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Graph operads: general construction and natural extensions of canonical operads

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Abstract

We propose a new way of defining and studying operads on multigraphs and similar objects. For this purpose, we use the combinatorial species setting. We study in particular two operads obtained with our method. The former is a direct generalization of the Kontsevich-Willwacher operad. This operad can be seen as a canonical operad on multigraphs, and has many interesting suboperads. The latter operad is a natural extension of the pre-Lie operad in a sense developed here and it is related to the multigraph operad. We also present various results on some of the finitely generated suboperads of the multigraph operad and establish links between them and the commutative operad and the commutative magmatic operad.

Introduction

Operads are mathematical structures which have been intensively studied in the context of topology, algebra [11] but also of combinatorics [3] —see for example [6, 13] for general references on symmetric and non-symmetric operads, set-operads through species, *etc.* In the last decades, several interesting operads on trees have been defined. Amongst these tree operads, maybe the most studied are the pre-Lie operad **PLie** [4] and the nonassociative permutative operad **NAP** [10].

However, it seems to us that a natural question to ask is what kind of operads can be defined on graphs and what are their properties? The need for defining appropriate graph operads comes from combinatorics, where graphs are, just like trees, natural objects to study. It comes also from physics, where it was recently proposed to use graph operads in order to encode the combinatorics of the renormalization of Feynman graphs in quantum field theory [9].

Other graph operads have been defined for example in [5, 8, 12, 13, 15]. In this paper, we go further in this direction and we define, using the combinatorial species setting [1], new graph operads. Moreover, we investigate several properties of these operads: we describe an explicit link with the pre-Lie tree operad mentioned above, and we study interesting (finitely generated) suboperads.

This paper is organized as follows. In Section 1 we give the definitions of species, operads and graphs as well as classical results on these objects. Moreover, we introduce here different notations used throughout the this paper. In Section 2 we propose new ways of constructing species and operads. We use these new constructions in Section 3 to define and study the main operads of interest of this paper. Section 4 is devoted to the study of some particularly interesting finitely generated suboperads.

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1 Definitions and reminders

Most definitions, results and proofs of this section can be found with more details in [13]. We refer the reader to [1] for the theory of species and to [11] for the theory of operads.

In all the following, $\mathbb{K}$ is a field of characteristic zero. For any positive integer n, $[n]$ stands for the set $\{1, \ldots, n\}$. For V a vector space and A a non empty finite set, we denote by $V \times A$ the vector space $\bigoplus_{a \in A} V$. We denote by (v, a) elements of $V \times A$ we thus have $(k_1 v_1 + k_2 v_2, a) = k_1(v_1, a) + k_2(v_2, a)$.

1.1 Species

Definition 1. A *(positive) set species* S consists of the following data.

- For each finite set V, a set $S[V]$, such that $S[\emptyset] = \emptyset$.
- For each bijection of finite sets $\sigma : V \to V'$, a map $S[\sigma] : S[V] \to S[V']$. These maps should be such that $S[\sigma \circ \tau] = S[\sigma] \circ S[\tau]$ and $S[\mathrm{id}] = \mathrm{id}$.

A *morphism of set species* $f : R \to S$ is a collection of map $f_V : R[V] \to S[V]$ such that for each bijection $\sigma : V \to V'$, $f_{V'} \circ R[\sigma] = S[\sigma] \circ f_V$.

A set species S is *connected* if $|S[\{v\}]| = 1$ for any singleton $\{v\}$.

In the previous definitions switching from sets to vector spaces, from maps to linear maps and cardinality with dimension, we obtain the definition of *(positive) linear species, morphism of linear species* and *connected linear species.*

We denote by $\mathcal{L}$ the functor from set species to linear species defined by $L(S)[V] = \mathbb{K}S[V]$, where $\mathbb{K}S[V]$ is the free $\mathbb{K}$-vector space on $S[V]$, and $L(f)_V$ the linear extension of f. We also denote by $\mathbb{K}S$ for $\mathcal{L}(S)$. For any set species S, and $w = \sum_{x \in S[V]} a_x x \in \mathbb{K}S[V]$ we call *support of* w the set of $x \in S[V]$ such that $a_x \neq 0$.

Example 2.

- We denote by X the set species defined by $X[V] = \{v\}$ if $V = \{v\}$ and $X[V] = \emptyset$ else.
- For V a non empty finite set, let $\mathrm{Pol}[V]$ be the set (and not the module) of polynomials on $\mathbb{Z}$ with variables in V. Then Pol is the set species of polynomials on $\mathbb{Z}$. When considering $\mathbb{K}\mathrm{Pol}$ one has to take into consideration the fact that we need to differentiate the plus of polynomials and the addition of vectors. We will thus denote by $\oplus$ the former and keep $+$ for the latter and we will denote by $0_V \in \mathrm{Pol}[V]$ the polynomial constant to 0 and keep the notation 0 for the null vector. For example, $ab \oplus c$ is an element of $\mathrm{Pol}[\{a, b, c\}]$, but $a \oplus b + c$ is a vector in $\mathbb{K}\mathrm{Pol}[\{a, b, c\}]$ with support $\{a \oplus b, c\}$.
- For any linear species S, we denote by $S^\vee$ the linear species defined by $S^\vee[V] = S[V]^*$ and $S^\vee[\sigma]x = \mathrm{sign}(\sigma) x \circ S[\sigma^{-1}]$. If S is a set species such that $S[V] = \{b_1, \ldots, b_n\}$, we denote by $b_i^\vee$, for $1 \leq i \leq n$, the element of $\mathbb{K}S^\vee[V]$ defined by $b_i^\vee(b_j) = 1$ if $i = j$ and $b_i^\vee(b_j) = 0$ else.

In all the following V denotes a non empty finite set.

Definition 3. Let R and S be two species. We can then construct new set species which are defined as follows:

$$\textit{Sum} \quad (R + S)[V] = R[V] \oplus S[V], \qquad \textit{Product} \quad R \cdot S[V] = \bigoplus_{V_1 \sqcup V_2 = V} R[V_1] \otimes S[V_2],$$

$$\textit{Hadamard product} \quad (R \times S)[V] = R[V] \otimes S[V], \qquad \textit{Derivative} \quad R'[V] = R[V + \{*\}] \text{ where } * \notin V,$$

$$\textit{n-th derivative} \quad R^{(n)} = R[V + \{*_1, \ldots, *_n\}] \text{ where } *_1, \ldots, *_n \notin V, \qquad \textit{Pointing} \quad R^\bullet[V] = R[V] \times V,$$

Assembly $\quad E(R)[V] = \bigoplus_{\cong} \bigotimes_{W \in V/\cong} R[W]$ where $\cong$ run over the set of equivalence relations on V.

We have the same definitions on set species by replacing sums by direct unions and tensor products by Cartesian products.

Note that these definitions are compatible with $\mathcal{L}$ i. e $\mathcal{L}(R+S) = \mathcal{L}(R) + \mathcal{L}(S)$, $\mathcal{L}(R \cdot S) = \mathcal{L}(R) \cdot \mathcal{L}(S)$ etc.

1.2 Operads

Definition 4. A *(symmetric) set* (resp *linear*) *operad* is a set (resp linear) species $\mathcal{O}$ together with a *unity* $e : $ (resp $\mathbb{K}$)$X \to \mathcal{O}$ and a set (resp linear) species morphism $\circ_* : \mathcal{O}' \cdot \mathcal{O} \to \mathcal{O}$ called *partial composition* such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{O}'' \cdot \mathcal{O}^2 & \xrightarrow{\circ_{*_1}} & \mathcal{O}' \cdot \mathcal{O} \\ \downarrow{\scriptstyle \circ_{*_2} \circ \mathrm{id} \cdot \tau} & & \downarrow{\scriptstyle \circ_{*_2}} \\ \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*_1}} & \mathcal{O} \end{array} \qquad \begin{array}{ccc} \mathcal{O}' \cdot \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*_1} \cdot \mathrm{id}} & \mathcal{O}' \cdot \mathcal{O} \\ \downarrow{\scriptstyle \mathrm{id} \cdot \circ_{*_2}} & & \downarrow{\scriptstyle \circ_{*_2}} \\ \mathcal{O}' \cdot \mathcal{O} & \xrightarrow{\circ_{*_1}} & \mathcal{O} \end{array}$$

$$\begin{array}{ccccc} \mathcal{O}' \cdot \mathbb{K}X & \xrightarrow{\mathcal{O}' \cdot e} & \mathcal{O}' \cdot \mathcal{O} & \xleftarrow{e' \cdot \mathcal{O}} & \mathbb{K}X' \cdot \mathcal{O} \\ & {\scriptstyle p} \searrow & \downarrow{\scriptstyle \circ_*} & \swarrow {\scriptstyle \cong} & \\ & & \mathcal{O} & & \end{array}$$

where $\tau_V : x \otimes y \in \mathcal{O}^2[V] \mapsto y \otimes x \in \mathcal{O}^2[V]$ and $p_V : x \otimes v \mapsto \mathcal{O}[* \mapsto v](x)$ with $* \mapsto v$ the bijection that sends $*$ on v and is the identity on $V \setminus \{v\}$.

An *operad morphism* is a species morphism compatible with unities and partial compositions.

Note also that if $(S, e, \circ_*)$ is a set operad, then extending e and $\circ_*$ linearly turns $(\mathbb{K}S, e, \circ_*)$ into a linear operad. In all the following, e will often be trivial and we will not mention it.

From now on we use species and operad for linear species and linear operad and only specify when we work with their set counterparts.

Example 5.

- The *singleton set species* E defined by $E[V] = \{V\}$ naturally has a set operad structure given by $\{V_1 + \{*\}\} \circ_* \{V_2\} = \{V_1 + V_2\}$.
- The *identity set species* Id given by $Id[V] = V$ has a set operad structure given by $v \circ_* w = v|_{* \leftarrow w}$ which is equal to v if $v \neq *$ and equal to w else.
- Let us recall the following operad structure on rooted trees: the units are the one vertex trees and for a rooted tree t_1 with vertex set $V_1 + \{*\}$ and a rooted tree t_2 with vertex set V_2 the partial composition $t_1 \circ_* t_2$ is the sum over all tree obtained as follows.

  1. Consider the forest obtained by removing $*$ from t_1 and take the union with t_2.
  2. Add an edge between the parent of $*$ in t_1 and the root of t_2.
  3. For each child of $*$ in t_1, add an edge between this vertex and any vertex of t_2.

  This operad is called the PreLie operad [4] and we will denote it by **PLie**.

