MEMGAMES

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ABSTRACT. In this article, we study the structure, and in particular the Grundy values, of a family of games known as memgames.

1. INTRODUCTION

Let $\mathscr{P}(\mathbb{N})$ denote the power set of the positive integers. For each function $F : \mathbb{N} \to \mathscr{P}(\mathbb{N})$, we define an impartial game, known as a *memgame*, as follows. The game is played with a single pile of stones, initially containing n stones. On the first move, the first player may remove any positive number of stones. On future moves, if the previous player removed k stones, then the next player may remove m stones if and only if $m \in F(k)$. (Of course, one may not remove more stones than are in the pile.) We represent the position with n stones where the last move was to remove k, by n_k. We use the term memgame, for the position reMEMbers a small part of its history. We call the function F the *memfunction*.

There are several classical memgames. For instance, if $F(k) = \{1, 2, \ldots, 2k\}$, then we recover the game of FIBONACCI NIM [Whi63], with the exception that in FIBONACCI NIM, the first player may not remove all the stones. More generally, if, for some $\alpha \geq 1$, we have $F(k) = \{1, 2, \ldots, \lfloor \alpha k \rfloor\}$, then we recover a class of take-away games studied for instance in [Sch70] and [RS18]. A more general class of memgames was studied in [HRR03]. In all these papers, the memfunction F has the form $F(k) = \{1, 2, \ldots, g(k)\}$ for some $g(k)$. In this paper, we focus on certain memfunctions of different forms.

In this article, we study the Grundy values [Gru39, Spr35] of certain memgames. We find that these Grundy values enjoy a rich structure, and we hope that this will be an avenue for much future work. The Grundy values for FIBONACCI NIM have already been investigated by the first two authors in [LR16].

In this article, if H is an impartial game, we denote its Grundy number by $\mathcal{G}(H)$. When the ruleset of a memgame (i.e. the memfunction $F : \mathbb{N} \to \mathscr{P}(\mathbb{N})$) is clear, we will write $\mathcal{G}(n_k)$.

2. MEM, MEM$^+$, AND MEM^0

We highlight three specific memgames. A memgame is determined entirely by its memfunction $F : \mathbb{N} \to \mathscr{P}(\mathbb{N})$. The game MEM is defined by the memfunction $F(k) = \{k, k+1, \ldots\}$. In other words, in MEM, one must remove at least as many stones as the previous player did. The game MEM$^+$ is defined by the memfunction $F^+(k) = \{k+1, k+2, \ldots\}$. In other words, in MEM$^+$, one must remove strictly more stones than the previous player did. The game MEM^0 is defined by the memfunction $F^0(k) = \mathbb{N} \setminus \{k\}$. In other words, in MEM^0, one may not remove exactly the same number of stones as the previous player did.

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$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	2	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
9	3	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
10	3	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
11	3	2	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
12	3	3	2	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
13	3	3	2	2	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
14	4	3	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
15	4	3	3	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0	0
16	4	3	3	2	2	2	1	1	1	1	1	1	1	1	1	0	0	0	0	0
17	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	0	0	0	0
18	4	4	3	3	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0	0
19	4	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	0	0
20	5	4	3	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	0

Table 1. Grundy values of MEM$^+$

A substantial amount of work has been done on games very closely related to MEM^0. For instance, Chapter 15 in Volume 3 of *Winning Ways for your Mathematical Plays* [BCG03] contains a discussion of the game D.U.D.E.N.E.Y., which is short for DEDUCTIONS UNFALLING, DISALLOWING ECHOES, NOT EXCEEDING Y. For a fixed value of Y, moves in D.U.D.E.N.E.Y. are the same as those of MEM^0, except that no more than Y stones may ever be removed on a single turn. The discussion in [BCG03] refers back to earlier work by Schuh, who discusses the game in [Sch68, Chapter XII, §217–224] and describes winning strategies when $Y = 3, 5, 7, 9$ (the case where Y is even is trivial, since the usual strategy for subtraction games still works).

3. GRUNDY VALUES OF MEM$^+$

The simplest of the three games to understand is MEM$^+$. See Table 1 and Figure 1 for the first few Grundy values.

