Width-k Eulerian polynomials of type A and B and its γ-positivity

Marwa Ben Abdelmaksoud[1] and Adel Hamdi[2]

[1,2]Faculty of Science of Gabes, Department of Mathematics,
Cité Erriadh 6072, Zrig, Gabes, Tunisia
[1]abdelmaksoud.marwa@hotmail.fr [2]aadel_hamdi@yahoo.fr.

Abstract

We define some generalizations of the classical descent and inversion statistics on signed permutations that arise from the work of Sack and Ulfarsson [20] and called after *width-k descents* and *width-k inversions* of type A in Davis's work [8]. Using the aforementioned new statistics, we derive some new generalizations of Eulerian polynomials of type A, B and D. It should also be noticed that we establish the γ-positivity of the *"width-k" Eulerian* polynomials and we give a combinatorial interpretation of finite sequences associated to these new polynomials using quasisymmetric functions and P-partition in Petersen's work [18].

Keywords: Coxeter groups, Eulerian polynomials, Unimodality, permutations, γ-positivity, (enriched) P-partition, quasisymmetric functions.

1 Introduction

The main purpose of this paper is to extend some fundamental aspects of the theory of Eulerian polynomials on Coxeter groups and their unimodality, symmetry and *Gamma*$-$positivity. Many polynomials with combinatorial meanings have been shown to be unimodal (see [5] or [14] for example). Let $\mathcal{A} = \{a_i\}_{i=0}^{d}$ be a finite sequence of nonnegative numbers. Recall that a polynomial $g(x) = \sum_{i=0}^{d} a_i x^i$ of degree d is said to be positive and unimodal, if the coefficients are increasing and then decreasing, i.e., there is a certain index $0 \leq j \leq d$ such that

$$0 \leq a_0 \leq a_1 \leq \ldots \leq a_{j-1} \leq a_j \geq a_{j+1} \geq \ldots \geq a_d \geq 0.$$

We will say that $g(x)$ is *palindromic* (or symmetric) with center of symmetry at $\lfloor d/2 \rfloor$, if $a_i = a_{d-i}$ for all $0 \le i \le d$.
The polynomial $g(x)$ is said to be *Gamma*-positive (or γ-positive) if

$$g(x) = \sum_{i=0}^{\lfloor d/2 \rfloor} \gamma_i x^i (1+x)^{d-2i},$$

for some $d \in \mathbb{N}$ and nonnegative reals $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor d/2 \rfloor}$. So, both palindromic and unimodal are two necessary conditions for the *Gamma*-positivity of $g(x)$. One of the most important polynomials in combinatorics is the nth Eulerian polynomials, defined as

$$A_n(x) = \sum_{\sigma \in S_n} x^{des_A(\sigma)}.$$

For example, $A_6(x) = x^5 + 57x^4 + 302x^3 + 302x^2 + 57x + 1$ is clearly symmetric, positive, and unimodal.

Given a set of combinatorial objects τ, a combinatorial statistics is an integer given to every element of the set. In other words, it is a function $st : \tau \to \mathbb{N}$.
For a statistics st on symmetric group S_n, one may form the generating function :

$$F_n^{st}(x) = \sum_{\pi \in S_n} x^{st(\pi)}.$$

Macmahon [15] considered four different statistics for a permutation σ in the group of all permutation S_n (it is also called type-A permutation) of the set $[n] := \{1, \ldots, n\}$. The number of descents ($des_A(\sigma)$), the number of excedances ($exc_A(\sigma)$), the number of inversions ($inv_A(\sigma)$), and the major index ($maj_A(\sigma)$). Given a permutation σ in S_n, we say that the pair $(i, j) \in [n]^2$ is an inversion of σ if $i < j$ and $\sigma(i) > \sigma(j)$, that $i \in \{1, 2, \ldots, n-1\}$ is a descent if $\sigma(i) > \sigma(i+1)$, and that $i \in \{1, \ldots, n\}$ is an excedance if $\sigma(i) > i$. The major index is the sum of all the descents. These four statistics have many generalization.

It is a well-known fact that the symmetric group S_n, is a *Coxeter group* with respect to the above generating set $\tau := \{\tau_i; 1 \le i \le n-1\}$ where $\tau_i := [1, 2, \ldots, i+1, i, \ldots, n]$. Therefore, the length of a permutation $\sigma \in S_n$ is defined to be

$$\ell^A(\sigma) := \min\{r \ge 0;\ \sigma = \tau_1, \ldots, \tau_r\ \tau_i \in \tau\}.$$

Note that, $\ell^A(\sigma) = inv_A(\sigma)$.

The paper is organized as follows : We start with some definitions which generalized the *width-k descents* and *width-k inversions* statistics on classical permutations studied by Davis [8] into signed permutations. In section 2, we will prove Proposition2.3, in which we improve the combinatorial formulas of these last statistics in signed permutations and give some examples. In section 3 and 4, we'll show Theorem 3.2, Theorem 4.2 and

Theorem 4.3, in which we define the *width-k Eulerian* polynomials of type A and B. So, we give some recurrence relations concerning the coefficients of these polynomials. Then, we will study the γ-positivity by specifying the combinatorial values of γ. Finally, in section 5, which is the same as section 3 and 4, we will define the *width-k Eulerian* polynomials of type D and we define two sets $WD_{n,k,p}$ and $W\bar{D}_{n,k,p}$ in order to find the recurrence relations for the coefficients of this polynomial. We will prove Theorem 5.4, so that we study the necessary condition for this polynomial to be γ-positive.

2 Width-k descents and width-k inversions on signed permutations

Recently, Sack and Ulfarsson [20] introduced a new natural generalizations of classical descents and inversions statistics for any permutation in S_n, and called after *width-k descents* and *width-k inversions* in Davis's work [8]. For each $1 \leq k < n$, the cardinals of these last statistics are defined as follows

$$des_k^A(\sigma) := |\{i \in [n-k]; \sigma(i) > \sigma(i+k)\}|,$$

$$inv_k^A(\sigma) := |\{(i,j) \in [n]^2; \sigma(i) > \sigma(j) \ \ and \ j - i = mk, \ m > 0\}|.$$

In this work, we study same analogness of these statistics on signed permutations. A signed permutation is a bijection of $[-n, n] := \{-n, \dots, -1, 1, \dots, n\}$ onto itself that satisfies $\pi(-i) = -\pi(i)$ for all $i \in [n]$. We denote by B_n the set of signed permutations of length n, also known as the hyperoctahedral groups.

Let $D_n \subset B_n$ be the subset consisting of the signed permutations with even number of negative entries. We denote by $neg(\pi)$ the number of negative entries in $\pi \in B_n$ and more precisely

$$D_n := \{\pi \in B_n; neg(\pi) \equiv 0 \ (mod\ 2)\}.$$

Adin, Brenti, and Roichman [1] defined a permutation statistics called the signed descent number (or type-B descent number) and, the $flag$ descent number. A signed descent of $\pi = (\pi(1), \pi(2), \dots, \pi(n)) \in B_n$ is an integer $0 \leq i \leq n-1$ satisfying $\pi(i) > \pi(i+1)$, where $\pi(0) = 0$ (a signed descent of $\pi \in D_n$ has the same notion in B_n). The signed descent number is defined by

$$des_B(\pi) := |\{0 \leq i \leq n-1; \ \pi(i) > \pi(i+1)\}|.$$

Whereas, the $flag$ descent statistics of a signed permutation π denoted by $fdes_B(\pi)$, counts a descent in position 0 once and all other descents twice. In other words,

$$fdes_B(\pi) := des_B(\pi) + des_A(\pi).$$

For example, let $\pi = (-2, 3, -1, -4) \in B_4$. Then the descents of π are $0, 2$ and 3. So, $des_B(\pi) = 3$ and $fdes_B(\pi) = 5$.

The *nth Eulerian* polynomials of signed permutations defined by

$$B_n(x) = \sum_{\pi \in B_n} x^{des_B(\pi)},$$

and the *nth flag descents* polynomials defined by

$$F_n(x) = \sum_{\pi \in B_n} x^{fdes_B(\pi)}.$$

Definition 2.1. *Let $\pi \in B_n$ be a permutation, the following statistics have defined as follows on π :*

1. *The inversion number of π : $inv_A(\pi) := |\{(i,j); 1 \leq i < j \leq n \text{ and } \pi(i) > \pi(j)\}|$,*
2. *$neg(\pi) := |\{i \in [n]; \pi(i) < 0\}|$,*
3. *$nsp(\pi) := |\{(i,j) \in [n]^2; i < j \text{ and } \pi(i) + \pi(j) < 0\}|$, is the number of negative sum pairs.*

The *Coxeter length* ℓ for π in B_n has the following combinatorial interpretation (see, for instance [6]) :

$$inv_B(\pi) := inv_A(\pi) + neg(\pi) + nsp(\pi).$$

Note that $\ell^B(\pi) = inv_B(\pi)$.

Now, we assume for each of the following definitions, let $n \in \mathbb{N}$, $k \in [n]$ and $\emptyset \neq K \subseteq [n]$. We give the same analogues of the width-k statistics on S_n defined in [8].

