

New Bounds on Antipowers in Binary Words

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Abstract

Fici et al. defined a word to be a k -power if it is the concatenation of k consecutive identical blocks, and an r -antipower if it is the concatenation of r pairwise distinct blocks of the same size. They defined $N(k, r)$ as the shortest length ℓ such that every binary word of length ℓ contains either a k -power or an r -antipower. In this note we obtain some new upper and lower bounds on $N(k, r)$.

1 Introduction

Regularities and repetitions have been studied extensively in the field of combinatorics on words. One of the early results in the area is Thue's observation that while every sufficiently long binary word contains a square, in contrast there are arbitrarily long words over a ternary alphabet avoiding squares [1, 9, 10]. In this context, *avoidance* of a certain set of words means that none of the words of this set appears as a factor. Thue's results show that avoidance of powers depends on the alphabet size. In this note, we focus solely on binary words. The study of avoidance of patterns has been extended to k -powers, i.e., nonempty words of the form u^k , and other variants of the problem; see e.g., [3, 4, 6].

In [5], Fici et al. introduced the notion of antipowers. Whereas a power is a sequence of adjacent blocks that are all the same, an antipower is a sequence of consecutive adjacent blocks of the same length that are pairwise different. Formally, a k -antipower is a word of the form $u_1 \cdots u_k$ such that $|u_1| = \cdots = |u_k|$ and the factors u_1, \dots, u_k are pairwise distinct, i.e., $|\{u_1, \dots, u_k\}| = k$.

Fici et al. also suggested investigating the simultaneous avoidance of powers and antipowers, which is the main topic of this work. They defined $N(k, r)$ to be the shortest

length ℓ such that every binary word of length ℓ contains either a k -power or an r -antipower as a factor. It is known that $N(k, r)$ is bounded polynomially in k and r [5]. The available numerical evidence (see [7]) suggests that for each fixed $r \geq 2$ we have

$$(2r - 4)k \leq N(k, r) \leq (2r - 4)k + f(r)$$

for some function $f: \mathbb{N} \rightarrow \mathbb{N}$. We first prove the lower bound $N(k, r) \geq (2r - 4)k$ for $k \geq 4$. Then we show that this bound is tight for $r = 3$.

This paper is based, in part, on the master's thesis of the second author [7].

2 A lower bound on $N(k, r)$

To prove the desired lower bound, we give an explicit family of binary words $(x_{k,r})$ and use combinatorial arguments to show that $x_{k,r}$ avoids k -powers and r -antipowers.

Theorem 1. *Let $r \geq 3$ and $k \geq \max\{r - 1, 4\}$. Define $x_{k,r} = ((01)^{k-1}00)^{r-3}(01)^{k-1}0$. Then $|x_{k,r}| = 2k(r - 2) - 1$, but $x_{k,r}$ has no k -powers, nor r -antipowers.*

Proof. First, we argue that $x_{k,r}$ contains no k -power y^k . Suppose it does. Clearly, $x_{k,r}$ does not contain 0000 or 1111, so $|y| \geq 2$. However, every factor of length $2k$ of $x_{k,r}$ contains the word 00, and so y^k contains 00. Thus, y^2 must also contain the factor 00 and y^3 must contain 00 at least twice. If two occurrences of 00 appear at a distance of less than $2k$ in y^3 , then y^k cannot be a factor of $x_{k,r}$. Therefore, y must have length at least $2k$. But then y^k has length at least $2k^2 \geq 2k(r - 1) = 2k(r - 2) + 2k > |x_{k,r}|$, contradicting our assumption.

Next, we argue that $x_{k,r}$ contains no r -antipower. First, observe that in any sequence of non-overlapping blocks in $x_{k,r}$, at most $r - 3$ blocks can contain 00 as a factor. All remaining blocks belong to the set $\{\varepsilon, 1\}(01)^*\{\varepsilon, 0\}$. Since the set $\{\varepsilon, 1\}(01)^*\{\varepsilon, 0\}$ only contains two words of each length, this implies that any sequence of r consecutive blocks of the same length contains at least two identical blocks. We conclude that $x_{k,r}$ contains no r -antipower. \square

This theorem immediately yields a lower bound that matches our conjectured upper bound up to an additive term that only depends on r .