- The set species of polynomials Pol has a natural partial composition given by the composition of polynomials: for $V_1 = \{v_1, \dots, v_k\}$ and $V_2 = \{v'_1, \dots, v'_l\}$ disjoint sets and $p_1(v_1, \dots, v_k, *) \in \mathrm{Pol}'[V_1]$ and $p_2(v'_1, \dots, v'_l) \in \mathrm{Pol}[V_2]$ define

$$(p_1 \circ_* p_2)(v_1, \dots, v_k, v'_1, \dots, v'_l) = p_1|_{* \leftarrow p_2} = p_1(v_1, \dots, v_k, p_2(v'_1, \dots, v'_l)) \in \mathrm{Pol}[V_1 + V_2]. \quad (1)$$

One can directly check that this partial composition satisfies the commutative diagrams of Definition 4. This turns Pol into a set operad where the units are the singleton polynomials $v \in \mathrm{Pol}[\{v\}]$. As mentioned previously, the linear extension of $\circ_*$ then turns $\mathbb{K}\mathrm{Pol}$ into a linear operad.

Both the set operads E and Id can be seen as set sub-operads of Pol respectively by the monomorphisms $\{V\} \mapsto \oplus_{v\in V} v$ and $v \mapsto v$. The operad $\mathbb{K}E$ can also be seen as a sub-operad of $\mathbb{K}\mathrm{Pol}$ by the monomorphism $\{V\} \mapsto \sum_{v\in V} v$ (which is not the linear extension of the previous monomorphism).

An *ideal* of an operad $\mathcal{O}$ is a subspecies S such that the image of the products $\mathcal{O}' \cdot S$ and $S' \cdot \mathcal{O}$ by the partial composition maps are in S. The *quotient species* $\mathcal{O}/S$ defined by $(\mathcal{O}/S)[V] = \mathcal{O}[V]/S[V]$ is then an operad with the natural partial composition and unit.

We now need to recall the notion of free operad [13]. For S a set species define the *free set operad* $\mathbf{Free}_S$ over S by $\mathbf{Free}_S[V]$ being the set of trees on V enriched with elements in S. Such a tree $\mathcal{T} \in \mathbf{Free}_S[V]$ is defined as follows.

- The leaves of $\mathcal{T}$ are the elements of V.
- Each internal vertex u of $\mathcal{T}$ is labelled with the set B_u of leaves that are descendants of u in $\mathcal{T}$.
- There is an element of $S[\pi_u]$ attached to each fiber (set of sons) of each internal vertex u.

The set π_u in the third item is defined as follows. To each leaf v we associate the set $B_v = \{v\}$. Then π_u is the set $\{B_w\,,\, w \in c(u)\}$ with $c(u)$ is the set of children of u.

The partial composition of $\mathbf{Free}_G$, which we denote by $\circ_*^\xi$ in order to not confuse it with an already existing operad structure on G, is the grafting of trees: for any disjoint sets V_1 and V_2 with $* \in V_1$, and $\mathcal{T}_1 \in \mathbf{Free}_G[V_1]$ and $\mathcal{T}_2 \in \mathbf{Free}_G[V_2]$, $\mathcal{T}_1 \circ^\xi * \mathcal{T}_2$ is the tree obtained by grafting $\mathcal{T}_2$ on the leaf $*$ of $\mathcal{T}_1$ and updating the labels accordingly, i. e for each vertex u of $\mathcal{T}_1$ with $*$ as descendant, update B_u to $B_u - \{*\} + V_2$.

In the linear case, for S a species, define the *free operad* $\mathbf{Free}_S$ over S by $\mathbf{Free}_S$ being the linear span of the set of trees on V enriched with elements in S. Such trees are defined in the same way as in the set species case and the partial composition $\circ^\xi *$ is also the grafting of trees. Remark that for S a set species we have that $\mathbb{K}\mathbf{Free}_S = \mathbf{Free}_{\mathbb{K}S}$.

For any $k \geq 0$, we denote by $\mathbf{Free}_S^{(k)}$ the subspecies of $\mathbf{Free}_S$ of trees with k exactly internal nodes.

If R is a subspecies of $\mathbf{Free}_S$, we denote by (R) the smallest ideal containing R and write that (R) is *generated by* R.

Definition 6. Let G be a species and R be a subspecies of $\mathbf{Free}_G$. Let $\mathrm{Ope}(G, R) = \mathbf{Free}_G/(R)$. The operad $\mathrm{Ope}(G, R)$ is *binary* if the species G of generators is concentrated in cardinality 2 (*i. e.*, for all $n \neq 2$, $G[[n]] = \{0\}$). This operad is *quadratic* if R is a subspecies of $\mathbf{Free}_G^{(2)}$.

Definition 7. Let $\mathcal{O} = \mathrm{Ope}(G, R)$ be a binary quadratic operad. Define the linear form $\langle -, - \rangle$ on $\mathbf{Free}_{G^\vee}^{(2)} \times \mathbf{Free}_G^{(2)}$ by

$$\langle f_1 \circ_* f_2, x_1 \circ_* x_2 \rangle = f_1(x_1) f_2(x_2), \tag{2}$$

The *Koszul dual* of $\mathcal{O}$ is then the operad $\mathcal{O}^! = \mathrm{Ope}(G^\vee, R^\perp)$ where $R^\perp$ is the orthogonal of R for $\langle -, - \rangle$.

When $\mathcal{O}$ is quadratic and its Koszul complex is acyclic, $\mathcal{O}$ is a Koszul operad [11]. In this case, the Hilbert series of $\mathcal{O}$ and of its Koszul dual are related by the identity

$$\mathcal{H}_{\mathcal{O}}(-\mathcal{H}_{\mathcal{O}^!}(-t)) = t. \tag{3}$$

1.3 Graphs and hypergraphs

In this subsection we present a formalism to define graphs and hypergraphs and their “multi” variants.

A *multiset* m over V is a set of couples $\{(v, m(v)) \mid v \in V\}$ in $V \times \mathbb{N}^*$. We denote by $D(m) = V$ the *domain* of m. We say that v is in m and denote by $v \in m$ if $v \in D(m)$. For any element v not in the domain of m, we have $m(v) = 0$.

We denote by $\mathcal{M}(V)$ the set of multisets with domain in $\mathcal{P}(V)$, $\mathcal{M}_k(V)$ the set of elements of $\mathcal{M}(V)$ of cardinality k (the cardinality of a multiset m over V being $\sum_{v \in V} m(v)$) and $\mathcal{M}(V)^*$ the set of multisets with domain in $\mathcal{P}(V)^* = \mathcal{P}(V) \setminus \{\emptyset\}$. We identify non empty sets with multisets constant equal to 1.

For m a multiset and V a set, we denote by $m \cap V = m \cap V \times \mathbb{N}^*$. If m' is another multiset, we call the union of m and m' the multiset $\{(v, m(v) + m'(v)) \mid v \in D(m) \cup D(m')\}$.

Definition 8. Let V be a set. A *multi-hypergraph* over V is a multiset with domain in $\mathcal{M}(V)^*$. In this context the elements of V are called *vertices*, the elements of a multi-hypergraph are called *edges* and the elements of an edge are called its *ends*. A vertex contained in the domain of no edge is called an *isolated vertex*. We denote by **MHG** the set species of multi-hypergraphs.

A *hypergraph* is a multi-hypergraph whose edges are sets. A *multigraph* is a multi-hypergraph whose edges have cardinality 2. A *graph* is a multi-hypergraph which is a hypergraph and a multigraph at the same time as well as a set. Denote by **HG**, **MG** and **G** the set species corresponding to these structures.

We also denote by **F** the species of *forests*, which is the subspecies of **G** such that for every $f \in \mathbf{F}[V]$ there are no sequences $e_1, \ldots, e_k$ of distinct edges such that $e_i \cap e_{i+1} \neq \emptyset$ for $1 \leq i < k$ and $e_k \cap e_1 \neq \emptyset$.

Finally, for a subspecies S of **MHG** we denote by S_c its sub-species of connected components that is to say elements such that for every pair of vertices v, v', there is a sequence of edge $e_1, \ldots, e_k$ such that $v \in e_1$, $v' \in e_k$ and $e_i \cup e_{i+1} \neq \emptyset$. We denote by $\mathbf{T} = \mathbf{F}_c$ the species of *trees*.

Note that for any sub-species S of **MHG** we have that $E(S_c) = S$.

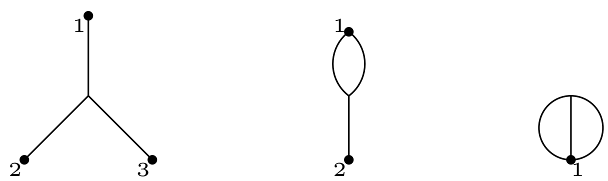


Figure 1: Three edges of cardinality 3.

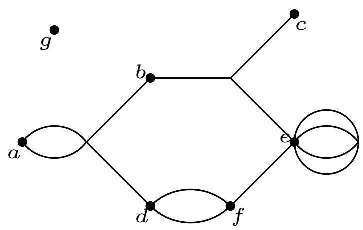


Figure 2: A multi-hypergraph over $\{a, b, c, d, e, f, g\}$.

Example 9. We represent the three edges $\{(1,1),(2,1),(3,1)\}$, $\{(1,2),(2,1)\}$ and $\{(1,3)\}$ in

Figure 1 and the multi-hypergraph

$$\{(\{(a,2),(b,1),(d,1)\},1),(\{(b,1),(c,1),(e,1)\},1),(\{e,4\},1),(\{(e,1),(f,1)\},1),(\{(d,1),(f,1)\},2)\} \tag{4}$$

over $\{a,b,c,d,e,f,g\}$ in Figure 2.

Remark 10. The set species **MHG** is isomorphic to the sub-species of Pol of polynomials with constant term equal to 0. This isomorphism is defined as follows. For V a finite set:

- the empty graph $\emptyset_V \in \mathbf{MHG}[V]$ is sent on the null polynomial 0_V,
- an edge e is sent on the monomial $\prod_{v\in e} v^{e(v)}$,
- an element $h \in \mathbf{MHG}[V]$ is sent on the polynomial $\bigoplus_{e\in h} e$.

We often identify **MHG** with this sub-species. This identification is very useful to do computations since it is easier to formally write operations on polynomials than on graphs. With this identification, hypergraphs can be seen as polynomials where each variable appears at most once in each monomial and multigraphs as homogeneous polynomials of degree 2.

Example 11. With this identification, the multi-hypergraph in Example 2 writes $a^2bd \oplus bce \oplus e^4 \oplus ef \oplus df \oplus df$.

2 Species and operad construction

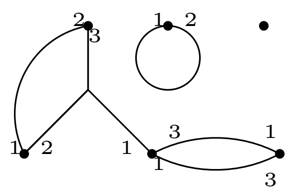
The goal of this section is to define new constructions of species and operads from already existing structures.

