sum_{d > x_0} \frac{1}{\varphi(d^2)} = 2a_0 + O\left(\frac{1}{(\log x)^B}\right).$$
(18)

Lemma 3.4. Let $x \ge 1$ be a large number, and let $x_0 = (\log x)^B$, with B > 0 an arbitrary constant. Then,

$$T_1 = 2 \sum_{\substack{x_0 < d \le x^{1/2} \\ n+1 \equiv 1 \mod d^2}} \Lambda(n+1) = O\left(\frac{x^{1/2}}{\log x}\right).$$
(19)

Proof. Taking an upper bound yields

$$\sum_{x_0 < d \le x^{1/2}} \lambda(d^2) \sum_{\substack{n \le x^{1/2} \\ d^2|n}} \Lambda(n+1) \ll (\log x) \sum_{x_0 < d \le x^{1/2}, \ n \le x^{1/2} \\ \ll (x^{1/2} \log x) \sum_{x_0 < d \le x^{1/2}} \frac{1}{d^2}.$$
(20)

The last finite sum in (20) has the upper bound

$$\sum_{x_0 < d \le x^{1/2}} \frac{1}{d^2} \ll \int_{x_0}^{x^{1/2}} \frac{1}{t^2} dt$$

$$\ll \frac{1}{x_0} - \frac{1}{x^{1/2}}$$

$$\ll \frac{1}{(\log x)^B},$$
(21)

since $x_0 = (\log x)^B$. Let B > 2. Substituting (21) into (20) verifies the estimate.

3.2 The Subsum S_1

The symbol [d, e] = lcm(d, e) denotes the lowest common multiple. Given a large number $x \ge 1$, define the subset of moduli

$$Q = \{q = [d, e] \le x^{1/2} : d \ne e \text{ and } d, e \le x^{1/4}\}.$$
(22)

The subset Q contains every integer $q \leq x^{1/2}$.

Lemma 3.5. Let $x \ge 1$ be a large number. Then,

$$S_1 = 2 \sum_{\substack{d, e \le x^{1/4} \\ d \ne e}} \lambda(d)\lambda(e) \sum_{\substack{n \le x^{1/2} \\ d|n, e|n}} \Lambda(n+1) = O\left(\frac{x^{1/2}(\log\log x)^2}{\log x}\right).$$
(23)

Proof. Use the subset of moduli Q in (22) to rewrite as

$$\sum_{\substack{d,e \le x^{1/4} \\ d \ne e}} \lambda(d)\lambda(e) \sum_{\substack{n \le x^{1/2} \\ d \mid n, e \mid n}} \Lambda(n+1) = \sum_{\substack{q \le x^{1/2} \\ q \mid n}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) + \sum_{\substack{x^{1/4} < q \le x^{1/2} \\ q \mid n}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) + \sum_{\substack{x^{1/4} < q \le x^{1/2} \\ q \mid n}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1)$$

$$= T_2 + T_3.$$
(24)

The terms $T_2 = T_2(x)$ and $T_3 = T_3(x)$ are estimated in Lemma 3.6, and Lemma 3.7. Summing the two terms yields

$$T_{2} + T_{3} = O\left(\frac{x^{1/2}(\log\log x)^{2}}{\log x}\right) + O\left(\frac{x^{1/2}}{\log x}\right)$$
(25)
$$= O\left(\frac{x^{1/2}(\log\log x)^{2}}{\log x}\right).$$

Lemma 3.6. If $x \ge 1$ is a large number, then,

$$T_2 = \sum_{q \le x^{1/4}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) = O\left(\frac{x(\log\log x)^2}{\log x}\right).$$
 (26)

Proof. Rewrite it in the form

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$$\sum_{q \le x^{1/4}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) = \sum_{q \le x^{1/4}} \lambda(q) \left(\sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) - \frac{x^{1/2}}{\varphi(q)} + \frac{x^{1/2}}{\varphi(q)} \right)$$
(27)
$$= x^{1/2} \sum_{q \le x^{1/4}} \frac{\lambda(q)}{\varphi(q)} + \sum_{q \le x^{1/4}} \lambda(q) \left(\sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) - \frac{x}{\varphi(q)} \right).$$

The first partial sum in (27) has the upper bound

$$\sum_{q \le x^{1/4}} \frac{\lambda(q)}{\varphi(q)} = O\left(\frac{x^{1/2}}{\log x}\right).$$
(28)

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This follows from Lemma 7.1. The second partial sum in (27) has the upper bound

$$\begin{vmatrix} \sum_{q \le x^{1/4}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) \\ \leq \sum_{q \le x^{1/4}} \left| \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) - \frac{x^{1/2}}{\varphi(q)} \right| \\ \leq \sum_{q \le x^{1/4+\delta(x)}} \left| \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) - \frac{x^{1/2}}{\varphi(q)} \right| \\ = O\left(\frac{x^{1/2} (\log \log x)^2}{\log x}\right).$$
(29)

This follows from Corollary 5.2 with $\delta(x) = 1/(\log x)^2$.

Lemma 3.7. If $x \ge 1$ is a large number, then,

$$T_3 = \sum_{\substack{x^{1/4} < q \le x^{1/2} \\ q|n}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q|n}} \Lambda(n+1) = O\left(\frac{x^{1/2}}{\log x}\right).$$
(30)

Proof. For $q \in [x^{1/4}, x^{1/2}]$, the $\psi(x, q, a)$ function is bounded by $2x^{1/2}/\varphi(q)$. Hence,

$$\left| \sum_{\substack{x^{1/4} < q \le x^{1/2} \\ q \mid n}} \lambda(q) \sum_{\substack{n \le x^{1/2} \\ q \mid n}} \Lambda(n+1) \right| \ll \left| x^{1/2} \sum_{\substack{x^{1/4} < q \le x^{1/2} \\ \varphi(q)}} \frac{\lambda(q)}{\varphi(q)} \right|$$

$$= O\left(\frac{x^{1/2}}{\log x}\right).$$
(31)

The estimate for the partial sum follows from Lemma 7.1.