Definition 2.2. *For any permutation $\pi = (\pi(1), \pi(2), \ldots, \pi(n)) \in B_n$, the numbers of all width-k descent, width-k flag descent, width-k negative descent, width-k inversion, width-k negative, width-k negative sum pairs are defined respectively by :*

1. $des_k^B(\pi) := |\{0 \leq i \leq n-k;\ \pi(i) > \pi(i+k)\}|, \quad \text{where} \quad \pi(0) = 0,$
2. $fdes_k^B := des_k^A(\pi) + des_k^B(\pi),$
3. $ndes_k^B(\pi) := |\{1 \leq i \leq n-k; \pi(-i) > \pi(i+k)\}|,$
4. $inv_k^A(\pi) := |\{(i,j); 1 \leq i < j \leq n; \pi(i) > \pi(j) \text{ and } j - i = mk, m > 0\}|,$
5. $neg_k(\pi) := |\{1 \leq i \leq \lfloor \frac{n}{k} \rfloor;\ \pi(ik) < 0\}|,$
6. $nsp_k(\pi) := |\{(i,j) \in [n]^2; \pi(i) + \pi(j) < 0 \text{ and } j - i = mk, m > 0\}|.$

The set of all *width-k* descents of $\pi \in D_n$ has the same notion as above in B_n. We let

$$des_k^D(\pi) := |\{0 \leq i \leq n-k; \pi(i) > \pi(i+k)\}|, \quad \text{where} \quad \pi(0) = 0.$$

Taking $k = 1$, we obtain the classical statistics of a signed permutation $\pi \in B_n$. So, the width-1 descent, the width-1 flag descent and the width-1 inversion are the usual descent, flag descent and inversion of a signed permutation.

This gives rise to another natural statistics on B_n, the *width-k* length statistics :

Definition 2.3. *The* Coxeter width-k length ℓ *for* $\pi \in B_n$ *has the following combinatorial interpretation,*

$$inv_k^B(\pi) := inv_k^A(\pi) + neg_k(\pi) + nsp_k(\pi). \tag{1}$$

Note that $\ell_k^B(\pi) = inv_k^B(\pi)$.

Let $\emptyset \neq K \subseteq [n]$ denote the set of widths under consideration. These latter statistics are also defined, respectively, by

$$Des_K^B(\pi) := \bigcup_{k\in K} Des_k^B(\pi) \text{ and } des_K^B(\pi) = |Des_K^B(\pi)|,$$

$$Inv_K^A(\pi) := \bigcup_{k\in K} Inv_k^A(\pi) \text{ and } inv_K^A(\pi) = |Inv_K^A(\pi)|,$$

$$\ell_K^B(\pi) := \bigcup_{k\in K} \ell_k(\pi).$$

Example 2.1. *Let* $\pi = (4, -1, -3, -6, 5, -7, 2) \in B_7$ *then :*
$Des_2^B(\pi) = \{0, 1, 2, 4, 5\}$, $Des_3^B(\pi) = \{0, 1, 3\}$, $fdes_2^B = 9$ *and* $fdes_3^B = 5$.
$Inv_2^A(\pi) = \{(1, 7), (5, 7), (2, 6), (4, 6), (2, 4), (1, 3)\}$,
$Inv_3^A(\pi) = \{((1, 7), (3, 6), (1, 4)\}$.
Thus, $Des_{\{2,3\}}^B(\pi) = \{0, 1, 2, 3, 4, 5\}$ *and,*
$Inv_{\{2,3\}}^A(\pi) = \{(1, 7), (5, 7), (2, 6), (4, 6), (2, 4), (1, 3), (3, 6), (1, 4)\}$.
Finally, $des_{\{2,3\}}^B(\pi) = 6$, $inv_{\{2,3\}}^A(\pi) = 8$.
$\ell_2^B(\pi) = 13$, $\ell_3^B(\pi) = 8$ *and* $\ell_{\{2,3\}}^B(\pi) = 19$.

The objective of the present study is to generalize the previous study of Davis [8], hence to analyse these new statistics and their relationships among each other. For $\sigma \in S_n$, we show the following propositions, with $K \subseteq [n]$, and these propositions remains valid to signed permutations.

Proposition 2.1. *For any nonempty* $K \subseteq [n]$, $k \in [n]$, *and* $j \in K$. *Let* $L = \{mj \in K, m > 0\}$ *and for each* $\pi \in B_n$, *we have*

$$\ell_k^B(\pi) = \sum_{m\geq 1} (des_{mk}^B(\pi) + ndes_{mk}^B(\pi)), \tag{2}$$

and

$$\ell_K^B(\pi) = \ell_{K\setminus L}^B(\pi). \tag{3}$$

Proof. Each element of $\ell_k^B(\pi)$ is pairs of the form $(i, i + mk)$ for $0 \leq i \leq n - 1$ or $(-i, i + mk)$ for $1 \leq i \leq n - 1$, for some $m > 0$. So, an element exists in $\ell_k^B(\pi)$ if and only if there is a width-mk descent of π at i or a width-mk ndescent of π at $(-i)$. Thus, $\ell_k^B(\pi)$ just counts the number of descents of length mk for every m possible.
For any $j \in K$, $\ell_j^B(\pi)$ contains all descents whose widths are multiple of j. Thus, mj is already accounted for in $\ell_j^B(\pi)$, for some positive integer m. □

Proposition 2.2. *For any nonempty $K \subseteq [n]$, we have*

$$\ell^B_K(\pi) = \sum_{\emptyset \subsetneq K' \subseteq K} (-1)^{|K'|+1}\, \ell^B_{lcm(K')}(\pi), \tag{4}$$

where we set $\ell^B_{lcm(K')}(\pi) = 0$ if $lcm(K') \geq n+1$.

Proof. This proof follows from the equations (2) and (3), since they are the same result for the classical permutations.
The proof of this proposition for the classical permutations remains true for the signed permutations. For more details see ([8], *Proposition 1.2*). □

Example 2.2. *Let $\pi = (4, -1, -3, -6, 5, -7, 2) \in B_7$ be the same permutation in the previous example. If $K = \{2, 3, 4, 6\}$ then $\ell^B_K(\pi) = \ell^B_{K\setminus\{4,6\}}(\pi)$.*
if $k = 2$, we have

$$\begin{aligned}
\ell^B_2(\pi) &= \sum_{m\geq 1}(des^B_{2m}(\pi) + ndes^B_{2m}) \\
&= (des^B_2(\pi) + ndes^B_2(\pi)) + (des^B_4(\pi) + ndes^B_4(\pi)) + (des^B_6(\pi) + ndes^B_6(\pi)) \\
&= (5 + 2) + (2 + 2) + (2 + 0) = 13.
\end{aligned}$$

We have $\ell^B_{\{2,3\}}(\pi) = 19$, where $\ell^B_2 = 13$ and $\ell^B_3 = 8$ but, $(0, 6)$ and $(1, 7)$ have the width both 2 and 3, it must also have the width-lcm$(2, 3)$. Thus, $\ell^B_6(\pi) = \{(0, 6), (1, 7)\}$. Finally, $\ell^B_{\{2,3\}}(\pi) = \ell^B_2(\pi) + \ell^B_3(\pi) - \ell^B_6(\pi)$.

In the following, we will generalize the search function described in [8], on the set of signed permutations, which helps to demonstrate the interaction between width-k statistics by changing its normalization map. This function is defined as follows :
Let n and k be positive integers for which $n = dk + r$ for some $(d, r) \in \mathbb{N}^2$ with $0 \leq r < k$, and for every π in B_n, we may then associate the set of disjoint substrings $\gamma_{n,k}(\pi) = \{\gamma^1_{n,k}(\pi), \gamma^2_{n,k}(\pi), \dots, \gamma^k_{n,k}(\pi)\}$ where

$$\gamma^i_{n,k}(\pi) = \begin{cases} (\pi(i), \pi(i+k), \pi(i+2k), \dots, \pi(i+dk)) & \text{if } i \leq r, \\ (\pi(i), \pi(i+k), \pi(i+2k), \dots, \pi(i+(d-1)k)) & \text{if } r < i \leq k. \end{cases}$$

Now, we define the following correspondence φ by

$$\varphi : B_n \to B^r_{d+1} \times B^{k-r}_d$$

$$\varphi(\pi) = (std\gamma^1_{n,k}(\pi), std\gamma^2_{n,k}(\pi), \dots, std\gamma^k_{n,k}(\pi)),$$

where std is the standardization map such that, for all $1 \leq i \leq k$, the permutation $std\gamma^i_{n,k}(\pi)$ obtained by replacing the smallest integer in absolute value of $\gamma^i_{n,k}(\pi)$ by 1, the second smallest integer in absolute value by 2, etc. Then, for each element of $\gamma^i_{n,k}(\pi)$, add a sign $(-)$ at each $\pi(i + jk) < 0$, where $0 \leq j \leq d$.
It gives that each $std\gamma^i_{n,k}(\pi)$ is a signed permutation of B_d or B_{d+1}.

Example 2.3. *If* $\pi = (4, -1, -3, -6, 5, -7, 2) \in B_7$*, suppose* $k = 3$*. We then have*

$$\begin{aligned}\gamma_{7,3}(\pi) &= (std\gamma^1_{7,3}(\pi), std\gamma^2_{7,3}(\pi), std\gamma^3_{7,3}(\pi))\\ &= (std(4, -6, 2), std(-1, 5), std(-3, -7))\\ &= ((2, -3, 1), (-1, 2), (-1, -2)).\end{aligned}$$

We come now to one of the main results in this work. Firstly, let n and k be positive integers such that $n = dk + r$, where $0 \le r < k$ and $d > 0$.
Let $M_{n,k}$, denote the multinomial coefficient defined by,

$$M_{n,k} = \binom{n}{(d+1)^r, d^{k-r}},$$

where i^m indicates i repeated m times.
Also let the x-analogue of the integer $n \ge 1$ be :

$$[n]_x := \frac{1 - x^n}{1 - x} = 1 + x + x^2 + \ldots + x^{n-1},$$

the x-analogue of the factorial $n \ge 1$ is then :

$$[n]_x! := [1]_x \ldots\ [n-1]_x[n]_x, \quad [0]! := 1,$$

and

$$[2n]_x!! := \prod_{i=1}^{n} [2i]_x, \quad [0]!! := 1.$$

Proposition 2.3. *For* $n \ge 1$*, we have*

$$\sum_{\pi \in B_n} x^{inv_A(\pi)+nsp(\pi)} t^{neg(\pi)} = [n]_x! \prod_{i=0}^{n-1} (1 + tx^i). \tag{5}$$