Definition 12. Let A be a set and S be a (resp set) species. An A-*augmentation* of S is a (resp set) species A-S such that $A\text{-}S[V] \cong S[V \times A]$ for every finite set V.

Example 13. Let A be a set. Instead of considering an A-augmented multi-hypergraph on V as a multi-hypergraph on $V \times A$, we consider them as multi-hypergraphs on V where the ends of the edges are labelled with elements of A. This is illustrated in Figure 5. In particular, the set species of oriented multigraphs $\mathbf{MG}_{or}$ is the set species $\{\_, >\} - \mathbf{MG}$ of multigraphs where each end of each edge is non a labelled end (i. e labeled by _) or labelled with an arrow head $>$.

Instead of seeing the variables of a polynomial in A-Pol$[V]$ as couples $(v,a) \in V \times A$, we consider them as elements of V indexed by elements of A: v_a.

Note that the identification presented in subsection 1.3 also holds between augmented multi-hypergraphs and augmented polynomials.



(5)

Proposition 14. Let S be a (resp set) species and $\mathcal{O}$ a set operad. Let φ be a collection of linear maps (resp maps) $\varphi_{V_1+\{*\},V_2} : (S[V_1+\{*\}]\otimes S[V_2])\times\mathcal{O}[V_2] \to S[V_1+V_2]$ (resp $S[V_1+\{*\}]\times S[V_2]$), where V_1 and V_2 are disjoint, such that:

- for $x \in S[V_1+\{*_1\}]$, $(y,f) \in S\times\mathcal{O}[V_2+\{*_2\}]$ and $(z,g) \in S\times\mathcal{O}[V_3]$:

$$\varphi_{V_1+\{*_1\},V_2+V_3}(x,\varphi_{V_2+\{*_2\},V_3}(y,z,g),f\circ_{*_2}g) = \varphi_{V_1+V_2+\{*_2\},V_3}(\varphi_{V_1+\{*_1\},V_2+\{*_2\}}(x,y,f),z,g) \tag{6}$$

- for $x \in S[V_1 + \{*_1, *_2\}]$, $(y, f) \in S \times \mathcal{O}[V_2]$ and $(z, g) \in S \times \mathcal{O}[V_3]$:

$$\varphi_{V_1+V_2+\{*_2\},V_3}(\varphi_{V_1+\{*_1,*2\},V_2}(x, y, f), z, g) = \varphi_{V_1+V_3+\{*_1\},V_2}(\varphi_{V_1+\{*_1,*2\},V_3}(x, z, g), y, f) \tag{7}$$

- there exists a map $e : X \to S$ such that for $(x, f) \in S \times \mathcal{O}[V]$ and $(y, g) \in S \times \mathcal{O}[V + \{*\}]$ we have $\varphi_{\{*\},V}(e(*), x, f) = x$ and $\varphi_{V+\{*\},\{v\}}(x, e(v), e_{\mathcal{O}}(v)) = S[\tau_{*,v}](x)$ where $e_{\mathcal{O}}$ is the unit of $\mathcal{O}$ and $\tau_{*,v}$ is the permutation which switches $*$ and v.

Then the partial composition $\circ_*^{\varphi}$ defined by

$$\begin{aligned} \circ_*^{\varphi} : S \times \mathcal{O}[V_1 + \{*\}] \otimes S \times \mathcal{O}[V_2] &\to S \times \mathcal{O}[V_1 + V_2] \\ (x, f) \otimes (y, g) &\mapsto (\varphi(x, y, g), f \circ_* g) \end{aligned} \tag{8}$$

makes $S \times \mathcal{O}$ an (resp set) operad with unit e. We call this operad the *semidirect product of S and $\mathcal{O}$ over φ* and we denote it by $S \ltimes_{\varphi} \mathcal{O}$.

Proof. This a rewriting of the axioms of Definition 4. □

When it is clear in the context we will not mention φ and just write semidirect product of S and $\mathcal{O}$ and denote by $S \ltimes \mathcal{O}$. The goal of this construction is to give an operad structure to S using the already known set operad structure on $\mathcal{O}$.

Example 15. Let C be a finite set and denote by C_{2+} the set species defined by $C_{2+}[V] = C$ if $|V| > 1$ and $C_{2+}[V] = \emptyset$ else. The species $\mathcal{C} = X + C_{2+}$ has a set operad structure with partial composition defined by, for $x \in \mathcal{C}'[V_1]$ and $y \in C[V_2]$: $x \circ_* y = x$ if $V_1 \neq \emptyset$ and $x \circ_* y = y$ else. Let $\mathcal{F}^C = X + \mathcal{F}^C_{2+}$ be the set species of maps with codomain C: $\mathcal{F}^C[V] = \{f : V \to C\}$ for $|V| > 1$. Then we have the semidirect product $\mathbb{K}\mathcal{F}^C \ltimes_{\varphi} \mathcal{C}$ given by, for $V_1 \neq \emptyset$, $|V_2| > 1$ and $f \in \mathcal{F}^C[V_1 + \{*\}]$ and $(g, x) \in \mathcal{F}^C \times \mathcal{C}[V_2]$: $\varphi(f, g, c) = 0$ if $f(*) \neq c$ and $\varphi(f, g, c)(v) = \begin{cases} f(v) & \text{if } v \in V_1 \\ g(v) & \text{if } v \in V_2 \end{cases}$ else. When $V_1 = \emptyset$ or $|V_2| = 1$ the partial composition is implied by the definition of the unit. We call this operad the *C-coloration* operad. Alone, one can see an element of $(f, c)\mathcal{F}^C \ltimes C[V]$ (with $|V| > 1$) as a corolla on V with its root colored by c and its leaves $v \in V$ colored by $f(v)$. The partial composition consists then in grafting two corollas if the root and the leaf on which it must be grafted share the same colors. However this operad is used more frequently in a Hadamard product with another operad as a way to color it.

Definition 16. Let A be a set and $\mathcal{O}$ be an (resp set) operad with unit e. The set species of *functions from A to $\mathcal{O}$* is defined by $\mathcal{F}^{\mathcal{O}}_A[V] = \{f : A \to \mathcal{O}[V]\}$. This set species has a set operad structure with the elements $f : A \to \{e(v)\}$ in $\mathcal{F}^{\mathcal{O}}_A[\{v\}]$ as units and partial composition defined by $f_1 \circ_* f_2(a) = f_1(a) \circ_* f_2(a)$.

Note that if A is a singleton then $\mathcal{F}^{\mathcal{O}}_A \cong \mathcal{O}$. Let A, B, C, D four sets such that A and B are disjoint and $f : A \to C$ and $g : B \to D$ two maps. We denote by $f \uplus g$ the map from $A \sqcup B$ to $C \cup D$ defined by $f \uplus g(a) = f(a)$ for $a \in A$ and $f \uplus g(b) = g(b)$ for $b \in B$.

Proposition 17. Let A and B be two disjoint sets and $\mathcal{O}_1$ and $\mathcal{O}_2$ be two operads. Then the set species $\mathcal{F}^{\mathcal{O}_1}_A \uplus \mathcal{F}^{\mathcal{O}_2}_B$ defined by $\mathcal{F}^{\mathcal{O}_1}_A \uplus \mathcal{F}^{\mathcal{O}_2}_B[V] = \{f \uplus g \mid f \in \mathcal{F}^{\mathcal{O}_1}_A, g \in \mathcal{F}^{\mathcal{O}_2}_B\}$ is a sub-operad of $\mathcal{F}^{\mathcal{O}_1+\mathcal{O}_2}_{A \sqcup B}$.

Proof. Since A and B are disjoint, the partial composition is well defined and stable on $\mathcal{F}^{\mathcal{O}_1}_A \uplus \mathcal{F}^{\mathcal{O}_2}_B$. □

3 Graph operads

In this section we use the construction of the previous section to define operad structures on $\mathbb{K}$**MHG** and its sub-species.

3.1 Graph insertion operads

Recall from Example 2 that we denote by $\oplus$ the addition of polynomials and 0_V the zero polynomials in order to distinguish them from the addition of vectors and the null vector. As announced in Remark 10, we identify the elements of **MHG** with polynomials with null constant term. We also identify A-augmented elements with polynomials with variables indexed by A.

We now consider that the addition and multiplication of polynomials are distributive on the addition of vectors.

Theorem 18. Let A be a set. Define the collection of maps $\varphi = \{\varphi_{V_1+\{*\},V_2} : (\mathbb{K}A\text{-}\mathbf{MHG}[V_1 + \{*\}] \otimes \mathbb{K}A\text{-}\mathbf{MHG}[V_2]) \times \mathcal{F}_A^{\mathbb{K}\mathbf{MHG}})[V_2] \to \mathbb{K}A\text{-}\mathbf{MHG}[V_1 + V_2]\}_{V_1\cap V_2=\emptyset}$ by

$$\varphi(h_1, h_2, f) = h_1|_{\{*_a \leftarrow f(a)_a\}} \oplus h_2, \tag{9}$$

where for a sum of polynomials $\sum P$, $(\sum P)_a = \sum P_a$ is the same sum of polynomials but with all the variables indexed by a.

We can then do the semidirect product of $\mathbb{K}A$-**MHG** and $\mathcal{F}_A^{\mathbb{K}\mathbf{MHG}}$ over φ.

We call any operad isomorphic to a sub-operad of A-$\mathbf{MHG} \ltimes \mathcal{F}_A^{\mathbf{MHG}}$ a *graph insertion* operad. The idea is to give a general construction of operads on (multi-)(hyper)graphs where the partial composition of two elements $g_1 \circ_* g_2$ is given by:

1. take the disjoint union of g_1 and g_2,
2. remove the vertex $*$ from g_1,
3. connect independently each loose ends of g_1 to g_2 in a certain way.

What we mean by independently is that the way of connecting one end does not depend on how we connect the other ends. Note that the "certain way" in which an end can be connected may include duplication of edges and augmentation of the number of vertices of edges. Examples are given after the proof of Proposition 18.

