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RECURSIVELY DIVISIBLE NUMBERS

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Divisible numbers are useful whenever a whole needs to be divided into equal parts. But sometimes it is necessary to divide the parts into subparts, and subparts into sub-subparts, and so on, in a recursive way. To understand numbers that are recursively divisible, I introduce the recursive divisor function: a recursive analog of the usual divisor function. I give a geometric interpretation of recursively divisible numbers and study their general properties—in particular the number and sum of the recursive divisors. I show that the number of the recursive divisors is equal to twice the number of ordered factorizations into integers greater than one, a problem that has been much studied in its own right. By computing those numbers which are more recursively divisible than all of their predecessors, I recover many of the grid sizes commonly used in graphic design and digital technologies, and suggest new grid sizes which have not yet been adopted but should be. These are especially relevant to recursively modular systems which must operate across multiple organizational length scales.

1. Introduction

1.1. **Plato's ideal city.** Consider one of the earliest references to numbers which can be divided into equal parts in many ways. Plato writes in his *Laws* [1] that the ideal population of a city is 5040, since this number has more divisors than any number less than it. He observes that 5040 is divisible by 60 numbers, including one to 10. A highly divisible population is useful for dividing the city into equal-sized sectors for administrative, social and military purposes.

This conception of divisibility can be extended. Once the city is divided into equal parts, it is often necessary to divide a part into equal subparts. For example, if 5040 is divided into 15 parts of 336, each part can in turn be divided into subparts in 20 ways, since 336 has 20 divisors. But if 5040 is divided into 16 parts of 315, each part can be divided into subparts in only 12 ways, since 315 has 12 divisors. Thus the division of the whole into 15 parts offers more optionality for further subdivisions than the division into 16 parts. Similar reasoning can be applied to the divisibility of the subparts into sub-subparts, and so on, in a hierarchical way.

The goal of this paper is to quantify the notion of recursive divisibility and understand the properties of numbers which possess it to a large degree.

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1.2. **Modular design.** In graphic and digital design, grid systems [2, 3] use a fixed number of columns or rows which form the primitive building blocks from which bigger columns or rows are made. For example, a grid of 24 columns is often used for books [2], and a grid of 96 columns is frequently used for websites [4]. In exchange for giving up some freedom to choose the size of parts (columns or rows), the space of possible designs gets smaller, making it easier to navigate. And the design elements become more interoperable, like how Lego bricks snap into place on a discrete grid, making it faster to build new designs.

What are the best grid sizes for modular design? The challenge is committing to a grid size now that provides the greatest optionality for an unknown future. Imagine, for example, that I have to cut a pie into slices, to be divided up later for an unknown number of colleagues. How many slices should I choose? The answer in this case is relatively straightforward: the best grids are the ones with the most divisors, such as the highly composite or super-abundant numbers [5, 6].

But the story gets more complicated when it is necessary to consider multiple steps into the future. For instance, imagine now that each colleague takes his share of pie home to further divide it amongst his family. In this case, not only does the whole need to be highly divisible, but the parts need to be highly divisible, too. This process can be extended in a hierarchical way.

Design across multiple length scales has long been a feature of graphic design. For example, newspapers are divided into columns for different stories, and columns into sub-columns of text. But with the rise of digital technologies, recursive modularity is becoming the rule. Columns of a website become the full screen of the same site on a phone. Different regions of the phone site offer different kinds of functionality, each with its own number of parts, which may change over time.

What are the design rules for recursive modularity? The state-of-the-art seems to be artisans' lore [2, 3, 7], and there is little in the way of quantitative reasoning.

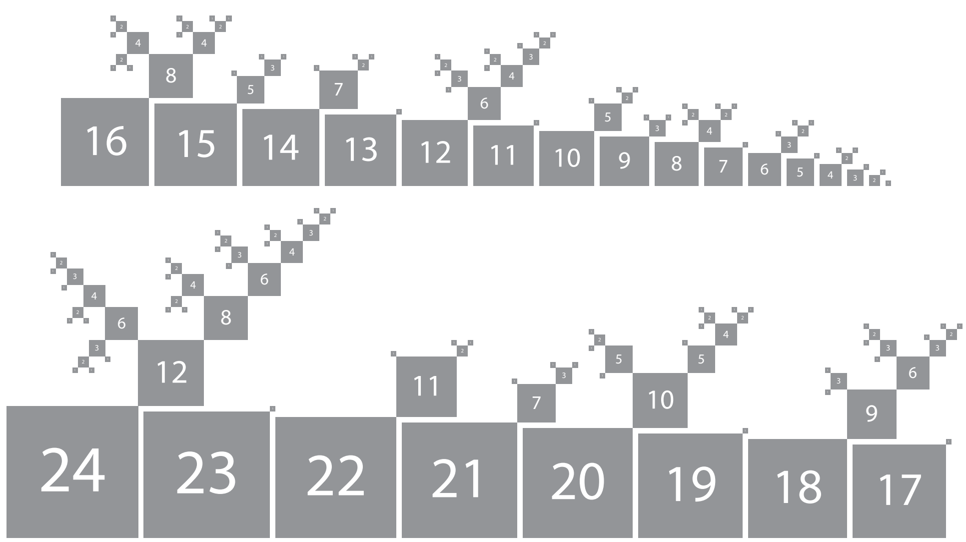

FIGURE 1. Divisor trees for 1 to 24. The number of recursive divisors $a(n)$ counts the number of squares in each tree and the sum of recursive divisors $b(n)$ adds up the side lengths of the squares in each tree. Divisor trees can be generated for any number n at lims.ac.uk/recursively-divisible-numbers.

Indeed, my interest in this problem originated in discussions with designers about the optimal grid size for the London Institute's website.

This paper gives a mathematical basis for choosing grids that are suitable for recursive design. It explains the preponderance of certain numbers in graphic design and display technologies, and predicts new numbers which have not been used but should be. More generally, it helps us understand recursively modular systems which must simultaneously operate across multiple organizational length scales.

1.3. **Outline of paper.** Including this introduction, this paper is divided into five parts. In part 2, I review the usual divisor function and define the recursive divisor function. I consider two specific instances of the recursive divisor function: the number of recursive divisors and the sum of recursive divisors. I introduce divisor trees (Figure 1), which give a geometrical interpretation of the recursive divisor function. Using this, I show that the number of recursive divisors is twice the number of ordered factorizations into integers greater than one, a problem which has been well-studied in its own right [8, 9, 10, 11, 12]. By examining the internal structure of divisor trees, I find a relation between the number and sum of recursive divisors.

In part 3, I investigate properties of the number of recursive divisors, taking advantage of their relation to the number of ordered factorizations. I give recursion relations for when n is the product of distinct primes, and for when n is the product of primes to a power. The latter can be written in closed form for one, two and three primes.

In part 4, I investigate properties of the sum of recursive divisors, which are more difficult to calculate than the number of recursive divisors in part 3. I give recursion relations for when n is the product of primes to a power. These can be written in closed form for one, two and three primes by making use of the relation between the sum and number of recursive divisors in part 2.

