de Finetti Lattices and Magog Triangles

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Abstract

Let $B_{n,2}$ denote the order ideal of the boolean lattice B_n consisting of all subsets of size at most 2. Let $F_{n,2}$ denote the poset extension of $B_{n,2}$ induced by the rule: $i < j$ implies $\{i\} \prec \{j\}$ and $\{i,k\} \prec \{j,k\}$. We give an elementary bijection from the set $\mathcal{F}_{n,2}$ of linear extensions of $F_{n,2}$ to the set of shifted standard Young tableau of shape $(n, n-1, \ldots, 1)$, which are counted by the strict-sense ballot numbers. We find a more surprising result when considering the set $\mathcal{F}_{n,2}^{(1)}$ of poset extensions so that each singleton is comparable with all of the doubletons. We show that $\mathcal{F}_{n,2}^{(1)}$ is in bijection with magog triangles, and therefore is equinumerous with alternating sign matrices. We adopt our proof techniques to show that row reversal of an alternating sign matrix corresponds to a natural involution on gog triangles.

Keywords: boolean lattice, poset extension, de Finetti's axiom, comparative probability order, completely separable preference, alternating sign matrix, total symmetric self-complementary plane partition, magog triangle, gog triangle, strict-sense ballot number

1 Introduction

The boolean lattice B_n consists of subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. Among the linear extensions of B_n, we focus on those that satisfy de Finetti's axiom [11]

$$X \preceq Y \text{ if and only if } X \cup Z \preceq Y \cup Z \text{ for all } Z \subset [n] \text{ such that } (X \cup Y) \cap Z = \varnothing.$$

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Such linear extensions appear under various names, including comparative probability orders, boolean term orders and completely separable preferences, see OEIS A005806 [27]. The number of such linear extensions for $1 \leq n \leq 7$ is

$$1, 1, 2, 14, 546, 169444, 560043206$$

but there is still no known general formula. Herein, we turn our attention to the order ideal $B_{n,2} \subset B_n$ of subsets of size at most 2, and count poset extensions of $B_{n,2}$ that adhere to de Finetti's axiom. We begin with the following definition.

Definition 1.1. *The de Finetti lattice F_n is the unique poset extension of the boolean lattice B_n induced by the two conditions*

(F1) $\varnothing \prec \{1\} \prec \{2\} \prec \ldots \prec \{n\}$, and

(F2) $X \preceq Y$ if and only if $X \cup Z \preceq Y \cup Z$ for all $Z \subset [n]$ such that $(X \cup Y) \cap Z = \varnothing$.

For $1 \leq m \leq n$, the de Finetti Lattice $F_{n,m}$ is the unique poset extension of $B_{n,m}$ induced by conditions (F1) and (F2), where $B_{n,m} \subset B_n$ is the order ideal of subsets of size at most m.

Figure 1.1 shows the de Finetti lattices $F_{4,m}$ for $1 \leq m \leq 4$. Condition (F1) enforces the natural ordering on the singletons. Condition (F2) is de Finetti's axiom [11] for a comparative probability order on the power set $\mathcal{P}([n])$, relaxed here to allow for incomparable sets. We make a few observations about de Finetti lattices. First, any poset extension of $B_{n,m}$ adhering to (F1) and (F2) contains $F_{n,m}$ as a sublattice. Second, it is evident that $F_n = F_{n,n}$ and that $F_{n,n-1} \cong F_n$. Third, when $m = 2$, condition (F2) is equivalent to the simpler statement that $i < j$ implies $\{i, k\} \prec \{j, k\}$ for $k \neq i, j$. Finally, we note that $F_{n,m}$ is not total order when $3 \leq n$ and $2 \leq m \leq n$, as certified by the incomparable sets $\{1, 2\}$ and $\{3\}$.

Definition 1.2. *A* de Finetti extension *$(E, \preceq_E)$ of lattice $F_{n,m}$ is a poset extension that adheres to de Finetti's condition (F2) for all sets $X, Y \subset [n]$ that are comparable in E. For $1 \leq k \leq m \leq n$, let $\mathcal{F}_{n,m}^{(k)}$ denote the collection of de Finetti extensions of $F_{n,m}$ such that each set $S \in F_{n,k}$ is comparable with every set $T \in F_{n,m}$. That is, if $|S| \leq k$, then S is comparable with every set in $F_{n,m}$. For convenience, we denote $\mathcal{F}_{n,m} = \mathcal{F}_{n,m}^{(m)}$. Finally, $\mathcal{F}_n = \mathcal{F}_{n,n} = \mathcal{F}_{n,n}^{(n)}$ is the set of de Finetti linear extensions of B_n.*

Figure 1.2 shows the boolean lattice B_3, the de Finetti lattice F_3 and the two total orders in $\mathcal{F}_3$. Figure 1.3, shows how the de Finetti extensions build upon one another to produce the 14 total orders in $\mathcal{F}_4$. Enumerations of these total orders can also be found in [16, 5, 10]. When $n > m \geq 3$, there are poset extensions of $F_{n,m}$ that do not adhere to (F2). For example, we can extend $F_{n,3}$ by adding the single comparison $\{4\} \prec \{3, 1\}$ without adding the comparison

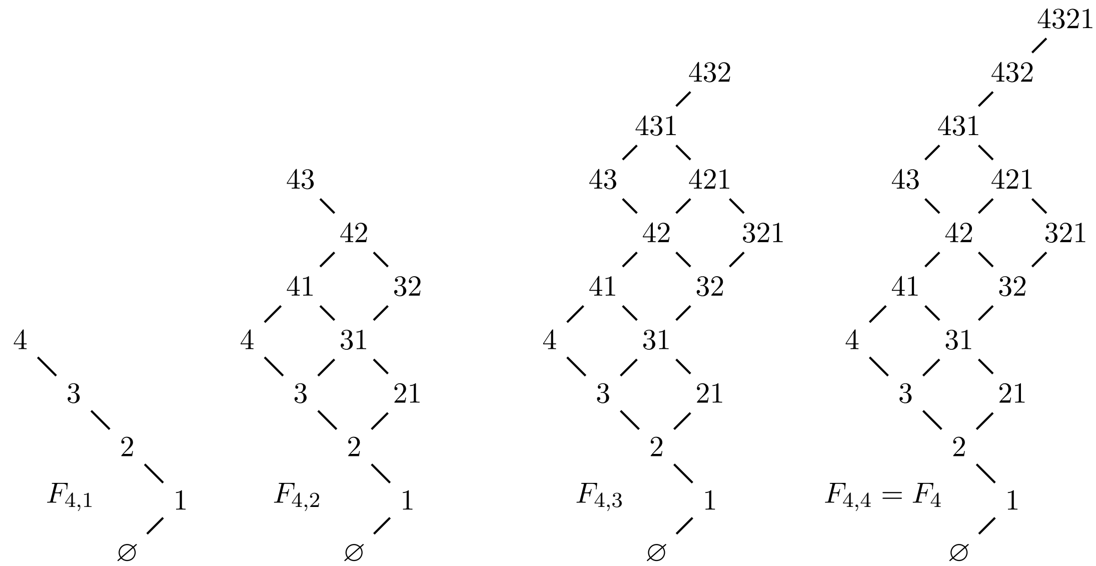


Figure 1.1: The de Finetti lattices $F_{4,m}$ for $1 \leq m \leq 4$, where ji to denotes the doubleton set $\{j, i\}$ where $j > i$.

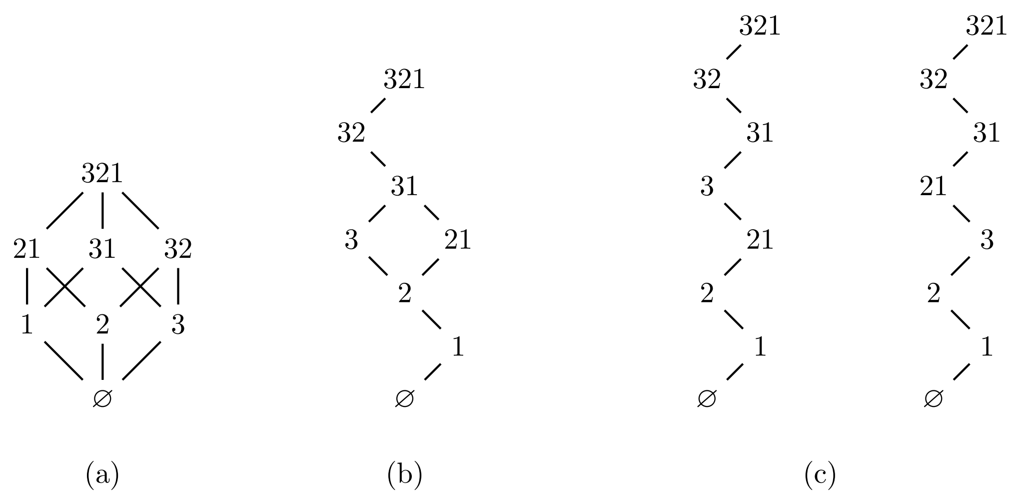


Figure 1.2: (a) The subset lattice B_3. (b) The de Finetti lattice F_3 extending B_3. (c) The two total orders in $\mathcal{F}_3$ extend F_3 by specifying $21 \prec 3$ and $3 \prec 21$, respectively.

$\{4, 2\} \prec \{3, 2, 1\}$. However, every poset extension of $F_{n,2}$ is a de Finetti extension. The $n = 4$ case is large enough to exhibit some complexity.