Theorem 3.1. *In the game of* MEM$^+$, $\mathcal{G}(n_k)$ *is the largest integer* m *for which*

$$mk + \frac{m(m+1)}{2} \leq n \tag{3.1}$$

Proof. Let the m^{th} triangular number be $T_m = \frac{m(m+1)}{2}$. We define the *m-front* to be the set of positions $F_m = \{F_m(k)\}$, where $F_m(k) = (km+T_m)_k$. We define the *m-sector* to be the space between the m-front (including the m-front) and the $(m+1)$-front, i.e. $\Delta_m = \bigcup_k \Delta_m(k)$, where

$$\Delta_m(k) = \{(km + T_m)_k, (km + T_m + 1)_k, \ldots, (k(m+1) + T_{m+1} - 1)_k\}.$$

For example, the region named '2's in Figure 1 is Δ_2. Observe that $|\Delta_m(k)| = k + m + 1$.

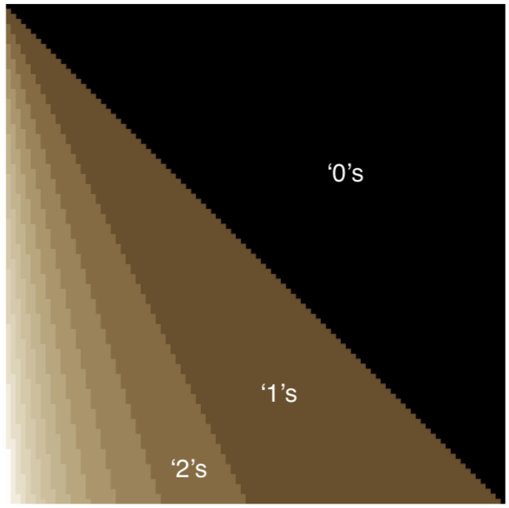

Figure 1. The Grundy values of the game MEM$^+$; the columns removal numbers k and the rows heap sizes n, with the upper left corner $n_k = 1_1$. Note that there is no move from this position, so $\mathcal{G}(1_1) = 0$, expanding into the black region, and the lighter shades symbolize increasing Grundy numbers.

We will prove by induction on n that $n_k \in \Delta_j$ if and only if $\mathcal{G}(n_k) = j$, for all $0 \le j < m$; the base case $n = 0$ is obvious. In order to prove this, it suffices to justify the following claim.

Claim. *For $n \in \Delta_m(k)$, then the set $\{\mathcal{G}((n-k-i)_{k+i}) \mid 1 \le i \le n-k\} = \{0, \dots, m-1\}$.*

First we prove that, for all i, $(n-k-i)_{k+i} \notin \Delta_m(k+i)$. This follows because

$$\min \Delta_m(k+i) = \min \Delta_m(k+1) = \max \Delta_m(k) - k,$$

so whenever a player removes more than the column number (here $k+i$) from the m-sector, then the resulting position is in an m'-sector with $m' < m$. Now we must prove that each such m'-sector appears. Firstly, note that $(n-k-1)_{k+1} \in \Delta_{m-1}(k+1)$, whenever $n \in \Delta_m$. Thus, it suffices to show that, for all $0 \le j < m-1$,

$$\{(n-k-1-i)_{k+1+i} \mid 1 \le i < n-k-1\} \cap \Delta_j \ne \varnothing.$$

But this holds, because, for any front position $x_y \in F_{j+1}$, we have $(x-1)_{y+1} \in \Delta_j$, and clearly, for each j, exactly one such pair of positions will be obtained as i ranges in the given interval. ∎

4. Grundy values of Mem

The ruleset of MEM is very similar to that of MEM$^+$. This might lead us to believe that its Grundy values should be closely related. Indeed, this is true, although there are also some surprises. See Table 2 as well as Figure 2 for some Grundy values of MEM.

Evidently, there is a lot of structure here. Most of the Grundy values are indeed very similar to those of MEM$^+$, but there is a small parabolic region with some more fractal-like behavior. We have written the numbers in this parabolic region as old style numbers 0123456789 to distinguish them from those in the main region.

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	2	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	4	3	2	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	3	3	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
8	2	2	2	2	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
9	4	4	3	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0
10	3	3	3	2	2	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
11	5	3	3	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0
12	4	4	4	3	2	2	1	1	1	1	1	1	0	0	0	0	0	0	0	0
13	5	4	4	3	2	2	1	1	1	1	1	1	1	0	0	0	0	0	0	0
14	6	5	4	3	2	2	2	1	1	1	1	1	1	1	0	0	0	0	0	0
15	3	3	3	3	3	2	2	1	1	1	1	1	1	1	1	0	0	0	0	0
16	6	6	5	4	3	2	2	2	1	1	1	1	1	1	1	1	0	0	0	0
17	5	5	5	4	3	2	2	2	1	1	1	1	1	1	1	1	1	0	0	0
18	4	4	4	4	3	3	2	2	2	1	1	1	1	1	1	1	1	1	0	0
19	6	4	4	4	3	3	2	2	2	1	1	1	1	1	1	1	1	1	1	0
20	7	6	6	5	4	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1