In part 5, I investigate numbers which are recursively divisible to a high degree. I call numbers with a record number of recursive divisors recursively highly composite, and list them up to one million. These have been studied in the context of the number of ordered factorizations [12]. I call numbers with a record sum of recursive divisors, normalized by n, recursively super-abundant, and also list them up to one million. I list applications of highly recursive numbers in technology and digital displays, and conclude with a list of open problems.

2. Recursive divisor function and divisor trees

Throughout this paper I write $m|n$ to indicate m divides n and $m\lfloor n$ to indicate m is a proper divisor of n.

2.1. **Divisor function.** In order to write down a recursive divisor function, I first recall the usual divisor function,

$$\sigma_x(n) = \sum_{m|n} m^x.$$

This sums the divisors of n raised to some integer power x.

When $x = 0$, the divisor function counts the number divisors of n and is generally written $d(n)$. It is well known that for

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j},$$

where $p_1, p_2, \ldots, p_j$ are prime,

$$d(n) = (1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_j). \tag{1}$$

Numbers n for which $d(n)$ is larger than that of all of the predecessors of n are called highly composite numbers and have been extensively studied [5, 6].

When $x = 1$, the divisor function sums the divisors of n and is generally written $\sigma(n)$. It is well known that

$$\sigma(n) = \frac{p_1^{\alpha_1+1}-1}{p_1-1}\,\frac{p_2^{\alpha_2+1}-1}{p_2-1}\ldots\frac{p_j^{\alpha_j+1}-1}{p_j-1}.$$

Numbers n for which $\sigma(n)/n$ is larger than that of all of the predecessors of n are called super-abundant numbers and have also been studied [6].

The quantity $\sigma(n)/n$ can be contrasted with $d(n)$ in the following way. Highly divisible numbers can be divided into equal-sized parts in many different ways. But when it comes to practical applications, not all divisions are equally useful. In general, divisions into fewer parts are more useful than divisions into many parts, because we are more likely to encounter the need for fewer parts. In other words, there is a greater need to divide a region into halves than into thirds, and into thirds than into fourths, and so on. To give preferential treatment to numbers with smaller divisors, consider that $d(n)$ awards a point for each divisor of n. Now let us give fewer points for larger divisors. In particular, let us award 1 point for numbers that can be divided into 1 part (namely, all numbers), 1/2 point for numbers that can be divided into 2 parts, 1/3 point for numbers that can be divided into 3 parts, and so on. This scheme gives the score of $\sigma(n)/n$.

2.2. **Recursive divisor function.** In contrast with the usual divisor function, in this paper I am concerned not only with the divisors of a number n but also the divisors of the resultant quotients, and the divisors of those resultant quotients, and so on. I introduce and study the recursive divisor function,

$$\kappa_x(n) = n^x + \sum_{m \lfloor n} \kappa_x(m),$$

where the sum is over the proper divisors of n. When $x = 0$, I call this the number of recursive divisors $a(n)$, and when $x = 1$, I call this the sum of recursive divisors $b(n)$.

Definition 1. *The number of recursive divisors is $a(1) = 1$ and*

$$a(n) = 1 + \sum_{m \lfloor n} a(m),$$

where $m \lfloor n$ means m is a proper divisor of n.

For example, $a(10) = 1 + a(1) + a(2) + a(5) = 6$. Note that $a(n)$ depends only on the set of exponents in the prime factorization of n and not on the primes themselves.

Definition 2. *The sum of recursive divisors is $b(1) = 1$ and*

$$b(n) = n + \sum_{m \lfloor n} b(m),$$

where $m \lfloor n$ means m is a proper divisor of n.

For example, $b(10) = 10 + b(1) + b(2) + b(5) = 20$. Unlike $a(n)$, $b(n)$ depends on both the exponents and the primes in the prime factorization of n. Ultimately I will be interested in $b(n)/n$, analogous to $\sigma(n)/n$ described above, but for now it is more natural to define and work with $b(n)$.

2.3. **Divisor trees.** A geometric interpretation of the recursive divisor function $\kappa(n)$ can be had by drawing the divisor tree for a given value of n. Divisor trees for 1 to 24 are shown in Figure 1. The number of recursive divisors $a(n)$ counts the number of squares in each tree, whereas $d(n)$ counts the number of squares in the main diagonal. The sum of recursive divisors $b(n)$ adds up the side lengths of the squares in each tree, whereas $\sigma(n)$ adds up the side lengths of the squares in the main diagonal. This can be extended to $\kappa_2(n)$, which adds up area, and so on, but in this paper I only consider $a(n) = \kappa_0(n)$ and $b(n) = \kappa_1(n)$.

A divisor tree is constructed as follows. First, draw a square of side length n. Let $m_1, m_2, \ldots$ be the proper divisors of n in descending order. Then draw squares of side length m_1, $m_2, \ldots$ with each consecutive square situated to the upper right of its predecessor, kitty-corner. This forms the main arm of a divisor tree. Now, for each of the squares of side length $m_1, m_2, \ldots$, repeat the process. Let $l_1, l_2, \ldots$ be the proper divisors of m_1 in descending order. Then draw squares of side length l_1, $l_2, \ldots$, but with the sub-arm rotated 90° counter-clockwise. Do the same for each of the remaining squares in the main arm. This forms the branches off of the main arm. Now, continue repeating this process, drawing arms off of arms off of arms, and so on, until the arms are single squares of size 1. Divisor trees can be generated for any number n at lims.ac.uk/recursively-divisible-numbers.

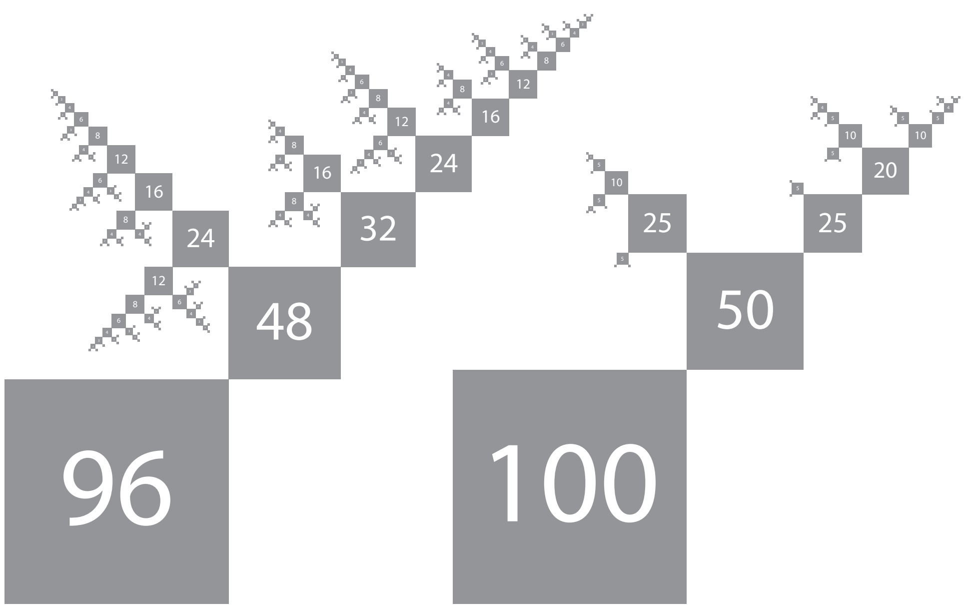

FIGURE 2. Divisor trees for 96 and 100. There are $a(96) = 224$ squares in the left tree and $a(100) = 52$ squares in the right. The sum of the side lengths of the squares, or one-fourth of the tree perimeter, is $b(96) = 768$ in the left tree and $b(100) = 340$ in the right. Divisor trees can be generated for any number n at lims.ac.uk/recursively-divisible-numbers.