Herein, we characterize $\mathcal{F}_{n,2} = \mathcal{F}^{(2)}_{n,2}$ and $\mathcal{F}^{(1)}_{n,2}$. We give a simple bijection between the total orders in $\mathcal{F}_{n,2}$ and shifted standard Young tableau (shifted SYT) of shape $(n, n-1, \ldots, 1)$, see OEIS A003121 [27]. In these shifted SYT of staircase shape, the first box in row $i > 1$ is located below the second box of row $i - 1$. The integers $1, 2 \ldots, n(n + 1)/2$ are arranged in the boxes so

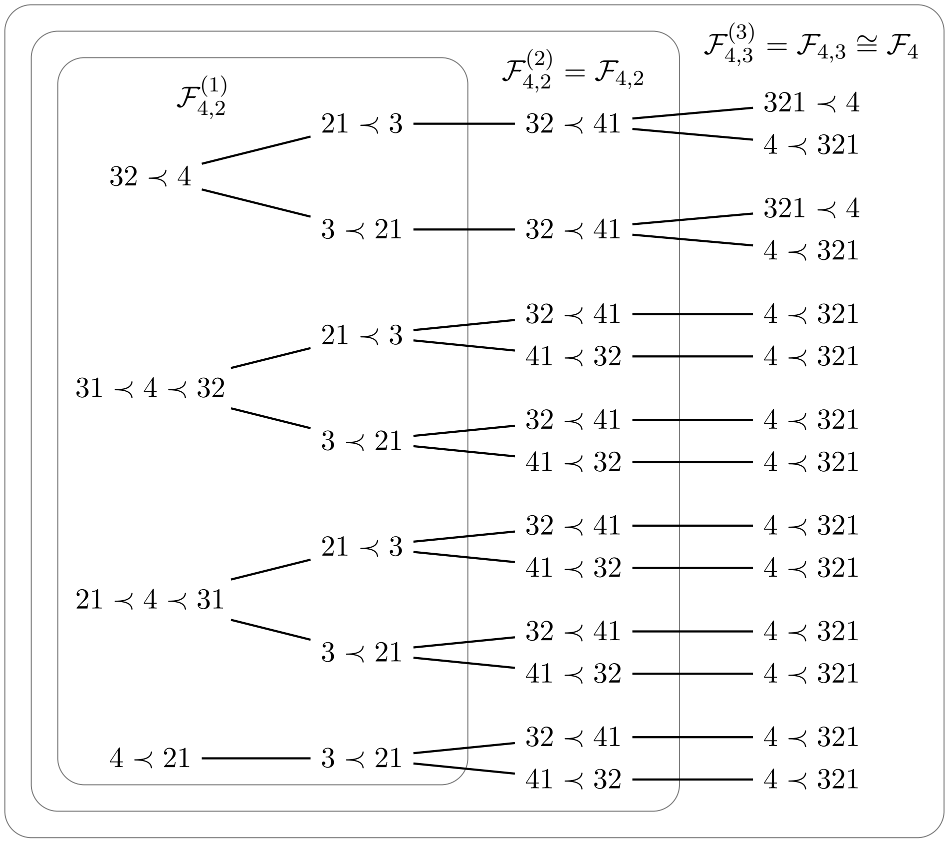


Figure 1.3: The de Finetti extensions of $\mathcal{F}_{4,2}$ and $\mathcal{F}_{4,3}$. Along with conditions (F1) and (F2), all other comparisons are determined by the ones shown. Two of the seven posets in $\mathcal{F}^{(1)}_{4,2}$ are total orders on $F_{4,2}$. Ten of the twelve posets in $\mathcal{F}_{4,2}$ induce total orders on $\mathcal{F}^{(3)}_{4,3}$. There are fourteen total orders in $\mathcal{F}^{(3)}_{4,3} \cong \mathcal{F}_4$.

that the rows and the columns are both increasing. These are equinumerous with the number of strict-sense ballots with n candidates, where candidate k gets k votes, candiate k never trails candidate ℓ for $n \geq k > \ell \geq 1$, see [2]. For $1 \leq n \leq 7$, the strict-sense ballot numbers are

$$1, 1, 2, 12, 286, 33592, 23178480$$

and the general formula for the nth strict-sense ballot number is

$$\binom{n+1}{2}! \, \frac{\prod_{k=1}^{n-1} k!}{\prod_{k=1}^{n} (2k-1)!}.$$

Proposition 1.3. *The set $\mathcal{F}_{n,2}$ is in bijection with shifted standard Young tableaux of shape $(n, n-1, \ldots, 1)$. Therefore $\mathcal{F}_{n,2}$ is enumerated by the strict-sense ballot numbers.*

Determining $|\mathcal{F}^{(1)}_{n,2}|$ uncovers a remarkable connection to alternating sign matrices (ASMs), see OEIS A005130 [27]. An ASM is a $n \times n$ matrix of 0's, 1's and -1's such that each or column sums to 1, and the nonzero entries in each row or column alternate in sign. The seven 3×3 ASMs are

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

Zielberger famously proved that alternating sign matrices (ASMs) are equinumerous with totally symmetric self-complimentary plane partitions (TSSCPPs) [33]. The first seven numbers in this sequence are

$$1, 2, 7, 42, 429, 7436, 218348$$

and the general formula is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}. \tag{2}$$

Zielberger's proof converts ASMs and TSSCPPs into gog and magog triangles, respectively, and then shows that these families of triangles are equinumerous. We prove that $\mathcal{F}_{n,2}^{(1)}$ is in bijection with magog triangles (and hence also in bijection with TSSCPPs), and are therefore equinumerous with ASMs.

Definition 1.4. *A magog triangle M is an array of positive integers $M(i,j)$ where*

(M1) $1 \le j \le i \le n$, so the array is triangular;

(M2) $1 \le M(i,j) \le j$, so entries in column j are at most j;

(M3) $M(i,j) \le M(i+1,j)$, so columns are weakly increasing; and

(M4) $M(i,j) \le M(i,j+1)$, so rows are weakly increasing.

We use $\mathcal{M}_n$ to denote the set of magog triangles of size n.

For example, the elements of $\mathcal{M}_3$ are

$$\begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 1 & 2 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 2 & 2 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 2 & \\ 1 & 2 & 2 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 1 & 3 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 1 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{lll} 1 & & \\ 1 & 2 & \\ 1 & 2 & 3 \end{array}. \tag{3}$$

Theorem 1.5. *The family $\mathcal{F}_{n,2}^{(1)}$ of de Finetti extensions of the lattice $F_{n,2}$ is in bijection with the family $\mathcal{M}_{n-1}$ of magog triangles of size $n-1$.*

This theorem follows immediately from two lemmas. First we prove that de Finetti extensions are in bijection with the (newly defined) family of *kagog triangles*. Then we give a bijection between kagog triangles and magog triangles.

Definition 1.6. *A kagog triangle K is an array of nonnegative integers $K(i,j)$ such that*

(K1) $1 \leq j \leq i \leq n-1$*, so the array is triangular;*

(K2) $0 \leq K(i,j) \leq j$*, so entries in column j are at most j;*

(K3) $K(i,j) \geq K(i+1,j)$*, so columns are weakly decreasing; and*

(K4) if $K(i,j) > 0$ then $K(i,j+1) > K(i,j)$, so rows can start with multiple zeros, but then the positive values are strictly increasing.

We use $\mathcal{K}_n$ to denote the set of kagog triangles of size n.

Note that a triangle in $\mathcal{K}_n$ only has $n-1$ rows and columns. The elements of $\mathcal{K}_3$ are

$$\begin{matrix} 1 & \\ 1 & 2 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 2 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & \\ 0 & 2 \end{matrix} \qquad \begin{matrix} 0 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 0 & \\ 0 & 0 \end{matrix} . \tag{4}$$

The kagog triangles in equation (4) are ordered so that that they biject to the magog triangles listed in equation (3).

Lemma 1.7. *The set of de Finetti extensions $\mathcal{F}^{(1)}_{n,2}$ is in bijection with the set of kagog triangles $\mathcal{K}_{n-1}$.*

Lemma 1.8. *The set of magog triangles $\mathcal{M}_n$ is in bijection with the set of kagog triangles $\mathcal{K}_n$.*

The key to Lemma 1.8 is to convert each of these triangles into a pyramid of stacked cubes, colored gray or white, so that gray cubes cannot appear above white cubes. We offer a generic definition for pyramid construction, which applies to any family $\mathcal{T}_n$ of triangular arrays that form a distributive lattice using the natural partial ordering $T_1 \prec T_2$ whenever $T_1(i,j) \leq T_2(i,j)$ for $1 \leq j \leq i \leq n$. This includes magog triangles $\mathcal{M}_n$ and kagog triangles $\mathcal{K}_n$, as well as gog triangles $\mathcal{G}_n$ (defined below).

Definition 1.9. *Let $\mathcal{T}_n$ be a finite distributive lattice of triangular arrays of positive integers $T = T(i,j)$ where $1 \leq j \leq i \leq n$ with minimal triangle $T_{\min}$ and maximal triangle $T_{\max}$. Define $\triangle T$ to be the two-color pyramid of cubes (i,j,k) where $1 \leq i \leq j \leq n$ and $1 \leq k \leq T_{\max}(i,j)$ where the tower of cubes at (i,j) consists of $T(i,j)$ white cubes below $T_{\max}(i,j) - T(i,j)$ gray cubes. Define $\triangle\mathcal{T}_n = \{\triangle T : T \in \mathcal{T}_n\}$ to be the collection of two-color pyramids.*

Figure 1.4 shows the seven magog pyramids, listed in the same order as in equation 3. To facilitate visualization, the pyramids have been sliced into layers of equal height.