Table 2. Grundy values of Mem

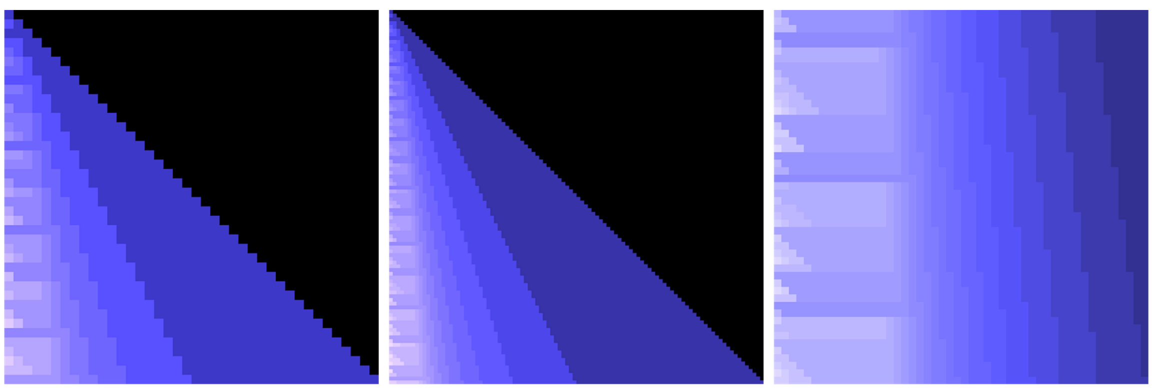

Figure 2. These pictures show some further Grundy values of the game Mem. In the middle picture, on the left, we note the emergence of a parabolic region with high Grundy values. In the rightmost picture, we have zoomed in on the fractal type behavior inside this region. Each number is a different shade, with lighter shades denoting larger Grundy values, so black cells are 0, dark blue cells are 1, and so forth.

Theorem 4.1. *In the game of* Mem, *if* $k^2 \geq n$, *then* $\mathcal{G}(n_k) = \lfloor \frac{n}{k} \rfloor$.

Proof. First, note that if $k^2 \geq n$, then any move, say to $n'_{k'}$, from n_k we have $k'^2 \geq n'$. This is clear, because $k' \geq k$ and $n' < n$, so $k'^2 \geq k^2 \geq n \geq n'$. Next, we must show, if $k^2 \geq n$ and $\lfloor \frac{n}{k} \rfloor = q$, then for any a with $k \leq a \leq n$, $\lfloor \frac{n-a}{a} \rfloor < q$. This is true because

$$\frac{n-a}{a} \leq \frac{n-a}{k} \leq \frac{n-k}{k} \leq \frac{n}{k} - 1 < \left\lfloor \frac{n}{k} \right\rfloor = q.$$

$n\backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	2	2	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
4	2	3	3	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
5	3	3	3	3	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
6	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
7	4	4	4	4	4	4	0	4	4	4	4	4	4	4	4	4	4	4	4	4
8	4	5	3	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5
9	5	5	5	5	5	5	5	5	0	5	5	5	5	5	5	5	5	5	5	5
10	6	6	4	6	6	3	6	6	6	6	6	6	6	6	6	6	6	6	6	6
11	6	5	7	4	7	7	7	7	7	7	0	7	7	7	7	7	7	7	7	7
12	7	7	7	7	4	7	7	7	7	7	7	0	7	7	7	7	7	7	7	7
13	6	6	6	6	6	6	6	6	6	6	6	6	0	6	6	6	6	6	6	6
14	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4
15	8	8	7	8	8	8	8	8	8	8	8	8	8	8	0	8	8	8	8	8
16	8	9	6	9	9	9	9	9	9	9	9	9	9	9	9	0	9	9	9	9
17	9	9	9	9	9	7	9	9	9	9	9	9	9	9	9	9	0	9	9	9
18	8	8	8	8	8	8	8	8	8	5	8	8	8	8	8	8	8	8	8	8
19	10	9	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	10	0	10
20	10	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	11	0