4 Estimate For The Error Term

Lemma 4.1. If $x \ge 1$ is a large number, then,

$$E(x) = 2 \sum_{\substack{d,e \le x^{1/4}}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \lambda(n)\Lambda(n+1) = O\left(\frac{x^{1/2}}{\log x}\right).$$
(32)

Proof. Applying Lemma 7.4 yields

$$2\sum_{d,e \le x^{1/4}} \lambda(d)\lambda(e) \sum_{\substack{n \le \sqrt{x} \\ d,e|n}} \lambda(n)\Lambda(n+1) = 2\sum_{d,e \le x^{1/4}} \lambda(d)\lambda(e) \left(O\left(\frac{x^{1/2}}{de(\log x)^B}\right)\right)$$
(33)
$$= O\left(\frac{x^{1/2}}{(\log x)^B} \sum_{d,e \le x^{1/4}} \frac{1}{de}\right)$$
$$= O\left(\frac{x^{1/2}}{(\log x)^{B-2}}\right).$$

Let B > 3. Now, since the last double partial sum is bounded by $O(\log x)^2$, it completes the proof.

5 Some Primes Numbers Theorems

Given a large number $x \leq 1$, and a pair of small fixed integers a < q such that gcd(a,q) = 1, let

$$\pi(x,q,a) = \sum_{\substack{p \le x \\ p \equiv a \bmod q}} 1 \tag{34}$$

be the prime counting function in arithmetic progression, and let

$$\psi(x,q,a) = \sum_{\substack{n \le x \\ n \equiv a \mod q}} \Lambda(n)$$
(35)

be the weighted prime counting function. A survey of the early developments of the prime number theorem over arithmetic progressions appears in [30]. Some later works for the asymptotic result of the prime counting function $\pi(x, q, a)$ are stated in [32], it also appears in [40, Lemma 3.2], but the moduli are restricted to $q \leq (\log x)^4$. **Lemma 5.1.** ([17, Lemma 1]) Denote by $\pi(x, a, q)$ the number of primes $p \le x$, $p \equiv a \mod q$ and gcd(a, q) = 1. Then

$$\pi(x,q,a) = \frac{x}{\varphi(q)\log x} \left(1 + O\left(\frac{1}{\log x}\right) \right)$$
(36)

uniformly for all $q < e^{c_1 \log x/\log \log x}$, except possibly for the multiples of certain $q > (\log x)^B$, but the implied constant depends on the arbitrary constant B > 0.

The last result and the next result are the same as [29, Corollary 11.22], which is well known as the Siegel-Walfisz theorem.

Corollary 5.1. Let $x \ge 1$ be a large number, and let $q \le (\log x)^B$, $B \ge 0$ an arbitrary constant. Then

$$\psi(x,q,a) = \frac{x}{\varphi(q)} \left(1 + O\left(\frac{1}{\log x}\right) \right).$$
(37)

Proof. Use partial summation and Lemma 5.1 to evaluate the summatory function

$$\psi(x,q,a) = \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n).$$
(38)

An extension of the prime number theorem in arithmetic progression to certain exponential size moduli $q < x^e$ is proved in [3]. The following result for the a large subset of moduli on average will be used to prove some estimate.

Theorem 5.1. ([18, Theorem 22.1]) Let $a \neq 0$. For any decreasing function $\delta(x) \geq 0$, and a large number $x \geq 1$,

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$$\sum_{q \le x^{1/2+\delta(x)}} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{x}{\varphi(q)} \right| \ll x \left(\delta(x) + \frac{\log \log x}{\log x} \right)^2.$$
(39)

Corollary 5.2. Let $x \ge 1$ be a large number, and let $\delta(x) = 1/(\log x)^2$. Then,

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$$\sum_{\substack{q \le x^{1/2+\delta(x)} \\ n \equiv a \mod q}} \Lambda(n) - \frac{x}{\varphi(q)} = O\left(\frac{x(\log\log x)^2}{\log x}\right).$$
(40)

Proof. Use partial summation and Theorem 5.1 to evaluate the summatory function

$$\psi(x,q,a) = \sum_{\substack{n \le x \\ n \equiv a \mod q}} \Lambda(n).$$
(41)

This short survey concludes with other important results and conjectures for primes in arithmetic progression. These results extend the range of moduli to all q < x on average.

Theorem 5.2. (Bombieri-Vinogradov) Given a constant C > 0, and a sufficiently large number $x \ge 1$. Then

$$\sum_{q \le x^{1/2}/(\log x)^B} \max_{\gcd(a,q)=1} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{x}{\varphi(q)} \right| = O\left(\frac{x}{(\log x)^C}\right),\tag{42}$$

where B = B(C) > 0 depends on C > 0.

Theorem 5.3. (Barban-Davenport-Halberstam) Given a sufficiently large number $x \geq 1$, let $1 \leq Q \leq x$. Then

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$$\sum_{q \le Q} \max_{\gcd(a,q)=1} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{x}{\varphi(q)} \right|^2 = O\left(Qx \log x\right) + O\left(\frac{x}{(\log x)^C}\right), \tag{43}$$

where C > 0 is a constant.

Proof. See [18, Corollary 9.15].

Conjecture 5.1. (Elliott-Halberstam) Given a pair of constants C > 0, and $\theta > 0$, let $x \ge 1$ be a sufficiently large number. Then

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$$\sum_{q \le x^{1-\theta}} \max_{\gcd(a,q)=1} \left| \sum_{\substack{n \le x \\ n \equiv a \bmod q}} \Lambda(n) - \frac{x}{\varphi(q)} \right| = O\left(\frac{x}{(\log x)^C}\right),\tag{44}$$

where B = B(C) > 0 depends on C > 0.

Extensive discussion on the level of distribution of the moduli is given in [18, p. 406]. The Montgomery conjecture for primes in arithmetic progression extends the range of moduli to all q < x.

Conjecture 5.2. ([29, Conjecture 13.9] Let a < q be integers, gcd(a,q) = 1, and $q \leq x$. Then,

$$\psi(x,a,q) = \frac{x}{\varphi(q)} + O\left(\frac{x^{1/2+\varepsilon}}{\varphi(q)q^{1/2}}\right).$$
(45)

Some Elementary Identities 6

These identities are sort of pre-hyperbola method technique. Nevertheless, these identities offer the same efficiency as the general hyperbola method, see [2, Theorem 3.17], and [29, Equation 2.9], et cetera.