Proof. Our proof will be by induction on n. Notice that the result is obvious for $n = 1$. So, suppose that the result holds for all natural numbers less than $n \ge 2$,

$$\text{i.e,} \qquad \sum_{\pi \in B_{n-1}} x^{inv_A(\pi)+nsp(\pi)} t^{neg(\pi)} = [n-1]_x! \prod_{i=0}^{n-2} (1 + tx^i). \tag{6}$$

For this, it is enough to account for the number of *inversions*, *nsp* and *negatives* for every integer $\pi(n)$ in $\pi = (\pi(1), \pi(2), \ldots, \pi(n-1), \pi(n)) \in B_n$.
Therefore, if $\pi(n) = l$, for $1 \le l \le n$ and for any $j \in [-n, n]$, such that $\pi(j) = k$ where $l < k \le n$, we have exactly an inversion or an nsp .
Then, $\pi(n)$ makes $(n - l)$ choices of *inversions* and *nsp*. It implies that identity (6) is multiplied by x^{n-l}, for all $\pi(n) = l$.
If $\pi(n) = -l$, $1 \le l \le n$, then for any $-l < k < l$ there exists $j \in [-n, n]$, such that $\pi(j) = k$. In that case, we have exactly $(2l - 2)$ *inversions* and *nsp*.
And for any $l < k \le n$ if $k > l$, we have exactly $(n - l)$ *inversions* or *nsp*. So, $\pi(n)$

makes $(n-2+l)$ choices of *inversions* and *nsp*. That gives the identity (6) is multiplied by tx^{n-2+l}. Finally, identity (6) is multiplied by $[n]_x(1+tx^{n-1})$. This completes the induction step. $\square$

An immediate consequence of the above result if $t=1$, the identity (5) becomes equal to :

$$\sum_{\pi\in B_n} x^{inv_A(\pi)+nsp(\pi)} = 2[n]_x![2(n-1)]_x!!. \tag{7}$$

We note that the sequences of coefficients of $\frac{1}{2}\sum_{\pi\in B_n} x^{inv_A(\pi)+nsp(\pi)}$ appear in the OEIS (sequence A162206).

3 Width-k Eulerian polynomials of type A

In this section, we will first present Davis's *width-k Eulerian* polynomials of type A. This description is very general, so we give its *Gamma-positivity* afterwards.

The concept of *Gamma-positivity* appeared first in the work of Foata and Schützenberger [9] and thereafter of Foata and Strehl ([11], [10]), on the classical Eulerian polynomials, one of the most important polynomials in combinatorics. *Gamma-positivity* is an elementary property that polynomials with symmetric coefficients may have, which directly implies their unimodality.

Working with *Eulerian descent* statistics is in a sense a generalization of the study of the Eulerian numbers. Just as there are Eulerian numbers, there is counting the number of permutations with the same descent number. For a permutation $\sigma\in S_n$, an index $i\in[n]$ is a double descent of σ if $\sigma(i-1)>\sigma(i)>\sigma(i+1)$, where $\sigma(0)=\sigma(n+1)=\infty$. We also have a *left peak* (resp. *peak*) of $\sigma\in S_n$ is any index $i\in[n-1]$ (resp. $2\le i\le n-1$) such that $\sigma(i-1)<\sigma(i)>\sigma(i+1)$, where $\sigma(0):=0$.

In the following, we need to define these statistics :

Definition 3.1. *For any permutation $\sigma=(\sigma(1),\sigma(2),\dots,\sigma(n))\in S_n$ and $1\le k\le n-1$, the numbers of all double width-k descents, width-k peaks (also, we say that a interior width-k peaks) and width-k left peaks are defined as follow :*

$$Ddes_k^A(\sigma) := \{i\in[n]; \sigma(i-k)>\sigma(i)>\sigma(i+k)\} \text{ and } \sigma(j)=\infty,\ \forall j>n \text{ or } j\le 0,$$

$$ddes_k^A(\sigma) := |Ddes_k^A(\sigma)|.$$

$$Peak_k(\sigma) := \{k+1\le i\le n-k; \sigma(i-k)<\sigma(i)>\sigma(i+k)\},$$

$$peak_k(\sigma) := |Peak_k(\sigma)|.$$

$$Lpeak_k(\sigma) := \{k\le i\le n-k; \sigma(i-k)<\sigma(i)>\sigma(i+k)\} \text{ and } \sigma(0):=0,$$

$$lpeak_k(\sigma) := |Lpeak_k(\sigma)|.$$

Now, we define the *width-k Eulerian* polynomials of type A by :

$$\mathfrak{WA}_{n,k}(x) = F_n^{des_k^A}(x) = \sum_{\sigma \in S_n} x^{des_k^A(\sigma)}. \tag{8}$$

We denote by $\mathrm{WA}_{n,k,p}$ the set $\{\sigma \in S_n; des_k^A(\sigma) = p\}$, and its cardinal by $a(n,k,p)$. For $k = 1$, we find the classical Eulerian polynomials, and its *nth* γ-positivity, $A_n(x) = \mathfrak{WA}_{n,1}(x)$, is given in ([9], Theorem 5.6) is defined by :

$$\mathfrak{WA}_{n,1}(x) = \sum_{p=0}^{\lfloor n-1/2 \rfloor} \gamma_{n,p}\ x^p(1+x)^{n-1-2p}, \tag{9}$$

where $\gamma_{n,p} = |\Gamma_{n,p}|$ and $\Gamma_{n,p}$ is the set of permutations $\sigma \in S_n$ with :

- ▷ $des_1^A(\sigma) = p$,
- ▷ $ddes_1^A(\sigma) = 0$.

The following identity was originally established in [20], but with a slightly different notation. Thereafter, Davis showed another proof ([8], Theorem 2.3).

Proposition 3.1. *[8]*

$$\mathfrak{WA}_{n,k}(x) = F_n^{des_k^A}(x) = M_{n,k} A_{d+1}^r(x) A_d^{k-r}(x). \tag{10}$$

Let $\alpha(n,k,p)$ be the coefficients of the polynomials $\mathfrak{WA}_{n,k}(x)$ such that $a(n,k,p) = M_{n,k}\alpha(n,k,p)$. For a clearer observation, we put, in the table below, a few coefficients of $\alpha(n,k,p)$, for $1 \leq n \leq 6,\ 1 \leq k \leq n-1$ and $0 \leq p \leq n-k$.

n	k	p					
		0	1	2	3	4	5
1	1	1					
2	1	1	1				
3	1	1	4	1			
	2	1	1				
4	1	1	11	11	1		
	2	1	2	1			
	3	1	1				
5	1	1	26	66	26	1	
	2	1	5	5	1		
	3	1	2	1			
	4	1	1				
6	1	1	57	302	302	57	1
	2	1	8	18	8	1	
	3	1	3	3	1		
	4	1	2	1			
	5	1	1				

TABLE 1 – A few values of $\alpha(n,k,p)$.

The polynomial $\mathfrak{WA}_{n,k}(x)$ is unimodal, symmetric with nonnegative coefficients and γ-positive with center of symmetry $\lfloor \frac{n-k}{2} \rfloor$ and $deg(\mathfrak{WA}_{n,k}(x)) = n-k$. Because, it is

the product of k unimodal, symmetric and γ-positive polynomials.
For example, $\mathfrak{WA}_{6,2}(x) = M_{6,2}(1+8x+18x^2+8x^3+x^4)$ is γ-positive since

$$\mathfrak{WA}_{6,2}(x) = 20x^0(1+x)^4 + 80x(1+x)^2 + 80x^2(1+x)^0.$$

$\alpha(n,k,p)$ is a new sequence integers on the Coxeter group of type A. So, it is natural to make the following problem on the recurrence relation of this sequence which has been by confirmed the fact that any $n \geq 4$, k be the smallest positive integer such that $n = 2k+r$ with $0 \leq r < k$ and $0 \leq p \leq n-k$, we have

$$\alpha(n,k,p) = \alpha(n-2,k-1,p-1) + \alpha(n-2,k-1,p),$$

$$\textit{with}\ \ \alpha(n,k,0) = \alpha(n,k,n-k) = 1 \text{ and } \ \alpha(n,k,-1) = 0.$$

Problem 3.1. *It is possible to find the recurrence relation of $\alpha(n,k,p)$, for all $1 \leq k \leq n$?*

Theorem 3.2. *For all $n \geq 1$ and $1 \leq k \leq n-1$,*

$$\mathfrak{WA}_{n,k}(x) = \sum_{\sigma \in S_n} x^{des_k^A(\sigma)} = \sum_{p=0}^{\lfloor n-k/2 \rfloor} \gamma_{n,k,p}^A \ x^p (1+x)^{n-k-2p},$$

where $\gamma_{n,k,p}^A = |\Gamma_{n,k,p}|$ and $\Gamma_{n,k,p}$ is the set of permutations $\sigma \in S_n$ with :
- ▷ $des_k^A(\sigma) = p$,
- ▷ $ddes_k^A(\sigma) = 0$.

For $1 \leq n \leq 6,\ 1 \leq k \leq n-1$ and $0 \leq p \leq \lfloor \frac{n-k}{2} \rfloor$, we record a few values of $\gamma_{n,k,p}^A$.

n	k	p		
		0	1	2
1	1	1		
2	1	1		
3	1	1	2	
	2	3		
4	1	1	8	
	2	6	0	
	3	12		
5	1	1	22	16
	2	10	20	
	3	30	0	
	4	60		
6	1	1	52	136
	2	20	80	80
	3	90	0	
	4	180	0	
	5	360		

TABLE 2 – A few values of $\gamma_{n,k,p}^A$.