Table 3. Grundy values of $\textsc{Mem}^0$

Next, we must show that for every t with $0 \le t < q$, there is some integer a with $k \le a \le n$ such that $\left\lfloor \frac{n-a}{a} \right\rfloor = t$. Let us temporarily omit the requirement that a be an integer. The value of a making $\frac{n-a}{a} = t$ is $a = \frac{n}{t+1}$, whereas the value of a making $\frac{n-a}{a} = t+1$ is $a = \frac{n}{t+2}$. It thus suffices to show that there is some *integer* a with

$$\frac{n}{t+2} < a \le \frac{n}{t+1}.$$

Now, since $t < q \le \sqrt{n}$, we have

$$\frac{n}{t+1} - \frac{n}{t+2} = \frac{n}{(t+1)(t+2)} \ge \frac{n}{\sqrt{n}(\sqrt{n}-1)} > 1,$$

so there is some integer in the range $\left(\frac{n}{t+2}, \frac{n}{t+1}\right]$. This completes the proof. ■

5. $\mathcal{P}$ positions in $\textsc{Mem}^0$

The most complex of these three games is $\textsc{Mem}^0$, and it is here that we see the richest structure. See Table 3 and Figure 3 for the first few Grundy values.

In order to characterize the $\mathcal{P}$ positions (and higher Grundy values) of $\textsc{Mem}^0$, we need to introduce the dyadic valuation.

Definition 5.1. Let n be a positive integer. We may uniquely write $n = 2^e m$, where e, m are nonnegative integers, and m is odd. We define its *dyadic valuation* to be $v_2(n) = e$.

By convention, we will say that $v_2(0)$ is even, without specifying its value.

Theorem 5.2. *The $\mathcal{P}$ positions of $\textsc{Mem}^0$ are of the form n_n, where $v_2(n) \equiv 0 \pmod 2$.*

Proof. Consider first a position of the form n_m, with $m \ne n$. Then there is a move to 0_0, so n_m is an $\mathcal{N}$ position. Therefore each $\mathcal{P}$ position must be of the form n_n. Note that $v_2(0) \equiv 0 \pmod 2$ by convention and that $0 = v_2(1) \equiv 0 \pmod 2$, but $1 = v_2(2) \equiv 0 \pmod 2$.

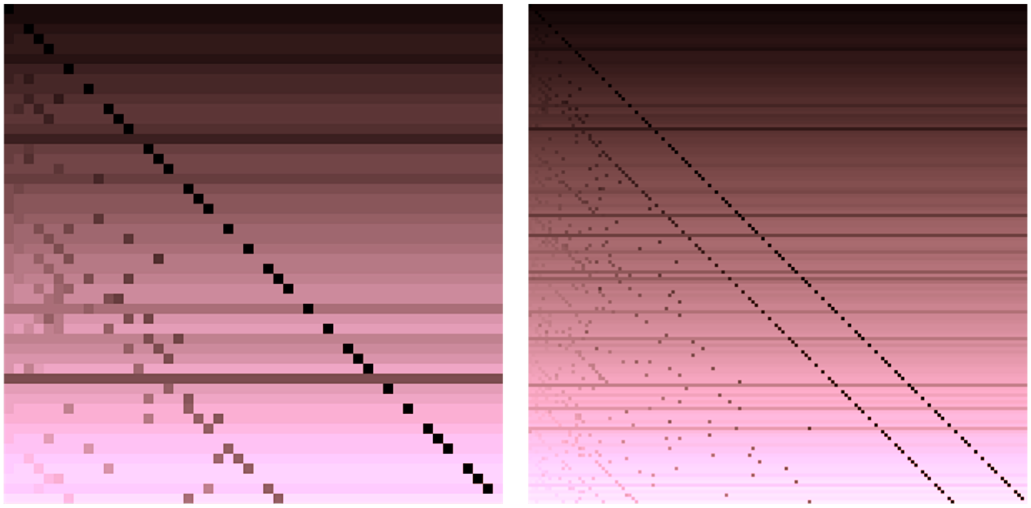

Figure 3. The pictures show the first few Grundy values of the game Mem^0. In the picture to the left, we see in particular the '0's on the main diagonal, and in the picture to the right, one can see the emergence of an accompanying 'left shifted' diagonal of Grundy values 12.