Proof. The linearity of φ is given by the fact that the addition and multiplication of polynomials are distributive on the addition of vectors. We need to verify that φ satisfies the three items of Proposition 14. The first two items are direct polynomials computations over polynomials:

$$\begin{aligned}\varphi_{V_1+\{*_1\},V_2+V_3}(h_1, \varphi_{V_2+\{*_2\},V_3}(h_2, h_3, g), f \circ_{*_2} g) &= h_1|_{\{*_{1a} \leftarrow f\circ_{*_2} g(a)_a\}} \oplus h_2|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3 \\ &= h_1|_{\{*_{1a} \leftarrow f(a)_a \circ_{*_2} g(a)_a\}} \oplus h_2|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3 \\ &= h_1|_{\{*_{1a} \leftarrow f(a)_a\}}|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_2|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3 \\ &= (h_1|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus h_2)|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3 \\ &= \varphi_{V_1+V_2+\{*_2\},V_3}(\varphi_{V_1+\{*_1\},V_2+\{*_2\}}(h_1, h_2, f), h_3, g)\end{aligned} \tag{10}$$

$$\begin{aligned}\varphi_{V_1+V_2+\{*_2\},V_3}(\varphi_{V_1+\{*_1,*2\},V_2}(h_1, h_2, f), h_3, g) &= (h_1|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus h_2)|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3 \\ &= h_1|_{\{*_{1a} \leftarrow f(a)_a\}}|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_2 \oplus h_3 \\ &= h_1|_{\{*_{2a} \leftarrow g(a)_a\}}|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus h_3 \oplus h_2 \\ &= (h_1|_{\{*_{2a} \leftarrow g(a)_a\}} \oplus h_3)|_{\{*_{1a} \leftarrow f(a)_a\}} \oplus h_2 \\ &= \varphi_{V_1+V_3+\{*_1\},V_2}(\varphi_{V_1+\{*_1,*2\},V_3}(h_1, h_3, g), h_2, f).\end{aligned} \tag{11}$$

For the last item, let $e : X \to$ -**MHG** be defined by $e(v) = \emptyset_{\{v\}}$. We then have, with $e_{\mathcal{F}}$ the unit of $\mathcal{F}_A^{\mathbf{MHG}}$:

$$\varphi_{\{*\},V}(e(*), h, f) = \emptyset_{\{*\}}|_{\{*_a \leftarrow f(a)_a\}} \oplus h = h. \tag{12}$$

Moreover, we have:

$$\begin{aligned}\varphi_{V+\{*\},\{v\}}(h, e(v), e_{\mathcal{F}}(v)) &= h|_{\{*_a \leftarrow e_{\mathcal{F}}(v)(a)_a\}} \oplus \emptyset_{\{v\}} \\ &= h|_{\{*_a \leftarrow v_a\}} = A\text{-}\mathbf{MHG}[\tau_{(*,v)}](h).\end{aligned} \tag{13}$$

This concludes the proof. □

In all the following when considering a semidirect product of a sub-species of $\mathbb{K}A$-**MHG** and a sub-operad of $\mathcal{F}_A^{\mathbf{MHG}}$, this product is over the map φ defined in the Proposition 18. We hence will omit the φ index.

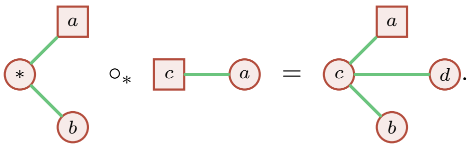
From now on we denote by $\sum V$ the sum $\sum_{v \in V} v$ in order to slightly lighten the notations.

Recall from Example 5 that we have natural embeddings of E and Id in Pol and a natural embedding of $\mathbb{K}E$ in $\mathbb{K}$Pol. Since the images of these embeddings have null constant term, these embeddings are in **MHG**.

Example 19. $G^\bullet$ has a natural set operad structure given by $G^\bullet \cong G \times Id \cong \{0\}\text{-}G \ltimes \mathcal{F}_{\{0\}}^{Id}$. For (g_1, v_1) and (g_2, v_2) two pointed graphs the partial composition $(g_1, v_1) \circ_* (g_2, v_2)$ is then equal to $(g_3, v_1|_{*\leftarrow v_2})$ where g_3 is the graph obtained by connecting all the ends on $*$ to v_2. More formally:

$$\begin{aligned}(g_1, v_1) \circ_* (g_2, v_2) &= (g_1|_{*\leftarrow v_2} \oplus g_2, v_1|_{*\leftarrow v_2}) \\ &= (G[\tau_{*,v_2}](g_1) \oplus g_2, v_1|_{*\leftarrow v_2}).\end{aligned} \tag{14}$$

For instance, one has:



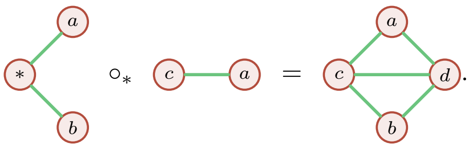
(15)

Remark that the set operad NAP [10] is a set sub-operad of the operad above and hence is a graph insertion set operad.

Example 20. G has a natural set operad structure given by $G \cong G \times E \cong \{0\}\text{-}G \ltimes \mathcal{F}_{\{0\}}^{E}$. For g_1 and g_2 two graphs the partial composition $g_1 \circ g_2$ is then the graph obtained by adding an edge between each neighbour of $*$ and each vertex of g_2. More formally, for $g_1 \in G'[V_1]$ and $g_2 \in G[V_2]$:

$$\begin{aligned}g_1 \circ_* g_2 &= g_1|_{*\leftarrow \oplus_{v\in V_2} v} \oplus g_2 \\ &= g_1 \cap V_1^2 \oplus \bigoplus_{v \in n(*)} v\Big(\bigoplus_{v' \in V_2} v'\Big) \oplus g_2 \\ &= g_1 \cap V_1^2 \oplus \bigoplus_{v \in n(*)} \bigoplus_{v' \in V_2} vv' \oplus g_2,\end{aligned} \tag{16}$$

where $n(*)$ is the set of neighbours of $*$. Note that we also consider g_1 as a set of edges in order to write $g_1 \cap V_1^2$ for the set of edges of g_1 not containing $*$. For instance, one has:



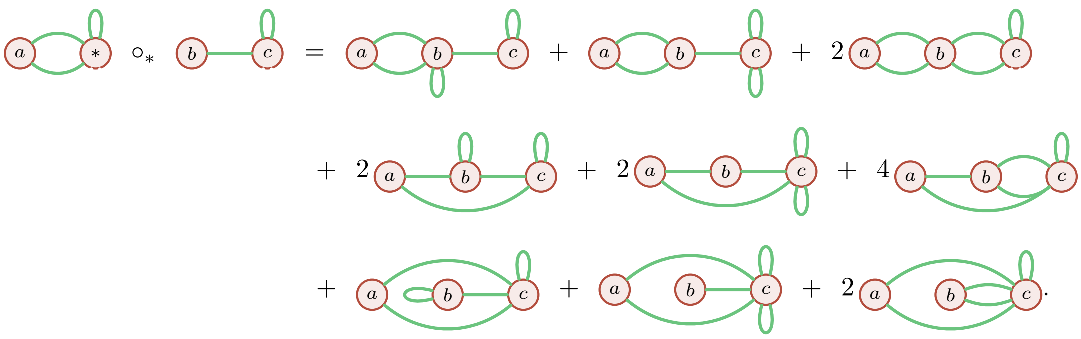
(17)

Let V_1 and V_2 be two disjoint sets. For any multigraphs $g_1 \in \mathbf{MG}'[V_1]$ and $g_2 \in \mathbf{MG}[V_2]$, define a partial composition of g_1 and g_2 as the sum of all the multigraphs of $\mathbf{MG}[V_1 \setminus \sqcup V_2]$ obtained by the following:

1. Take the disjoint union of g_1 and g_2;
2. Remove the vertex $*$. We then have some edges with one (or two if $*$ has loops) loose end(s);

3. Connect each loose end to any vertex in V_2.

For instance, one has:



(18)

Let us now state the main results of this subsection:

Theorem 21. The species $\mathbb{K}\mathbf{MG}$, endowed with the preceding partial composition, is an operad.

Proof. This is the operad structure on $\mathbb{K}MG$ implied by the isomorphism of species $\mathbb{K}\mathbf{MG} \to \{0\}\text{-}MG \ltimes \mathcal{F}^{\mathbb{K}E}_{\{0\}}$. □

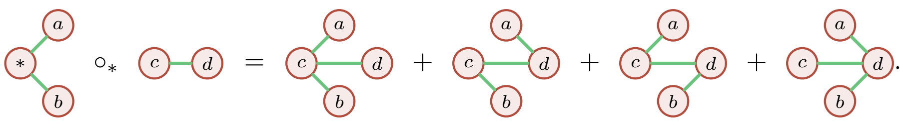
One notes that the species $\mathbb{K}\mathbf{G}$ and $\mathbb{K}\mathbf{MG}_c$ are suboperads of $\mathbb{K}\mathbf{MG}$, that $\mathbb{K}\mathbf{G}_c$ a suboperad of $\mathbb{K}\mathbf{G}$, and that $\mathbb{K}\mathbf{T}$ is a suboperad of $\mathbb{K}\mathbf{G}_c$. In particular, this structure on $\mathbb{K}\mathbf{G}$ is known as the Kontsevich-Willwacher operad [12]. This partial composition can be formally written as follows. For any $g_1 \in \mathbf{MG}[V_1]$ and $g_2 \in \mathbf{MG}[V_2]$ such that V_1 and V_2 are two disjoint sets and $* \in V_1$,

$$\begin{aligned} g_1 \circ_* g_2 &= g_1|_{*\leftarrow \sum V_2} \oplus g_2 \\ &= g_1 \cap V_1 \oplus \bigoplus_{v \in n(*)} v(\sum V_2) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2 \\ &= \sum_{f:n(*)\to V_2} \sum_{l:[g_1(**)]\to V_2V_2} g_1 \cap V_1^2 \oplus \bigoplus_{v\in n(*)} vf(v) \oplus \bigoplus_{i=1}^{g_1(**)} l(i) \oplus g_2, \end{aligned} \tag{19}$$

where $n(*)$ is the multiset of neighbours of $*$ in g_1 and $g_1(**)$ is the number of loops on $*$ in g_1. This partial composition reformulates in a simpler way on $\mathbb{K}\mathbf{G}$. For any $g_1 \in \mathbf{G}[V_1]$ and $g_2 \in \mathbf{G}[V_2]$ such that V_1 and V_2 are two disjoint sets and $* \in V_1$,

$$\begin{aligned} g_1 \circ_* g_2 &= g_1|_{*\leftarrow \sum V_2} \oplus g_2 \\ &= g_1 \cap V_1 \oplus \bigoplus_{v\in n(*)} v(\sum V_2) \oplus g_2 \\ &= \sum_{f:n(*)\to V_2} g_1 \cap V_1^2 \oplus \bigoplus_{v\in n(*)} vf(v) \oplus g_2, \end{aligned} \tag{20}$$

where $n(*)$ is now the set of neighbour of $*$ in g_1. For instance, one has:



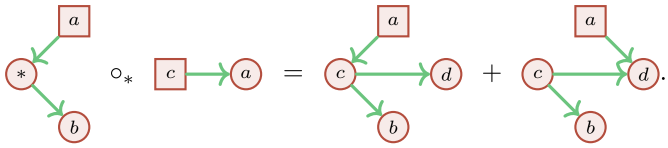
(21)

We observe that all the graphs appearing in $g_1 \circ_* g_2$ have 1 as coefficient.