Corollary 2. $N(k, r) \geq (2r - 4)k$ for all $k \geq \max\{r - 1, 4\}$.

Remark 3. In particular we get $N(k, 4) \geq 4k$ for $k \geq 4$. Computations show that equality holds for $11 \leq k \leq 30$. We conjecture that equality holds for all $k \geq 11$.

3 Binary words avoiding 3-antipowers

For a language $L \subseteq \{0, 1\}^*$, we let $\mathcal{C}_n(L)$ denote the (transitive) closure of $L \cap \{0, 1\}^n$ under bitwise complementation and reversal. It consists of all length- n words from L , all bitwise complements of length- n words from L , all reversed length- n words from L and all

bitwise complements of reversed length- n words from L . It is easy to see that if a binary word contains a k -power, then both its bitwise complement and its reversal also contain a k -power. The same argument applies to r -antipowers. Together with the fact that bitwise complementation and reversal are involutions, this implies that $\mathcal{C}_n(L)$ avoids k -powers (resp., r -antipowers) if and only if $L \cap \{0, 1\}^n$ avoids k -powers (resp., r -antipowers).

The definition of \mathcal{C}_n allows us to give a simple description of all binary words avoiding 3-antipowers. Before giving the general result, we only characterize words avoiding 3-antipowers whose lengths are multiples of 3.

Theorem 4. *Let $n \geq 18$ be divisible by 3. Let A_n be the set of binary words of length n avoiding 3-antipowers. Then $A_n = \mathcal{C}_n(0^* \cup (01)^* \cup (01)^*0 \cup 0^*10^* \cup 0^*011 \cup 0^*101)$.*

Proof. Note that it is easy to verify the claim for $n = 18$ by enumerating all words of this length. For larger values, we prove the claim by induction.

Let $n > 18$ with $n \equiv 0 \pmod{3}$. By induction, all factors of length $n - 3$ avoid 3-antipowers. Moreover, it is easy to see that when splitting any word from A_n into three blocks of equal length, at least two of these blocks coincide. Therefore, all words in A_n avoid 3-antipowers. It remains to show that if $w \in \{0, 1\}^n$ avoids 3-antipowers, then $w \in A_n$. Since avoidance of 3-antipowers is invariant under bitwise complementation, we may assume that $|w|_0 \geq |w|_1$. Moreover, if $|w|_0 = |w|_1$, we may assume that w starts with a 0.

We factorize $w = xyz$ with $|x| = |z| = 3$. By induction, xy and yz belong to A_{n-3} . Since w contains at least as many zeroes as ones, and starts with a 0 if the number of zeroes equals the number of ones, this implies that $y \in 0^* \cup (10)^* \cup (10)^*1 \cup 0^*10^*$.

If $y \in 0^*10^*$, then $x = z = 000$, otherwise either $xy \notin A_{n-3}$ or $yz \notin A_{n-3}$. This implies $w \in 0^*10^*$, thus $w \in A_n$. Similarly, if $y \in (10)^* \cup (10)^*1$, then $w \in (01)^* \cup (01)^*1$ and $w \in A_n$.

The remaining case is $y \in 0^*$. Since at least two of the factors w_1, w_2, w_3 in the unique factorization $w = w_1w_2w_3$ with $|w_1| = |w_2| = |w_3|$ must coincide, either $x = 000$ or $z = 000$. Avoidance of 3-antipowers is invariant under reversal, so we may assume $x = 000$. Since $y \in 0^*$ implies that yz belongs to $0^* \cup 0^*10^* \cup 0^*011 \cup 0^*101$ and this set is closed under prepending zeroes, we obtain that w has the desired form. This concludes the proof. \square

We now extend this characterization to words whose lengths are not divisible by 3.