Suppose the result holds for all $n' < n$. Now, from n_n, with $n > 1$ such that $v_2(n) \not\equiv 0 \pmod 2$, clearly $(n/2)_{(n/2)}$ is the desired move option (because $v_2(n/2) \equiv 0 \pmod 2$). On the other hand, if $v_2(n) \equiv 0 \pmod 2$, then either a player must move away from the main diagonal, or leave a position with odd dyadic valuation. ■

6. Higher Grundy values of Mem^0

Before computing any higher Grundy values, we introduce the notion of a *frontier* in Mem^0. Note that the positions $n_{n+1}, n_{n+2}, n_{n+3}, \ldots$ all have the same moves available and thus have the same Grundy values. Thus, we denote the position $n_{n+1}, n_{n+2}, n_{n+3}, \ldots$ by n_∞. We call positions of the form n_∞ *frontier positions*, and we call the Grundy value $\mathcal{G}(n_\infty)$ the n^{th} *frontier value.*

The first few frontier values are 0,1,1,2,3,3,2,4,5,5,6,7,7,6,4,8,9,9,8,10. We will prove the following results:

Theorem 6.1. *The frontier values are unbounded.*

Theorem 6.2. *Every integer appears at least once as a frontier value.*

Theorem 6.3. *If $f(m)$ denotes the least n so that $\mathcal{G}(n_\infty) = m$, then $f(m) < f(m')$ whenever $m < m'$.*

Observe that the positions n_k for $k \le n$ and n_∞ are very similar: they have all the same options, except for one: the position $(n-k)_k$ is an option from n_∞, but not from n_k. As a result, we have the following important lemma:

Lemma 6.4. *$\mathcal{G}(n_k)$ is equal to $\mathcal{G}(n_\infty)$ or to $\mathcal{G}((n-k)_k)$.*

Definition 6.5. We call n_k an *exceptional position* if $\mathcal{G}(n_k) = \mathcal{G}((n-k)_k)$.

As a consequence, we immediately have the following proposition:

Proposition 6.6. *Suppose n is the smallest integer for which there exists some k with $\mathcal{G}(n_k) = m$. Then $\mathcal{G}(n_\infty) = m$.*

Proof. By Lemma 6.4, either $\mathcal{G}(n_k) = \mathcal{G}(n_\infty)$ or $\mathcal{G}(n_k) = \mathcal{G}((n-k)_k)$. By minimality of n, we exclude the second possibility. ■

Theorem 6.7 (The Final Frontier Theorem)**.** *If $\mathcal{G}(n_\infty) = m$ and $a > 2n$, then $\mathcal{G}(a_\infty) \neq m$.*

Proof. If $a > 2n$, then there is a move from a_∞ to $n_{a-n} = n_\infty$. Thus $m = \mathcal{G}(n_\infty) \neq \mathcal{G}(a_\infty)$. ■

Thus $2n$ is the *final* (possible) *frontier* for the Grundy value m.

Theorem 6.7 allows us to prove Theorem 6.1.

Proof of Theorem 6.1. By Theorem 6.7, each integer only appears finitely many times on the frontier. Thus, there must be infinitely many (and hence unbounded) numbers on the frontier. ■

In conjunction with Proposition 6.6, we can also establish Theorem 6.2:

Proof of Theorem 6.2. Since the frontier values are unbounded, every integer must appear as some Grundy value $\mathcal{G}(n_k)$. By Proposition 6.6, the first time m appears as $\mathcal{G}(n_k)$, it establishes itself on the frontier. Thus every nonnegative integer appears on the frontier. ■

Proof of Theorem 6.3. If $m < m'$, then the first instance of m must occur before the first instance of m'. Thus by Proposition 6.6, the first frontier value equal to m must be less than the first frontier value equal to m'. ■

What happens to a Grundy value *after* the final frontier? It turns out that there is a curious dichotomy here:

Theorem 6.8 (The Mortality Theorem)**.** *Suppose that m appears at least twice on the frontier, say as $\mathcal{G}(n_\infty) = \mathcal{G}(n'_\infty) = m$ with $n < n'$. Then if $a > 2n'$, we have $\mathcal{G}(a_k) \neq m$ for all k. Thus the value m dies out after row $2n'$.*

Proof. If $a > 2n'$, then from a_∞, there are moves to both $n_{a-n} = n_\infty$ and to $n'_{a-n'} = n'_\infty$. From a_k, at least one of these is a legal move: the legal moves are to $(a-i)_i$ for $i \neq k$, and k cannot be equal to both n and n' simultaneously. Thus m is an excludant for a_k, so $\mathcal{G}(a_k) \neq m$. ■

Example. Let $m = 11$. The first frontier value for m is $n = 20$, so that $\mathcal{G}(20_\infty) = 11$. There is also a second frontier value of 11, namely $\mathcal{G}(21_\infty) = 11$. Thus Theorem 6.8 implies that 11 never appears as a Grundy value of a_k for $a > 42$. It turns out that there are several Grundy values equal to 11 with $a > 21$, namely $\mathcal{G}(22_2) = \mathcal{G}(40_{19}) = \mathcal{G}(42_{22}) = 11$.