Proof. By the standardization map defined on S_n in [8] and for all $1 \leq k \leq n-1$,

$$\begin{array}{lccl} \varphi: & S_n & \longrightarrow & S_{d+1}^{r} \times S_d^{k-r} \\ & \sigma & \longmapsto & \varphi(\sigma) = (std\gamma_{n,k}^{1}(\sigma), std\gamma_{n,k}^{2}(\sigma), \ldots, std\gamma_{n,k}^{k}(\sigma)). \end{array}$$

Let $\varphi(\sigma) = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ such that $std\gamma_{n,k}^{i}(\sigma) = \sigma_i$ for all i. Thanks to this application, we note that :

$$des_k^A = \sum_{i=1}^{k} des_A(\sigma_i).$$

It is clear that each *width-k* descent and *width-k* double descent in σ are usual descent and double descent in some unique σ_i. $\square$

4 Width-k Eulerian polynomials of type B

In this section, we give a new generalized of type B Eulerian polynomials and its γ-positivities which is inspired to type A. We define the *width-k Eulerian* polynomials of type B by :

$$\mathfrak{WB}_{n,k}(x) = F_n^{des_k^B}(x) = \sum_{\pi \in B_n} x^{des_k^B(\pi)}.$$

We denote by $\mathrm{WB}_{n,k,p}$ the set $\{\pi \in B_n; des_k^B(\pi) = p\}$ and its cardinal by $b(n,k,p)$.

For $k = 1$, we find the classical Eulerian polynomials of type B, and its *nth* γ-positive, $B_n(x) = \mathfrak{WB}_{n,1}(x)$ in the following result.

Theorem 4.1. ([*18*], Proposition 4.15) *For all $n \geq 1$,*

$$\mathfrak{WB}_{n,1}(x) = \sum_{p=0}^{\lfloor n/2 \rfloor} \gamma_{n,p}^{B}\ x^p (1+x)^{n-2p},$$

where $\gamma_{n,p}^{B}$ is equal to the number of permutations $\sigma \in S_n$ with p left peaks, multiplied by 4^p.

Now, we need to define the statistics descent of type A over the set of signed permutations by, for all $\pi \in B_n$,

$$des_A(\pi) := |\{i \in [n-1]; \pi(i) > \pi(i+1)\}|.$$

We then have the following identities :

Theorem 4.2. *For any $n \geq k \geq 1$ and $d \geq 0$ such that $n = dk + r$, $0 \leq r < k$, we have*

$$F_n^{des_k^B}(x) = 2^{n-d} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^{r}(x), \tag{11}$$

$$F_n^{fdes_k^B}(x) = 2^{n-d} M_{n,k}\ F_d(x)\ A_d^{k-r-1}(x^2)\ A_{d+1}^{r}(x^2), \tag{12}$$

$$F_n^{\ell_k^B}(x) = 2^{k-1} M_{n,k}\ [d]_x^{k-r-1}\ [d+1]_x^{r}\ [2d]_x^{r+1}!!\ [2(d-1)]_x^{k-r-1}!!. \tag{13}$$

Proof. We consider the correspondence φ defined in section 2 by :

$$\varphi : B_n \to B_{d+1}^r \times B_d^{k-r}$$
$$\varphi(\pi) = (std\gamma_{n,k}^1(\pi), std\gamma_{n,k}^2(\pi), \ldots, std\gamma_{n,k}^k(\pi)).$$

We fix k in $[n]$ and $\varphi(\pi) = (\pi_1, \pi_2, \ldots, \pi_k) \in B_{d+1}^r \times B_d^{k-r}$ such that, $std\gamma_{n,k}^i(\pi) = \pi_i$ for all i. There exists $M_{n,k}$ choice to partition $[n]$ in the subsequences $\gamma_{n,k}^i(\pi)$.
We define,

$$\varepsilon(\pi_i) = \begin{cases} 1 & \text{if } \pi_i(1) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

It is also important to note that,

$$des_k^B(\pi) = des_B(\pi_k) + \sum_{i=1}^{k-1}(des_B(\pi_i) - \varepsilon(\pi_i)),$$

then for all $\pi \in B_n$,

$$\sum_{i=1}^{k-1}(des_B(\pi_i) - \varepsilon(\pi_i)) = \sum_{i=1}^{k-1} des_A(\pi_i).$$

Thus,

$$\begin{aligned}
F_n^{des_k^B}(x) &= \sum_{\pi\in B_n} x^{des_k^B}(\pi) \\
&= M_{n,k} \sum_{\pi_k\in B_d,\ (\pi_1,\ldots,\pi_{k-1})\in B_d^{k-r-1}\times B_{d+1}^r} x^{des_B(\pi_k)} x^{des_A(\pi_1)} \ldots x^{des_A(\pi_{k-1})} \\
&= M_{n,k}\ B_d(x)\ (2^d)^{k-r-1} A_d^{k-r-1}(x)\ (2^{d+1})^r\ A_{d+1}^r(x) \\
&= 2^{n-d}\ M_{n,k}\ B_d(x)\ A_d^{k-r-1}(x)\ A_{d+1}^r(x).
\end{aligned}$$

This proves the first identity.

Using the definition of width-k descent, we get

$$\begin{aligned}
fdes_k^B(\pi) &= des_k^A(\pi) + des_k^B(\pi) \\
&= \sum_{i=1}^{k} des_A(\pi_i) + des_B(\pi_k) + \sum_{i=1}^{k-1} des_A(\pi_i) \\
&= \sum_{i=1}^{k-1} 2des_A(\pi_i) + des_A(\pi_k) + des_B(\pi_k) \\
&= fdes_B(\pi_k) + \sum_{i=1}^{k-1} 2des_A(\pi_i).
\end{aligned}$$

On the other hand, we have

$$\sum_{\pi\in B_n} x^{2des_A(\pi)} = 2^n A_n(x^2).$$

Thus,

$$\begin{aligned}
F_n^{fdes_k^B}(x) &= \sum_{\pi \in B_n} x^{fdes_k^B(\pi)} \\
&= M_{n,k} \sum_{\pi_k \in B_d \ ,(\pi_1,\ldots,\pi_{k-1}) \in B_d^{k-r-1} \times B_{d+1}^r} x^{fdes_B(\pi_k)} x^{2des_A(\pi_1)} \ldots x^{2des_A(\pi_{k-1})} \\
&= 2^{n-d} \ M_{n,k} \ F_d(x) \ A_d^{k-r-1}(x^2) \ A_{d+1}^r(x^2).
\end{aligned}$$

This proves the second identity.

Now, we observe that the *Coxeter width-k* length ℓ, is the sum of

$$\ell_k^B(\pi) = \ell^B(\pi_k) + \sum_{i=1}^{k-1}(inv_A(\pi_i) + nsp(\pi_i)).$$

On the other hand, the generating function of the Coxeter length ℓ can be presented in the following manner (see, for instance in section 3.15 [13]).

$$\sum_{\pi \in B_n} x^{inv_B(\pi)} = \sum_{\pi \in B_n} x^{\ell^B(\pi)} = [2n]_x!!.$$

$$\begin{aligned}
F_n^{\ell_k^B}(x) &= \sum_{\pi \in B_n} x_k^{\ell^B}(\pi) \\
&= M_{n,k} \sum_{\pi_k \in B_d, \ (\pi_1,\ldots,\pi_{k-1}) \in B_d^{k-r-1} \times B_{d+1}^r} x^{\ell^B(\pi_k)} x^{(inv_A(\pi_1)+nsp(\pi_1))} \ldots x^{(inv_A(\pi_{k-1})+nsp(\pi_{k-1}))} \\
&= 2^{k-1} M_{n,k} [2d]_x^{r+1}!! [2(d-1)]_x^{k-r-1}!! [d]_x^{k-r-1} [d+1]_x^r.
\end{aligned}$$

This proves the third identity. □

Using identity (11), we find that $\mathfrak{WB}_{n,k}(x) = 2^{n-d} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^r(x)$. Let $\beta(n,k,p)$ be the coefficient of the polynomial $\mathfrak{WB}_{n,k}(x)$ such that, $b(n,k,p) = 2^{n-d} M_{n,k} \beta(n,k,p)$. For a clearer observation, we put in the table below a few coefficients of $\beta(n,k,p)$, for $1 \le n \le 6, 1 \le k \le n$ and $0 \le p \le n-k+1$.

As $\beta(n,k,p)$ is a new sequence integers on the Coxeter group of type B, it natural to make the following problem on the recurrence relation of this sequence. It has been by confirmed the fact that any $n \ge 3$, k be the smallest positive integer such that $n+1 = 2k+r$ with $0 \le r < k$ and $0 \le p \le n-k+1$, we have the following recurrence relation

$$\beta(n,k,p) = \beta(n-2,k-1,p-1) + \beta(n-2,k-1,p)$$

where,

$$\beta(n,k,0) = \beta(n,k,n-k+1) = 1, \quad \text{and } \beta(n,k,-1) = 0.$$

Problem 4.1. *It is possible to find the recurrence relation of $\beta(n,k,p)$, for all $1 \le k \le n$?*

n	k	p						
		0	1	2	3	4	5	6
1	1	1	1					
2	1	1	6	1				
	2	1	1					
3	1	1	23	23	1			
	2	1	2	1				
	3	1	1					
4	1	1	76	230	76	1		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	1	237	1682	1682	237	1	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	1	722	10543	23548	10543	722	1
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

TABLE 3 – The first few values of $\beta(n,k,p)$.

$\mathfrak{WB}_{n,k}(x)$ is unimodal and symmetric with nonnegative coefficients and it is γ-positive (as the product of k γ-positive polynomials) with center of symmetry $\lfloor \frac{n-k+1}{2} \rfloor$ and $deg(\mathfrak{WB}_{n,k}(x)) = n-k+1$.
For example, $\mathfrak{WB}_{6,2}(x) = 2^3 M_{6,2}(1+27x+116x^2+116x^3+27x^4+x^5)$ is γ-positive since

$$\mathfrak{WB}_{6,2}(x) = 160x^0(1+x)^5 + 3520x(1+x)^3 + 6400x^2(1+x).$$

So, we have the following theorem,

Theorem 4.3. *For any* $1 \leq k \leq n$,

$$\mathfrak{WB}_{n,k}(x) = \sum_{\pi \in B_n} x^{des_k^B(\pi)} = \sum_{p=0}^{\lfloor n-k+1/2 \rfloor} \gamma_{n,k,p}^B \; x^p(1+x)^{n-k+1-2p},$$

where $\gamma_{n,k,p}^B = 2^{2p+k-1}|\Gamma_{n,k,p}^{(\ell)}|$ *and* $\Gamma_{n,k,p}^{(\ell)}$ *is the set of permutations* $\sigma \in S_n$ *with* p *width-*k *left peaks.*

To prove this theorem, we need to generalize some results of the work of Petersen [18] on P-partitions and enriched P-partitions.