2.4. **Properties of divisor trees.** In order to establish properties of the number and sum of recursive divisors, it is helpful to consider a more fine-grained description of divisor trees, namely, the number and sum of divisors—or squares in the divisor tree—of a given size.

Proposition 1. *Let the number of recursive divisors of size k be $a(n,k)$. Then $a(n,n)=1$ and for $k\lfloor n$,*

$$a(n,k)=\sum_{m\lfloor n} a(m,k),$$

and $a(n,k)=0$ otherwise, where $k\lfloor n$ means k is a proper divisor of n.

Proof. The Proposition follows immediately from consideration of the divisor trees. □

Lemma 1. *The number of recursive divisors of size k satisifes*

$$a(k\,n,k)=a(n,1).$$

Proof. By Proposition 1,

$$a(n,1)=\sum_{m\lfloor n} a(m,1) \tag{2}$$

and

$$a(k\,n,k)=\sum_{m\lfloor k\,n} a(m,k).$$

Since $a(m,k)=0$ if k does not divide m, this can be rewritten as

$$a(k\,n,k)=\sum_{m\lfloor n} a(k\,m,k). \tag{3}$$

Let the prime omega function $\Omega(n)$ sum the exponents in the prime factorization of n, that is, for $n=p_1^{\alpha_1}\,p_2^{\alpha_2}\ldots p_j^{\alpha_j}$, $\Omega(n)=\alpha_1+\alpha_2+\ldots+\alpha_j$. I prove the theorem by induction on $\Omega(n)$. The base case $\Omega(n)=0$, or $n=1$, holds by Proposition 1: $a(k\cdot 1,k)=a(1,1)$. I now show that if $a(k\,n,k)=a(n,1)$ for all n such that $\Omega(n)<i$, then $a(k\,n,k)=a(n,1)$ for all n such that $\Omega(n)<i+1$. To see why, observe that in (3) all of the proper divisors m of n must have $\Omega(m)$ at most $\Omega(n)-1$. Therefore by assumption all of the $a(k\,m,k)$ in (3) reduce to $a(m,1)$, and the right side of (3) takes the form of the right side of (2) and thus equals $a(n,1)$. □

Lemma 2. *For $n>1$, the number of recursive divisors of size 1 is equal to half the total number of recursive divisors, that is,*

$$a(n,1)=a(n)/2.$$

Note that recursive divisors of size 1 are equivalent to squares of size 1 in the divisor trees.

Proof. Clearly

$$a(n)=\sum_{m|n} a(n,m).$$

By Lemma 1, this becomes

$$a(n)=\sum_{m|n} a(n/m,1)=\sum_{m|n} a(m,1), \tag{4}$$

which we can equally express as

$$a(n) = a(n,1) + \sum_{m \lfloor n} a(m,1). \tag{5}$$

Inserting Proposition 1 with $k = 1$ into the above,

$$a(n) = 2a(n,1). \quad \square$$

2.5. **Relation to the number of ordered factorizations.** Here I show that for $n > 1$, the number of recursive divisors $a(n)$ is twice the number of ordered factorizations into integers greater than one, which I call $g(n)$. But before getting to that, I first introduce $g(n)$ and mention some of the work on it.

The number of ordered factorizations $g(n)$ satisifes $g(1) = 1$ and

$$g(n) = \sum_{m \lfloor n} g(m).$$

For example, 12 can be written as the product of integers greater than one in eight ways: $12 = 6 \cdot 2 = 2 \cdot 6 = 4 \cdot 3 = 3 \cdot 4 = 3 \cdot 2 \cdot 2 = 2 \cdot 3 \cdot 2 = 2 \cdot 2 \cdot 3$. So $g(12) = 8$.

Kalmar [8] was the first to consider $g(n)$, and it was later studied more systematically by Hille [9]. Over the last 80 years several authors have extended Hille's results [10, 11, 12], some of which we will mention later.

Theorem 1. *Let $g(n)$ be the number of ordered factorizations into integers greater than one and set $g(1) = 1$. Then for $n > 1$,*

$$a(n) = 2g(n).$$

Proof. The definition of $g(n)$ is identical to Proposition 1 for $k = 1$, that is, identical to $a(n,1)$. Since $g(1) = a(1,1) = 1$, the proof follows directly from Lemma 2. $\square$

2.6. **Relation between the number and sum of recursive divisors.** The $b(n)$ are more difficult to calculate than the $a(n)$, and it would be helpful to have an expression relating the $b(n)$ to the $a(n)$. Here I give just such a relation. I will use it later to explicitly determine $b(n)$ for certain values of n.

Theorem 2. *Let $B(n) = b(n)/n$ and $A(n) = a(n)/n$. Then*

$$B(n) = \frac{1}{2} + \frac{1}{2}\sum_{m|n} A(m).$$

Proof. Let $b(n,m)$ be the sum of the side lengths of squares of size m in the nth divisor tree. Then

$$b(n) = \sum_{m|n} b(n,m) = \sum_{m|n} m\, a(n,m).$$

By Lemma 1,

$$b(n) = \sum_{m|n} m\, a(n/m,1) = n + \sum_{m \lfloor n} m\, a(n/m,1).$$

By Lemma 2,

$$b(n) = n + \frac{1}{2}\sum_{m \lfloor n} m\, a(n/m) = \frac{n}{2} + \frac{1}{2}\sum_{m|n} m\, a(n/m) = \frac{n}{2} + \frac{n}{2}\sum_{m|n} a(m)/m.$$

With $B(n) = b(n)/n$ and $A(n) = a(n)/n$, the theorem follows. $\square$

3. Number of recursive divisors

The number of recursive divisors $a(n)$ is not as readily determined as the number of divisors $d(n)$. The first 96 values of $a(n)$ are shown in Table 1, which also gives a concise Mathematica algorithm for generating them. The first 100,000 values are plotted in Figure 3.

3.1. **Distinct primes.** Let $n = p_1\, p_2 \ldots p_k$ be the product of k distinct primes. By (1), the number of divisors is simply $d(p_1\, p_2 \ldots p_k) = 2^k$. Here I calculate the less straightforward $a(p_1\, p_2 \ldots p_k)$.