This two-color pyramid mapping is a variation of the standard interpretation triangular array T as a stack of cubes where the tower at (i,j) has height $T(i,j)$. Indeed, we can view

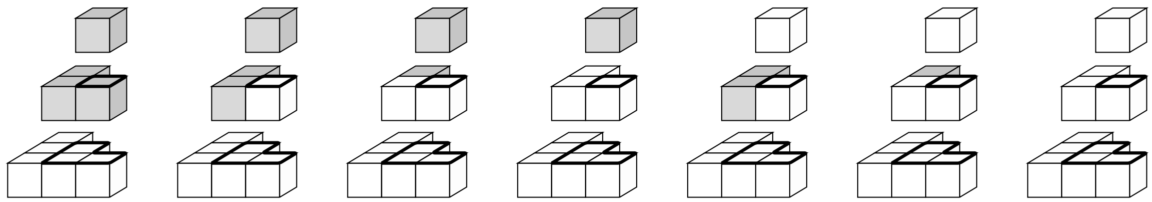


Figure 1.4: The two-color pyramids from $\triangle\mathcal{M}_3$, sliced into horizontal layers. The shadow of each layer is thickly drawn on the layer below.

the white cubes as present and the gray cubes as absent. In our proof, tracking the absent cubes is essential, so the two-color pyramids are more illuminating. Intuitively, the bijection from magog triangles to kagog triangles corresponds to removing the bottom layer of the magog pyramid, then swapping the colors of the cubes and finally performing an appropriate affine transformation.

Given the success of the two-color pyramid view of magog triangles, we conclude the paper by investigating two-color pyramids of gog triangles, which are also known as monotone triangles.

Definition 1.10. *A gog triangle G is a triangular array of positive integers $G(i,j)$ such that*

(G1) $1 \le j \le i \le n$, *so the array is triangular;*

(G2) $1 \le G(i,j) \le n-i+j$, *so entry j in row i is at most $n-i+j$;*

(G3) $G(i,j) < G(i,j+1)$, *so rows are strictly increasing;*

(G4) $G(i,j) \ge G(i+1,j)$, *so columns are weakly decreasing; and*

(G5) $G(i,j) \le G(i+1,j+1)$, *so diagonals are weakly increasing.*

We use $\mathcal{G}_n$ to denote the set of gog triangles of size n.

The gog triangles of $\mathcal{G}_3$ are

$$\begin{array}{ccc} 1 & & \\ 1 & 2 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 1 & & \\ 1 & 3 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 & & \\ 1 & 2 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 & & \\ 1 & 3 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 2 & & \\ 2 & 3 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 3 & & \\ 1 & 3 & \\ 1 & 2 & 3 \end{array} \quad \begin{array}{ccc} 3 & & \\ 2 & 3 & \\ 1 & 2 & 3 \end{array} . \tag{5}$$

Gog triangles are in bijection with alternating sign matrices; we have listed these seven triangles in the same order as the 3×3 ASMs in equation (1). The jth row of the gog triangle records the locations of the 1's in the vector obtained by adding the first j rows of the corresponding ASM.

Theorem 1.11. *There is a natural gog triangle involution $f : \mathcal{G}_n \to \mathcal{G}_n$ that corresponds to both (1) an affine transformation of two-color pyramids, and (2) reversing the order of the rows of the corresponding ASM.*

This theorem is a satisfying observation for those interested in ASMs.

2 Background

A *partially ordered set* (or *poset* for short) consists of a set P and a binary relation $\preceq$ that is reflexive ($x \preceq x$), antisymmetric (if $x \preceq y$ and $y \preceq x$ then $x = y$) and transitive (if $x \preceq y$ and $y \preceq z$ then $x \preceq z$). A *lattice* is a poset such that every pair of elements have a least upper bound and a greatest lower bound. A *totally ordered set* is a poset where every pair of elements is comparable. A *linear extension* of a partial order is a totally ordered set that is compatible with the partial order. For an introduction to posets and lattices, see Chapter 3 of Stanley [28].

For a poset P, let $\mathcal{L}(P)$ denote the set of linear extensions of P. Brightwell and Tetali [8] determined an accurate asymptotic formula for $|\mathcal{L}(B_n)|$, improving on work of Sha and Kleitman [25]. The value of $|\mathcal{L}(B_n)|$ is known for $1 \leq n \leq 7$, see OEIS A046873 [27]. The $n = 7$ case was recently determined by Brower and Christensen [9] using machinery developed to study the game of Chomp played on the boolean lattice. Pruesse and Ruskey [24] introduced the linear extension graph $G(B_n)$ whose vertex set is $\mathcal{L}(B_n)$ and whose edge set consists of pairs of linear extensions that differ by a single adjacent transposition. Felsner and Massow [12] determined the diameter of $G(B_n)$.

Researchers have also studied linear extensions of subposets of B_n, including the order ideal $B_{n,m}$ of subsets of size at most m. Fink and Gregor [14] determined the linear extension diameter of the subposet $B_n^{1,k}$ of B_n that is induced by levels 1 and k. Brouwer and Christensen [9] determined that

$$|\mathcal{L}(B_{n,2})| = \frac{n!\left(\binom{n}{2}+n\right)!}{\prod_{i=1}^{n}\left(in - \binom{i}{2}\right)} = \binom{n+1}{2}!\frac{1}{\prod_{i=1}^{n}\left(n - \frac{i-1}{2}\right)}$$

and computed $|\mathcal{L}(B_{n,3})|$ for $n \leq 7$. Comparing this formula with our Proposition 1.3 shows that $n! \cdot |\mathcal{F}_{n,2}|$ is $o(|\mathcal{L}(B_{n,2})|)$. It is no surprise that de Finetti extensions of $B_{n,2}$ are exceptionally rare among linear extensions of $B_{n,2}$.

The de Finetti orders $\mathcal{F}_n$ are orderings of $\mathcal{P}([n])$ that satisfy both (F1) and (F2). The value of $|\mathcal{F}_n|$ is known for $1 \leq n \leq 7$, see OEIS A005806 [27]. These orderings appear in a variety of settings with names that reflect the application at hand [13, 22, 5, 10]. To emphasize the common poset context, we opt for the generic name "de Finetti order," which also pays homage to de Finetti's axiom [11]. It is pleasing that (F1) and (F2) lead to common extensions of the boolean lattice B_n and the order on $[n]$ induced by the standard ordering on the integers. Indeed, when $x_i \le y_i$ for $1 \le i \le k$ then (F1) and (F2) lead to the (intuitive) conclusion that $\{x_1, x_2, \ldots, x_k\} \preceq \{y_1, y_2, \ldots, y_k\}$.

In probability theory, the orderings in $\mathcal{F}_n$ are known as *comparative probability orders*, and they enjoy applications in decision theory and economics [20, 13, 15, 26]. A comparatively probability order $\preceq$ is *additively representable* when there is a probability measure $p : [n] \to [0, 1]$ that induces the order, namely $p(X) \le p(Y)$ if and only if $X \preceq Y$. In a more algebraic context, Maclagan [22] refered to orderings in $\mathcal{F}_n$ as *boolean term orders* and studied their combinatorial properties. A single adjacent transposition results in a total order that violates ($F2$), so the linear extension graph on $\mathcal{F}_n$ is totally disconnected. Instead, Maclagan introduced a *flip* operation between boolean term orders, which consists of multiple (related) adjacent transpositions so that (F2) still holds. It is an open question whether the flip graph is connected for $n \ge 9$. Christian et al. [10] further studied flippable pairs of orders and their relation to the polytope of an additively representable order.

In social choice theory, these orderings are called *completely separable preferences* [17, 5]. In this setting, de Finetti's condition ensures that a voter's preference for the outcomes on a subset $S \subset [n]$ of proposals is independent of the outcome of the proposals in $\overline{S}$. Hodge and TerHaar [19] showed that the number of de Finetti extensions satisfies $n! \cdot |\mathcal{F}_n| = o(\mathcal{L}(B_n))$. In fact, they proved the stronger condition that linear extensions with at least one pair X, Y of proper nontrivial subsets satisfying condition (F2) are vanishingly rare. Other research on separable preferences focuses on the *admissibility problem*: which collection of subsets can occur as the collection of *separable* sets S, meaning that (F2) holds for any subsets $X, Y \subset S$ and any $Z \subset \overline{S}$, see [19, 18, 3].

Theorem 1.3 establishes a bijection between the de Finetti extensions $\mathcal{F}^{(1)}_{n,2}$ and magog triangles $\mathcal{M}_{n-1}$. This connects our poset extension problem to the illustrious family of *alternating sign matrices*. See [6, 7], respectively, for a brief or an extended recounting of the history of the famous alternating sign matrix conjecture. Magog triangles of $\mathcal{M}_n$ are in bijection with totally the symmetric self-complementary plane partitions (TSSCPP) in a $2n \times 2n \times 2n$ box. Andrews [1] proved that the number of such TSSCPP is given by equation (2). Meanwhile, gog triangles $\mathcal{G}_n$ are in bijection with $n \times n$ alternating sign matrices (ASM). Zielberger [33] proved that $|\mathcal{M}_n| = |\mathcal{G}_n|$, which confirmed that TSSCPPs and ASMs are equinumerous. Kuperberg [21] later gave a more streamlined proof using the 6-vertex model from statistical mechanics.

There are many combinatorial manifestations of the ASM sequence (2), see [7, 23]. A natural bijective proof between TSSCPPs and ASMs (or equivalently, between magog and gog triangles) remains elusive, though progress on subfamilies has been achieved [4, 31]. Triangular arrays of numbers (such as gog, magog and kagog triangles) continue to play an essential role in ASM and TSSCPP research. For example, Striker [30] defined a tetrahedral poset T_n whose subposets trace connections between TSSCPPs, ASMs and other combinatorial sequences. In particular, T_n has one subposet whose order ideals can be described via families of triangular arrays. The order ideals of one such subposet is in bijection with gog triangles (and hence with ASMs). There are six distinct subposets whose order ideals (with associated triangular families) are in bijection with magog triangles (and hence with TSSCPPs). We note that our kagog triangles are not among the triangular families described in [30], so the family of TSSCPP triangles continues to grow.