The Mortality Theorem allows for the possibility that a number can appear exactly once along the frontier. Indeed, this happens with $m = 0$: we have $\mathcal{G}(0_\infty) = 0$, but 0 does not occur again along the frontier. When a number occurs only once on a frontier, then it does *not* die out at any point. Indeed, there are arbitrarily large values of n for which $\mathcal{G}(n_n) = 0$.

It turns out that whenever a number occurs exactly once along the frontier, it exhibits a very similar pattern to that of 0. The first instance of this is $m = 12$.

Example. $m = 12$ occurs exactly once along the frontier, with $\mathcal{G}(22_\infty) = 12$. However, it also appears along a *diagonal*. There are infinitely many values of a for which $\mathcal{G}((a+22)_a) = 12$. First, if 12 reestablishes itself a second time along the frontier, it must do so by row $2 \times 22 = 44$. By computation, we observe that it does not.

Now, suppose that $n > 44$. Then there is a move from n_k to 22_∞, *unless* $k = n - 22$. So, if $\mathcal{G}(n_k)$ is going to be equal to 12, we *must* have $k = n - 22$. By a finite check, we also observe that there are several exceptional position: $\mathcal{G}(22_k) = 12$ for $k \neq 2, 10$, and $\mathcal{G}(24_1) = \mathcal{G}(32_5) = 12$.

If $k + 22 > 88$, there are moves from $(k+22)_k$ to a_∞ for $a = 0, 1, \ldots, 21$. Thus $0, 1, \ldots, 11$ are all excludents of $(k+22)_k$. Thus $\mathcal{G}((k+22)_k) = 12$ iff 12 is not an excludent.

The options of $(k+22)_k$ are a_{k+22-a} for $a \neq 22$. If k is even, then we can stay on the diagonal $(k+22)_k$ by removing $k/2$ stones, to $(k/2+22)_{k/2}$. The Grundy value here might or might not be 12. The Grundy value of every other option is definitely *not* 12 though: for $a \leq 44$ the option is a_∞ (since $k + 22 > 88$), and for $a > 44$ the option a_{k+22-a} has a move to 22_∞.

Thus we have shown that for $k + 22 > 88$, $\mathcal{G}((k+22)_k) = 12$ if and only if k is odd and $\mathcal{G}((k/2+22)_{k/2}) \neq 12$. This leads to the following characterization: $\mathcal{G}((k+22)_k) = 12$ if and only if $v_2(k) \equiv 0 \pmod 2$, with the following exceptions:

- $k = 2^e$, $e \geq 4$,
- $k = 3 \times 2^e$, $e \geq 0$,
- $k = 15 \times 2^e$, $e \geq 0$.

For these exceptional cases, $\mathcal{G}((k+22)_k) = 12$ iff $v_2(k) \equiv 1 \pmod 2$.

7. Questions

Our analysis of memgames leaves many questions open.

(1) Mem^0 has simple parameterless rules. So, why is $m = 12$ special?
(2) Are $m = 0$ and $m = 12$ the only immortal numbers in Mem^0? (There are no others up to $m = 2000$.) If there are others, are there infinitely many?
(3) How many times can a number m appear on the frontier? We have found numbers that appear three times on the frontier; the smallest is 17, which appears on the frontier as $\mathcal{G}(n_\infty)$ for $n \in \{29, 30, 35\}$. There are many others as well, the first few of which are 24, 38, 42, and 50. We have not yet found any numbers that appear more than three times on the frontier. Still, we conjecture that a number can appear arbitrarily many times on the frontier.
(4) Are there other memgames with interesting Grundy structure? Specifically, in [Lar09], the memory is extended to include the include k previous moves by the other player, where k is a given game parameter. How do the Grundy values change, if we extend the definition of Mem^0 to allow up to $k-1$ consecutive mimics of the other player's move, but not the k^{th} one?

Remark 7.1. Mem^0 has been considered previously. It appears as #22 in their 2002 list of unsolved problems (see [GN02]), under the name of "short local nim." The Grundy values of the Frontier values appear on the Online Encyclopedia of Integer Sequences as A131469 [Slo]. Curiously, on OEIS, Mem^0 is referred to as "short global nim."

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