Lemma 6.1. If $n \ge 1$ is an integer, $\mu(n)$ is the Mobius function, and $\Lambda(n)$ is the vonMangoldt function, then,

$$\Lambda(n) = -\sum_{d|n} \mu(d) \log d.$$
(46)

Proof. Let $\log n = \sum_{d|n} \log(d) = \sum_{d|n} \Lambda(d)$, and use the Mobius inversion formula to compute its inverse.

Lemma 6.2. If $n \ge 1$ is an integer, $\mu(n)$ is the Mobius function, and $\Lambda(n)$ is the vonMangoldt function, then,

$$\Lambda(n) = -\sum_{\substack{d|n\\d < \sqrt{x}}} \mu(d) \log d - \sum_{\substack{d|n\\d \le \sqrt{x}}} \mu(n/d) \log(n/d).$$
(47)

Lemma 6.3. If $n \ge 1$ is an integer, and $\lambda(n)$ is the Liouville function, then,

$$\sum_{d|n} \lambda(d) = \sum_{\substack{d|n\\d < \sqrt{x}}} \lambda(d) + \sum_{\substack{d|n\\d \le \sqrt{x}}} \lambda(n/d).$$
(48)

Lemma 6.4. If $n \ge 1$ is an integer, and $\lambda(n)$ is the Liouville function, then,

$$\left(\sum_{d|n}\lambda(d)\right)^2 = 2\sum_{\substack{d,e|n\\d,e\leq\sqrt{n}}}\lambda(d)\lambda(e) + 2\lambda(n)\sum_{\substack{d,e|n\\d,e\leq\sqrt{n}}}\lambda(n/d)\lambda(n/e).$$
(49)

Proof. As per Lemma 6.3, and expand the expression

$$\left(\sum_{d|n}\lambda(d)\right)^2 = \left(\sum_{\substack{d|n\\d<\sqrt{x}}}\lambda(d) + \sum_{\substack{d|n\\d\le\sqrt{x}}}\lambda(n/d)\right) \left(\sum_{\substack{e|n\\e<\sqrt{x}}}\lambda(e) + \sum_{\substack{e|n\\e\le\sqrt{x}}}\lambda(n/e)\right),\tag{50}$$

and simplify it.

7 Some Elementary Partial Sums

A few details on the finite sum $\sum_{n \leq x} \lambda(n) / \varphi(n)$ and the finite sum $\sum_{n \leq x} \lambda(n) \Lambda(n)$ are investigated here. Both, conditional and unconditional results are provided.

Lemma 7.1. If C > 0 is a constant, and $x \ge 1$ is a sufficiently large number, then,

(i)
$$\sum_{n \le x} \frac{\mu(n)}{\varphi(n)} = O\left(\frac{1}{(\log x)^C}\right),$$

(ii)
$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} = O\left(\frac{1}{(\log x)^C}\right).$$

Proof. (ii) Substitute the identity $n/\varphi(n) = \sum_{d|n} \mu^2(d)/\varphi(d)$ in the partial sum, see [2, p. 47], and reverse the order of summation:

$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} = \sum_{n \le x} \frac{\lambda(n)}{n} \sum_{d|n} \frac{\mu^2(d)}{\varphi(d)}$$

$$= \sum_{d \le x} \frac{\mu^2(d)}{\varphi(d)} \sum_{\substack{n \le x \\ d|n}} \frac{\lambda(n)}{n}$$

$$= \sum_{d \le x} \frac{\mu^2(d)\lambda(d)}{d\varphi(d)} \sum_{m \le x/d} \frac{\lambda(m)}{m}.$$
(51)

The last line in (51) follows from $\lambda(dm) = \lambda(d)\lambda(m)$ for n = dm. Apply the well known estimate $\sum_{n \leq x} \lambda(n)/n = O\left(1/(\log x)^C\right)$, with C > 0 constant, see [29, Exercise 11, p. 184], to the inner sum:

$$\sum_{d \le x} \frac{\mu^2(d)\lambda(d)}{d\varphi(d)} \sum_{m \le x/d} \frac{\lambda(m)}{m} \ll \frac{1}{(\log x)^C} \sum_{d \le x} \frac{\mu^2(d)\lambda(d)}{d\varphi(d)}$$

$$= O\left(\frac{1}{(\log x)^C}\right)$$
(52)

The partial sum in the right side of (52),

$$\sum_{d \le x} \frac{\mu^2(d)\lambda(d)}{d\varphi(d)} = \sum_{d \ge 1} \frac{\mu(d)}{d\varphi(d)} - \sum_{d > x} \frac{\mu(d)}{d\varphi(d)} = a_0 + O\left(\frac{1}{(\log x)^B}\right),\tag{53}$$

B > 0 constaant, converges to a nonnegative constant $a_0 > 0$.

A different approach using product is presented in Lemma 16.1. The estimate of the partial sum $\sum_{n \leq x} \lambda(n)/\varphi(n)$ in Lemma 16.1 is geared for the short interval $[1, x_0]$, and the estimate in Lemma 7.1 is geared for the shifted short interval $[x_0, x_1]$, where $1 < x_0 < x_1 \leq x$.

Lemma 7.2. Assume the RH. If $\varepsilon > 0$ is a small constant, and $x \ge 1$ is a sufficiently large number, then,

(i)
$$\sum_{n \le x} \frac{\mu(n)}{\varphi(n)} = O\left(\frac{1}{x^{1/2-\varepsilon}}\right),$$

(ii) $\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} = O\left(\frac{1}{x^{1/2-\varepsilon}}\right).$

Proof. (i) Similar to the previous proof, but use the conditional upper bound of the Mertens sum $\sum_{n \leq x} \mu(n)/n = O\left(1/x^{1/2+\varepsilon}\right)$, see [38, Theorem 1], and [9].

Lemma 7.3. If B > 2 is a constant, and $x \ge 1$ is a sufficiently large number, then,

(i)
$$\sum_{n \le x} \mu(n)\Lambda(n) = O\left(\frac{x}{(\log x)^{B-2}}\right)$$
,
(ii) $\sum_{n \le x} \lambda(n)\Lambda(n) = O\left(\frac{x}{(\log x)^{B-2}}\right)$.