3 Proof of Proposition 1.3

This brief section offers a simple bijection between $\mathcal{F}_{n,2}$ and shifted standard Young tableaux (shifted SYT) of shape $(n, n-1, \ldots, 1)$. Figure 3.1 exemplifies the mapping for $n = 4$. To ease exposition, we identify the singleton $\{i\}$ with the doubleton $\{i, 0\}$. Ignore the set $\varnothing$ and lay out the lattice $F_{n,2}$ in a shifted staircase grid so that row k contains the sets $\{i, k-1\}$ for $k \leq i \leq n$ in increasing order. This grid induces a shifted staircase Ferrers diagram $(n, n-1, \ldots, 1)$ whose boxes are indexed the $n(n+1)/2$ nontrivial members of $F_{n,2}$.

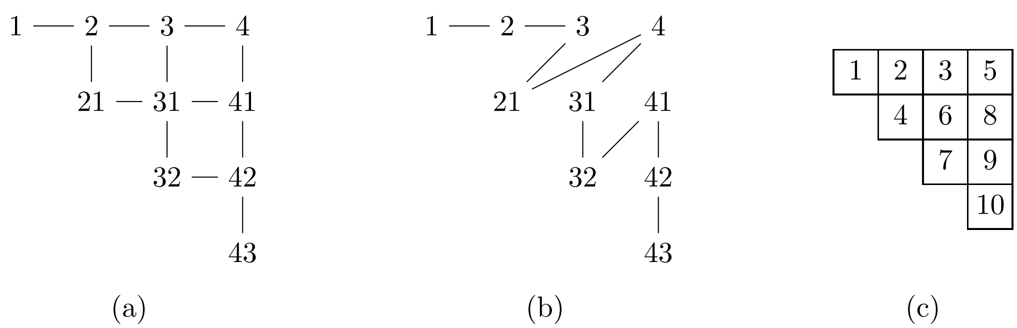


Figure 3.1: (a) The nontrivial sets in $F_{4,2}$ laid out in a shifted staircase grid. (b) A de Finetti total order and (c) its corresponding standard shifted Young tableau.

Consider a total ordering $E \in \mathcal{F}_{4,2}$. Place the integer ℓ in the box corresponding to the ℓth set in total ordering E. The result is a shifted SYT of staircase shape: the rows and columns of the resulting tableau are both increasing because the total ordering satisfies properties (F1) and (F2) of Definition 1.1. This mapping is surjective: starting from a shifted SYT, we can reverse the process to find a total ordering $E \in \mathcal{F}_{4,2}$ that maps to it. This completes the proof of Proposition 1.3. $\square$

4 The bijection from $\mathcal{F}_{n,2}^{(1)}$ to $\mathcal{K}_{n-1}$

In this section, we prove Lemma 1.7. Figure 4.1 shows the de Finetti lattice $F_{n,2}$ for $n = 3, 4, 5$ and also indicates the sublattice

$$I_k = \{\{j, i\} : 1 \leq i < j < k\} \tag{6}$$

of doubletons that are incomparable with singleton $\{k\}$.

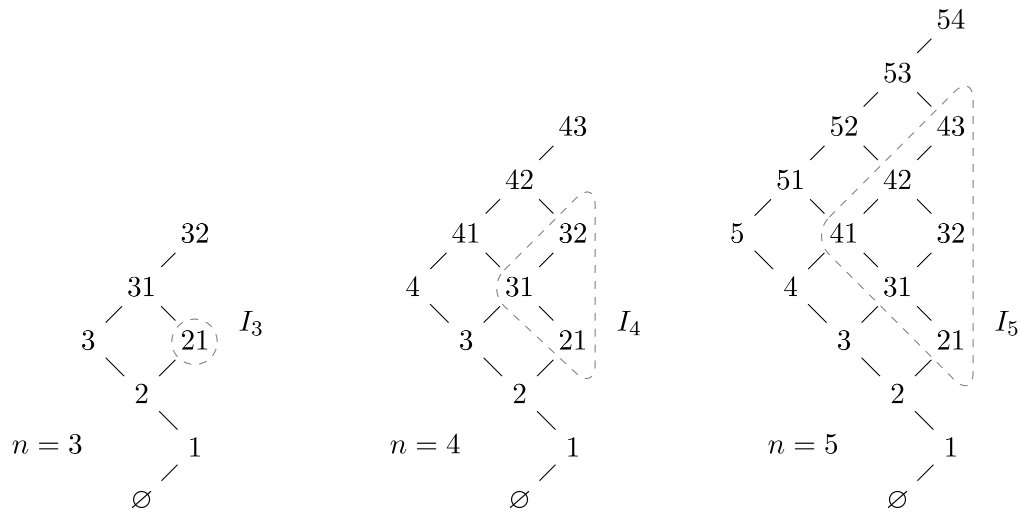


Figure 4.1: The lattice $F_{n,2}$ induced by $1 \prec 2 \prec \cdots \prec n$ and de Finetti's condition for $n = 3, 4, 5$. The set I_n contains the doubletons whose comparison with the singleton n is not determined by de Finetti's condition.

For $k \geq 3$, let $\Phi(I_k)$ be the collection of de Finetti extensions of $I_k \cup \{k\}$ for which the singleton $\{k\}$ is comparable with every doubleton of I_k (and no additional extraneous relations). When we restrict a poset extension $E \in \mathcal{F}_{n,2}^{(1)}$ to the set $I_k \cup \{k\}$, we obtain some $E_k \in \Phi(I_k)$. Similarly, we can induce a unique poset extension E of $F_{n,2}$ from a list $(E_3, E_4, \ldots, E_n)$ where $E_k \in \Phi(I_k)$. We will have $E \in \mathcal{F}_{n,2}^{(1)}$ provided that the union of these orderings does not violate de Finetti's condition (F2). Figure 4.2 gives an example of a poset extention $E \in \mathcal{F}_{n,2}^{(1)}$ and its collection of $E_k \in \Phi(I_k)$.

Our bijection from the poset extensions of $F_{n,2}$ in $\mathcal{F}_{n,2}^{(1)}$ to the kagog triangles in $\mathcal{K}_{n-1}$ proceeds as follows. Given a de Finetti extension $E \in \mathcal{F}_{n,2}^{(1)}$, we create the corresponding list $(E_3, \ldots E_n)$ where $E_k \in \Phi(I_k)$. We then map extension E to a kagog triangle $K \in \mathcal{K}_{n-1}$ so that extension E_j maps to row $j-2$ of triangle K for $3 \leq j \leq n$. The row constraint (K3) of the kagog triangle will correspond to the internal structure of each E_k. The column constraint (K2)

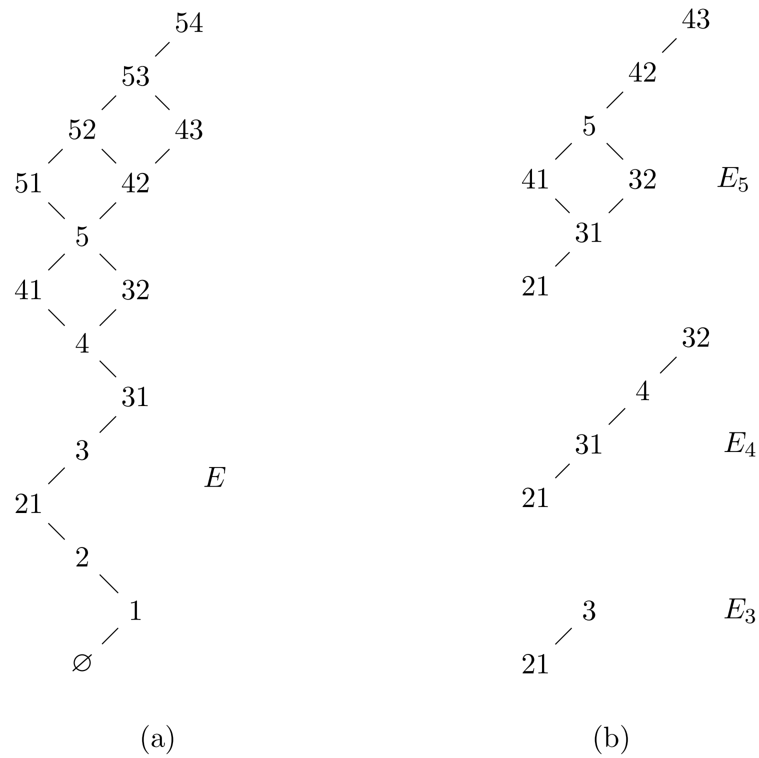


Figure 4.2: (a) A poset extension E from $\mathcal{F}_{n,2}^{(1)}$. Each singleton is comparable with every other set. (b) The subposets E_3, E_4 and E_5 of E.

of the kagog triangle will correspond to having singleton de Finetti extensions $(E_3, E_4, \ldots, E_n)$ whose union also abides by de Finetti's condition.

We begin by introducing a convenient k-list version of the power set $\mathcal{P}([k])$. Let

$$\mathcal{L}([k]) = \big\{(\underbrace{0, \ldots, 0}_{k-j}, s_1, s_2, \ldots, s_j) : 0 \leq j \leq k \text{ and } 1 \leq s_1 < s_2 < \cdots < s_j \leq k\big\}$$

be the set of k-lists produced by listing the elements of $S \subset [k]$ in increasing order and then prepending $k - |S|$ zeros.