Theorem 3. *Let $n = p_1\, p_2 \ldots p_k$ be the product of k distinct primes. Then the exponential generating function of $a(p_1 p_2 \ldots p_k)$ is*

$$\mathrm{EG}(a(p_1 p_2 \ldots p_k), x) = \frac{e^x}{2 - e^x}.$$

Proof. This theorem is equivalent to

$$a(p_1 p_2 \ldots p_k) = 1 + \sum_{i=0}^{k-1} \binom{k}{i} a(p_1 p_2 \ldots p_i). \tag{6}$$

I prove it by induction. First note that $a(p_1) = 1 + \binom{1}{0} a(1) = 2$. I now show that if (6) is true for k, then it is true for $k+1$. I do so by adding to $a(p_1 p_2 \ldots p_k)$ all of the

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	17	2	33	6	49	4	65	6	81	16
2	2	18	16	34	6	50	16	66	26	82	6
3	2	19	2	35	6	51	6	67	2	83	2
4	4	20	16	36	52	52	16	68	16	84	88
5	2	21	6	37	2	53	2	69	6	85	6
6	6	22	6	38	6	54	40	70	26	86	6
7	2	23	2	39	6	55	6	71	2	87	6
8	8	24	40	40	40	56	40	72	152	88	40
9	4	25	4	41	2	57	6	73	2	89	2
10	6	26	6	42	26	58	6	74	6	90	88
11	2	27	8	43	2	59	2	75	16	91	6
12	16	28	16	44	16	60	88	76	16	92	16
13	2	29	2	45	16	61	2	77	6	93	6
14	6	30	26	46	6	62	6	78	26	94	6
15	6	31	2	47	2	63	16	79	2	95	6
16	16	32	32	48	96	64	64	80	96	96	224

Table 1. The first 96 values of the number of recursive divisors $a(n)$. A concise Mathematica algorithm for the $a(n)$ is as follows: `n = 2; max = 96; a = {1}; While[n <= max, a = Append[a, 1 + Total[Part[a, Delete[Divisors[n], -1]]]]; n++]; a`

as for divisors which include p_{k+1}, apart from $a(p_1p_2\dots p_{k+1})$, since $p_1p_2\dots p_{k+1}$ is not a proper divisor of itself. I also must add $a(p_1p_2\dots p_k)$, which is left out of the expression for $a(p_1p_2\dots p_k)$. This adds up to $a(p_1p_2\dots p_{k+1})$, and we can write

$$\begin{aligned}
a(p_1p_2\dots p_{k+1}) &= a(p_1p_2\dots p_k) + a(p_1p_2\dots p_k)\\
&\quad + \binom{k}{0}a(p_1) + \binom{k}{1}a(p_1p_2) + \dots + \binom{k}{k-1}a(p_1p_2\dots p_k)\\
&= 1 + a_1(p_1p_2\dots p_k) + \sum_{i=0}^{k-1}\binom{k}{i}a(p_1p_2\dots p_i) + \sum_{i=0}^{k-1}\binom{k}{i}a(p_1p_2\dots p_{i+1})\\
&= 1 + (k+1)a_1(p_1p_2\dots p_k) + \sum_{i=0}^{k-1}\left(\binom{k}{i} + \binom{k}{i-1}\right)a(p_1p_2\dots p_i)\\
&= 1 + (k+1)a_1(p_1p_2\dots p_k) + \sum_{i=0}^{k-1}\binom{k+1}{i}a(p_1p_2\dots p_i)\\
&= 1 + \sum_{i=0}^{k}\binom{k+1}{i}a(p_1p_2\dots p_i). \quad \square
\end{aligned}$$

So for the product of $k = 1, 2, \dots$ distinct primes, $a(k) = 2, 6, 26, 150, 1082, 9366, \dots$, which is the number of ordered set partitions of subsets of $\{1, \dots, n\}$ (OEIS A000629 [13]).

3.2. **Primes to a power.** When n is equal to the product of primes to powers, $a(n)$ is governed by recursion relations relating it to values of $a(n)$ for primes to lower powers.

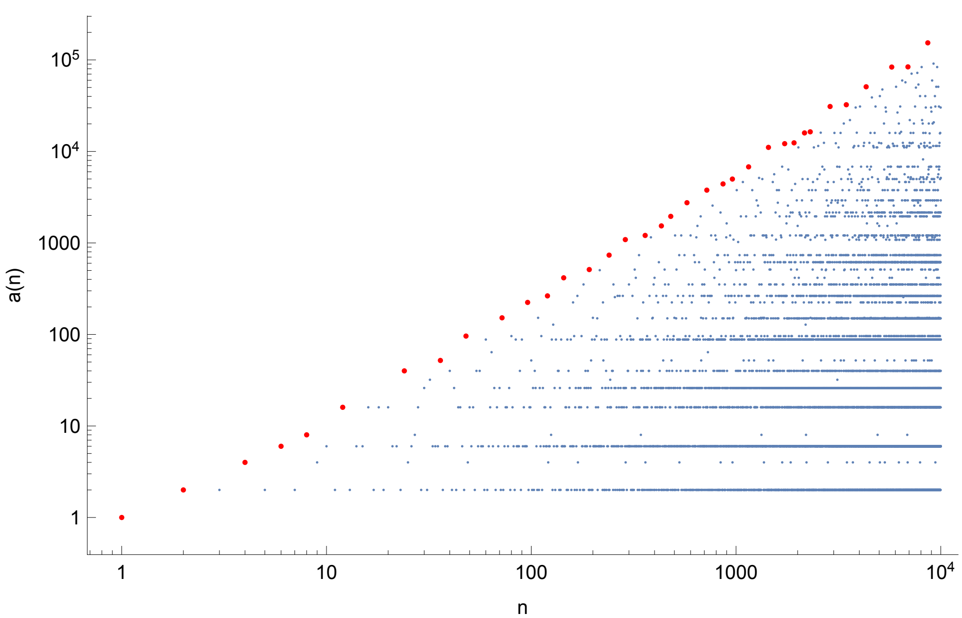

FIGURE 3. The number of recursive divisors $a(n)$. The recursively highly composite numbers, which satisfy $a(n) > a(m)$ for all $m < n$, are the big red points.

Theorem 4. *Let p, q and r be prime. Then*

$$\begin{aligned}
a(p^c) &= 2a(p^{c-1}),\\
a(p^cq^d) &= 2\Big(a(p^{c-1}q^d) + a(p^cq^{d-1}) - a(p^{c-1}q^{d-1})\Big),\\
a(p^cq^dr^e) &= 2\Big(a(p^{c-1}q^dr^e) + a(p^cq^{d-1}r^e) + a(p^cq^dr^{e-1}),\\
&- a(p^cq^{d-1}r^{e-1}) - a(p^{c-1}q^dr^{e-1}) - a(p^{c-1}q^{d-1}r^e),\\
&+ a(p^{c-1}q^{d-1}r^{e-1})\Big).
\end{aligned}$$

Analogous recursion relations apply for the product of four and more primes to powers.