Let us turn to the oriented case (cf Example 13). Let V_1 and V_2 be two disjoint sets such that $* \in V_1$. For any rooted oriented multigraphs $(g_1, v_1) \in \mathbf{MG}^\bullet_{or}[V_1]$ and $(g_2, v_2) \in \mathbf{MG}_{or}[V_2]^\bullet$, define a partial composition of (g_1, v_1) and (g_2, v_2) as the sum of all the rooted multigraphs of $\mathbf{MG}^\bullet_{or}[V_1 \setminus \{*\} \sqcup V_2]$ obtained by the following:

1. Take the disjoint union of g_1 and g_2;
2. Remove the vertex $*$. We then have some edges with a loose end;
3. Connect each non labelled loose end to v_2;
4. Connect each labelled loose end to any vertex in V_2;
5. The new root is v_1 if $v_1 \neq *$ and is v_2 otherwise.

For instance, by depicting by squares the roots of the multigraphs, one has:



(22)

Theorem 22. The species $\mathbb{K}\mathbf{MG}^{\bullet}_{orc}$, endowed with the preceding partial composition, is an operad.

Proof. This is the operad structure on $\mathbb{K}MG^{\bullet}_{or}$ implied by the monomorphism $\mathbb{K}G^{\bullet}_{or} \hookrightarrow \{\_,>\}$-$G \ltimes \mathcal{F}^{\mathbb{K}Id}_{\_} \uplus \mathcal{F}^{\mathbb{K}E}_{>}$ defined by:

$$\begin{aligned} \mathbb{K}\mathbf{G}^{\bullet}_{or}[V] &\hookrightarrow \{\_,>\}\text{-}\mathbf{G} \ltimes \mathcal{F}^{\mathbb{K}Id}_{\_} \uplus \mathcal{F}^{\mathbb{K}E}_{>}[V] \\ (g,r) &\mapsto (g, f : \begin{cases} \_ \mapsto r \\ > \mapsto (\sum V)_{>} \end{cases}). \end{aligned} \tag{23}$$

This concludes the proof □

It is straightforward to note that the subspecies of connected components $\mathbb{K}\mathbf{MG}^{\bullet}_{orc}$ and the species $\mathbb{K}\mathbf{G}^{\bullet}_{or}$ are suboperads of $\mathbb{K}\mathbf{MG}$ and that $\mathbb{K}\mathbf{G}^{\bullet}_{orc}$ is a suboperad of $\mathbb{K}\mathbf{G}^{\bullet}_{or}$.

In a rooted tree, each edge has a parent end and a child end. Given a rooted tree t with root r, denote by t_r the oriented tree where each parent end of t is labelled and each child end is non labelled. Then, the monomorphism $\mathbf{T}^{\bullet} \hookrightarrow \mathbf{G}^{\bullet}_{orc}$ which sends each ordered pair (t,r), where t is a tree and r is its root, on (t_r, r) induces an operad structure on the species of rooted trees which is exactly the operad **PLie**. Hence **PLie** is a graph insertion operad.

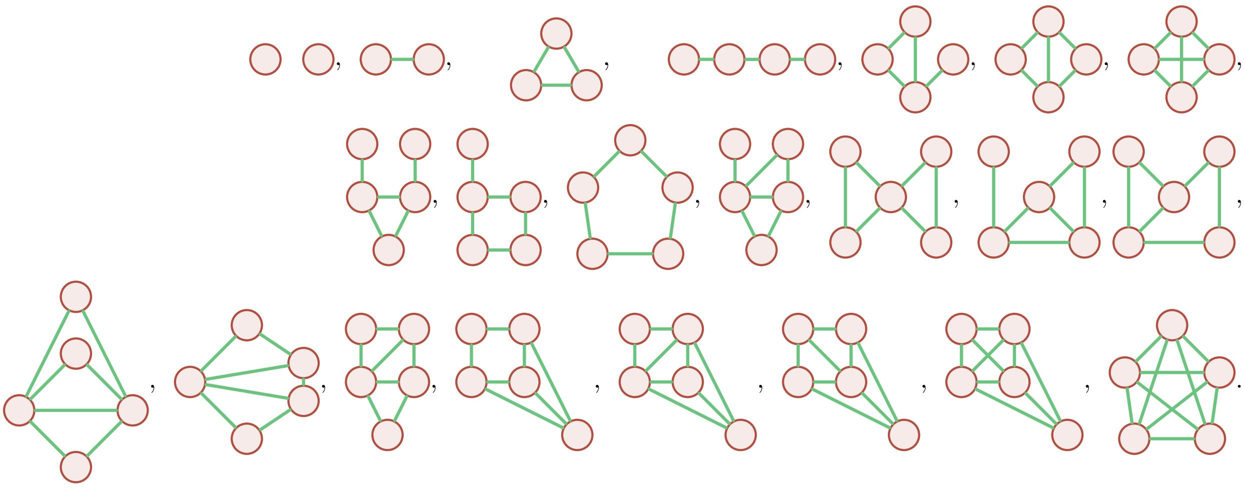
For the sake of completeness, let us end this section by mentioning that the notion of graph insertion operad introduced here is different than the one mentioned in [9], in the context of Feynman graph insertions in quantum field theory.

3.2 Canonical graph operad

We study here in more details the operad structure on $\mathbb{K}\mathbf{G}$ implied by the one on $\mathbb{K}\mathbf{MG}$ given in Theorem 21. We will see that while $\mathbb{K}\mathbf{G}$ itself has an involved operadic structure, it has many interesting sub-operads.

Before explaining how $\mathbb{K}G$ has an involved operadic structure, let us first introduce some notations. Let S be a species, I be a set, $\{V_i\}_{i\in I}$ be a family of finite sets, and $x_i \in S[V_i]$ for all $i \in I$. We call *subspecies of S generated by* $\{x_i\}_{i\in I}$ the smallest subspecies of S containing the family $\{x_i\}_{i\in I}$. If S is furthermore an operad, we call *suboperad of S generated by* $\{x_i\}_{i\in I}$ the smallest suboperad of S containing the family $\{x_i\}_{i\in I}$. We write that x *is generated by* $\{x_i\}_{i\in I}$ if x is in the suboperad generated by $\{x_i\}_{i\in I}$.

These definitions given, it is natural to search for a smallest family of generators of $\mathbb{K}\mathbf{G}$. The search of such a family is computationally hard. Using computer algebra, we obtain a family of generators of $\mathbb{K}\mathbf{G}$ of arity less than 5:



(24)

Due to the symmetric group action on $\mathbb{K}\mathbf{G}$, only the knowledge of the shapes of the graphs is significant. While (24) does not provide to us any particular insight on a possible characterisation of the generators, it does suggest that any graph with "enough" edges must be a generator. This is confirmed by the following lemma.

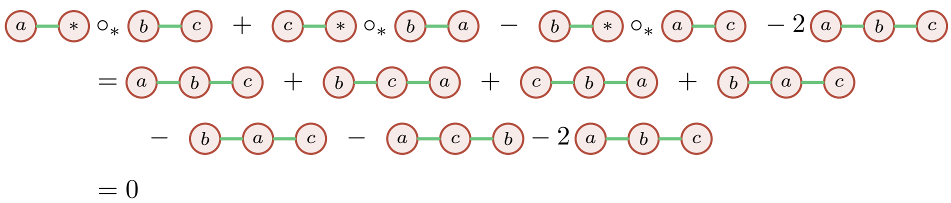
Lemma 23. Let $\{V_i\}_{i\in I}$ be a family of non empty finite sets, $\{g_i\}_{i\in I}$ be a family of graphs such that $g_i \in \mathbf{G}[V_i]$, and let g be a graph in $\mathbf{G}[V]$ with at least $\binom{n-1}{2}+1$ edges, where $n = |V|$. Then g is generated by $\{g_i\}_{i\in I}$ if and only if $g = g_i$ for some $i \in I$.

Proof. Suppose that $g \notin \{g_i\}_{i\in I}$. It is sufficient to show that g can not appear in the support of any vector of the form $g_1 \circ_* g_2$ for any g_1 and g_2 different of g. Hence let V_1 and V_2 be two disjoint finite sets such that $V_1 \sqcup V_2 = V$, $g_1 \in G'[V_1]$ and $g_2 \in G[V_2]$, and denote by e_1 the number of edges of g_1 and by e_2 the number of edges of g_2. Then the graphs in the support of $g_1 \circ_* g_2$ have $e_1 + e_2$ edges. This is maximal when g_1 and g_2 are both complete graphs and is then equal to $\binom{x}{2} + \binom{n-x}{2} = x^2 - nx + \binom{n}{2}$ where $0 \leq x = |V_1| \leq n-1$.

If $x = 0$ then necessarily $g_1 = \emptyset_*$ and $g \in \mathrm{Supp}(g_1 \circ_* g_2) = \mathrm{Supp}(g_2)$ if and only if $g = g_2$. This is impossible, hence $x \neq 0$. The expression $x^2 - nx + \binom{n}{2}$ is then maximal for $x = 1$ or $x = n-1$ and is equal in both cases to $\binom{n-1}{2} < \binom{n-1}{2} + 1$. This implies that g can not be in the support of $g_1 \circ_* g_2$. This concludes the proof. $\square$

Proposition 24. The operad $\mathbb{K}\mathbf{G}$ is not free and has an infinite number of generators.

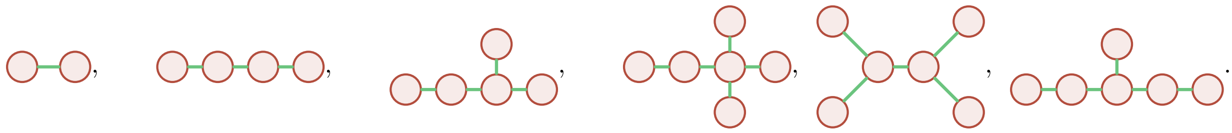
Proof. The fact that $\mathbb{K}\mathbf{G}$ has an infinite number of generators is a direct consequence of Lemma 23. Moreover, the relation



(25)

shows that $\mathbb{K}\mathbf{G}$ is not free. $\square$

As a consequence of Proposition 24, it seems particularly involved to further investigate the structure of $\mathbb{K}\mathbf{G}$. Let us then restrict further to its suboperad $\mathbb{K}\mathbf{T}$ of trees. A family of generators of $\mathbb{K}\mathbf{T}$ with arity less than 6 is:



(26)

This operad $\mathbb{K}\mathbf{T}$ has a non trivial link with the pre-Lie operad **PLie** [4]. This link is given by the following result.