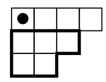
Lemma 4.1. *Each row $1 \leq k \leq n-1$ of a kagog triangle in $\mathcal{K}_n$ is an element of $\mathcal{L}([k])$.*

Proof. The constraint (K4) on row k of a kagog triangle in $\mathcal{K}_n$ is identical to the conditions on a list in $\mathcal{L}([k])$. □

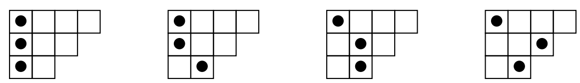
Lemma 4.2. *For $n \geq 3$, $\Phi(I_n)$ is in bijection with $\mathcal{L}([n-2])$.*

Let us build some intuition with two examples. First, we consider extensions in $\Phi(I_5)$. We must determine the comparisons of the singleton $\{5\}$ with the doubletons in the lattice I_5. By interweaving empty boxes among the doubletons, we obtain the template

where omitting the doubletons gives the Ferrers diagram for the integer partition $(4, 3, 2)$. Specifying the comparisons with singleton $\{5\}$ is equivalent to placing a dot in each row of $(4, 3, 2)$. Looking only at the top row, placing a 5 in the first box



means that $5 \prec 41$. This puts no further de Finetti restrictions on the remaining two rows. The four ways to complete this configuration are

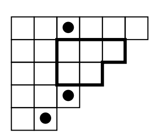


which correspond to the comparisons

$$5 \prec 32 \qquad 21 \prec 5 \prec 31 \qquad 31 \prec 5 \prec 32 \qquad 32 \prec 5 \prec 41.$$

Our final step is to count the boxes to the right of these dots, starting from the bottom row and moving up. This results in the lists $(1, 2, 3)$, $(0, 2, 3)$, $(0, 1, 3)$ and $(0, 0, 3)$ from $\mathcal{L}([3])$.

Next, we consider extensions in $\Phi(I_7)$. Specifying the comparisons of singleton $\{7\}$ with the doubletons in I_7 is equivalent to placing a dot in each row of the integer partition $(6, 5, 4, 3, 2)$. Suppose that we place a 7 in the third box of the first row, corresponding to $62 \prec 7 \prec 63$. Now de Finetti's condition leads to $21 \prec 32 \prec 42 \prec 52 \prec 62 \prec 7$, which yields the partially filled diagram



which contains a shifted copy of partition $(3, 2)$ whose rows must each be assigned a dot. This can be done in four ways, and counting the boxes to the right of the dots gives the lists $(0, 0, 1, 2, 3)$, $(0, 0, 0, 2, 3)$, $(0, 0, 0, 1, 3)$ and $(0, 0, 0, 0, 3)$ from $\mathcal{L}([5])$. We now prove Lemma 4.2 by strong induction.

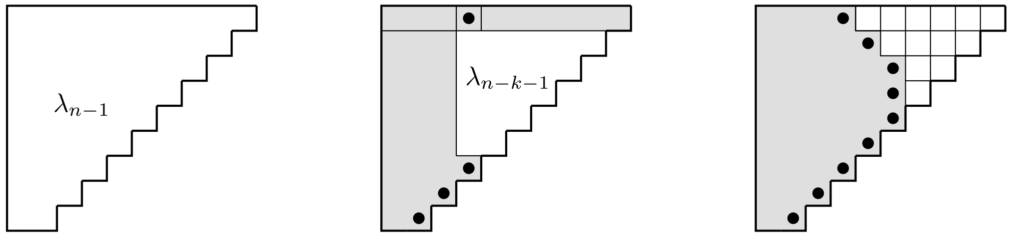


Figure 4.3: (a) The Ferrers diagram $\lambda_{n-1} = (n-1, n-2, \dots, 2)$ for ordering singleton $\{n\}$ with the doubletons. (b) When $\{n-1, k-1\} \prec n \prec \{n-1, k\}$, we place a dot in the kth position. This places de Finetti restrictions on the remaining rows. Completing the order is equivalent to choosing an order inside tempate λ_{n-k-1}. (c) Counting the boxes to the right of the dots gives the subset $\{6, 4, 2, 1\} \subset [n-2]$.

Proof of Lemma 4.2. We recursively define the bijection $f : \Phi(I_n) \to \mathcal{L}([n-2])$. For $n = 3$, we map the ordering with $21 \prec 3$ to the list (0) and the ordering with $3 \prec 21$ to the list (1). Assume that we have specified the bijection $f : \Phi(I_\ell) \to \mathcal{L}([\ell-2])$ for $2 \leq \ell < n$. We determine the image of an extension $E \in \Phi(I_n)$. As in the above examples, we represent E by placing a dot in each row of the Ferrers diagram $\lambda_{n-1} = (n-1, n-2, \dots, 2)$, see Figure 4.3(a). Our target list in $L \in \mathcal{L}([n-2])$ will be obtained by counting the boxes to the right of the dot in each row.

Placing a dot in position $1 \leq k \leq n-1$ of the first row of the template

$$\square \quad \{n-1, 1\} \quad \square \quad \{n-1, 2\} \quad \square \quad \cdots \quad \square \quad \{n-1, n-3\} \quad \square \quad \{n-1, n-2\} \quad \square$$

resolves the ordering of singleton $\{n\}$ with the doubletons $\{n-1, j\}$ for $1 \leq j \leq n-2$. If $k = 1$ then $\{n\} \prec \{n-1, 1\}$; if $1 < k < n-1$ then $\{n-1, k-1\} \prec \{n\} \prec \{n-1, k\}$; and if $k = n-1$ then $\{n-1, n-2\} \prec \{n\}$. De Finetti's condition (F2) puts constraints on the remaining rows. For rows $2 \leq i \leq n-k-1$, we must place n in position k or higher. For rows $i > n-k-1$, we must place n in the rightmost (diagonal) position. Therefore, we can restrict our attention to rows $2 \leq i \leq n-k-1$ and positions $k \leq j \leq n-2$. But this is simply a translation of the mapping $f : \Phi(I_{n-k}) \to \mathcal{L}([n-k-2])$ via a copy of λ_{n-k-1}, see Figure 4.3(b). Let $(a_1, a_2, \dots, a_{n-k-2}) \in \mathcal{L}([n-k-2])$ be the image of this mapping. We set

$$f(E) = (\underbrace{0, \dots, 0}_{k}, a_1, a_2, \dots, a_{n-k-2}, n-k-1).$$

The values in this list are the number of boxes to the right of the dots in Figure 4.3(c), when ordered from bottom to top. $\square$

We can now prove that the set of de Finetti extensions $\mathcal{F}_{n,2}^{(1)}$ is in bijection with the set of kagog triangles $\mathcal{K}_{n-1}$.

Proof of Lemma 1.7. Let $E \in \mathcal{F}_{n,2}^{(1)}$ be a de Finetti extension of $F_{n,2}$ so that every singleton is universally comparable in E. Consider $(E_3, E_4, \ldots, E_n)$ where $E_k \in \Phi(I_k)$ is the poset extension of $I_k \cup \{k\}$ induced by E. Create a triangular array $T = T(i,j)$ for $1 \leq j \leq i \leq n-2$ by applying the mapping f from Lemma 4.2 to each element in this list of extensions, using the indexing convention

$$f(E_k) = \big(T(k-2,1), T(k-2,2), \cdots, T(k-2,k-2)\big), \quad 3 \leq k \leq n.$$

By Lemma 4.2, each row satisfies the kagog row constraint. Meanwhile, the extension E satisfies de Finetti's condition (F2). In particular, for any $1 \leq i < j < k \leq n$, if $\{k\} \prec \{j,i\}$ then $\{k-1\} \prec \{j,i\}$. In terms of triangle T, this means that $T(k-2, n-j) \geq T(k-3, n-j)$. For $2 \leq j \leq n-1$, this is precisely the contraint that column $n-k$ of a kagog triangle must be weakly decreasing constraint on column. Column $n-1$ has a single entry, so the final column is (vacuously) weakly increasing. □

5 The bijection from $\mathcal{M}_n$ to $\mathcal{K}_n$

We now prove Lemma 1.8. Along with Lemma 1.7, this completes the proof of Theorem 1.5. Recall that each triangle family $\mathcal{T}_n$ forms a distributive lattice and that Definition 1.9 constructs two-color pyramids in relation to the maximum and minimum triangle of $\mathcal{T}_n$.

The minimum magog triangle has $M_{\min}(i,j) = 1$ for every entry (i,j) and the maximum magog triangle has $M_{\max}(i,j) = j$ for every entry (i,j). Our first transformation is to subtract $M_{\min}$ from each magog triangle. The rightmost column becomes all-zero, so we omit it and reindex. This leads to the family of omagog triangles (short for "zeroed-magog" triangles).

Definition 5.1. *An omagog triangle M° is an array of nonnegative integers $M^\circ(i,j)$ such that*

(OM1) $1 \leq j \leq i \leq n-1$*, so the array is triangular*

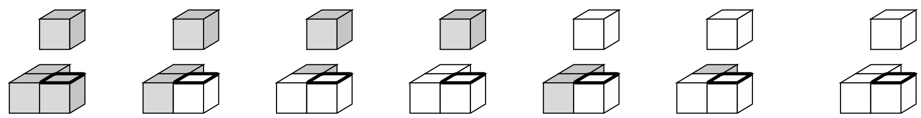
(OM2) $M^\circ(i,j) \leq j$*, so the entries in column k are at most j;*

(OM3) $M^\circ(i,j) \leq M^\circ(i+1,j)$*, so columns are weakly increasing; and*

(OM4) $M^\circ(i,j) \leq M^\circ(i,j+1)$*, so rows are weakly increasing.*

We use $\mathcal{M}_n^\circ$ to denote the set of all omagog triangles with $n-1$ rows.

The set $\mathcal{M}_3^\circ$ appears in Figure 5.1, with elements ordered so that they biject to the magog triangles of equation (3). The minimum omagog triangle satisfies $M^\circ_{\text{min}}(i,j) = 0$ and the maximum omagog triangle satisfies $M^\circ_{\text{max}}(i,j) = j$ for all entries (i,j).