Corollary 5. *Let A_n be the set of binary words of length n avoiding 3-antipowers. Let $k \geq 6$. Then*

- $A_{3k} = \mathcal{C}_{3k}(0^* \cup (01)^* \cup (01)^*0 \cup 0^*10^* \cup 0^*011 \cup 0^*101)$,
- $A_{3k+1} = \mathcal{C}_{3k+1}(0^* \cup (01)^* \cup (01)^*0 \cup 0^*10^* \cup 0^*011 \cup 0^*101 \cup 10^*1)$ and
- $A_{3k+2} = \mathcal{C}_{3k+2}(0^* \cup (01)^* \cup (01)^*0 \cup 0^*10^* \cup 0^*011 \cup 0^*101 \cup 10^*1 \cup 10^*10 \cup 10^*11)$.

In particular, there exist $6k + 12$ (resp., $6k + 16$, $6k + 26$) binary words of length $3k$ (resp., $3k + 1$, $3k + 2$) avoiding 3-antipowers.

Proof. For words of length $3k + 1$, it suffices to investigate their two factors of length $3k$ and apply the previous theorem. Similarly, for words of length $3k + 2$, we investigate all three factors of length $3k$.

We now fix some $k \geq 6$ and count the number of words of length $3k$. Words from 0^* and $(01)^*0$ coincide with their reverses and words from $(01)^*$ coincide with the bitwise complements of their reverses. Thus, $\mathcal{C}_{3k}(0^* \cup (01)^* \cup (01)^*0)$ contains exactly four words. The set $\mathcal{C}_{3k}(0^*10^*)$ contains $6k$ words: all words of the form 0^i10^j or 1^i01^j with $0 \leq i < 3k$ and $i + j = 3k - 1$. The sets 0^*011 and 0^*101 contain exactly one word of length $3k$; and the reverse, the bitwise complement and the bitwise complement of the reverse of each of these words are distinct from one another. Therefore, $\mathcal{C}_{3k}(0^*011 \cup 0^*101)$ contains eight words. Together, this shows that $|A_{3k}| = 4 + 6k + 8 = 6k + 12$.

Since every word from 10^*1 coincides with its reverse but not with its bitwise complement, we obtain $|A_{3k+1}| = 4 + (6k + 2) + 8 + 2 = 6k + 16$. Moreover, every word from $10^*10 \cup 10^*11$ does not coincide with its reverse, its bitwise complement or the bitwise complement of its reverse, so $|A_{3k+2}| = 4 + (6k + 4) + 8 + 2 + 8 = 6k + 26$. \square

Remark 6. Corollary 5 shows that there are only linearly many binary words avoiding 3-antipowers. In contrast, the construction given in [5, Prop. 12] can be adapted to show that $u(n)$, the number of length- n binary words avoiding 4-antipowers, grows faster than any polynomial in n . This can be proved as follows: let $\mathbf{a} = (a_i)_{i \geq 1}$ be any infinite sequence of integers satisfying $a_1 = 1$ and $a_{i+1} \geq 4a_i$. For $j \geq 1$ define the characteristic sequence

$$c_{\mathbf{a}}(j) = \begin{cases} 1, & \text{if } j = a_i \text{ for some } i ; \\ 0, & \text{otherwise,} \end{cases}$$

and $\mathbf{c}_{\mathbf{a}} = c_{\mathbf{a}}(1)c_{\mathbf{a}}(2) \cdots$. Then it is not hard to show, using the ideas in [5, Prop. 12], that every length- n prefix of every $\mathbf{c}_{\mathbf{a}}$ avoids 4-antipowers. Let $t(n)$ be the number of length- n prefixes of words of this form; then $u(n) \geq t(n)$. It is not hard to see that $t(n) = t(n - 1) + t(\lfloor n/4 \rfloor)$ for $n \geq 4$. Then, according to [2], we have $t(n) = n^{\Theta(\log n)}$.