Proof. I first prove the case of $n = p^c$. From Definition 1,

$$a(p^c) = 1 + \sum_{i=0}^{c-1} a(p^i). \tag{7}$$

Adding $a(p^c)$ to both sides,

$$2a(p^c) = 1 + \sum_{i=0}^{c} a(p^i).$$

With $c \to c-1$,

$$\sum_{i=0}^{c-1} a(p^i) = 2a(p^{c-1}) - 1,$$

which, when inserted into (7), gives

$$a(p^c) = 2a(p^{c-1}).$$

Using this, the recursion relation for $n = p^cq^d$ can be proved, which can in turn be used to prove the case for $n = p^cq^dr^e$, and so on. The approach is similar to, but somewhat simpler than, that used to prove Theorem 6. However, Hille [9] and Chor *et al.* [10] proved that identical recursion relations govern $g(n)$, the number of ordered factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and inserting this into Hille's and Chor's recursion relations gives the desired results. □

Corollary 1. *Let τ be the maximum exponent in the prime factorization of n. Then 2^τ divides $a(n)$.*

Proof. Notice that all of the recursion relations in Theorem 4 have a factor of 2 on the right-hand side. The corollary is implied by iterating the recursion relation τ times. On each iteration, the exponents on the right are reduced by at most 1. Now iterate until the smallest exponent is reduced to 0. Then the exponent disappears since, for example, $a(p^cq^0) = a(p^c)$. Continuing this process, the recursion relations are iterated τ times, giving τ factors of 2. □

In Table 3, which shows the high water marks for $a(n)$, the $a(n)$ are expressed as a product of 2^τ and an integer. Note that there are in general no other guaranteed divisors of $a(n)$, since in many cases $a(n)$ is the product of 2^τ and a prime.

The recursion relations in Theorem 4 can be solved for one, two and three primes to powers.

Theorem 5. *Let p, q and r be prime. Then*

$$\begin{aligned} a(p^c) &= 2^c \\ a(p^c q^d) &= 2^c \sum_{i=0}^{d} \binom{d}{i}\binom{c+i}{i}. \\ a(p^c q^d r^e) &= \sum_{j=0}^{d} (-1)^j \binom{d}{j}\binom{c+d-j}{d} a(p^{c+d-j} r^e). \end{aligned}$$

Proof. The result for $n = p^c$ follows by inspection. For $n = p^c q^d$ and $n = p^c q^d r^e$, Chor *et al.* [10] give the analogous results for $g(n)$, the number of ordered factorizations into integers greater than one. From Theorem 1, $a(n) = 2g(n)$, and applying this to Chor's results gives the desired recurrence relations. □

4. Sum of recursive divisors

The first 96 values of the sum of recursive divisors $b(n)$ are shown in Table 2, which also gives a concise Mathematica algorithm for generating them. The first 100,000 values are plotted in Figure 4.

4.1. **Primes to a power.** When n is equal to the product of primes to powers, $b(n)$ is governed by recursion relations relating it to values of $b(n)$ for primes to lower powers. The recursion relations are similar to those for $a(n)$, but are more complex.

Theorem 6. *Let p, q and r be prime. Then*

$$b(p^c) = 2b(p^{c-1}) + (p-1)p^{c-1},$$

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	17	18	33	50	49	58	65	86	81	146
2	3	18	54	34	56	50	112	66	188	82	128
3	4	19	20	35	50	51	74	67	68	83	84
4	8	20	58	36	176	52	122	68	154	84	430
5	6	21	34	37	38	53	54	69	98	85	110
6	14	22	38	38	62	54	190	70	184	86	134
7	8	23	24	39	58	55	74	71	72	87	122
8	20	24	116	40	156	56	196	72	524	88	276
9	14	25	32	41	42	57	82	73	74	89	90
10	20	26	44	42	132	58	92	74	116	90	432
11	12	27	46	43	44	59	60	75	144	91	114
12	42	28	74	44	106	60	346	76	170	92	202
13	14	29	30	45	96	61	62	77	98	93	130
14	26	30	104	46	74	62	98	78	216	94	146
15	26	31	32	47	48	63	124	79	80	95	122
16	48	32	112	48	304	64	256	80	400	96	768

Table 2. The first 96 values of the sum of recursive divisors $b(n)$. A concise Mathematica algorithm for the $b(n)$ is as follows: `n = 2; max = 100; b = {1}; While[n <= max, b = Append[b, n + Total[Part[b, Delete[Divisors[n], -1]]]]; n++ ]; b`

$$\begin{aligned}
b(p^c q^d) &= 2\Big(b(p^{c-1}q^d) + b(p^c q^{d-1}) - b(p^{c-1}q^{d-1})\Big) + (p-1)(q-1)p^{c-1}q^{d-1}, \\
b(p^c q^d r^e) &= 2\Big(b(p^{c-1}q^d r^e) + a(p^c q^{d-1} r^e) + a(p^c q^d r^{e-1}) \\
&- b(p^c q^{d-1} r^{e-1}) - a(p^{c-1} q^d r^{e-1}) - a(p^{c-1} q^{d-1} r^e) \\
&+ b(p^{c-1} q^{d-1} r^{e-1})\Big) \\
&+ (p-1)(q-1)(r-1)p^{c-1}q^{d-1}r^{e-1}.
\end{aligned}$$

Proof. I first prove the case of $n = p^c$. From Definition 2,

$$b(p^c) = p^c + \sum_{i=0}^{c-1} b(p^i). \tag{8}$$

Adding $b(p^c)$ to both sides,

$$2b(p^c) = p^c + \sum_{i=0}^{c} b(p^i).$$

With $c \to c-1$,

$$\sum_{i=0}^{c-1} b(p^i) = 2b(p^{c-1}) - p^{c-1},$$

which, when inserted into (8), gives the desired recurrence relation.

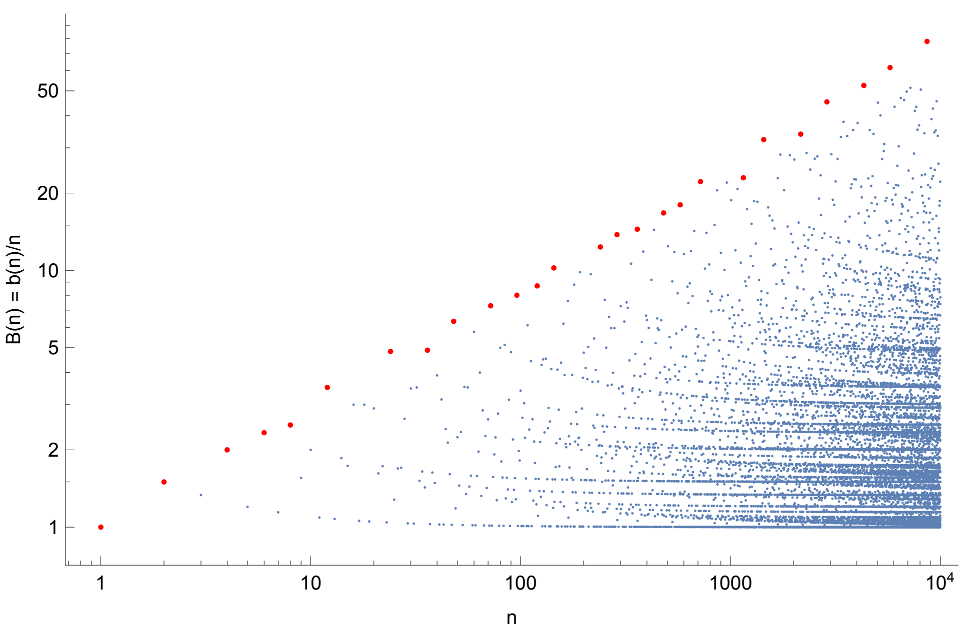

FIGURE 4. The sum of recursive divisors $b(n)$, normalized by n. The recursively superabundant numbers, which satisfy $b(n)/n > b(m)/m$ for all $m < n$, are the big red points.