0	0	0	1	0	0	1
0 0	0 1	1 1	1 1	0 2	1 2	1 2

Figure 5.1: The omagog triangles $\mathcal{M}_3^\circ$ and the corresponding two-color omagog pyramids $\triangle\mathcal{M}_3^\circ$. The pyramids are sliced into horizontal layers. The shadow of a layer is thickly drawn on the layer below.

Proof of Lemma 1.8. We create a bijection ψ from omagog pyramids $\triangle\mathcal{M}_n^\circ$ to kagog pyramids $\triangle\mathcal{K}_n$ via a sequence of elementary transformations.

Recall that a two-color pyramid $\triangle T$ is a collection of cubes (i,j,k) that are colored white or gray. Renaming these colors as color 1 and color 0, respectively, then the two-color pyramid becomes a binary function on the set of admissible coordinates, that is $\triangle M^\circ : (i,j,k) \mapsto \{0,1\}$. Viewing $\triangle M^\circ$ as a function allows us to describe the collection $\triangle\mathcal{M}_n^\circ$ of two-color pyramids with a system of inequalities. We have

- $\triangle M^\circ(i,j,k)$ is defined for $1 \leq k \leq j \leq i \leq n-1$.
- $\triangle M^\circ(i,j,k) \leq \triangle M^\circ(i+1,j,k)$: the columns of the magog triangle are nondecreasing,
- $\triangle M^\circ(i,j,k) \leq \triangle M^\circ(i,j+1,k)$: the rows of the magog triangle are nondecreasing, and
- $\triangle M^\circ(i,j,k+1) \leq \triangle M^\circ(i,j,k)$: color 1 (white, present) cubes are below color 0 (gray, absent) cubes, so the cubes that are present obey "gravity."

We now perform our four step transformation ψ.

- **Step 1:** Invert the colors, or exchange color 0 for color 1 and vice versa. This reverses the inequalities.
- **Step 2:** Push all cubes north in their respective column so that row 1 has length $n-1$. This is equivalent to moving the cube (i,j,k) to $(i-(j-1),j,k)$.
- **Step 3:** Tip the entire stack over the y-axis via a clockwise rotation by $\pi/2$. This is equivalent to moving the cube (i,j,k) to $(n-k,j,i)$.

- **Step 4:** Reflect the stack through the plane $y = (n+1)/2$. This is equivalent to moving the cube (i, j, k) to $(i, n-j, k)$.

After composing these four steps, cube (i, j, k) switches color and moves to $(n-k, n-j, i-j+1)$. Figure 5.2 shows the mapping ψ for an omagog pyramid in $\triangle\mathcal{M}_4^{\circ}$.

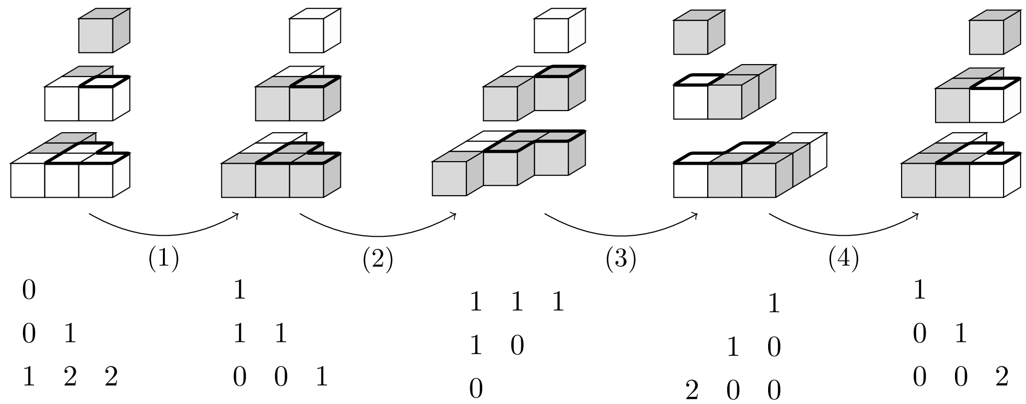


Figure 5.2: Example of the omagog to kagog bijection ψ. (1) Invert the colors. (2) push the cubes northward along the columns. (3) Tip the stack around the y-axis. (4) Reflect through to the plane $y = (n+1)/2$.

Updating the omagog pyramid inequalities at every step leads to the following algebraic constraints for some pyramid $\triangle P$:

(P1) $\triangle P(i, j, k)$ is defined for $1 \leq k \leq j \leq i \leq n-1$.

(P2) $\triangle P(i, j, k) \geq \triangle P(i+1, j, k)$,

(P3) $\triangle P(i, j, k) \geq \triangle P(i, j-1, k-1)$, and

(P4) $\triangle P(i, j, k) \geq \triangle P(i, j, k+1)$.

These pyramid inequalities correspond to the kagog triangle constraints of Definition 1.6, where we must recall that color 1 (white) cubes are present and color 0 (gray) cubes are absent. Condition (P1) ensures that the domain for admissible cubes (i, j, k) is correct and that the height of tower (i, j) is at most j, so (K1) and (K2) hold. Condition (P4) states that the cubes adhere to gravity: color 1 blocks must appear below color 0 blocks. Conditions (P2) and (P4) ensure that the columns are weakly decreasing, so (K3) holds. Conditions (P3) and (P4) ensure that the rows are strictly increasing after the first nonzero entry, so (K4) holds. Indeed, if cube $(i, j-1, k-1)$ is color 1, then (i, j, k) is color 1, so the tower at (i, j) must be taller than the tower at $(i, j-1)$. □

5.1 A Catalan Submapping

In this brief section, we show that the mapping $\psi : \triangle\mathcal{M}_n^\circ \to \triangle\mathcal{K}_n$ induces a natural bijection between Catalan subfamilies of these pyramids. We start by describing two known Catalan families [29]. Let $\mathcal{S}_n$ denote the set of nondecreasing sequences $(s_0, s_1, \ldots, s_{n-1})$ where $0 \le s_i \le i$ for $0 \le i \le n-1$ and $s_i \le s_{i+1}$ for $0 \le i \le n-2$. Let $\mathcal{C}_n$ denote the set of coin pyramids whose bottom row contains n consecutive coins.

Next, we define our associated pyramid families. Let $\mathcal{S}'_n \subset \mathcal{M}_n^\circ$ be the set of omagog triangles whose first $n-2$ rows are all zero. Let $\mathcal{C}'_n \subset \mathcal{K}_n$ be the set of kagog triangles such that every entry in column j is either $j-1$ or j.

Proposition 5.2. *Let $\mathcal{S}_n, \mathcal{C}_n, \mathcal{S}'_n$ and $\mathcal{C}'_n$ be the families defined above.*

(a) There is an elementary bijection $\sigma : \mathcal{S}_n \to \mathcal{C}_n$.

(b) There is an elementary bijection $\rho : \mathcal{S}_n \to \mathcal{S}'_n$.

(c) There is an elementary bijection $\tau : \mathcal{C}_n \to \mathcal{C}'_n$.

(d) Restricting the bijection $\psi : \triangle\mathcal{M}_n^\circ \to \triangle\mathcal{K}_n$ from Lemma 1.8 to $\triangle\mathcal{S}'_n$ gives a bijection to $\triangle\mathcal{C}'_n$. Furthermore, this bijection has a natural interpretation in terms of monotone sequences and coin pyramids. Namely, $\sigma = \tau^{-1} \circ \psi \circ \rho$.

Proof. Figure 5.3 shows the families $\mathcal{S}_3$, $\mathcal{C}_3, \mathcal{S}'_3$, $\mathcal{C}'_3$. It also shows two families $\mathcal{H}_3, \mathcal{H}'_3$ of hybrid configurations that are essential in multiple stages of the proof.

Proof of (a). Our bijection joins $\mathcal{S}_n$ with $\mathcal{C}_n$ via the set $\mathcal{P}_n$ of lattice paths from $(0,0)$ to (n,n) that never travel above the diagonal $y = x$, composing mappings described in [29]. First, we map sequence $s = (s_0, s_1, \ldots, s_{n-1}) \in \mathcal{S}_n$ to the lattice path $p \in \mathcal{P}_n$ whose kth horizontal step is at height s_k. Next, we place gray (missing) coins in each square below p, and place white coins in each square above the path p, up to and including the squares along the diagonal $y = x$. Let $\mathcal{H}_n$ denote the hybrid family of configurations of paths and coins, where missing coins are gray. To complete the mapping σ, reflect the white coins in the hybrid configuration through $\theta = \pi/8$ to obtain the corresponding coin pyramid.

Proof of (b). The monotone sequences $\mathcal{S}_n$ map quite simply to $\mathcal{S}'_n$. The sequence $s \in \mathcal{S}_n$ maps to the omagog triangle in $\mathcal{M}_n^\circ$ whose final row is $(s_1, \ldots, s_{n-1})$, and whose other rows are all-zero. This mapping is clearly a bijection.