Proof. (ii) By Lemma 6.1, the partial sum can be converted to

$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} = -\sum_{n \le x} \lambda(n) \sum_{d|n} \mu(d) \log d$$

$$= -\sum_{d \le x} (\mu(d) \log d) \sum_{\substack{n \le x \\ d|n}} \lambda(n)$$

$$= -\sum_{d \le x} (\mu(d)\lambda(d) \log d) \sum_{m \le x/d} \lambda(m).$$
(54)

Apply the well known estimate $\sum_{n \le x} \lambda(n) = O\left(x/(\log x)^B\right)$, with C > 0 constant, see [29, Exersice 11, p. 184], to the inner sum:

$$-\sum_{d \le x} (\mu(d)\lambda(d)\log d) \sum_{m \le x/d} \lambda(m) = O\left(\frac{x}{(\log x)^B} \sum_{d \le x} \frac{\log d}{d}\right)$$
(55)
$$= O\left(\frac{x}{(\log x)^{B-2}}\right),$$

for any constant B-2 > 0.

Lemma 7.4. Assume the RH. If $\varepsilon > 0$ is a small constant, and $x \ge 1$ is a sufficiently large number, then,

(i)
$$\sum_{n \le x} \mu(n)\Lambda(n) = O\left(x^{1/2+\varepsilon}\right)$$
,
(ii) $\sum_{n \le x} \lambda(n)\Lambda(n) = O\left(x^{1/2+\varepsilon}\right)$.

Proof. (i) Similar to the previous proof, but use the conditional upper bound of the Mertens sum $\sum_{n \le x} \mu(n) = O\left(x^{1/2+\varepsilon}\right)$, see [38, Theorem 1], and [9].

8 Zeta Function And Its Plane Of Convergence

The set of prime powers is denoted by

$$\mathbb{P}_{\infty} = \left\{ 2, 2^2, 3, 5, 7, 2^3, 3^2, 11, 13, 2^4, 17, 19, 23, 5^2, 3^3, 29, 31, 2^5, 37, \ldots \right\},\tag{56}$$

and the subset of primes $n^2 + 1$, $n \ge 1$, is denoted by

$$\mathcal{A} = \{2, 5, 17, 37, 101, 197, 401, \ldots\} \subset \mathbb{P}_{\infty}.$$
(57)

The subset \mathcal{A} contains at most a finite number of prime powers $p^k = m^2 + 1$ for $k, m \in \mathbb{N}$, see Exercise 17.14. The multiplicative set generated by $\mathcal{A} \cup \{1\}$ is the subset of integers

$$\mathcal{B} = \{n = ab : a, b \in \mathcal{A} \cup \{1\}\} = \{1, 2, 2^2, 5, 2^3, 2 \cdot 5, 2^4, 17, 2^2 \cdot 5, 5^2, 2^5, 37, \ldots\}.$$
 (58)

The zeta function over the subset of integers ${\mathcal B}$ is defined by the function

$$\zeta_{\mathcal{A}}(s) = \prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^s} \right)^{-1} = \sum_{n \in \mathcal{B}} \frac{1}{n^s},\tag{59}$$

of a complex number $s \in \mathbb{C}$. Furthermore, the restricted vonMangoldt function is defined by

$$\Lambda_{\mathcal{A}}(n) = \begin{cases} \log(m^2 + 1) & \text{if } n = (m^2 + 1)^k \text{ is a prime power,} \\ 0 & \text{if } n \neq (m^2 + 1)^k \text{ is not a prime power,} \end{cases}$$
(60)

where the exponent $k \geq 1$.

Theorem 8.1. Let $s \in \mathbb{C}$ be a complex number. Then,

(i)
$$\zeta_{\mathcal{A}}(s) = \prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^s}\right)^{-1}$$
, is convergent on the half plane $\Re e(s) = \sigma > 1/2$.
(ii) $\zeta_{\mathcal{A}}(s) = \prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^s}\right)^{-1}$, has a pole at $\Re e(s) = 1/2$.

Proof. (i) Given a complex number $\Re e(s) = \sigma > 1/2$, it is sufficient to show it is bounded by a convergent product. Therefore, as the primes are of the form $p = n^2 + 1$, the relations

$$\prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^{\sigma}} \right)^{-1} \leq \prod_{n \ge 1} \left(1 - \frac{1}{(n^2 + 1)^{\sigma}} \right)^{-1} = O(1)$$

$$(61)$$

implies convergence on the upper half plane $\mathcal{H} = \{s \in \mathbb{C} : \Re e(s) > 1/2\}$. (ii) For a complex number $\Re e(s) = \sigma > 1/2$, the logarithm derivative of the zeta function is

$$\frac{\zeta'_{\mathcal{A}}(s)}{\zeta_{\mathcal{A}}(s)} = -\frac{d}{ds} \log \zeta_{\mathcal{A}}(s)$$

$$= -\frac{d}{ds} \log \prod_{p \in \mathcal{A}} \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$= \sum_{p \in \mathcal{A}} \frac{d}{ds} \log \left(1 - \frac{1}{p^s}\right)$$

$$= \sum_{n \in \mathcal{B}} \frac{\Lambda_{\mathcal{A}}(n)}{n^s},$$
(62)

confer the literature on the logarithm derivatives of zeta functions, L-functions, et cetera. Now, Theorem 1.1 implies that

$$\psi_{\mathcal{A}}(x) = \sum_{\substack{n \le x \\ n \in \mathcal{B}}} \Lambda_{\mathcal{A}}(n) = \sum_{m \le x^{1/2}} \Lambda(m^2 + 1) \gg x^{1/2}.$$
(63)

for any real number $x \ge 1$. Continue to use (63) to derive a lower bound of the partial sum of (62) at s = 1/2. Specifically,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{B}}} \frac{\Lambda_{\mathcal{A}}(n)}{n^{1/2}} = \int_{1}^{x} \frac{1}{t^{1/2}} d\psi_{\mathcal{A}}(t)$$

$$= C_{0} + \frac{1}{2} \int_{1}^{x} \frac{\psi_{\mathcal{A}}(t)}{t^{3/2}} dt$$

$$\gg \log x,$$
(64)

where C_0 is a constant. As the partial sum in (64) is unbounded at s = 1/2, it immediately follows that $\zeta_{\mathcal{A}}(s)$ has a pole at s = 1/2. In addition, it is a holomorphic function on the upper half plane $\mathcal{H} = \{s \in \mathbb{C} : \Re e(s) > 1/2\}.$

The power series expansion at $s_0 = 1/2$ should have a simple pole and the shape

$$\zeta_{\mathcal{A}}(s) = \frac{c_{-1}}{s - 1/2} + c_0 + c_1 \left(s - 1/2\right) + c_2 \left(s - 1/2\right)^2 + \cdots,$$
(65)

where the residue $c_{-1} = a_2$.