The sequence $t(n)$ is sequence [A330513](#) in the On-Line Encyclopedia of Integer Sequences [8], and the sequence $u(n)$ is sequence [A275061](#). The exact growth rate of $u(n)$ is apparently still not known.

To extend Corollary 5 to infinite words, it suffices to determine all words whose factors belong to the set A_n defined in Theorem 4.

Corollary 7. *The only infinite binary words avoiding 3-antipowers are of the form 0^ω , $(01)^\omega$, 110^ω , 1010^ω , 0^i10^ω for $i \geq 0$, and their binary complements.*

Using Corollary 5, we can prove a tight upper bound for $N(k, 3)$.

Theorem 8. $N(k, 3) = 2k$ for all $k \geq 9$.

Proof. The lower bound follows from Corollary 2. Thus, it suffices to show that every binary word of length $2k$ contains either a k -power or a 3-antipower.

Let $w \in \{0, 1\}^{2k}$ be a word that avoids 3-antipowers. By closure of the sets of k -powers and 3-antipowers under bitwise complementation, we may assume that $|w|_0 \geq |w|_1$. If $|w|_1 \leq 1$, then 0^k is a factor of w . If $|w|_1 > 3$, then $w \in \{(01)^k, (10)^k\}$ by Corollary 5. Moreover, if $|w|_1 \in \{2, 3\}$, then $w = xyz$ for some $x, y, z \in \{0, 1\}^*$ with $|x|, |z| \leq 3$ and $y = 0^{2k-|xz|}$. In any case, w contains a k -power. \square

Our results above only hold for words of length at least 18. With a slightly more technical proof or by computing all words of length less than 18 avoiding 3-antipowers, it can be shown that the characterization given in Corollary 5 can be extended to all words of length at least 12. Moreover, it is easy to see that all words of length ≤ 5 avoid 3-antipowers. For the remaining lengths, the following table lists all words avoiding 3-antipowers up to bitwise complementation and reversal.

n	Words of length n avoiding 3-antipowers, up to complementation and reversal
6	000000, 000100, 001100, 000010, 001010, 010110, 000001, 010001, 011001, 000101, 100101, 010101, 000011
7	0000000, 0001000, 0000100, 0010100, 0000010, 0100010, 0001010, 0101010, 0000001, 1000001, 0010001, 0101001, 0011001, 0000101, 1000101, 0010101, 0000011
8	00000000, 00001000, 00000100, 00010100, 00000010, 00100010, 00001010, 00101010, 01100110, 01010110, 00000001, 10000001, 01000001, 00010001, 01010001, 00101001, 10011001, 00000101, 01000101, 00010101, 10010101, 01010101, 00000011, 10000011, 00110011, 00101011
9	000000000, 000010000, 000001000, 000101000, 000000100, 000000010, 010000010, 010001010, 010101010, 000000001, 001000001, 001010001, 000000101, 101000101, 000000011
10	0000000000, 0000010000, 0000001000, 0000000100, 0000000010, 0010000010, 0010100010, 0000000001, 1000000001, 0001000001, 0001010001, 0000000101, 0101000101, 0101010101, 0000000011
11	00000000000, 00000100000, 00000010000, 00000001000, 00000000100, 00100000100, 00000000010, 00010000010, 00010100010, 01010001010, 01010101010, 00000000001, 10000000001, 01000000001, 00001000001, 10001010001, 00000000101, 00101000101, 00000000011, 10000000011

4 A lower bound for $r = 5$

While the lower bound in Corollary 2 is tight up to an additive constant that only depends on r , better bounds are possible for specific values of r . One such example is given below.