In the theory of partition, there are two definitions of P-partition. The first one is due to Stanley [22] which defines them as order reversing maps while the second definition defined by Gessel [12] as order preserving maps. In the current paper, we will adopt the second definition.

Definition 4.1. *(P-Partition) Let* $X = \{x_1, x_2, \ldots\}$ *be a countable and totally ordered set. For a given poset* P *with partial order* $<_p$*, a P-partition is an order-preserving map* $f : [n] \to X$ *such that :*

1. if $i <_p j$ then $f(i) \leq f(j)$,

2. if $i <_p j$ and $i > j$ dans $\mathbb{Z}$ then $f(i) < f(j)$.

We assume that X as a subset of the positive integers. We let $\mathfrak{L}(P)$ denote the set of all permutations of $[n]$ which extend P to a total order. When X has finite cardinality p, the number of P-partitions must also be finite. In this case, define the order polynomial, denoted $\Omega(P;p)$, to be the number of P-partitions $f : [n] \to X$. In our case, when studying P-partitions it is enough to consider P is a permutation and $\Omega_A(P,p)$ is the type A order polynomial.
Notice that for any permutation π and any positive integer p, we can write :

$$\Omega_A(\pi, p) = |\{f : [n] \to [p]; 1 \leq f(\pi(1)) \leq f(\pi(2)) \leq \ldots \leq f(\pi(n)) \leq p,$$

$$\text{and if } s \in Des_A(\pi) \text{ then} f(\pi(s)) < f(\pi(s+1))\}|.$$

For the group of signed permutations, the only difference from the symmetric group is, if $\pi(1) < 0$, then 0 is a descent of π. Let $\Omega_B(\pi,p)$ be the order polynomial for any signed permutation. For fixed n, the order polynomials prove to be : $\Omega_A(i,x) = \Omega_B(i,x) = \binom{x+n-i}{n}$, for any permutation of type A or type B with $i-1$ descents.
Similarly of corollary 2.4 of [18], we will given the relation between the width-k order polynomial of a poset P and the sum of the width-k order polynomials of its linear extensions.

Corollary 4.4. *The width-k order polynomial of a poset P is the sum of the width-k order polynomials of its linear extensions :*

$$\Omega(P,k,p) = \sum_{\pi \in \mathfrak{L}(P)} \Omega(\pi,k,p),$$

where, for any permutation π, the width-k order polynomial

$$\Omega(\pi,k,p) := |\{f : [n-k+1] \to [p]/ \tag{14}$$

$$\begin{gathered} 1 \leq f(\pi(1)) \leq f(\pi(1+k)) \leq f(\pi(1+2k)) \leq \ldots f(\pi(1+\tau_d k)) < \\ f(\pi(2)) \leq f(\pi(2+k)) \leq f(\pi(2+2k)) \leq \ldots f(\pi(2+\tau_d k)) < \ldots < \\ f(\pi(k)) \leq f(\pi(k+k)) \leq f(\pi(k+2k)) \leq \ldots \leq f(\pi(k+\tau_d k)) \leq p, \\ and\ f(\pi(s)) < f(\pi(s+k)),\ if\ s \in des_k^B(\pi)\}|, \end{gathered}$$

with

$$\tau_d = \begin{cases} d & if\ i \leq r, \\ d-1 & if\ \ r < i \leq k, \end{cases}$$

and $n = dk + r$ is the Euclidean division of n by k.

The proof of the above corollary follows from the set of all P-partitions is the disjoint union of all π-partitions for linear extensions π of P.

We can think of any permutation $\pi \in B_n$ as a poset with the total order $\pi(s) < \pi(s+k)$. In this case, P is an antichain of $n-k+1$ elements. So, we have $|P| = n-k+1$. Now, we give the analog generating function of type B order polynomials in [19] in the following in term of width-k.

Theorem 4.5. *For a given permutation $\pi \in B_n$. The generating function for width-k type B order polynomials is :*

$$\sum_{p\geq 0} \Omega_B(\pi, k, p)x^p = \frac{x^{des_k^B(\pi)}}{(1-x)^{n-k+2}}. \tag{15}$$

Proof. Let $\mathfrak{L}(P) = \{\pi\}$, where π has width-k descents counting an extra width-k descent at the end (we assume $\pi(k+\tau_d k) > \pi(n+1)$). Then, $\Omega_B(\pi; k; p)$ is the number of solutions of equation (14) :

$$\begin{gathered} 1 \leq f(\pi(1)) \leq f(\pi(1+k)) \leq f(\pi(1+2k)) \leq \ldots f(\pi(1+\tau_d k)) < \\ f(\pi(2)) \leq f(\pi(2+k)) \leq f(\pi(2+2k)) \leq \ldots f(\pi(2+\tau_d k)) < \ldots < \\ f(\pi(k)) \leq f(\pi(k+k)) \leq f(\pi(k+2k)) \leq \ldots \leq f(\pi(k+\tau_d k)) \leq p - (des_k^B(\pi) - 1), \end{gathered}$$

which is equal to the number of ways choosing $n-k+1$ things from $p - des_k^B(\pi) + 1$ with repetitions. Therefore, the number is $\binom{p-des_k^B(\pi)+1+n-k}{n-k+1} = \binom{p-des_k^B(\pi)+n-k+1}{n-k+1}$. So the result becomes :

$$\sum_{p\geq 0} \Omega_B(\pi, k, p)x^p = \sum_{p\geq 0} \binom{p - des_k^B(\pi) + n - k + 1}{n-k+1} x^p = \frac{x^{des_k^B(\pi)}}{(1-x)^{n-k+2}} .$$

□

Petersen, in [18], make a relation between enriched P-partitions and quasisymmetric functions. Hence, we can see this link in terms of width-k statistic consequently, this connection helps us prove the theorem 4.3.
The basic theory of enriched P-partitions is due to Stembridge [24]. An enriched P-partitions and a left enriched P-partitions of type A defined as follow :
Let $\mathbb{P}'$ denote the set of nonzero integers, totally ordered so that :

$$-1 < +1 < -2 < +2 < -3 < +3 < \ldots,$$

and $\mathbb{P}^{(\ell)}$ to be the integers with the following total order :

$$0 < -1 < +1 < -2 < +2 < -3 < +3 < \ldots$$

In general, for any totally ordered set $X = \{x_1, x_2, \ldots\}$, we define $\mathbb{X}'$ and $\mathbb{X}^{(\ell)}$ to be the set :

$$\{-x_1, x_1, -x_2, \ldots\}$$

and

$$\{x_0, -x_1, x_1, -x_2, \ldots\},$$

with total order

$$x_0 < -x_1 < x_1 < -x_2 < x_2 < \ldots.$$

Definition 4.2. *An enriched P-partition (resp. left enriched P-partition) is an order-preserving map $f : P \to \mathbb{X}'$ (resp. $\mathbb{X}^{(\ell)}$) such that for all $i <_P j$ in P,*

1. *if $i <_P j$ in $\mathbb{Z}$ then $f(i) \leq^+ f(j)$,*
2. *if $i <_P j$ and $i > j$ in $\mathbb{Z}$ then $f(i) \leq^- f(j)$.*

Let $\varepsilon(P)$, denote the set of all enriched P-partitions and $\varepsilon^{(\ell)}(P)$, denotes the set of left enriched P-partitions. The number of (left) enriched P-partitions is finite if we assume that $|X| = p$. In this case, define *the enriched order polynomial*, denoted $\Omega'(P, p)$, to be the number of enriched P-partitions $f : P \to X'$ and, *the left enriched order polynomial*, denoted $\Omega^{(\ell)}(P, p)$, to be the number of left enriched P-partitions $f : P \to X^{(\ell)}$.

Following Gessel [12], a quasisymmetric function is one for which the coefficient of $x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_p}^{\alpha_p}$ is the same for all fixed tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_p)$ and all $i_1 < i_2 < \ldots < i_p$.
For any subset $D = \{d_1 < d_2 < \ldots < d_{p-1}\}$ of $[n]$, the quasisymmetric functions is characterized by two common bases, defined by the monomial quasisymmetric functions, M_D, and the fundamental quasisymmetric functions F_D :

$$M_D = \sum_{i_1 < i_2 < \ldots < i_p} x_{i_1}^{d_1} x_{i_2}^{d_2 - d_1} \ldots x_{i_p}^{n - d_{p-1}} = \sum_{i_1 < i_2 < \ldots < i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_p}^{\alpha_p},$$

and

$$F_D = \sum_{D \subset T \subset [n-1]} M_T = \sum_{\substack{i_1 \leq i_2 \leq \ldots \leq i_p \\ d \in D \Rightarrow i_d < i_{d+1}}} \prod_{d=1}^{n} x_{i_d} = \Gamma_A(\pi),$$

where $\Gamma_A(\pi)$ is the generating function for the type A P-partitions of a permutation with descent set D. It is possible to recover the order polynomial of π by specializing :

$$\Omega_A(\pi, p) = \Gamma_A(\pi)(1^p). \tag{16}$$

For all $D \subset [n-1]$, the functions M_D and F_D span the quasisymmetric functions of degree n, indicated $Qsym_n$. The ring of quasisymmetric functions defined by : $Qsym := \bigoplus_{n \geq 0} Qsym_n$.