9 Gaussian Numbers Field

The Gaussian numbers field is the set $\mathbb{Q}[i] = \{\alpha = a + ib : a, b \in \mathbb{Q}\}$. Each algebraic integer $\alpha \in \mathbb{Z}[i]$ has a unique factorization as

$$\alpha = \pi_1^{u_1} \cdot \pi_2^{u_2} \cdots q_1^{v_1} \cdot q_2^{v_2} \cdots,$$
(66)

where $\pi_k = a_k + ib_k \in \mathbb{Z}[i], q_k = 4c_k + 3 \in \mathbb{N}$ are primes, and $u_k, v_k \ge 0$ are integers. The trace and norm $T, N : \mathcal{K} \longrightarrow \mathbb{Z}$ on the numbers field $\mathcal{K} = \mathbb{Q}[i]$ are defined by

$$Tr(a+ib) = 2a$$
 and $N(a+ib) = a^2 + b^2$. (67)

The subset of primes $p = n^2 + 1$ is a special case of the more general primes $p = a^2 + b^2$. Modulo the generalized Riemann hypothesis, the case $p = a^2 + b^2$, with $b = O(\log p)$, was proved in [1]. There are other related results, both conditional, and unconditional, in the literature.

The zeta function of the numbers field \mathcal{K} is defined by the function

$$\zeta_{\mathcal{K}}(s) = \prod_{\pi \in \mathcal{K}} \left(1 - \frac{1}{N(\pi)^s} \right)^{-1} = \sum_{\alpha \in \mathcal{K}} \frac{1}{N(\alpha)^s},\tag{68}$$

and its inverse is defined by

$$\frac{1}{\zeta_{\mathcal{K}}(s)} = \prod_{\pi \in \mathcal{K}} \left(1 - \frac{1}{N(\pi)^s} \right) = \sum_{\alpha \in \mathcal{K}} \frac{\mu_{\mathcal{K}}(\alpha)}{N(\alpha)^s},\tag{69}$$

of a complex number $s \in \mathbb{C}$. Here, the Mobius function is defined by

$$\mu_{\mathcal{K}}(\alpha) = \begin{cases} \pm 1 & \text{if } \alpha \text{ is a squarefree algebraic integer,} \\ 0 & \text{if } \alpha \text{ is not a squarefree algebraic integer.} \end{cases}$$
(70)

Furthermore, the vonMangoldt function is defined by

$$\Lambda_{\mathcal{K}}(n) = \begin{cases} \log(N(\pi)) & \text{if } N(\pi) = (a^2 + b^2)^k \text{ is a prime power,} \\ 0 & \text{if } N(\pi) \neq (a^2 + b^2)^k \text{ is not a prime power,} \end{cases}$$
(71)

where the exponent $k \ge 1$.

10 Fixed Divisors Of Polynomials

Definition 10.1. The fixed divisor $\operatorname{div}(f) = \operatorname{gcd}(f(\mathbb{Z}))$ of a polynomial $f(x) \in \mathbb{Z}[x]$ over the integers is the greatest common divisor of its image $f(\mathbb{Z}) = \{f(n) : n \in \mathbb{Z}\}$.

The fixed divisor $\operatorname{div}(f) = 1$ if and only if the congruence equation $f(n) \equiv 0 \mod p$ has $\nu_f(p) < p$ solutions for every prime $p < \operatorname{deg}(f)$, see [18, p. 395]. An irreducible polynomial can represent infinitely many primes if and only if it has a fixed divisor $\operatorname{div}(f) = 1$.

The function $\nu_f(q) = \#\{n : f(n) \equiv 0 \mod q\}$ is multiplicative, and can be written in the form

$$\nu_f(q) = \prod_{p^b \mid\mid q} \nu_f(p^b),\tag{72}$$

where $p^b || q$ is the maximal prime power divisor of q, and $\nu_f(p^b) \leq \deg(f)$, see [33, p. 82].

Example 10.1. A few well known polynomials are listed here.

- 1. The polynomials $g_1(x) = x^2 + 1$ and $g_2(x) = x^2 + 3$ are irreducible over the integers and have the fixed divisors $\operatorname{div}(g_1) = 1$, and $\operatorname{div}(g_2) = 1$ respectively. Thus, these polynomials can represent infinitely many primes.
- 2. The polynomials $g_3(x) = x(x+1) + 2$ and $g_4(x) = x(x+1)(x+2) + 3$ are irreducible over the integers. But, have the fixed divisors $div(g_3) = 2$, and $div(g_4) = 3$ respectively. Thus, these polynomials cannot represent infinitely many primes.
- 3. The polynomial $g_5(x) = x^p x + p$, with $p \ge 2$ prime, is irreducible over the integers. But, has the fixed divisor $\operatorname{div}(g_5) = p$. Thus, this polynomial cannot represent infinitely many primes.

11 Admissible Quadratic Polynomials

Basically, this technique employed in the proof of Theorem 1.1 is restricted to square integers and associated collection of irreducible polynomials $f_1(n) \in \mathbb{Z}[x]$ of fixed divisor div $f_1 = 1$, see Definition 10.1. The basic technique can be extended to other related collections of irreducible polynomials such as $f_2(x)$, see below, provided that it can be mapped into $f_1(x)$ and its important properties, (irreducibility, and fixed divisors), remain invariant. However, it is not applicable to cubic polynomials as $f_3(x)$.