I now prove the case of $n = p^c q^d$. From Definition 2,

$$b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d-1} b(p^c q^j). \tag{9}$$

Adding $b(p^c q^d)$ to both sides,

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) + \sum_{j=0}^{d} b(p^c q^j), \tag{10}$$

which we can equally write

$$2b(p^c q^d) = p^c q^d + \sum_{i=0}^{c} \sum_{j=0}^{d} b(p^i q^j). \tag{11}$$

With $d \to d-1$ in (10), we find

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - p^c q^{d-1} - \sum_{j=0}^{c-1} \sum_{i=0}^{d-1} b(p^i q^j). \tag{12}$$

With $c \to c-1$ and $d \to d-1$ in (11), we find

$$\sum_{i=0}^{c-1} \sum_{j=0}^{d-1} b(p^i q^j) = 2b(p^{c-1} q^{d-1}) - p^{c-1} q^{d-1},$$

which inserting into (12) yields

$$\sum_{j=0}^{d-1} b(p^c q^j) = 2b(p^c q^{d-1}) - 2b(p^{c-1} q^{d-1}) + (1-p) p^{c-1} q^{d-1}. \tag{13}$$

With $c \to c-1$ in (11), we find

$$\sum_{i=0}^{c-1} \sum_{j=0}^{d} b(p^i q^j) = 2b(p^{c-1} q^d) - p^{c-1} q^d. \tag{14}$$

Inserting (13) and (14) into (9) gives the desired recursion relation.

For $n = p^c q^d r^e$, the proof is similar to that for $n = p^c q^d$ and is omitted here. □

4.2. **Explicit values.** The recursion relations in Theorem 6 can be solved. I only give the results for $n = p^c$ and $n = p^c q^d$. For $n = p^c q^d r^e$, the solution is more intricate but can be solved in a similar way to that for $n = p^c q^d$. The expressions simplify when $p = 2$.

Theorem 7. *Let p and q be prime, and $B(n) = b(n)/n$. Then*

$$\begin{aligned}
B(p^c) &= \frac{p-1-(2/p)^c}{p-2} \quad \text{for } p \text{ odd}, \\
B(2^c) &= (c+2)/2, \\
B(p^c q^d) &= \frac{1}{2} + \frac{1}{2} \sum_{i=0}^{c} \frac{2^i}{p^i} \sum_{j=0}^{d} \frac{1}{q^j} \sum_{k=0}^{j} \binom{i+k}{k} \binom{j}{k}, \\
B(2^c q^d) &= \frac{1}{2} + \frac{1}{2} \sum_{j=0}^{d} \frac{1}{q^j} \sum_{k=0}^{j} \binom{j}{k} \binom{c+k+1}{k+1}.
\end{aligned}$$

Proof. I first prove the case of $n = p^c$. From Theorem 2,

$$B(p^c) = \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c} A(p^i) \tag{15}$$

From Theorem 4, $a(p^c) = 2^c$ and $A(p^i) = a(p^i)/p^i = (2/p)^i$. Inserting this into (15), we find

$$B(p^c) \quad = \quad \frac{p - 1 - (2/p)^c}{p-2}.$$

For $p = 2$ this is indeterminate but, by L'Hôpital's rule,

$$B(2^c) = (c+2)/2.$$

I now prove the case of $n = p^c q^d$. From Theorem 2,

$$B(p^c q^d) = \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c}\sum_{j=0}^{d} A(p^i q^j). \tag{16}$$

Theorem 5 gives $a(p^c q^d)$ explicitly. Inserting $A(p^i q^j) = a(p^i q^j)/(p^i q^j)$ into (16) yields

$$B(p^c q^d) = \frac{1}{2} + \frac{1}{2}\sum_{i=0}^{c}\frac{2^i}{p^i}\sum_{j=0}^{d}\frac{1}{q^j}\sum_{k=0}^{j}\binom{i+k}{k}\binom{j}{k}.$$

For $p = 2$, this simplies:

$$\begin{aligned} B(2^c q^d) \quad &= \quad \frac{1}{2} + \frac{1}{2}\sum_{j=0}^{d}\frac{1}{q^j}\sum_{k=0}^{j}\binom{j}{k}\sum_{i=0}^{c}\binom{i+k}{k} \\ &= \quad \frac{1}{2} + \frac{1}{2}\sum_{j=0}^{d}\frac{1}{q^j}\sum_{k=0}^{j}\binom{j}{k}\binom{c+k+1}{k+1}. \quad \square \end{aligned}$$

5. Recursively highly composite and recursively super-abundant numbers

5.1. **Highly composite and super-abundant numbers.** I briefly review highly composite and super-abundant numbers before considering their recursive analogues. A number n is highly composite if it has more divisors than any of its predecessors, that is, $d(n) > d(m)$ for all $m < n$. These are shown in the right-hand column of Table 3. A number n is super-abundant if the sum of its divisors, normalized by n, is greater than that of any of its predecessors, that is, $\sigma(n)/n > \sigma(m)/m$ for all $m < n$. These are the starred numbers in the right-hand column of Table 3. Both types of numbers have been extensively studied by Ramanujan and others [5, 6]. For sufficiently small values of n, super-abundant numbers are also highly composite, but later this ceases to be the case. The first super-abundant number that is not highly composite is 1,163,962,800 (A166735 [13]), and in fact only 449 numbers are both super-abundant and highly composite (A166981 [13]).