Recall that **PLie** can be seen as an operad structure on $\mathbb{K}T^{\bullet}$.

Proposition 25. *The monomorphism of species $\psi : \mathbb{K}\mathbf{T} \to \mathbb{K}\mathbf{T}^{\bullet}$ defined, for any tree $t \in \mathbf{T}[V]$ by*

$$\psi(t) = \sum_{r \in V} (t, r), \tag{27}$$

is a monomorphism of operads from $\mathbb{K}\mathbf{T}$ to **PLie***.*

Proof. Let $t \in T[V]$ be a tree and $r, v \in V$. Denote by $n_t(v)$ the set of neighbours of v in t and denote by $c_{t,r}(v)$ the set of children of v when t is rooted on r, i. e $c_{t,r}(v) = n_>(v)$ in t_r. If $r \neq v$, further denote by $p_{t,r}(v)$ the parent of v in t when t is rooted on r i. e $\{p_{t,r}(v)\} = n\_(v)$ in t_r.

Let V_1 and V_2 be two disjoint sets and $t_1 \in T'[V_1]$ and $t_2 \in T[V_2]$. We have:

$$\begin{aligned}
\psi_{V_1}(t_1) \circ_* \psi_{V_2}(t_2) &= \sum_{r_1 \in V_1 + \{*\}} (t_1, r_1) \circ_* \sum_{r_2 \in V_2} (t_2, r_2) \\
&= \sum_{r_1 \in V_1 + \{*\}} \sum_{r_2 \in V_2} (t_1, r_1) \circ_* (t_2, r_2) \\
&= \sum_{r_1 \in V_1} \sum_{r_2 \in V_2} \left(t_1 \cap V_1^2 \oplus p_{t_1,r_1}(*) r_2 \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,r_1}(*)} \left(\sum V_2\right) v, r_1 \right) \\
&\quad + \sum_{r_2 \in V_2} \left(t_1 \cap V_1^2 \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,*}(*)} \left(\sum V_2\right) v, r_2 \right) \\
&= \sum_{r_1 \in V_1} \left(t_1 \cap V_1^2 \oplus p_{t_1,r_1}(*) \left(\sum V_2\right) \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,r_1}(*)} \left(\sum V_2\right) v, r_1 \right) \\
&\quad + \sum_{r_2 \in V_2} \left(t_1 \cap V_1^2 \oplus t_2 \oplus \bigoplus_{v \in c_{t_1,*}(*)} \left(\sum V_2\right) v, r_2 \right) \\
&= \sum_{r \in V_1 + V_2} \left(t_1 \cap V_1^2 \oplus \bigoplus_{v \in n_{t_1}(*)} \left(\sum V_2\right) v \oplus t_2, r \right) \\
&= \sum_{r \in V_1 + V_2} (t_1|_{* \leftarrow \sum V_2} \oplus t_2, r) \\
&= \psi_{V_1 + V_2}(t_1 \circ_* t_2)
\end{aligned} \tag{28}$$

□

A natural question to ask is how to extend this morphism to $\mathbb{K}\mathbf{G}_c$ and $\mathbb{K}\mathbf{MG}_c$. Let us introduce some notations in order to answer this question. For $g \in \mathbf{MG}_c[V]$, $r \in V$, and $t \in \mathbf{T}[V]$ a spanning tree of g, let $\overrightarrow{g}^{(t,r)} \in \mathbf{MG}_{orc}$ be the oriented multigraph obtained by labelling the edges of g in t in the same way as the edges of t_r, and by labelling both ends of the edges in g not in t. More formally, we have $\overrightarrow{g}^{(t,r)} = t_r \oplus \iota_{\mathbf{G}}(g \setminus t)$, where $\iota : \mathbb{K}\mathbf{MG} \to \mathbb{K}\mathbf{MG}_{or}$ sends a multigraph to the oriented multigraph obtained by labelling all the edges ends.

Define $\mathbb{K}\mathcal{O}_2 \subset \mathbb{K}\mathcal{O}_1 \subset \mathbb{K}\mathbf{ST}$ three subspecies of $\mathbb{K}\mathbf{MG}^{\bullet}_{orc}$ by

$$\mathbf{ST}[V] = \left\{(\overrightarrow{g}^{(t,r)}, r) : g \in \mathbf{MG}_c[V], r \in V \text{ and } t \text{ a spanning tree of } g\right\}, \tag{29}$$

$$\mathcal{O}_1[V] = \left\{\sum_{r\in V}(\overrightarrow{g}^{(t(r),r)}, r) : g \in \mathbf{MG}_c[V] \text{ and for each } r,\ t(r) \text{ a spanning tree of } g\right\}, \tag{30}$$

$$\mathcal{O}_2[V] = \Big\{(\overrightarrow{g}^{(t_1,r)}, r) - (\overrightarrow{g}^{(t_2,r)}, r) : g \in \mathbf{MG}_c[V], r \in V, \text{ and } t_1 \text{ and } t_2 \text{ two spanning trees of } g\Big\}. \tag{31}$$

Lemma 26. The following properties hold:

(i) $\mathbb{K}\mathbf{ST}$ is a suboperad of $\mathbb{K}\mathbf{MG}^{\bullet}_{orc}$ isomorphic to $\mathbb{K}\mathbf{MG} \times \mathbf{PLie}$,

(ii) $\mathbb{K}\mathcal{O}_1$ is a suboperad of $\mathbb{K}\mathbf{ST}$,

(iii) $\mathbb{K}\mathcal{O}_2$ is an ideal of $\mathbb{K}\mathcal{O}_1$.

Proof. Proof of i. The species morphism $\mathbb{K}\mathbf{MG} \times \mathbf{PLie} \hookrightarrow \mathbb{K}\mathbf{MG}^{\bullet}_{orc}$ given by $(g, (t, r)) \mapsto (\overrightarrow{g}^{(t,r)}, r)$ is an operad morphism and hence its image $\mathbf{ST}$ is a suboperad of $\mathbb{K}MG^{\bullet}_{orc}$.

In order to prove the next two items we first give two equalities. Let $U : \mathbb{K}MG_{or} \to \mathbb{K}MG$ be the forgetful functor which sends an oriented graph on the graph obtained by forgetting the orientation (i. e the labels). Let V_1 and V_2 be two disjoint sets, $g_1 \in \mathbf{MG}'_c[V_1]$ and $g_2 \in \mathbf{MG}_c[V_2]$ be two connected multigraphs, t a spanning tree of g_1 and for each $v \in V_2$, $t(v)$ a spanning tree of g_2. Then, for $r \in V_2$

$$\begin{aligned} &U \times \mathrm{id}\left((\overrightarrow{g_1}^{(t,*)}, *) \circ_* (\overrightarrow{g_2}^{(t(r),r)}, r)\right) \\ &= \left(g_1 \cap V_1 \oplus \bigoplus_{v\in n(*)} v(\textstyle\sum V_2) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r\right) \\ &= (g_1 \circ_* g_2, r). \end{aligned} \tag{32}$$

Let now r be a vertex in V_1. Denote by p the parent of $*$ in t_r, by $c(*)$ the children of $*$ in t_r, by $n_{g_1\setminus t}(*)$ the multiset of neighbours of $*$ in $g_1 \setminus t$ and by $n(*)$ the multiset of neighbours of $*$ in g_1, so that $n(*) = n_{g_1\setminus t}(*) \cup c(*) \cup \{p\}$. Then

$$\begin{aligned} &U \times \mathrm{id}\left((\overrightarrow{g_1}^{(t,r)}, r) \circ_* \sum_{v\in V_2}(\overrightarrow{g_2}^{(t(v),v)}, v)\right) \\ &= \sum_{v\in V_2} U \times \mathrm{id}\left((\overrightarrow{g_1}^{(t,r)}, r) \circ_* \sum_{v\in V_2}(\overrightarrow{g_2}^{(t(v),v)}, v)\right) \\ &= \sum_{v\in V_2}\left(g_1 \cap V_1^2 \oplus pv \oplus \bigoplus_{v'\in c(*)} v'\left(\textstyle\sum V_2\right) \oplus \bigoplus_{v'\in n_{g_1\setminus t}(*)} v'\left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r\right) \\ &= \left(g_1 \cap V_1^2 \oplus p\left(\textstyle\sum V_2\right) \oplus \bigoplus_{v'\in c(*)} v'\left(\sum V_2\right) \oplus \bigoplus_{v'\in n_{g_1\setminus t}(*)} v'\left(\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r\right) \\ &= \left(g_1 \cap V_1^2 \oplus \bigoplus_{v\in n(*)} v\left(\textstyle\sum V_2\right) \oplus ((\sum V_2)^2)^{\oplus g_1(**)} \oplus g_2, r\right) \\ &= (g_1 \circ_* g_2, r). \end{aligned} \tag{33}$$

Proof of ii. Let V_1 and V_2 be two disjoint sets, $g_1 \in \mathbf{MG}'_c[V_1]$ and $g_2 \in \mathbf{MG}_c[V_2]$ be two connected multigraphs and for each $v \in V_1 + \{*\}$, $t(v)$ a spanning tree of g_1 and for each $v \in V_2$, $t(v)$ a spanning tree of g_2. We have

$$\begin{aligned}\sum_{r_1 \in V_1+\{*\}} \overrightarrow{g_1}^{(t(r_1),r_1)} \circ_* \sum_{r_2 \in V_2} \overrightarrow{g_2}^{(t(r_2),r_2)} &= \sum_{r_1 \in V_1+\{*\}} \sum_{r_2 \in V_2} \overrightarrow{g_1}^{(t(r_1),r_1)} \circ_* \overrightarrow{g_2}^{(t(r_2),r_2)} \\ &= \sum_{r_1 \in V_1+\{*\}} \sum_{r_2 \in V_2} \left(\overrightarrow{g_1}^{(t(r_1),r_1)}|_{*\_\leftarrow r_2\_,*_> \leftarrow (\sum V_2)_>} \oplus \overrightarrow{g_2}^{(t(r_2),r_2)}, r_1|_{*\leftarrow r_2} \right).\end{aligned} \tag{34}$$

Then from 32 and 33 we know that applying $U \times \mathrm{id}$ to the preceding sum gives us:

$$\sum_{r \in V_1+V_2} (g_1 \circ_* g_2, r). \tag{35}$$

To conclude remark that $\mathbb{K}\mathcal{O}_1[V]$ can be defined as the reciprocal image of $\mathbb{K}\{\sum_{v\in V}(g,v) \,|\, g \in \mathbf{MG}_c[V]\}$ by $U \times \mathrm{id} : \mathbb{K}\mathbf{ST} \to \mathbb{K}\mathbf{MG}_c^\bullet$.