1.
$$f_1(x) = ax^2 + c$$
,

- 2. $f_2(x) = ax^2 + bx + c$,
- 3. $f_3(x) = ax^3 + bx^2 + cx + d$.

12 Primes In Quadratic Arithmetic Progressions

The cardinality of the subset of linear primes $\{p = an + b : n \ge 1\}$ defined by a linear polynomial $f(x) = ax + b \in \mathbb{Z}[x]$ over the integers, gcd(a, b) = 1, was settled by Dirichlet as an infinite subset of prime numbers. The cardinality of the subset of quadratic primes $\{p = an^2 + bn + c : n \ge 1\}$ defined by certain irreducible quadratic polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ over the integers, gcd(a, b, c) = 1, is believed to be an infinite subset of prime numbers.

The Quadratic Primes Conjecture claims that certain Diophantine equations $y = ax^2 + bx + c$ have infinitely many prime solutions y = p, as the integer $x = n \in \mathbb{Z}$ varies. More generally, the Bouniakowsky conjecture, [35, p. 386], claims that for any irreducible $f(x) \in \mathbb{Z}[x]$ over the integers of fixed divisor div(f) = 1, and degree deg $(f) \ge 1$, the Diophantine equation y = f(x) has infinitely many prime solutions y = p as the integers $x = n \in \mathbb{Z}$ varies.

The expected distribution of the quadratic primes has its systematic development in the middle of the last century, and it is described in the well-known series of lectures Partitio Numerorum of Hardy and Littlewood. This claim for quadratic polynomials will be referred to as the quadratic primes conjecture. This conjecture and other generalizations are discussed in [35, p. 406], [19, p. 25], [30, p. 342], and similar sources. Related works are given in [10], [27], [20], and [21].

Conjecture 12.1. ([23]) Let $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ be an irreducible polynomial over the integers, and assume that

- (i) gcd(a, b, c) = 1,
- (ii) $b^2 4ac$ is not a square in \mathbb{Z} ,
- (iii) gcd(a+b,c) = 2k+1 is odd.

Then, for a sufficiently large real number $x \ge 1$, the number of primes of the form $p = an^2 + bn + c \le x$ has the asymptotic counting function

$$\pi_f(x) = C_f \frac{x^{1/2}}{\log x} + O\left(\frac{x^{1/2}}{\log^2 x}\right),\tag{73}$$

where the constant is given by

$$C_f = \frac{\epsilon}{\sqrt{a}} \prod_{p \mid \gcd(a,b)} \left(1 + \frac{1}{p-1} \right) \prod_{2$$

Here, the quadratic symbol is defined by

$$\chi(u) = \left(\frac{u}{p}\right). \tag{75}$$

Conditions (i) and (ii) imply that f(x) is irreducible over the integers \mathbb{Z} , and condition (iii) implies that the fixed divisor $\operatorname{div}(f) = \operatorname{gcd}(f(\mathbb{Z})) = 1$, see Definition 10.1.

The distribution of the roots of quadratic polynomials modulo $m \ge 2$ are studied in [18], and [39].

13 Optimization Problem

The problem of finding quadratic polynomials with high densities of prime numbers in very short interval [0, x] has intrigued people for a long time. The Euler polynomial $f(x) = x^2 + x + 41$ has the highest prime density known for small x < 41. The optimization problem, which seeks the best constant C_f in equation (74) for some in $f(x) \in \mathbb{Z}[x]$, is studied in [21], [20] and by many other authors. A survey of record setting polynomials and some of the early results and experiments appears in [35, p. 196].

14 Least Primes in Quadratic Arithmetic Progressions

Given an admissible fixed triple (a, b, c), the least prime $p(a, b, c) \ge 2$ in a quadratic arithmetic progression $\{p = an^2 + bn + c : n \ge 1, \gcd(a, b, c) = 1\}$, satisfies

$$an^2 + bn + c \le p \tag{76}$$

for n = 0 or n > 0. If c > 1 is not a prime, then the determination of the least prime p(a, b, c) > c in a quadratic arithmetic progression is an open problem.

It is very rare, but the first prime value f(n) of an irreducible polynomial $f(x) \in \mathbb{Z}[x]$ of fixed divisor div f = 1 can be arbitrarily large, the theoretical details are proved in [28]. For example, the first prime value of the polynomial $x^{12} + 488669$ is extremely large, probably infeasable for computer calculations, but it exists.

15 Distribution of the Fractional Parts

Let the fractional part function be defined by $\{x\} = x - [x]$, where [x] is the largest integer function. The statistical properties of the fractional parts of various sequences of real numbers are of interest in the mathematical sciences. In the case of the sequence of quadratic primes $\{p = n^2 + 1, n \ge 1\}$, and the fractional part of these primes, the existence problems are equivalent. The best known result claims that

$$\{\sqrt{p}\} < \frac{c}{p^{1/4+\varepsilon}},\tag{77}$$

with c > 0 constant, and $\varepsilon > 0$ arbitrarily small. This is proved in [6], and [22]. A continuation of this result is the following.

Corollary 15.1. There are infinitely many primes $p \ge 2$ such that the fractional parts satisfy the inequality

$$\{\sqrt{p}\} < \frac{c}{\sqrt{p}},\tag{78}$$

with c > 1/2 constant.

Proof. Take $p = n^2 + 1, n \ge 1$. By definition, this is precisely

$$\{\sqrt{p}\} = \sqrt{p} - [\sqrt{p}] \\ = \sqrt{n^2 + 1} - \left[\sqrt{n^2 + 1}\right] \\ = \sqrt{n^2(1 + 1/n^2)} - n$$
(79)
$$= n\left(1 + \frac{1}{2n^2} + O\left(\frac{1}{n^4}\right)\right) - n \\ = \frac{1}{2n} + O\left(\frac{1}{n^3}\right) \\ < \frac{c}{\sqrt{p}},$$

where c > 1/2 is a constant. By Theorem 1.1, it follows that this inequality occurs infinitely often.