n		.	$a(n)$	n			$d(n)$
$*1$	$=$	1	1	$*1$	$=$	1	1
$*2$	$=$	2	$1 \cdot 2$	$*2$	$=$	2	2
$*4$	$=$	2^2	$1 \cdot 2^2$	$*4$	$=$	2^2	3
$*6$	$=$	$2 \cdot 3$	$3 \cdot 2$	$*6$	$=$	$2 \cdot 3$	4
8	$=$	2^3	$1 \cdot 2^3$				
$*12$	$=$	$2^2 \cdot 3$	$4 \cdot 2^2$	$*12$	$=$	$2^2 \cdot 3$	6
$*24$	$=$	$2^3 \cdot 3$	$5 \cdot 2^3$	$*24$	$=$	$2^3 \cdot 3$	8
$*36$	$=$	$2^2 \cdot 3^2$	$13 \cdot 2^2$	$*36$	$=$	$2^2 \cdot 3^2$	9
$*48$	$=$	$2^4 \cdot 3$	$6 \cdot 2^4$	$*48$	$=$	$2^4 \cdot 3$	10
				$*60$	$=$	$2^2 \cdot 3 \cdot 5$	12
72	$=$	$2^3 \cdot 3^2$	$19 \cdot 2^3$				
96	$=$	$2^5 \cdot 3$	$7 \cdot 2^5$				
$*120$	$=$	$2^3 \cdot 3 \cdot 5$	$33 \cdot 2^3$	$*120$	$=$	$2^3 \cdot 3 \cdot 5$	16
144	$=$	$2^4 \cdot 3^2$	$26 \cdot 2^4$				
				$*180$	$=$	$2^2 \cdot 3^2 \cdot 5$	18
192	$=$	$2^6 \cdot 3$	$8 \cdot 2^6$				
$*240$	$=$	$2^4 \cdot 3 \cdot 5$	$46 \cdot 2^4$	$*240$	$=$	$2^4 \cdot 3 \cdot 5$	20
288	$=$	$2^5 \cdot 3^2$	$34 \cdot 2^5$				
$*360$	$=$	$2^3 \cdot 3^2 \cdot 5$	$151 \cdot 2^3$	$*360$	$=$	$2^3 \cdot 3^2 \cdot 5$	24
432	$=$	$2^4 \cdot 3^3$	$96 \cdot 2^4$				
480	$=$	$2^5 \cdot 3 \cdot 5$	$61 \cdot 2^5$				
576	$=$	$2^6 \cdot 3^2$	$43 \cdot 2^6$				
$*720$	$=$	$2^4 \cdot 3^2 \cdot 5$	$236 \cdot 2^4$	$*720$	$=$	$2^4 \cdot 3^2 \cdot 5$	30
				$*840$	$=$	$2^3 \cdot 3 \cdot 5 \cdot 7$	32
864	$=$	$2^5 \cdot 3^3$	$138 \cdot 2^5$				
960	$=$	$2^6 \cdot 3 \cdot 5$	$78 \cdot 2^6$				
$*1152$	$=$	$2^7 \cdot 3^2$	$53 \cdot 2^7$				
				$*1260$	$=$	$2^2 \cdot 3^2 \cdot 5 \cdot 7$	36
$*1440$	$=$	$2^5 \cdot 3^2 \cdot 5$	$346 \cdot 2^5$				
				$*1680$	$=$	$2^4 \cdot 3 \cdot 5 \cdot 7$	40
1728	$=$	$2^6 \cdot 3^3$	$190 \cdot 2^6$				
1920	$=$	$2^7 \cdot 3 \cdot 5$	$97 \cdot 2^7$				
$*2160$	$=$	$2^4 \cdot 3^3 \cdot 5$	$996 \cdot 2^4$				
2304	$=$	$2^8 \cdot 3^2$	$64 \cdot 2^8$				
				$*2520$	$=$	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	48
$*2880$	$=$	$2^6 \cdot 3^2 \cdot 5$	$484 \cdot 2^6$				
3456	$=$	$2^7 \cdot 3^3$	$253 \cdot 2^7$				
$*4320$	$=$	$2^5 \cdot 3^3 \cdot 5$	$1590 \cdot 2^5$				
				$*5040$	$=$	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	60
$*5760$	$=$	$2^7 \cdot 3^2 \cdot 5$	$653 \cdot 2^7$				
6912	$=$	$2^8 \cdot 3^3$	$328 \cdot 2^8$				
				7560	$=$	$2^3 \cdot 3^3 \cdot 5 \cdot 7$	64
$*8640$	$=$	$2^6 \cdot 3^3 \cdot 5$	$2402 \cdot 2^6$				
				$*10080$	$=$	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	72
$*11520$	$=$	$2^8 \cdot 3^2 \cdot 5$	$856 \cdot 2^8$				

TABLE 3. The left column shows the recursively highly composite numbers and the recursively super-abundant numbers (which are starred) up to one million. All of the recursively super-abundant numbers shown here are also recursively highly composite, apart from one, 181,440. The right column shows the highly composite numbers and the super-abundant numbers (which are starred) up to one million. All of the super-abundant numbers shown here are also highly composite.

n			$a(n)$
$*17280$	$=$	$2^7 \cdot 3^3 \cdot 5$	$3477 \cdot 2^7$
23040	$=$	$2^9 \cdot 3^2 \cdot 5$	$1096 \cdot 2^9$
$*25920$	$=$	$2^6 \cdot 3^4 \cdot 5$	$10368 \cdot 2^6$
$*30240$	$=$	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	$20874 \cdot 2^5$
$*34560$	$=$	$2^8 \cdot 3^3 \cdot 5$	$4864 \cdot 2^8$
46080	$=$	$2^{10} \cdot 3^2 \cdot 5$	$1376 \cdot 2^{10}$
$*51840$	$=$	$2^7 \cdot 3^4 \cdot 5$	$15979 \cdot 2^7$
$*60480$	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	$34266 \cdot 2^6$
$*69120$	$=$	$2^9 \cdot 3^3 \cdot 5$	$6616 \cdot 2^9$
86400	$=$	$2^7 \cdot 3^3 \cdot 5^2$	$28481 \cdot 2^7$
$*103680$	$=$	$2^8 \cdot 3^4 \cdot 5$	$23692 \cdot 2^8$
$*120960$	$=$	$2^7 \cdot 3^3 \cdot 5 \cdot 7$	$53485 \cdot 2^7$
138240	$=$	$2^{10} \cdot 3^3 \cdot 5$	$8790 \cdot 2^{10}$
161280	$=$	$2^9 \cdot 3^2 \cdot 5 \cdot 7$	$17656 \cdot 2^9$
$*172800$	$=$	$2^8 \cdot 3^3 \cdot 5^2$	$42520 \cdot 2^8$
$*207360$	$=$	$2^9 \cdot 3^4 \cdot 5$	$34026 \cdot 2^9$
$*241920$	$=$	$2^8 \cdot 3^3 \cdot 5 \cdot 7$	$80176 \cdot 2^8$
276480	$=$	$2^{11} \cdot 3^3 \cdot 5$	$11447 \cdot 2^{11}$
311040	$=$	$2^8 \cdot 3^5 \cdot 5$	$103540 \cdot 2^8$
$*345600$	$=$	$2^9 \cdot 3^3 \cdot 5^2$	$61436 \cdot 2^9$
$*362880$	$=$	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	$267219 \cdot 2^7$
$*414720$	$=$	$2^{10} \cdot 3^4 \cdot 5$	$47576 \cdot 2^{10}$
$*483840$	$=$	$2^9 \cdot 3^3 \cdot 5 \cdot 7$	$116256 \cdot 2^9$
552960	$=$	$2^{12} \cdot 3^3 \cdot 5$	$14652 \cdot 2^{12}$
604800	$=$	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	$480953 \cdot 2^7$
622080	$=$	$2^9 \cdot 3^5 \cdot 5$	$156278 \cdot 2^9$
691200	$=$	$2^{10} \cdot 3^3 \cdot 5^2$	$86362 \cdot 2^{10}$
$*725760$	$=$	$2^8 \cdot 3^4 \cdot 5 \cdot 7$	$422932 \cdot 2^8$
829440	$=$	$2^{11} \cdot 3^4 \cdot 5$	$65018 \cdot 2^{11}$
$*967680$	$=$	$2^{10} \cdot 3^3 \cdot 5 \cdot 7$	$163934 \cdot 2^{10}$