Proof of iii. It is easy to see that $\mathbb{K}\mathcal{O}_2$ is a left ideal of $\mathbb{K}\mathbf{ST}$ and hence of $\mathbb{K}\mathcal{O}_1$. Let V_1 and V_2 be two disjoint finite sets, $g_1 \in \mathbf{MG}'_c[V_1]$ and $g_2 \in \mathbf{MG}_c[V_2]$, $r \in V_1$, t a spanning tree of g_1 and for every $v \in V_2$, $t(v)$ a spanning tree of g_2. Then from 32 and 33 we know that $U \times \mathrm{id}(\overrightarrow{g_1}^{(t,r)} \circ_* \sum_{v\in V_2} \overrightarrow{g_2}^{(t(v),v)})$ is of the form $(g_1 \circ_* g_2, r)$ if $r \neq *$, and of the form $\sum_{v\in V_2}(g_1 \circ_* g_2, v)$ otherwise. In both cases it does not depend on t. This concludes this proof since $\mathbb{K}\mathcal{O}_2[V]$ is the kernel of $(U \times \mathrm{id})_V : \mathbb{K}\mathbf{ST}[V] \to \mathbb{K}G_c^\bullet[V]$. ☐

We can see **PLie** as a suboperad of **ST** by the monomorphism $(t, r) \mapsto (t_r, r)$. The image of the operad morphism ψ of Proposition 25 is then $\mathbb{K}\mathcal{O}_1 \cap \mathbf{PLie}$ and we have that $\mathbb{K}\mathcal{O}_2 \cap \mathbf{PLie} = \{0\}$ and hence $\mathbb{K}\mathcal{O}_1 \cap \mathbf{PLie}/\mathbb{K}\mathcal{O}_2 \cap \mathbf{PLie} = \mathbb{K}\mathcal{O}_1 \cap \mathbf{PLie}$.

Proposition 27. The operad isomorphism $\psi : \mathbb{K}\mathbf{T} \to \mathbf{PLie} \cap \mathbb{K}\mathcal{O}_1$ extends into an operad isomorphism $\psi : \mathbb{K}\mathbf{MG}_c \to \mathbb{K}\mathcal{O}_1/\mathbb{K}\mathcal{O}_2$ satisfying, for any $g \in \mathbf{MG}_c[V]$,

$$\psi(g) = \sum_{r \in V} \overrightarrow{g}^{(t(r),r)}, \tag{36}$$

where for each $r \in V$, $t(r)$ is a spanning tree of g. Furthermore, this isomorphism restricts itself to an isomorphism $\mathbb{K}\mathbf{G}_c \to \mathbb{K}\mathcal{O}_1 \cap \mathbb{K}\mathbf{G}^\bullet_{orc}/\mathbb{K}\mathcal{O}_2 \cap \mathbb{K}\mathbf{G}^\bullet_{orc}$.

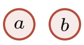
Proof. This statement is a direct consequence of Lemma 26 and its proof. ☐

The last results are summarized in the following commutative diagram of operad morphisms.

$$\begin{array}{ccccccc} \mathbb{K}\mathbf{T} & \xrightarrow{\sim} & \mathbf{PLie} \cap \mathbb{K}\mathcal{O}_1/\mathbb{K}\mathcal{O}_2 & = & \mathbf{PLie} \cap \mathbb{K}\mathcal{O}_1 & \hookrightarrow & \mathbf{PLie} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{K}\mathbf{G}_c & \xrightarrow{\sim} & \mathbb{K}\mathcal{O}_1 \cap \mathbb{K}\mathbf{G}^\bullet_{orc}/\mathbb{K}\mathcal{O}_2 \cap \mathbb{K}\mathbf{G}^\bullet_{orc} & \twoheadleftarrow & \mathbb{K}\mathbf{G}^\bullet_{orc} \cap \mathbb{K}\mathcal{O}_1 & \hookrightarrow & \mathbb{K}\mathbf{G}^\bullet_{orc} \cap \mathbb{K}\mathbf{ST} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{K}\mathbf{MG}_c & \xrightarrow{\sim} & \mathbb{K}\mathcal{O}_1/\mathcal{O}_2 & \twoheadleftarrow & \mathbb{K}\mathcal{O}_1 & \hookrightarrow & \mathbb{K}\mathbf{MG} \times \mathbf{PLie} \end{array} \tag{37}$$

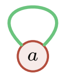
4 Finitely generated suboperads

Let us now focus on finitely generated suboperads of $\mathbb{K}\mathbf{MG}$. In particular we will study the operads generated by:

1. $\{$$\}$ which is isomorphic to **Com**,

2. {(a)—(b)} which is isomorphic to **ComMag**,

3. {(a) (b), (a)—(b)} which we will denote by **SP**,

4. {  , (a) (b)} which we will denote by **LP**.

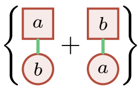
First remark that the suboperad of $\mathbb{K}\mathbf{G}$ generated by {(a) (b)} is isomorphic to the commutative operad **Com**. Indeed,

$$(a)\ (*) \circ_* (b)\ (c) = (a)\ (b)\ (c) = (*)\ (c) \circ_* (a)\ (b). \tag{38}$$

Now recall that the set operad **ComMag** is the free operad generated by one binary and symmetric element [2]. More formally, $\mathbf{ComMag}[V]$ is spanned by nonplanar binary trees with set of leaves equal to V. Let s be the connected species defined by $\dim(s[V]) = 1$ if $|V| = 2$, $\dim(s[V]) = 0$ otherwise. The action of transposition on the sole element of $s[\{a, b\}]$ is trivial. Then $\mathbf{ComMag} = \mathbf{Free}_s$.

Proposition 28. The suboperad of $\mathbb{K}\mathbf{G}$ generated by {(a)—(b)} is isomorphic to **ComMag**.

Proof. We know from Proposition 25 that the operad of the statement is isomorphic to the suboperad of **PLie** generated by


(39)

Then [2] gives us that this suboperad is isomorphic to **ComMag**. This concludes the proof □

The fact that we can see both **Com** and **ComMag** as disjoint suboperads of $\mathbb{K}\mathbf{G}$ gives us a natural way to define the smallest operad containing these two as disjoint suboperads. Denote by G the subspecies of $\mathbb{K}\mathbf{G}$ generated by {(a) (b), (a)—(b)} and **SP** the suboperad generated by G. This operad has some nice properties.

Proposition 29. The operad **SP** is isomorphic to the operad $\mathrm{Ope}(G, R)$ where R is the subspecies of $\mathbf{Free}_G$ generated by

$$(c)\ (*) \circ^{\xi}_* (a)\ (b) - (a)\ (*) \circ^{\xi}_* (b)\ (c), \tag{40a}$$

and

$$(a)—(*) \circ^{\xi}_* (b)\ (c) - (c)\ (*) \circ^{\xi}_* (a)—(b) - (b)\ (*) \circ^{\xi}_* (a)—(c). \tag{40b}$$

Therefore, **SP** is binary and quadratic.

Proof. There is a natural epimorphism ϕ from $\mathbf{Free}_G$ to **SP** which is the identity on (a) (b) and (a)—(b) and which sends a partial composition $g_1 \circ^{\xi}_* g_2$ on the partial composition $g_1 \circ_* g_2$. The fact that (R) is included in the kernel of ϕ is straightforward. Let now be $w \in \mathbf{Free}_G/(R)[V]$. A possible representant of w is of the form $\sum_{i=1}^{l} a_i w_i$ where for each $1 \leq i \leq l$ $a_i \in \mathbb{K}$ and there is a partition $V = V_{i,1} \sqcup \cdots \sqcup V_{i,k_i}$ such that $w_i = (\ldots(\mu_i \circ_{*_{i,1}} t_{i,1})\ldots) \circ_{*_{i,k_i}} t_{i,k_i}$ with μ_i the sole element in $\in \mathbf{Com}[\{*_{i,1}, \ldots, *_{i,k_i}\}]$ and $t_{i,j}$ is in the basis of $\mathbf{ComMag}[V_{i,j}]$. Here we use the identification of **ComMag** and **Com** as suboperads of $\mathbb{K}\mathbf{G}$ done previously. Without loss of generality, we can suppose that all the w_i are on the same partition of w i. e $V = V_1 \sqcup \cdots \sqcup V_k$ and for all i, j, $k_i = k$ and $V_{i,j} = V_j$.

With these notations we now have

$$\phi(w) = \sum_{i=1}^{l} a_i \bigoplus_{j=1}^{k_i} \phi(t_{i,j}). \tag{41}$$

Denote by $\mathbf{G}[V_1, \ldots, V_k] = \{g_1 \oplus \cdots \oplus g_k\,,\ g_i \in \mathbf{G}[V_i]\}$. Then there is an isomorphism from $\mathbb{K}\mathbf{G}[V_1, \ldots, V_k]$ to $\mathbb{K}\mathbf{G}[V_1] \otimes \cdots \otimes \mathbb{K}\mathbf{G}[V_k]$ defined by $g_1 \oplus \cdots \oplus g_k \mapsto g_1 \otimes \cdots \otimes g_k$. This isomorphism sends $\phi(w)$ on $\sum_{i=1}^{l} a_i \bigotimes_{j=1}^{k_i} \phi(t_{i,j})$. Since for all $1 \leq i \leq k$ the basis of $\mathbf{ComMag}[V_i]$ is a free family, the family $\{v_1 \otimes \cdots \otimes v_k\,,\ v_i$ is in the basis of $\mathbf{ComMag}[V_i]\}$ is also free and hence $\phi(w) = 0$ implies $a_i = 0$ for all $1 \leq i \leq k$. This shows that the epimorphism ϕ is also a monomorphism and hence an isomorphism, which concludes this proof. □

Proposition 30. The operad $\mathbf{SP}$ admits as Koszul dual the operad $\mathbf{SP}^!$ which is isomorphic to the operad $\mathrm{Ope}(G^\vee, R)$ where R is the subspecies of $\mathbf{Free}_{G^\vee}$ generated by

$$\left(\textcircled{a}\!-\!\textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{b}\!-\!\textcircled{c}\right)^\vee, \tag{42a}$$

$$\left(\textcircled{a}\ \textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{b}\!-\!\textcircled{c}\right)^\vee + \left(\textcircled{c}\!-\!\textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{a}\ \textcircled{b}\right)^\vee + \left(\textcircled{b}\!-\!\textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{a}\ \textcircled{c}\right)^\vee, \tag{42b}$$

$$\left(\textcircled{a}\ \textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{b}\ \textcircled{c}\right)^\vee + \left(\textcircled{c}\ \textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{a}\ \textcircled{b}\right)^\vee + \left(\textcircled{b}\ \textcircled{*}\right)^\vee \circ_*^\xi \left(\textcircled{c}\ \textcircled{a}\right)^\vee. \tag{42c}$$

Proof. Let us respectively denote by r_1 and r_2 and r_1', r_2', and r_3' the vectors (40a), (40b), (42a), (42b), and (42c). Denote by I the operad ideal generated by r_1 and r_2. Then as a vector space, $I[[\{a,b,c\}]]$ is the linear span of the set

$$\{r_1, r_1 \cdot (ab), r_2, r_2 \cdot (abc), r_2 \cdot (acb)\}, \tag{43}$$

where $\cdot$ is the action of the symmetric group, e.g $r_1 \cdot (ab) = \mathbf{Free}_G[(ab)](r_1)$. This space is a sub-space of dimension 5 of $\mathbf{Free}_G[\{a,b,c\}]$, which is of dimension 12. Hence, since as a vector space

$$\mathbf{Free}_{G^\vee}[\{a,b,c\}] \cong \mathbf{Free}_{G^*}[\{a,b,c\}] \cong \mathbf{Free}_G[\{a,b,c\}], \tag{44}$$

$I^\perp[\{a,b,c\}]$ must be of dimension 7.