16 Some Partial And Infinite Products

The results for these related products are optional. The estimate of the partial sum $\sum_{n \leq x} \lambda(n)/\varphi(n)$ in Lemma 16.1 is geared for the short interval $[1, x_0]$, and the estimate in Lemma 7.1 is geared for the shifted short interval $[x_0, x_1]$, where $1 < x_0 < x_1 \leq x$.

Lemma 16.1. For a large number $x \ge 1$,

(i)
$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} \asymp \prod_{p \le x} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{(p^2 - 1)(p - 1)}\right),$$

(ii) $\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} \asymp c_0 \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right)\right),$

where $c_0 > 0$ is a constant.

Proof. (i) Use the multiplicative properties of the two arithmetic functions $\lambda(p^k) = (-1)^k$ and $\varphi(p^k) = p^{k-1}(p-1)$ respectively, to convert the sum to a product.

$$\begin{split} \sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} &\asymp \prod_{p \le x} \left(1 + \sum_{p^k \le x} \frac{\lambda(p^k)}{\varphi(p^k)} \right) \\ &= \prod_{p \le x} \left(1 - \frac{1}{p-1} + \frac{1}{p(p-1)} - \frac{1}{p^2(p-1)} + \cdots \right) \\ &= \prod_{p \le x} \left(1 - \frac{1}{p} - \frac{1}{p^2(p-1)} + \frac{1}{p^3(p-1)} - \cdots \right) \\ &= \prod_{p \le x} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p^2(p-1)} \frac{p}{p-1} + \frac{1}{p^3(p-1)} \frac{p}{p-1} - \cdots \right) \\ &= \prod_{p \le x} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{p(p-1)^2} + \frac{1}{p^2(p-1)^2} - \cdots \right) \\ &= \prod_{p \le x} \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{(p^2-1)(p-1)} \right). \end{split}$$
(80)

(ii) Use asymptotic estimates for the products to obtain

$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} \approx \prod_{p \le x} \left(1 - \frac{1}{p} \right) \prod_{p \le x} \left(1 - \frac{1}{(p^2 - 1)(p - 1)} \right)$$

$$= \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right) \right) \prod_{p \le x} \left(1 - \frac{1}{(p^2 - 1)(p - 1)} \right)$$

$$= \frac{e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right) \right) \left(c_0 + O\left(\frac{1}{x}\right) \right)$$

$$= \frac{c_0 e^{-\gamma}}{\log x} \left(1 + O\left(\frac{1}{\log x}\right) \right) ,$$
(81)

where the inner product converges to a constant. A small scale calculation, using 10^6 primes, the constant has the approximate value

$$c_{0} = \prod_{p \ge 2} \left(1 - \frac{1}{p(p-1)^{2}} + \frac{1}{p^{2}(p-1)^{2}} - \cdots \right)$$

$$= \prod_{p \ge 2} \left(1 - \frac{1}{(p^{2}-1)(p-1)} \right)$$

$$= 0.615132657318171877819725438740602....$$
(82)

Lemma 16.2. The following product has an explicit evaluation and numerical value:

(i)
$$\sum_{n\geq 1} \frac{\lambda(n)}{n\varphi(n)} = \prod_{p\geq 2} \left(1 - \frac{p}{(p^2+1)(p-1)}\right),$$

(ii) $\prod_{p\geq 2} \left(1 - \frac{p}{(p^2+1)(p-1)}\right) = 0.458937522009...$

Proof. (i) Use the multiplicative properties of the two arithmetic functions $\lambda(p^k) = (-1)^k$ and $\varphi(p^k) = p^{k-1}(p-1)$ respectively, to convert the sum to a product.

$$\sum_{n \le x} \frac{\lambda(n)}{n\varphi(n)} = \prod_{p \ge 2} \left(1 + \sum_{p^k \le x} \frac{\lambda(p^k)}{p^k \varphi(p^k)} \right)$$

$$= \prod_{p \ge 2} \left(1 - \frac{1}{p(p-1)} + \frac{1}{p^3(p-1)} - \frac{1}{p^5(p-1)} + \cdots \right)$$

$$= \prod_{p \ge 2} \left(1 - \frac{p}{(p^2+1)(p-1)} \right).$$
(83)

(ii) A limited numerical experiment gives

$$\prod_{p\geq 2} \left(1 - \frac{p}{(p^2+1)(p-1)} \right) = 0.458937522009147570415895603071402\dots,$$
(84)

for $p \le 10^6 \log 10^6$.

Lemma 16.3. The following product has an explicit evaluation and numerical value:

(i)
$$\sum_{n \ge 1} \frac{\mu^2(n)\lambda(n)}{n\varphi(n)} = \prod_{p \ge 2} \left(1 - \frac{1}{p(p-1)}\right),$$

(ii) $\prod_{p \ge 2} \left(1 - \frac{1}{p(p-1)}\right) = .373955832771....$

17 Problems

Exercise 17.1. Verify the decomposition

$$\sum_{d|n} \lambda(d) = \sum_{\substack{d|n \\ d < \sqrt{n}}} \lambda(d) + \sum_{\substack{d|n \\ d \le \sqrt{n}}} \lambda(n/d)$$

Exercise 17.2. Verify the identity

$$\sum_{d^2|n} \mu(n/d^2) = \lambda(n)$$

Exercise 17.3. Verify the decomposition

$$-\Lambda(n) = \sum_{d|n} \mu(d) \log d = \sum_{\substack{d|n \\ d < \sqrt{n}}} \mu(d) \log d + \sum_{\substack{d|n \\ d \le \sqrt{n}}} \mu(n/d) \log(n/d)$$

Exercise 17.4. Evaluate the finite sum, [2, p. 73],

$$\sum_{n \le x} \lambda(n) \left[\frac{x}{n} \right] = \left[x^{1/2} \right].$$

Exercise 17.5. Evaluate the finite sum,

$$\sum_{n \le x} \left(\sum_{d|n} \lambda(d) \right)^2 = \sum_{d,e \le x} \lambda(d)\lambda(d) \left[\frac{x}{de} \right] = \left[x^{1/2} \right].$$

Exercise 17.6. Evaluate the finite sum, [29, p. 43],

$$\sum_{n \le x} \frac{\mu^2(n)}{\varphi(n)} = a_0 + \log x + O(x^{-1/2} \log x).$$

Exercise 17.7. Let $p \ge 2$ be a prime, and let $a \approx \sqrt{p}$ be an integer. Show that

$$\left(\frac{p-1}{2}\right)! \equiv \pm a \bmod p$$

if and only if $p = a^2 + 1$. For example, $p = 6^2 + 1$ implies that $18! \equiv \pm 6 \mod 37$.