Also, the generating function for enriched P-partitions $f : P \to \mathbb{P}'$ is defined by :

$$\Delta_A(P) := \sum_{f \in \varepsilon(P)} \prod_{i=1}^{n} x_{|f(i)|}.$$

Evidently, $\Delta_A(P)$ is a quasisymmetric function and we can specialize it as :

$$\Omega'(P, p) = \Delta_A(P)(1^p).$$

Chow [7] gave a connection between ordinary type B P-partitions and type B quasisymmetric functions. Furthermore, Petersen [18] related the type B quasisymmetric

functions to left enriched P-partitions and type B enriched P-partitions.
For fixed n and for any subset $D = \{d_1 < d_2 < \ldots < d_{p-1}\}$ of $[0, n]$, the monomial N_D and the fundamental quasisymmetric functions of $type_B$, L_D, are defined by :

$$N_D = \sum_{0<i_2<\ldots<i_p} x_0^{d_1} x_{i_2}^{d_2-d_1} \ldots x_{i_p}^{n-d_{p-1}} = \sum_{0<i_2<\ldots<i_p} x_0^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_p}^{\alpha_p},$$

and

$$L_D = \sum_{D\subset T\subset[0,n-1]} N_T = \sum_{\substack{0\leq i_2\leq\ldots\leq i_p \\ d\in D\Rightarrow i_d<i_{d+1}}} \prod_{d=1}^{n} x_{i_d} = \Gamma_B(\pi),$$

where $\Gamma_B(\pi)$ is the generating function for the ordinary type B P-partitions of any signed permutation with descent set D. Once more, we can particularize it as :

$$\Omega_B(\pi, p) = \Gamma_B(\pi)(1^{p+1}). \tag{17}$$

Similar of type A, these functions form a basis for the type B quasisymmetric functions of degree n, defined by $BQsym := \bigoplus_{n\geq 0} BQsym_n$.
The generating function for left enriched P-partitions f : $P \to \mathbb{P}^{(\ell)}$, defined by :

$$\Delta^{(\ell)}(P) := \sum_{f\in\varepsilon^{(\ell)}(P)} \prod_{i=1}^{n} x_{|f(i)|}.$$

It is also clear that $\Delta^{(\ell)}(P)$ is a quasisymmetric function and we have :

$$\Omega^{(\ell)}(P, p) = \Delta^{(\ell)}(P)(1^{p+1}).$$

Recalling the standardization map $std\ \phi$ defined in [7] on S_n by :
For any $n - 1 \geq k \geq 1$ and $d > 0$ such that $n = dk + r$ and $0 \leq r < k$,

$$\phi : S_n \to S_{d+1}^r \times S_d^{k-r}$$

$$\phi(\pi) = (std\gamma_{n,k}^1(\pi), std\gamma_{n,k}^2(\pi), \ldots, std\gamma_{n,k}^k(\pi)).$$

Fix $k \in [n - 1]$ and $\phi(\pi) = (\pi_1, \pi_2, \ldots, \pi_k) \in S_{d+1}^r \times S_d^{k-r}$ such that, $std\gamma_{n,k}^i(\pi) = \pi_i$ for all i. In the remainder of this paragraph, we define τ_d by :

$$\tau_d = \begin{cases} d & \text{if } i \leq r , \\ d-1 & \text{if } r < i \leq k. \end{cases}$$

For any two subsets of the integers D ant T, define the set $D + k = \{d + k | d \in D\}$ with, $1 \leq k \leq n$ and define the symmetric set difference by : $D\Delta T = (D \cup T)\backslash(D \cap T)$.
As indicated in the second section that, we can write width-k descent as follows : $des_k^B(\pi) = des_B(\pi_k) + \sum_{i=1}^{k-1}(des_B(\pi_i) - \epsilon(\pi_i))$. So, we count 0 as a descent just in π_k,

and all other permutations, we see it as a descent of type A. In addition, we can define $\Omega_B(D,k,p)$ by the ordinary type B order polynomial of any signed permutation with width-k descent set D by :

$$\Omega_B(D,k,p) = \Omega_B(D_k,p_k)\prod_{i=1}^{k-1}\Omega'_B(D_i,p_i), \tag{18}$$

with $\Omega_B(D_k,p_k)$ is the ordinary type B order polynomial of signed permutation with descent set D_k and $\Omega'_B(D_i,p_i)$ is the ordinary type B order polynomial of signed permutation with descent set $D_i\backslash\{0\}$, whereD_i is the set of descents in each π_i and $p_i = p-k+i$, for any $1 \leq i \leq k$. Thus, $\cup_{i=1}^{k} D_i = D$.
In the following, to study the left enriched P-partitions it is sufficient to consider the case where P is a permutation. So, it is possible to characterize the set of all left enriched π-partition in term of descent set. Left peaks in S_n are a special case of type B peaks, which are naturally related to type B descents. Thus, in terms of width-k descent sets, we can define the set of all width-k left enriched π-permutation, but we consider 0 as a descent only in π_k. Then, for $\pi \in S_n$, we can describe :

$$\Omega^{(\ell)}(\pi,k,p) = \Omega^{(\ell)}(\pi_k,p_k)\prod_{i=1}^{k-1}\Omega^{(\ell)'}(\pi_i,p_i), \tag{19}$$

where $\Omega^{(\ell)'}(\pi_i,p_i)$ is the left enriched order polynomial with $Des(\pi_i) \subset [\tau_d]$.
It is important to observe that,

$$Lpeak_k(\pi) = Lpeak(\pi_k)\bigcup_{i=1}^{k-1} Peak(\pi_i),$$

thus,

$$lpeak_k(\pi) = lpeak(\pi_k) + \sum_{i=1}^{k-1} peak(\pi_i).$$

The generating function for enriched π-partitions depends on the set of peaks and the generating function for left enriched π-partitions depends on the set of left peaks. As we can write $\Delta^{(\ell)}(\pi)$ according to the monomial and fundamental quasisymmetric functions of type B. In this case, if we using the application ϕ above, we can define the width-k left enriched P-partitions by :

$$\Delta^{(\ell)}(\pi,k) = \Delta^{(\ell)}(\pi_k)\prod_{i=1}^{k-1}\Delta_A(\pi_i),$$

with nonnegative coefficients.
This coefficient is equal to the number of product of enriched and left enriched π-partitions f such that :

$$(|f(\pi_1(1))|, |f(\pi_1(2))|, \ldots, |f(\pi_1(1+\tau_d k))|) = (1,\ldots 1,\ldots, p_1, \ldots p_1),$$

$$(|f(\pi_2(1))|, |f(\pi_2(2))|, \ldots, |f(\pi_2(1+\tau_d k))|) = (1, \ldots 1, \ldots, p_2, \ldots p_2),$$
$$\vdots$$
$$(|f(\pi_k(1))|, |f(\pi_k(2))|, \ldots, |f(\pi_k(1+dk))|) = (0, \ldots, 0, 1, \ldots 1, \ldots, p_k, \ldots p_k).$$

Applying the results of (Stembridge[24], Proposition 3.5), (Petersen [18], Theorem 6.6) and the Eq. (18), we find :

$$\Delta^{(\ell)}(\pi, k) = 2^{lpeak(\pi_k)} \sum_{\substack{D_k \subset [0,d-1] \\ Lpeak(\pi_k) \subset D_k \Delta (D_k+1)}} L_{D_k} \prod_{i=1}^{k-1} 2^{peak(\pi_i)+1} \sum_{\substack{D_i \subset [\tau_d] \\ Peak(\pi_i) \subset D_i \Delta (D_i+1)}} F_{D_i}.$$

We can also write,

$$\begin{aligned} \Omega^{(\ell)}(\pi, k, p) &= 2^{lpeak(\pi_k)} \sum_{\substack{D_k \subset [0,d-1] \\ Lpeak(\pi_k) \subset D_k \Delta (D_k+1)}} \Omega_B(D_k, p_k) \prod_{i=1}^{k-1} 2^{peak(\pi_i)+1} \sum_{\substack{D_i \subset [\tau_d] \\ Peak(\pi_i) \subset D_i \Delta (D_i+1)}} \Omega'_B(D_i, p_i), \\ &= 2^{lpeak_k(\pi)+k-1} \sum_{\substack{D \subset [0,n-k] \\ Lpeak_k(\pi) \subset D \Delta (D+k)}} \Omega_B(D, k, p). \end{aligned}$$

From this result, we can find the following Theorem.

Theorem 4.6. *We have the following generating function for width-k left enriched order polynomials,*

$$\sum_{p \geq 0} \Omega^{(\ell)}(\pi; k; p) x^p = 2^{k-1} \frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}} \left(\frac{4x}{(1+x)^2} \right)^{lpeak_k(\pi)}.$$

Proof. For any permutation π in S_n, we have

$$\sum_{p \geq 0} \Omega^{(\ell)}(\pi, k, p) x^p = \sum_{p \geq 0} 2^{lpeak_k(\pi)+k-1} \sum_{\substack{D \subset [0,n-k] \\ Lpeak_k(\pi) \subset D \Delta (D+k)}} \Omega_B(D, k, p) x^p.$$

Applying the generating function for width-k type B order polynomials defined in Theorem (4.5), we obtain also :

$$\sum_{p \geq 0} \Omega^{(\ell)}(\pi, k, p) x^p = \frac{2^{lpeak_k(\pi)+k-1}}{(1-x)^{n-k+2}} \sum_{\substack{D \subset [0,n-k] \\ Lpeak_k(\pi) \subset D \Delta (D+k)}} x^{|D|}.$$