Proof of (c). We map coin pyramids $\mathcal{C}_n$ to the triangles in $\mathcal{C}'_n$ via the hybrid configurations in $\mathcal{H}_n$. After mapping a coin pyramid to its hybrid configuration in $\mathcal{H}_n$, we ignore the white coins on the diagonal (which correspond to the fixed base of the coin pyramid), and reflect the

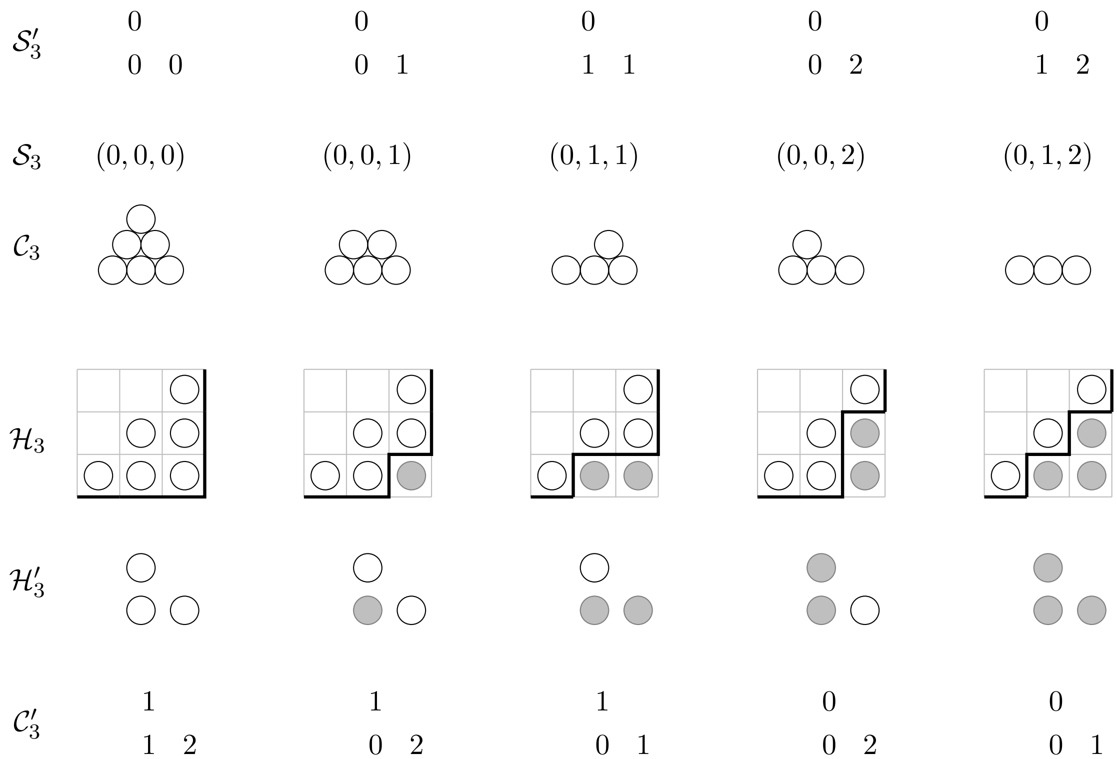


Figure 5.3: Six Catalan families used in the proof of Proposition 5.2. The family $\mathcal{S}_3$ of monotone sequences (s_0, s_1, s_2) maps simply to the subfamily $\mathcal{S}'_3$ of omagog pyramids whose first row is zero. The family $\mathcal{C}_3$ of coin pyramids is in bijection with $\mathcal{S}_3$ via the hybrid family $\mathcal{H}_3$ consisting of lattice paths and coins, where s_k is the height of the horizontal step starting at $x = k$. We mapping $\mathcal{C}_3$ to the subfamily $\mathcal{C}'_3$ of kagog pyramids via family $\mathcal{H}'_3$, the mirror image of the non-diagonal coins of $\mathcal{H}_3$.

remaining coins across the vertical axis to get a triangular array of the appropriate shape. Let $\mathcal{H}'_n$ denote the resulting family of triangular arrays of two-colored coins. Replace each white coin with a 1 and each gray coin with a 0. Finally, add $j-1$ to the entries in column j for $1 \le j \le n-1$. The result is a kagog triangle in $\mathcal{C}'_n$. This invertible mapping is a bijection.

Proof of (d). First, we show that the bijection $\psi : \triangle\mathcal{M}^\circ_n \to \triangle\mathcal{K}_n$ maps $\triangle\mathcal{S}'_n$ to $\triangle\mathcal{C}'_n$. All of the white (present) blocks of $\triangle S \in \triangle\mathcal{S}'_n$ are in row $n-1$. Let $\triangle K = \psi(\triangle S)$ where ψ is the mapping in the proof of Lemma 1.8. Recall that in this mapping, the block $\triangle S(i,j,k)$ flips colors and moves to $\triangle K(n-k, n-j, i-j+1)$. In particular, the gray block $\triangle S(n-2, j, k)$ maps to the white block $\triangle K(n-k, n-j, n-j-1)$. This proves that every tower in column $\ell = n - j$ has height at least $\ell - 1 = n - j - 1$; in other words, ψ bijects $\triangle\mathcal{S}'_n$ to $\triangle\mathcal{C}'_n$.

It remains to show that the mapping ψ corresponds to the mapping $\sigma : \mathcal{S}_n \to \mathcal{C}_n$. The key is to take a bird's eye view of a kagog pyramid $\triangle K \in \triangle\mathcal{C}'_n$. This view only shows the topmost blocks; this is sufficient, since the blocks in the lower layers are all white. We will see that the coin colors of $h' \in \mathcal{H}'_n$ correspond to the block colors of the top layer of a unique $\triangle K = \psi(\triangle S)$. Keeping this intuition in mind, we conclude the proof. After mapping, the block $\triangle S(n-1, j, k)$ flips color and maps to the top-layer block $\triangle K(n-k, n-j, n-j)$. Suppose that $\triangle S(n-1, j, k)$ is white for $1 \leq k \leq \ell$ and gray for $\ell + 1 \leq k \leq j$. This means that $\triangle K(n-k, n-j, n-j)$ is gray for $1 \leq k \leq \ell$ and white for $\ell + 1 \leq k \leq j$. In other words, $\triangle K(k', j', j')$ is gray for $n - \ell \leq k' \leq n-1$ and white for $j' \leq k' \leq n - \ell - 1$. The bird's eye view of the pyramids of $\triangle\mathcal{K}_n$ bijects to the hybrid configurations of $\mathcal{H}'_n$, where we replace the blocks with coins. □

6 The involution of $\mathcal{G}_n$

In this section, we prove Theorem 1.11. Analogous to the previous section, we start by defining ogog triangles. The minimum gog triangle has $G_{\min}(i,j) = j$ for all entries (i,j), while the maximum gog triangle has $G_{\max}(i,j) = n - i + j$. For every gog triangle, we construct its ogog counterpart by subtracting the minimum gog triangle. The last row in every gog triangle is always $[1\ 2\ \cdots\ n]$ since it has length n and is strictly increasing. As such, every ogog triangle has a final row of zeros, which we omit from ogog triangle.

Definition 6.1. *An ogog triangle G° is a triangular array of nonnegative integers $G^\circ(i,j)$ such that*

(OG1) $1 \leq j \leq i \leq n-1$*, so the array is triangular;*

(OG2) $0 \leq G^\circ(i,j) \leq n - i$*, so values in row i are at most $n - i$;*

(OG3) $G^\circ(i,j) \leq G^\circ(i, j+1)$*, so rows are weakly increasing;*

(OG4) $G^\circ(i,j) \geq G^\circ(i+1, j)$*, so columns are weakly decreasing; and*

(OG5) $G^\circ(i,j) \leq G^\circ(i+1, j+1) + 1$*, so diagonals cannot decrease by more than 1.*

We use $\mathcal{G}^\circ_n$ to denote the set of all ogog triangles with $n-1$ rows.

For example, the ogog triangles of of $\mathcal{G}^\circ_3$ are

$$\begin{matrix} 0 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 0 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 1 & \\ 1 & 1 \end{matrix} \qquad \begin{matrix} 2 & \\ 0 & 1 \end{matrix} \qquad \begin{matrix} 2 & \\ 1 & 1 \end{matrix} \ ,$$

where these ogog triangles are ordered so that they biject to the gog triangles in equation (5). As constructed, the color 1 (white) cubes are present in the ogog triangle, while the color 0 (gray) cubes are absent. Our next lemma states that the gray cubes also represent a gog triangle.

Lemma 6.2. *Let G° be an ogog triangle and let $\triangle G^\circ$ be its two-color cube pyramid representation. The color 0 cubes of $\triangle G^\circ$ are an affine transformation of another ogog triangle.*

Note that this correspondence is an involution on the set of ogog triangles: swapping the colors twice leads us back to the original two-coloring of the cube pyramid.

Proof. Similar to the proof of Lemma 1.8, we describe ogog pyramids via a set of inequalities, perform a multistep transformation and then check that the resulting inequalities also describe the set of ogog pyramids. The inequalities for ogog pyramids are:

- $\triangle G^\circ(i,j,k)$ is defined $1 \leq j \leq i \leq n-1$: length of row i is at most i,
- $\triangle G^\circ(i,j,k)$ is defined $1 \leq k \leq n-i$: height of row i is at most $n-i$,
- if $j < i$, then $\triangle G^\circ(i,j+1,k) \geq \triangle G^\circ(i,j,k)$: rows are weakly increasing,
- if $i > 1$, then $\triangle G^\circ(i-1,j,k) \geq \triangle G^\circ(i,j,k)$: columns are weakly decreasing
- if $i < n$, then $\triangle G^\circ(i+1,j+1,k-1) \geq \triangle G^\circ(i,j,k)$: diagonals cannot decrease by more than 1, and
- if $k > 1$, then $\triangle G^\circ(i,j,k-1) \geq \triangle G^\circ(i,j,k)$: the present cubes obey gravity.

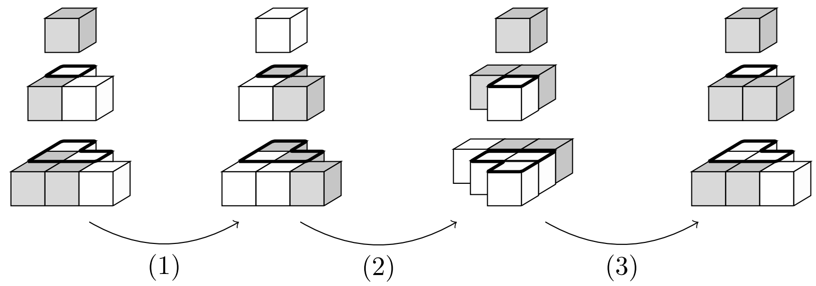
The three-step mapping φ is:

- **Step One:** Invert the colors, or exchange color 1 for color 0 and vice versa. This reverses the inequalities.
- **Step Two:** Perform a quarter rotation of $\mathbb{R}^3$ about the x-axis. This moves the cube (i,j,k) to position $(i,-k,j)$. This tips the two-color cube pyramid onto its side.
- **Step Three:** Rotate by π around the z-axis and then translate by $(n,0,0)$. This moves cube (i,j,k) to $(n-i,-j,k)$.