Theorem 9. *Let $k \geq 14$. Then $N(k, 5) \geq 6k + 4$. If, additionally, $k \equiv 5 \pmod{10}$, then $N(k, 5) \geq 6k + 5$.*

Proof. The following table gives, for each $k \geq 14$, a word w_k that contains no k power and no 5-antipower. The construction depends on the equivalence class of $k \pmod{10}$. Therefore, the table contains separate rows for each $i \in \{0, \dots, 9\}$ where $k \equiv i \pmod{10}$.

i	w_k
0	$00(01)^{k-1}00(01)^{k-1}00(01)^{k-1}000$
1, 3, 7, 9	$0^8(01)^{k-3}00(01)^{k-1}00(01)^{k-1}0$
2, 4, 8	$(01)^{(k+2)/2}(10)^{k-1}11(10)^{k-1}1(10)^{(k+2)/2}$
5	$0000(01)^{k-1}00(01)^{k-1}00(01)^{k-1}00$
6	$000(01)^{k-1}11(01)^{k-1}11(01)^{k-1}10$

To see that none of these words contains a k -power, we can use exactly the same argument as in the proof of Theorem 1.

To prove avoidance of 5-antipowers, we also resort to ideas from the proof of Theorem 1 but the arguments are slightly more technical. Let us first investigate the case $k \equiv 0 \pmod{10}$, i.e., words of the form

$$00(01)^{k-1}00(01)^{k-1}00(01)^{k-1}000.$$

By contradiction, assume that such a word contains a factor of the form $u_1u_2u_3u_4u_5$ where all the u_i have the same lengths and are pairwise distinct. In the following, we say that a binary word is *alternating* if (and only if) it belongs to the set $\{\varepsilon, 1\}(01)^*\{\varepsilon, 0\}$.

Note that since the $(01)^{k-1}$ patterns in w_k appear at even distances and since $|u_1u_2|$ and $|u_3u_4|$ have even lengths, only one of the three words u_1, u_3, u_5 can be alternating; otherwise, two of these words would coincide. Similarly, either u_2 or u_4 must not be alternating. On the contrary, since $|w_k| = 6k + 3 \equiv 3 \pmod{5}$, either the prefix 00 or the suffix 00 of w_k does not overlap with the occurrence of $u_1u_2u_3u_4u_5$. Considering all possible factorizations of w_k into five consecutive blocks of equal lengths, it is easy to see that at least two of the words u_1, u_2, u_3, u_4, u_5 must be alternating. Together with the previous observation, this means that exactly one of the words u_1, u_3, u_5 and exactly one of the words u_2, u_4 is alternating. We now analyze the six possible cases.

First, suppose that u_1 and u_2 are alternating. This implies that the u_i have length at most $k - 1$ because every factor of length $2k$ of w_k contains 00 as a prefix or 00 as a suffix or 000 as a factor. But then, it is easy to see that at either u_3 or u_4 are alternating as well, a contradiction. A similar argument applies whenever two adjacent blocks are alternating. The remaining cases are that u_1, u_4 are alternating or u_2, u_5 are alternating. Since they are symmetrical, it suffices to investigate the first case. Considering the possible positions of u_1 and u_4 in w_k , it becomes obvious that then, either u_2 or u_3 has to be alternating as well, again contradicting the observation above.

This concludes the correctness proof of the construction. Similar arguments can be used to prove avoidance of 5-antipowers for the lower bound witnesses given for $i \in \{1, \dots, 9\}$. \square

5 Open problems

The exact asymptotic behavior of $N(k, r)$ remains open. To the best of our knowledge, the best general upper bound known to date is linear in k and cubic in r . However, it is

consistent with this result and our conjecture that the asymptotic behavior is of the form $(2r - 4)k + f(r)$ for some function $f(r) \in \mathcal{O}(r^3)$. While still linear in k and cubic in c , proving an upper bound of the form $ckr + f(r)$ for some constant $c \in \mathbb{N}$ and some function $f(r) \in \mathcal{O}(r^3)$ would be a big step towards proving our conjecture.

It would also be interesting to investigate the growth of f more carefully. Can our lower bound be improved to show that f is unbounded? Can we prove a subcubic upper bound? More numerical evidence might help establish a conjecture on the growth rate of f .

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