It is not complicated to specify the generation function for D sets by size. For all $s \in Lpeak_k(\pi)$, we have precisely s or $s-k$ is in D. So, there is still $n-k+1-2lpeak_k(\pi)$ elements in $[0,\, n-k]$ can be contained in D or not. Thus, we obtain

$$\begin{aligned} \sum_{\substack{D \subset [0,n-k] \\ Lpeak_k(\pi) \subset D \Delta (D+k)}} x^{|D|} &= \underbrace{(x+x)(x+x)\ldots(x+x)}_{lpeak_k(\pi)} \underbrace{(1+x)(1+x)\ldots(1+x)}_{n-k+1-2lpeak_k(\pi)} \\ &= (2x)^{lpeak_k(\pi)} (1+x)^{n-k+1-2lpeak_k(\pi)}. \end{aligned}$$

By combining them all together, we deduce the desired result

$$\begin{aligned}\sum_{p\geq 0}\Omega^{(\ell)}(\pi;k;p)x^p &= \frac{2^{lpeak_k(\pi)+k-1}}{(1-x)^{n-k+2}}(2x)^{lpeak_k(\pi)}(1+x)^{n-k+1-2lpeak_k(\pi)}\\ &= 2^{k-1}\frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}}\left(\frac{4x}{(1+x)^2}\right)^{lpeak_k(\pi)}.\end{aligned}$$

□

Recalling that the number of permutations of n with p width-k left peaks is $|\Gamma^{(\ell)}_{n,k,p}|$. Then, the width-k left peak polynomial is defined as :

$$W^{(\ell)}_{n,k}(x) = \sum_{\pi\in S_n} x^{lpeak_k(\pi)} = \sum_{p=0}^{\lfloor\frac{n-k+1}{2}\rfloor} |\Gamma^{(\ell)}_{n,k,p}|x^p. \tag{20}$$

We need also the following identity for the type B Eulerian polynomials given by Stembridge ([23], Prop 7.1 b) :

$$\sum_{p\geq 0}(2p+1)^n x^p = \frac{B_n(x)}{(1-x)^{n+1}}.$$

Proposition 4.7. *We have the following relation between the width-k left peak polynomials and the width-k Eulerian polynomials of type B :*

$$W^{(\ell)}_{n,k}\left(\frac{4x}{(1+x)^2}\right) = \frac{\mathfrak{WB}_{n,k}(x)}{2^{k-1}(1+x)^{n-k+1}}. \tag{21}$$

Proof. Using the relation of identity (19), the number of width-k left enriched P-partitions f is

$$(2p+1)^d\ \left((2p+1)^{(d-1)}\right)^r\ \left((2p+1)^d\right)^{k-r-1}.$$

Since, $\pi_k \in S_d$ and $(\pi_1,\dots,\pi_{k-1}) \in S^r_{d+1}\times S^{k-r-1}_d$. In fact, the number in total of $f:[n-k+1]\to[p]^{(\ell)}$ is $(2p+1)^{n-k+1}$. Consequently, $\Omega^{(\ell)}(\pi,k,p) = (2p+1)^{n-k+1}$.
A type B poset is a poset P_B whose elements are $\pm[n]$. In the present case, P_B is an antichain of $\pm[n-k+1]$ elements. Thus, the order polynomial $\Omega_B(\pi,k,p)$ is the same of $\Omega^{(\ell)}(\pi,k,p)$.
By reason of $\mathfrak{L}(P_B) = \mathfrak{WB}_{n,k}$, we have,

$$\sum_{p\geq 0}(2p+1)^{n-k+1}x^p = \frac{\mathfrak{WB}_{n,k}(x)}{(1-x)^{n-k+2}}.$$

Using Theorem 4.6, we have

$$\begin{aligned}2^{k-1}\frac{(1+x)^{n-k+1}}{(1-x)^{n-k+2}}W^{(\ell)}_{n,k}\left(\frac{4x}{(1+x)^2}\right) &= \sum_{p\geq 0}(2p+1)^{n-k+1}x^p\\ &= \frac{\mathfrak{WB}_{n,k}(x)}{(1-x)^{n-k+2}}.\end{aligned}$$

The reorganization of the terms gives the desired result :

$$2^{k-1}W_{n,k}^{(\ell)}\left(\frac{4x}{(1+x)^2}\right)=\frac{\mathfrak{WB}_{n,k}(x)}{(1+x)^{n-k+1}}.$$

□

Now, we can prove Theorem 4.3.

Proof of Theorem 4.3. Substituting x by $\frac{4x}{(1+x)^2}$ in Eq. (20) and using Eq. (4), we obtain

$$\sum_{p=0}^{\lfloor n-k+1/2\rfloor}\gamma_{n,k,p}^{B}\ x^p(1+x)^{n-k+1-2p}=\sum_{p=0}^{\lfloor\frac{n-k+1}{2}\rfloor}2^{2p+k-1}|\Gamma_{n,k,p}^{(\ell)}|x^p(1+x)^{n-k+1-2p}$$

the desired result. □

For $1\leq n\leq 6,\ 1\leq k\leq n$ and $0\leq p\leq\lfloor\frac{n-k+1}{2}\rfloor$, we record a few values of $\gamma_{n,k,p}^{B}$ in the following table.

n	k	p			
		0	1	2	3
1	1	1			
2	1	1	4		
	2	4			
3	1	1	20		
	2	12	0		
	3	24			
4	1	1	72	0	
	2	24	96		
	3	96	0		
	4	192			
5	1	1	232	976	
	2	80	480	640	
	3	480	0		
	4	960	0		
	5	1920			
6	1	1	716	7664	3904
	2	160	3520	6400	
	3	1440	5760	0	
	4	5760	0		
	5	11520	0		
	6	23040			

Table 4 – The first few values of $\gamma_{n,k,p}^{B}$.

5 Width-k Eulerian polynomials of type D

Now we will study the *Gamma*-positivity of the width-k Eulerian polynomials on the set D_n. Borowiec and Mlotkowski [4] have introduced a new array of type D Eulerian numbers. They found in particular a recurrence relation for this array. In this section, we will generalize these numbers and we give a new generalization of the Eulerian polynomials of type D, $D_n(x)$, and its Gamma-positivity with the statistics *width-k descent*.

The case $k = 1$ corresponds to the classical Euleurian polynomials $D_n(x)$.
We start by defining the width-k Eulerian polynomials of type D by :

$$\mathfrak{WD}_{n,k}(x) = F_n^{des_k^D}(x) = \sum_{\pi \in D_n} x^{des_k^D(\pi)}.$$

In the following theorem, we give the mean result of this section on the set D_n.

Theorem 5.1. *For any $n \geq 1$, we have*
if $k = 1$ then

$$F_n^{des_1^D}(x) = F_n^{des_D}(x) = D_n(x),$$

and if $2 \leq k \leq n$ then

$$F_n^{des_k^D}(x) = 2^{n-d-1} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^r(x). \tag{22}$$

Proof. This identity follows completely in the same way as the identity (11) of Theorem4.2, with the same reasoning except the fact that $|D_n| = \frac{|B_n|}{2}$. □

Denote $\bar{D}_n = B_n \backslash D_n$ and,

$$\mathrm{WD}_{n,k,p} = \{\pi \in D_n; des_k^D(\pi) = p\},$$

$$\mathrm{W\bar{D}}_{n,k,p} = \{\pi \in \bar{D}_n; des_k^{\bar{D}}(\pi) = p\}.$$

So that $\mathrm{WD}_{n,k,p} = \mathrm{WB}_{n,k,p} \bigcap D_n$ and $\mathrm{W\bar{D}}_{n,k,p} = \mathrm{WB}_{n,k,p} \backslash D_n$. The cardinalities of these sets will be denoted by $d(n,k,p)$ and $\bar{d}(n,k,p)$, respectively. Since $\mathrm{WB}_{n,k,p} = \mathrm{WD}_{n,k,p} \bigcup \mathrm{W\bar{D}}_{n,k,p}$, we have

$$b(n,k,p) = d(n,k,p) + \bar{d}(n,k,p).$$

Now, we give the following symmetry which generalizes the Proposition 4.1 of Borowiec and Mlotkowski in [4] for $k = 1$.

Proposition 5.2. *For $0 \leq p \leq n$ we have*
if n is even with $1 \leq k \leq n$, if n is odd with $2 \leq k \leq n$

$$\delta(n,k,p) = \delta(n,k,n-k+1-p), \quad \bar{\delta}(n,k,p) = \bar{\delta}(n,k,n-k+1-p),$$

if n is odd with $k = 1$

$$\delta(n,k,p) = \bar{\delta}(n,k,n-k+1-p), \quad \bar{\delta}(n,k,p) = \delta(n,k,n-k+1-p).$$

Proof. Let $\phi_{(k,p)} : B_n \to B_n$, for $\pi \in B_n$ define $-\pi \in B_n$ by $\phi_{(k,p)}(\pi) = -\pi$. Since the map $\phi_{(k,p)}$ defined a bijection $\mathrm{WD}_{n,k,p} \to \mathrm{WD}_{n,k,n-k+1-p}$, $\mathrm{W\bar{D}}_{n,k,p} \to \mathrm{W\bar{D}}_{n,k,n-k+1-p}$ if n is even with $1 \leq k \leq n$ and if n is odd with $2 \leq k \leq n$ and $\mathrm{WD}_{n,k,p} \to \mathrm{W\bar{D}}_{n,k,n-k+1-p}$, if n is odd with $k = 1$. □

For $k = 1$, Borowiec and Mlotkowski [4] showed the following recurrence relations :

Proposition 5.3. *([4], Proposition 4.5) For* $0 \leq p \leq n$

$$d(n,1,p) = (2p+1)d(n-1,1,p) + (2n-2p+1)d(n-1,1,p-1) + (-1)^p \binom{n-1}{p-1},$$

$$\bar{d}(n,1,p) = (2p+1)\bar{d}(n-1,1,p) + (2n-2p+1)\bar{d}(n-1,1,p-1) - (-1)^p \binom{n-1}{p-1}.$$

Using identity (22), we find that $\mathfrak{WD}_{n,k}(x) = 2^{n-d-1} M_{n,k} B_d(x) A_d^{k-r-1}(x) A_{d+1}^r(x)$, for $2 \leq k \leq n$. Then, for a clearer observation we put in the table below the coefficients of the product polynomials $B_d(x) A_d^{k-r-1}(x) A_{d+1}^r(x)$, denoted by $\delta(n,k,p)$, for $1 \leq n \leq 6$, $1 < k \leq n$ and $0 \leq p \leq n-k+1$. We record also a few values of $\bar{\delta}(n,k,p)$ of $\mathrm{W\bar{D}}_{n,k,p}$.