Recursively super-abundant but not recursively highly composite

$*181440 = 2^6 \cdot 3^4 \cdot 5 \cdot 7$

n			$d(n)$
$*15120$	$=$	$2^4 \cdot 3^3 \cdot 5 \cdot 7$	80
20160	$=$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	84
$*25200$	$=$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7$	90
$*27720$	$=$	$2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	96
45360	$=$	$2^4 \cdot 3^4 \cdot 5 \cdot 7$	100
50400	$=$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7$	108
$*55440$	$=$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	120
83160	$=$	$2^3 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	128
$*110880$	$=$	$2^5 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	144
$*166320$	$=$	$2^4 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	160
221760	$=$	$2^6 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$	168
$*277200$	$=$	$2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	180
$*332640$	$=$	$2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	192
498960	$=$	$2^4 \cdot 3^4 \cdot 5 \cdot 7 \cdot 11$	200
$*554400$	$=$	$2^5 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11$	216
$*665280$	$=$	$2^6 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11$	224
$*720720$	$=$	$2^4 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$	240

5.2. **Recursively highly composite numbers.** By analogy with highly composite numbers, a number n is recursively highly composite if it has more recursive divisors than any of its predecessors.

Definition 3. *A number is recursively highly composite if it satisfies*

$$a(n) > a(m) \text{ for all } m < n.$$

These numbers are shown in the left-hand column of Table 3 up to one million. In terms of divisor trees, a number is recursively highly composite if its divisor tree has more squares than any of its predecessors' divisor trees. Because $a(n)$ depends only on the exponents in the prime factorization of $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_j^{\alpha_j}$, the exponents in recursively highly composite numbers must be non-increasing. For assume that $\alpha_{i+k} < \alpha_i$ in a recursively highly composite number. But $p_i^{\alpha_{i+k}} p_{i+k}^{\alpha_i} < p_i^{\alpha_i} p_{i+k}^{\alpha_{i+k}}$, so there exists an $m < n$ with the same value of $a(n)$, contradicting the assumption. (I take the primes to be in ascending order.)

n	*Technology*		*Standard displays*	
*24	10×24	Tetris		
	24×16	384-well assay		
*48	128×48	TRS 80		
72	72 points/in	Adobe point		
96	96×64	TI-81 calculator		
	96×65	Nokia 1100		
*120	120×160	Nokia 100	160×120	QQVGA
144	144×168	Pebble Time watch		
*240	240×64	Atari Portfolio	320×240	Quarter VGA
288	352×288	Video CD	352×288	CIF
*360	360×360	LG Watch Style	640×360	nHD
480	320×480	iPhone 1–3	640×480	VGA
	480×480	LG Watch Sport		
576	576 lines	PAL analog television	1024×576	WSVGA
*720	720×364	Macintosh XL, Hercules	1280×720	HD
864			1152×864	XGA+
960		Facebook website		
*1152			1152×2048	QWXGA
*1440		3.5" disk block size	2560×1440	Quad HD
1920			1920×1080	Full HD
*2160	2160×1440	Microsoft Surface Pro 3	4096×2160	DCI 4K
			3840×2160	4K Ultra HD
2304	2304×1440	MacBook Retina display		
	4096×2304	21.5" 4K Retina iMac		
*2880	2880×1800	15 in MacBook Pro	5120×2880	5K
3456		Canon EOS 1100D		
*4320			7680×4320	8K Ultra HD
*8640			15360×8640	16K Ultra HD

TABLE 4. Some applications of recursively highly composite numbers and recursively super-abundant numbers (which are starred) in technology and standard displays.

5.3. **Recursively super-abundant numbers.** By analogy with super-abundant numbers, a number n is recursively super-abundant if the sum of its recursive divisors, normalized by n, is greater than that of any of its predecessors.

Definition 4. *A number is recursively super-abundant if it satisfies*

$$b(n)/n > b(m)/m \text{ for all } m < n.$$

The $b(n)$ are the starred numbers in the left-hand column of Table 3. In terms of divisor trees, a number is recursively super-abundant if the perimeter of its divisor tree, normalized such that the largest square has length unity, is bigger than that of any of its predecessors' divisor trees.

For sufficiently small values of n, recursively super-abundant numbers are recursively highly composite. The first exception is at 181,440, and there are likely more. (See the list of open questions at the end.)

5.4. **Applications.** Recursively divisible numbers are especially well suited to design across multiple length scales, in which the whole must be divided into parts, the parts into subparts, and so on. Recursively highly composite and recursively super-abundant numbers are frequently found in technology and standard displays, examples of which are shown in Table 4.

In technology, these numbers are used for the screen resolutions of watches, phones, cameras and computers. They appear in games, such as Tetris, and in high-throughput technologies, such as test tube microplates. As a result, users of these technologies have maximal optionality for dividing the space into parts in a hierarchical way when, for example, building a website, designing a game or planning an experiment.

In display technologies, many standard resolutions use these numbers in the height or width of the display, measured in pixels. Because standard displays tend to preserve certain aspect ratios, such as 16:9, it is usually not possible for both numbers to be either recursively highly composite or recursively super-abundant.

5.5. **Ten open questions.** In this paper I introduced and studied the recursive divisor function and recursively divisible numbers. There are many open questions on this topic, and I list 10 here.

Question 1. *For what values of n does a divisor tree overlap itself?*

Question 2. *For what values of n do divisor trees have an exact or approximate fractal dimension?*

Question 3. *For $x = 2$ in the definition of the recursive divisor function, is $\sigma_2(n)/n^2$ bounded? If so, what is the bound?*

Question 4. *What are the recursion relations for $\sigma_2(n)$ for n equal to primes to a power?*

Question 5. *Can the relation between $a(n)/n$ and $b(n)/n$ in Theorem 2 be generalized to other values of x in the definition of the recursive divisor function?*

Question 6. *What is the recursion relation for $b(n)$ when n is the product of k distinct primes?*

Question 7. *Are there a finite number of numbers that are both recursively highly composite and recursively super-abundant, like the highly composite and super-abundant numbers (OEIS A166981* [13]*)?*

Question 8. *If the former statement is true, what is the largest number that is both recursively highly composite and recursively super-abundant?*

Question 9. *What is the largest number that is both highly composite and recursively highly composite? And that is both super-abundant and recursively super-abundant? (Table 3 suggests both might be 720.)*

Question 10. *How do the shapes of recursively highly composite and recursively super-abundant numbers differ from each other, in terms of the distribution of exponents in their prime factorization?*

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