Exercise 17.8. Determine the characteristic function of cubic integers $n \in \mathbb{N}$ in terms of number theoretical functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, similar to the characteristic function of square integers in (4), see the reference arXiv:1701.02286 for some details. For instance,

$$\sum_{d|n} f(d) = \begin{cases} 1 & \text{if } n = m^3, \\ 0 & \text{if } n \neq m^3. \end{cases}$$

Exercise 17.9. Determine the characteristic function of quartic integers $n \in \mathbb{N}$ in terms of number theoretical functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, similar to the characteristic function of square integers in (4). For instance,

$$\sum_{d|n} f(d) = \begin{cases} 1 & \text{if } n = m^4, \\ 0 & \text{if } n \neq m^4. \end{cases}$$

Exercise 17.10. Determine the characteristic function of quintic integers $n \in \mathbb{N}$ in terms of number theoretical functions $f : \mathbb{N} \longrightarrow \mathbb{N}$, similar to the characteristic function of square integers in (4). For instance,

$$\sum_{d|n} f(d) = \begin{cases} 1 & \text{if } n = m^5, \\ 0 & \text{if } n \neq m^5. \end{cases}$$

Exercise 17.11. Let $SL_2[\mathbb{Z}]$ be the subset of invertible 2×2 matrices. Show that an irreducible polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ remains irreducible under the map $F(x) = f(\gamma x)$, where $\gamma \in SL_2[\mathbb{Z}]$.

Exercise 17.12. Let $SL_2[\mathbb{Z}]$ be the subset of invertible 2×2 matrices, and let $F(x) = f(\gamma x)$, where $\gamma \in SL_2[\mathbb{Z}]$. Determine whether or not the fixed divisor div f of an irreducible polynomial $f(x) = ax^2 + bx + c \in \mathbb{Z}[x]$ remains invariant under the group action of $SL_2[\mathbb{Z}]$.

Exercise 17.13. Let $\mathcal{A} = \{2, 5, 17, 37, 101, 197, 401, \ldots\}$ be the subset of primes. Show that the prime harmonic series

$$\sum_{p \in \mathcal{A}} \frac{1}{p}$$

converges.

Exercise 17.14. Show that the subset of primes $\mathcal{A} = \{2, 5, 17, 37, 101, 197, 401, \ldots\}$ contains at most finitely many prime powers $p^k = m^2 + 1$ with $k \ge 2, m \ge 1$. Consult the literature on the Catalan conjecture for more details.

Exercise 17.15. Prove that the subset of primes $\mathcal{E} = \{p = n^2 + n + 41 : n \ge 0\}$ contains infinitely many primes. Hint: Modify the proof of Theorem 1.1, and consult the literature for some theory and numerical data, for example, the paper arxiv.1207.7291.

Exercise 17.16. Given a large number $x \ge 1$, show that the partial sum and partial product are proportional:

$$\sum_{n \le x} \frac{\mu^2(n)}{n} \asymp \prod_{p \le x} \left(1 + \frac{1}{p} \right).$$

Exercise 17.17. Given a large number $x \ge 1$, employ Lemma 16.1 to show that the partial sum and partial product are proportional:

$$\sum_{n \le x} \frac{\lambda(n)}{\varphi(n)} \asymp \prod_{p \le x} \left(1 + \sum_{p^k \le x} \frac{\lambda(p^k)}{p^k \varphi(p^k)} \right).$$

Exercise 17.18. Explain why the infinite series and the infinite product have different rate of convergence. For example, these pairs:

$$\sum_{n\geq 1} \frac{\mu(n)}{n} = \prod_{p\geq 2} \left(1 - \frac{1}{p}\right), \qquad \sum_{n\geq 1} \frac{\lambda(n)}{\varphi(n)} = \prod_{p\geq 2} \left(1 + \sum_{p^k\leq x} \frac{\lambda(p^k)}{p^k \varphi(p^k)}\right), etc$$

Exercise 17.19. Develop the analytic number theory of the least prime $p(a, b, c) \ge c > 1$ in a quadratic arithmetic progression $\{p = an^2 + bn + c : n \ge 1, gcd(a, b, c) = 1\}$.

Exercise 17.20. Develop the analytic number theory of the least primitive root $r(a, b, c) \ge 2$ in a quadratic arithmetic progression $\{p = an^2 + bn + c : n \ge 1, gcd(a, b, c) = 1\}$.

Exercise 17.21. Show that quadratic arithmetic progressions $\{p = an^2 + bn + c : n \ge 1, gcd(a, b, c) = 1\}$ contain finite number of pseudo primes. Reference: Study the paper Arxiv: 1305.3580.

Exercise 17.22. Show that the sequence of quadratic primes $\{p = an^2 + bn + c : n \ge 1, gcd(a, b, c) = 1\}$ contain finite number of Wilson primes.

Exercise 17.23. Are the fractional parts uniformly distributed? How is the cumulative density function $F(x) = \#\{p \le x : \{\sqrt{p}\} \le 1/\sqrt{p}\}/x$ computed?

Exercise 17.24. Fix a pair $0 \le a < b \le 1$. Estimate or compute order of magnitute of the discrepancy of the sequence of fractional parts. The discrepancy is defined by

$$D_N = \sup_{0 \le a < b \le 1} \left| \frac{\#\{p \le N : \{\sqrt{p}\} \in [a, b]\}}{N} - (b - a) \right|$$

for any large number $N \geq 1$.

Exercise 17.25. Fix a pair of integers $2k, k \ge 1$ and $u \ne \pm 1, b^m, m \ge 1$. Determine whether there are finitely many or infinitely many prime pairs p and p + 2k with a common primitive root u parts.

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