Denote by J the ideal generated by r_1', r_2' and r_3'. Then as a vector space $J[\{a,b,c\}]$ is the linear span of the set

$$\{r_1', r_1' \cdot (ab), r_1' \cdot (ac), r_2', r_2' \cdot (abc), r_2' \cdot (acb), r_3'\}. \tag{45}$$

This space is of dimension 7. To conclude we need to show that for any $f \in J[\{a,b,c\}]$ and $x \in I[\{a,b,c\}]$ we have $< f, x >= 0$. Denote by $p_{a,b} = \textcircled{a}\ \textcircled{b}$ and $s_{a,b} = \textcircled{a}\!-\!\textcircled{b}$. Among the 21 cases to check, we have for example:

$$\begin{aligned} < r_1', r_1 > &=< s_{a,*}^\vee \circ_*^\xi s_{b,c}^\vee\,,\ p_{*,c} \circ_*^\xi p_{a,b} - p_{a,*} \circ_*^\xi p_{b,c} > \\ &=< s_{a,*}^\vee \circ_*^\xi s_{b,c}^\vee\,,\ p_{*,c} \circ_*^\xi p_{a,b} > - < s_{a,*}^\vee \circ_*^\xi s_{b,c}^\vee\,,\ p_{a,*} \circ_*^\xi p_{b,c} > \\ &= s_{a,*}^\vee(p_{*,c}) s_{b,c}^\vee(p_{a,b}) - s_{a,*}^\vee(p_{a,*}) s_{b,c}^\vee(p_{b,c}) = 0, \end{aligned} \tag{46}$$

and

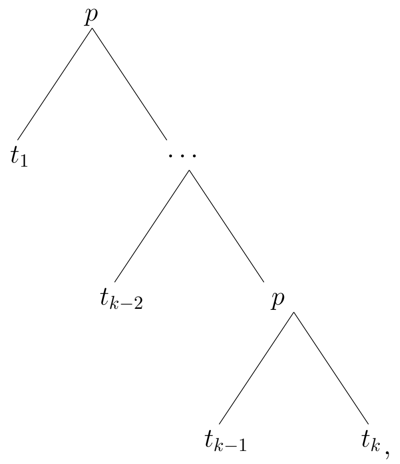
$$\begin{aligned} &< r_2' \cdot (abc), r_2 >= \\ &\qquad < p_{b,*}^\vee \circ_*^\xi s_{c,a}^\vee + s_{a,*}^\vee \circ_*^\xi p_{b,c}^\vee + s_{c,*}^\vee \circ_*^\xi p_{a,b}^\vee, s_{a,*} \circ_* p_{b,c} - p_{c,*} \circ_* s_{a,b} - p_{b,*} \circ_* s_{c,a} > \\ &\qquad = p_{b,*}^\vee(s_{a,*}) s_{c,a}^\vee(p_{b,c}) - p_{b,*}^\vee(p_{c,*}) s_{c,a}^\vee(s_{a,b}) - p_{b,*}^\vee(p_{b,*}) s_{c,a}^\vee(s_{c,a}) \\ &\qquad + s_{a,*}^\vee(s_{a,*}) p_{b,c}^\vee(p_{b,c}) - s_{a,*}^\vee(p_{c,*}) p_{b,c}^\vee(s_{a,b}) - s_{a,*}^\vee(p_{b,*}) p_{b,c}^\vee(s_{c,a}) \\ &\qquad + s_{c,*}^\vee(s_{a,*}) p_{a,b}^\vee(p_{b,c}) - s_{c,*}^\vee(p_{c,*}) p_{a,b}^\vee(s_{a,b}) - s_{c,*}^\vee(p_{b,*}) p_{a,b}^\vee(s_{c,a}) \\ &\qquad = -1 + 1 = 0. \end{aligned} \tag{47}$$

We leave the verification of the 19 remaining cases to the interested reader. □

In order to compute the Hilbert series of $\mathbf{SP}^!$ we need to use identity (3) and hence to prove that $\mathbf{SP}$ is Koszul. Providing the necessary background to define Koszul operads and their properties is out of the scope of this article and the interested reader is referred to [11] and [13]. We use here the characterisation given in [7].

Proposition 31. The operad $\mathbf{SP}$ is Koszul.

Proof. The rooted trees in $\mathbf{Free}_G[n]$ are in bijection with planar tree following the process described in [7]. Consider the trees of the form



(48)

where p is the graph with two vertices and no edge, t the graph with two vertices and one edge and, for $1 \leq i \leq k$, t_i is a tree with internal vertices labelled by t and set of leaves $V_i \subseteq [n]$ such that $\bigsqcup V_i = [n]$. Then choosing $t < p$ and an order on planar trees similar to the suitable order presented in section 3.4 of [7] makes the considered trees a PBW basis of the operad (over an $\mathbb{S}$-module) $\bigoplus_{n \geq 0} \mathbf{SP}[n]$. This concludes the proof. □

Proposition 32. The Hilbert series of $\mathbf{SP}^!$ is

$$\mathcal{H}_{\mathbf{SP}^!}(x) = \frac{(1 - \log(1-x))^2 - 1}{2}. \tag{49}$$

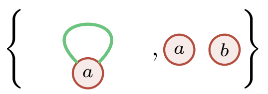
Proof. The Hilbert series of $\mathbf{ComMag}$ is $\mathcal{H}_{\mathbf{ComMag}}(x) = 1 - \sqrt{1-2x}$ hence the Hilbert series of $\mathbf{SP} \cong E(\mathbf{ComMag})$ is $\mathcal{H}_{\mathbf{SP}}(x) = e^{1-\sqrt{1-2x}} - 1$, where the -1 comes from the fact that we consider positive species. We deduce the Hilbert series of $\mathbf{SP}^!$ from $\mathcal{H}_{\mathbf{SP}}$ and the identity (3). □

The first dimensions $\dim \mathbf{SP}^![[n]]$ for $n \geq 1$ are

$$1, 2, 5, 17, 74, 394, 2484, 18108, 149904. \tag{50}$$

This is sequence **A000774** of [14]. This sequence is in particular linked to some pattern avoiding signed permutations and mesh patterns.

Before ending this section let us mention the suboperad **LP** of $\mathbb{K}\mathbf{MG}$ generated by

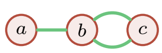


(51)

This operad presents a clear interest since its two generators can be considered as minimal elements in the sense that a partial composition with the two isolated vertices adds exactly one vertex and no edge, while a partial composition with the loop adds exactly one edge and no vertex. A natural question to ask at this point concerns the description of the multigraphs generated by these two minimal elements.

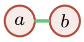
Proposition 33. The following properties hold:

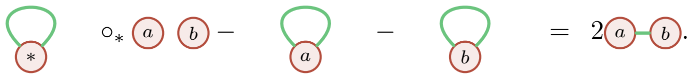
- the operad **SP** is a suboperad of **LP**;
- the operad **LP** is a strict suboperad of $\mathbb{K}\mathbf{MG}$. In particular, the multigraph



(52)

is in $\mathbb{K}\mathbf{MG}$ but is not in **LP**.

Proof. • The following shows that  is in $\mathbf{LP}[\{a, b\}]$ and hence that $\mathbf{SP}$ is a suboperad of $\mathbf{LP}$:

 (53)

- Using computer algebra, we generated all vectors in $\mathbf{LP}[\{a, b, c\}]$ with three edges and showed that the announced multigraph is not a linear combination of these.

□

References

[1] F. Bergeron, G. Labelle, and P. Leroux. *Combinatorial species and tree-like structures*, volume 67 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1998.

[2] N. Bergeron and J.-L. Loday. The symmetric operation in a free pre-Lie algebra is magmatic. *Proc. Amer. Math. Soc.*, 139(5):1585–1597, 2011.

[3] F. Chapoton. Operads and algebraic combinatorics of trees. *Séminaire Lotharingien de Combinatoire*, 58, 2008.

[4] F. Chapoton and M. Livernet. Pre-Lie algebras and the rooted trees operad. *Int. Math. Res. Notices*, 8:395–408, 2001.

[5] S. Giraudo. Combalgebraic structures on decorated cliques. *Formal Power Series and Algebraic Combinatorics*, 2017.

[6] S. Giraudo. *Nonsymmetric Operads in Combinatorics*. Springer Nature Switzerland AG, 2018.

[7] Eric Hoffbeck. A Poincaré–Birkhoff–Witt criterion for Koszul operads. *manuscripta mathematica*, 131(1):87, 2009.

[8] M. Kontsevich. Operads and motives in deformation quantization. volume 48, pages 35–72. 1999.

[9] D. Kreimer. Combinatorics of (perturbative) quantum field theory. *Phys. Rept.*, 363:387–424, 2002.

[10] M. Livernet. A rigidity theorem for pre-Lie algebras. *J. Pure Appl. Algebra*, 207(1):1–18, 2006.

[11] J.-L. Loday and B. Vallette. *Algebraic Operads*, volume 346 of *Grundlehren der mathematischen Wissenschaften*. Springer, Heidelberg, 2012. Pages xxiv+634.

[12] Y. I. Manin and B. Vallette. Monoidal structures on the categories of quadratic data. *Arxiv1902.03778v2*, 2019.

[13] M. A. Méndez. *Set operads in combinatorics and computer science*. SpringerBriefs in Mathematics. Springer, Cham, 2015. Pages xvi+129.

[14] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. `https://oeis.org/`.

[15] T. Willwacher. M. Kontsevich's graph complex and the Grothendieck–Teichmüller Lie algebra. *Invent. Math.*, 200(3):671–760, 2015.