We can remark that for $0 \leq p \leq n-k+1$, where k is the smallest positive integer such that $n+1 = 2k+r$ with $0 \leq r < k$, we have the following recurrence relations : For any $n \geq 3$,

$$\delta(n,k,p) = \delta(n-2,k-1,p-1) + \delta(n-2,k-1,p),$$

for any $n \geq 4$,

$$\bar{\delta}(n,k,p) = \bar{\delta}(n-2,k-1,p-1) + \bar{\delta}(n-2,k-1,p),$$

where,

$$\delta(n,k,-1) = \bar{\delta}(n,k,-1) = 0,$$

if $2 \leq k \leq n$

$$\delta(n,k,0) = \bar{\delta}(n,k,0) = \delta(n,k,n-k+1) = \bar{\delta}(n,k,n-k+1) = 1,$$

if $k = 1$

$$\delta(n,1,0) = 1, \quad \bar{\delta}(n,1,0) = 0,$$

and

$$\delta(n,1,n) = \begin{cases} 1 & \text{if n is even,} \\ 0 & \text{if n is odd,} \end{cases}$$

$$\bar{\delta}(n,1,n) = \begin{cases} 0 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$$

Problem 5.1. *Is it possible to find recurrence relations for* $\delta(n,k,p)$ *and* $\bar{\delta}(n,k,p)$ *for all* $1 \leq k \leq n$*?*

Remark 5.1. *If* n *is even,*

$$deg(\mathfrak{WD}_{n,k}(x)) = n-k+1 \quad for\ all\ 1 \leq k \leq n.$$

If n *is odd,*

$$deg(\mathfrak{WD}_{n,k}(x)) = \begin{cases} n-1 & if\ k = 1, \\ n-k+1 & if\ 2 \leq k \leq n. \end{cases}$$

In table 5, we observe that if n *is odd and in the case where* $k = 1$*, the polynomial* $\mathfrak{WD}_{n,k}(x)$ *is not unimodal and not symmetric. For example,* $\mathfrak{WD}_{5,1}(x) = 1 + 116x + 846x^2 + 836x^3 + 121x^4$.

n	k	p						
		0	1	2	3	4	5	6
1	1	1	1					
2	1	1	2	1				
	2	1	1					
3	1	1	10	13	0			
	2	1	2	1				
	3	1	1					
4	1	1	36	118	36	1		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	1	116	846	836	121	0	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	1	358	5279	11764	5279	358	1
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

TABLE 5 – A few values of $\delta(n,k,p)$.

n	k	p						
		0	1	2	3	4	5	6
1	1	0	1					
2	1	0	4	0				
	2	1	1					
3	1	0	13	10	1			
	2	1	2	1				
	3	1	1					
4	1	0	40	112	40	0		
	2	1	7	7	1			
	3	1	2	1				
	4	1	1					
5	1	0	121	836	846	116	1	
	2	1	10	26	10	1		
	3	1	3	3	1			
	4	1	2	1				
	5	1	1					
6	1	0	364	5264	11784	5264	364	0
	2	1	27	116	116	27	1	
	3	1	8	14	8	1		
	4	1	3	3	1			
	5	1	2	1				
	6	1	1					

TABLE 6 – A few values of $\bar{\delta}(n,k,p)$.

Theorem 5.4. *For any $1 \le k \le n$, (unless if n is odd and $k = 1$), we have*

$$\mathfrak{WD}_{n,k}(x) = \sum_{\pi \in D_n} x^{des_k^D(\pi)} = \sum_{p=0}^{\lfloor n-k+1/2 \rfloor} \gamma_{n,k,p}^D \; x^p (1+x)^{n-k+1-2p},$$

where for all $2 \le k \le n$, $\gamma_{n,k,p}^D = \frac{\gamma_{n,k,p}^B}{2}$.

Proof. Firstly, if n is odd and for $k = 1$, there are no permutations π in D_n such as $des_1^D(\pi) = n$. Thus, $\mathfrak{WD}_{n,1}(x)$ is not symmetric and therefore $\mathfrak{WD}_{n,1}(x)$ is not γ-positive.
If n is even and for all $1 \le k \le n$, the number of permutations whose $des_k^D(\pi) = n - k + 1 - p$ is equivalent to the number of permutations of which $des_k^D(\pi) = p$ with $0 \le p \le \lfloor \frac{n-k+1}{2} \rfloor$. Thus, $\mathfrak{WD}_{n,k}(x)$ is symmetric with center of symmetry $\lfloor \frac{n-k+1}{2} \rfloor$. Moreover, it is easy to see that $\mathfrak{WD}_{n,k}(x)$ is unimodal.
Therefore, since for all $2 \le k \le n$ the width-k Eulerian polynomials of type D is γ-positive, and the cardinal of this group is equal to $\frac{\lfloor B_{n,k} \rfloor}{2}$, thus $\gamma_{n,k,p}^D = \frac{\gamma_{n,k,p}^B}{2}$. □

For $2 \le n \le 6,\ 1 \le k \le n$ and $0 \le p \le \lfloor \frac{n-k+1}{2} \rfloor$, we record a few values of $\gamma_{n,k,p}^D$.

n	k	p			
		0	1	2	3
2	1	1	0		
	2	2			
3	1				
	2	6	0		
	3	12			
4	1	1	32	48	
	2	12	48		
	3	48	0		
	4	96			
5	1				
	2	40	240	320	
	3	240	0		
	4	480	0		
	5	960			
6	1	1	352	3856	1920
	2	80	1760	3200	
	3	720	2880	0	
	4	2880	0		
	5	5760	0		
	6	11520			

TABLE 7 – A few values of $\gamma_{n,k,p}^D$.

Problem 5.2. *Is it possible to find a recurrence relation for $\gamma_{n,1,p}^D$?, if n is even and $k = 1$.*

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References

[1] R. M. Adin, F. Brenti and Y. Roichman, *Descent numbers and major indices for the hyperoctahedral group.* Advances in Applied Mathematics 27 (2001), 210-224.

[2] CA. Athanasiadis, *Gamma-positivity in combinatorics and geometry.* arXiv:1711.05983v2.

[3] M. Aguiar, N. Bergeron and K. Nyman, *The peak algebra and the descent algebras of types B and D.* Trans. Amer. Math. Soc. 356 (2004), 2781-2824.

[4] A. Borowiec and W. Mlotkowski, *New Eulerian numbers of type D.* Electr. J. Comb. 23(1) : P1.38 (2016).

[5] P. Brändén, *Unimodality, log-concavity, real-rootedness and beyond.* Handbook of Enumerative Combinatorics, arXiv:1410.6601.

[6] A. Björner and F. Brenti, *Combinatorics of Coxeter groups.* Graduate Texts in Mathematics, vol. 231, Springer, New York, 2005.

[7] C.O. Chow, *Noncommutative symmetric functions of type B,* Ph.D. thesis (2001)

[8] R. Davis, *Width-k Generalizations of Classical Permutation Statistics.* J. Integer Seq. 20 (2017), no. 6, Art. 17.6.3, 14 pp.

[9] D. Foata and M. P. Schützenberger, *Théorie Géométrique des Polynômes Eulériens.* Lecture Notes in Mathematics 138, Springer-Verlag, Berlin-New York, 1970.

[10] D. Foata and V. Strehl, *Euler numbers and variations of permutations,* in Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Tomo I, Atti dei Convegni Lincei, No. 17, Accad. Naz. Lincei, Rome, 1976, pp. 119-131.

[11] D. Foata and V. Strehl, *Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers,* Math. Z. 137 (1974), 257-264.

[12] I. Gessel, *Multipartite P -partitions and inner products of skew Schur functions,* in : Contemp. Math., vol. 34, 1984, pp. 289-317.

[13] J. E. Humphreys, *Reflection Groups and Coxeter Groups,* Cambridge Studies in Advanced Mathematics, no. 29, Cambridge Univ. Press, Cambridge, 1990.

[14] Z. Lin and J. Zeng, *The γ-positivity of basic Eulerian polynomials via group actions,* J. Combin. Theory, Ser. A, 135 (2015), 112-129.

[15] Percy A. Macmahon, *Combinatory Analysic,* Chelsea Publishing Co., New York, 1960.

[16] K. Nyman, *Enumeration in Geometric Lattices and the Symmetric Group,* Ph.D. Thesis, Cornell University, 2001.

[17] K. Nyman, *The peak algebra of the symmetric group,* J. Algebraic Combin. 17 (2003), 309-322.

[18] T.K. Petersen, *Enriched P-partitions and peak algebras,* Adv. Math. 209 (2007), 561-610.

[19] V. Reiner, *Signed posets,* J. Combin. Theory, Ser. A, 62 (1993), 324-360.

[20] J. Sack and H. Ulfarsson, *Refined inversion statistics on permutations,* Electron. J. Combin,19(1) :Paper 29, 27, 2012.

[21] L. Solomon, *A Mackeyformula in the group ring of a Coxeter group,* J. Algebra 41 (1976) 255Ű268.

[22] R. P. Stanley, *Ordered structures and partitions,* Mem. Amer. Math. Soc. 119 (1972).

[23] R. Stanley, *Enumerative Combinatorics,* Volume I, Cambridge University Press, 1997.

[24] J. Stembridge, *Enriched P-partitions,* Transactions of the American Mathematical Society 349 (1997), 763-788.

[25] J. Stembridge, *Some permutation representations of Weyl groups associated with the cohomology of toric varieties,* Adv. Math., 106(2) :244-301, 1994.