After composing these three steps, cube (i,j,k) switches color and moves to $(n-i,k,j)$. Figure 6.1 exemplifies the mapping φ for an ogog pyramid from $\triangle\mathcal{G}_4^\circ$.

Careful algebra shows that the resulting constraints are a permutation of the algebraic inequalities for an ogog cube pyramid. As such, this mapping takes one gog triangle to another gog triangle. This affine mapping is an involution, so it is bijective. $\square$

The ogog pyramids are in bijection with gog triangles, and hence also in bijection with alternating sign matrices. Our next corollary shows that the involution φ of Lemma 6.2 reverses the rows of the associated ASM.



2	1	1 0 0	2
0 2	2 0	1 1	1 1
0 0 1	1 1 0	2	0 0 1

Figure 6.1: Example of the ogog to ogog bijection φ. (1) Invert the colors. (2) Tip the stack around the x-axis by a quarter turn. (3) Rotate one half turn about the z-axis, then translate by $(n, 0, 0)$.

Corollary 6.3. *Let φ be the ogog pyramid involution of Lemma 6.2. Let A be an $n \times n$ alternating sign matrix corresponding to ogog triangle G° with two-color cube pyramid $\triangle G^\circ$. Let $\triangle H^\circ = \varphi(\triangle G^\circ)$ and let H° be the associated ogog triangle. Then H° is the ogog triangle corresponding to processing the rows of A in reverse order.*

Figure 6.2 shows an example of how the alternating sign matrices A and B corresponding to ogog pyramids $\triangle G^\circ$ and $\triangle H^\circ = \varphi(\triangle G^\circ)$ are the row reversals of one another.

Proof. Starting with the alternating sign matrix A, we obtain the ogog triangle G° as follows. First, we create the matrix A' whose ith row is the sum of the first i rows of A. This is a 0-1 matrix whose ith row contains exactly i ones. We convert A' into a gog triangle G by reporting the indices of the ones in each row. We then set $G^\circ = G - G_{\min}$, which corresponds to subtracting $[1, 2, \ldots, i]$ from row i of G for $1 \le i \le n$ and then deleting the final row (which is all-zero).

Let A_i denote the ith row of A and let $A'_i = \sum_{k=1}^{k} A_i$ denote the ith row of the partial sum matrix A'. Let $1 \le a'_1 < a'_2 < \cdots < a'_i \le n$ denote the locations of the ones in row A'_i. Then $G(i,j) = a'_j$, or equivalently $[a'_1, a'_2, \cdots, a'_i]$ is the ith row of the gog triangle G. The entries satisfy

$$\begin{array}{rclcl} 1 & \le & a'_1 & \le & n - i + 1, \\ a'_{j-1} & < & a'_j & \le & n - i + j, \qquad 2 \le j \le i. \end{array}$$

Row i of ogog triangle G° is

$$[a'_1 - 1, a'_2 - 2 \cdots, a'_i - i]. \tag{7}$$

We start with row $n - 1$ of our triangle, as it is the simplest row to comprehend. Row

$$A \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \overset{\text{row}}{\underset{\text{reversal}}{\longleftrightarrow}} \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B \qquad \text{ASM}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$A' \quad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad B' \qquad \text{ASM partial sums}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

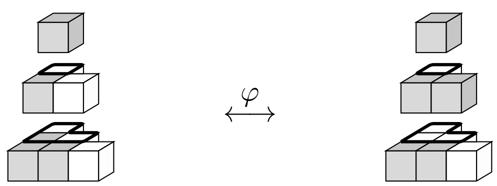
$$G \quad \begin{array}{llll} 3 & & & \\ 1 & 4 & & \\ 1 & 2 & 4 & \\ 1 & 2 & 3 & 4 \end{array} \qquad\qquad \begin{array}{llll} 3 & & & \\ 2 & 3 & & \\ 1 & 2 & 4 & \\ 1 & 2 & 3 & 4 \end{array} \quad H \qquad \text{gog triangle}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$$G^\circ \quad \begin{array}{lll} 2 & & \\ 0 & 2 & \\ 0 & 0 & 1 \end{array} \qquad\qquad \begin{array}{lll} 2 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{array} \quad H^\circ \qquad \text{ogog triangle}$$

$$\updownarrow \qquad\qquad\qquad \updownarrow$$

$\triangle G^\circ$



$\triangle H^\circ$ two-color cube pyramid

Figure 6.2: The alternating sign matrices A and B corresponding to the two-color cube pyramids $\triangle G^\circ$ and $\triangle H^\circ = \varphi(\triangle G^\circ)$ are the row reversals of one another.

$n-1$ of gog triangle G is $[a'_1, a'_2, \ldots, a'_{n-1}]$, which is missing a single number $\ell \in [n]$, namely the location ℓ of the unique one in row n of A. By equation (7), the corresponding ogog row consists of $\ell - 1$ zeros followed by $n - \ell$ ones.

Consider this row in the context of the two-color ogog pyramid $\triangle G^\circ$ and its image $\triangle H^\circ = \varphi(\triangle G^\circ)$. Row $n-1$ of pyramid $\triangle G^\circ$ has height 1. It contains $\ell - 1$ cubes of color 0, followed by $n-\ell$ cubes of color 1. After transformation φ, the cube $(n-1, j, 1)$ switches color and moves to $(1, 1, j)$. So $\triangle H^\circ$ has a tower of blocks at $(1, 1)$ of height $n-1$, with $\ell - 1$ cubes of color 1 below $n - \ell$ cubes of color 0. It follows that ogog triangle H° has $H^\circ(1,1) = \ell - 1$, and thus the corresponding gog triangle H has $H(1,1) = \ell$. This confirms that the first row of gog triangle

H corresponds to the last row of matrix A, as desired.

We now handle a generic row i of ogog triangle G°; Figure 6.3 shows an example. The entries of row i are a weakly increasing list of length i, drawn from $\{0, 1, \ldots, n-i\}$. Let $0 \leq s_m \leq i$ be the number of consecutive m's in this list, so that

$$\sum_{m=0}^{n-i} s_m = i.$$

In the corresponding gog triangle G, row i is missing the integers

$$1 + p + \sum_{k=0}^{p} s_k \quad \text{where} \quad 0 \leq p \leq n - i - 1. \tag{8}$$

Let us pause to make some key observations. The missing integers in row i of G are precisely the locations of the zeros in the partial sum $A'_i = \sum_{\ell=1}^{i} A_\ell$. Since the sum of all the rows yeilds the all-ones vector, these are also the locations of the ones in the partial sum $\sum_{\ell=i+1}^{n} A_\ell$. Of course, summing the last $n-i$ rows of A is the same as summing the first $n-i$ rows of the row reversal of A.

Next, we translate our observations into statements about two-color pyramids. When we convert ogog triangle G° into pyramid $\triangle G^\circ$, row i of G° maps to the $i \times (n-i)$ wall of cubes

$$\triangle G^\circ_i = \{(i, j, k) : 1 \leq j \leq i \text{ and } 1 \leq k \leq n - i\}.$$

The layer of wall $\triangle G^\circ_i$ at height k consists of $\sum_{m=0}^{k-1} s_m$ cubes of color 0 followed by $\sum_{m=k}^{n-i} s_m$ cubes of color 1. The transformation $\varphi : \triangle G^\circ \mapsto \triangle H^\circ$ maps $\triangle G^\circ_i$ to the $(n-i) \times i$ wall

$$\triangle H^\circ_{n-i} = \{(n-i, k, j) : 1 \leq k \leq n-i \text{ and } 1 \leq j \leq i\}.$$

We have inverted the colors and exchanged vertical and horizontal, so the tower of wall $\triangle H^\circ_{n-i}$ at $(n-i, k)$ consists of $\sum_{m=0}^{k-1} s_m$ cubes of color 1, stacked below $\sum_{m=k}^{n-i} s_m$ cubes of color 0.

We now translate the structure of pyramid $\triangle H^\circ$ into the triangle setting. Ogog triangle H° has $H^\circ(n-i, k) = \sum_{m=0}^{k-1} s_m$ for $1 \leq h \leq n-i$, so its corresponding gog triangle H has

$$H(n-i, k) = k + \sum_{m=0}^{k-1} s_m \quad \text{where} \quad 1 \leq k \leq n - i. \tag{9}$$

The formulas in equations (8) and (9) are equivalent (taking $k = p + 1$). Therefore, row $n-i$ of gog triangle H contains the locations of the ones in the partial sum $\sum_{k=i+1}^{n} A_k$. In other words, gog triangle H is constructed by considering the rows of alternating sign matrix A in reverse order. $\square$

Proof of Theorem 1.11. Let $\gamma : \mathcal{G}_n \to \mathcal{G}^\circ_n$ be the bijection $\gamma(G) = G - G_{\min}$. Let $\pi : \mathcal{G}^\circ_n \to \triangle\mathcal{G}^\circ_n$ be the bjiection $\pi(G^\circ) = \triangle G^\circ$. By Corollary 6.3, the desired involution $f : \mathcal{G}_n \to \mathcal{G}_n$ is $f = \gamma^{-1} \circ \pi^{-1} \circ \varphi \circ \pi \circ \gamma$. $\square$

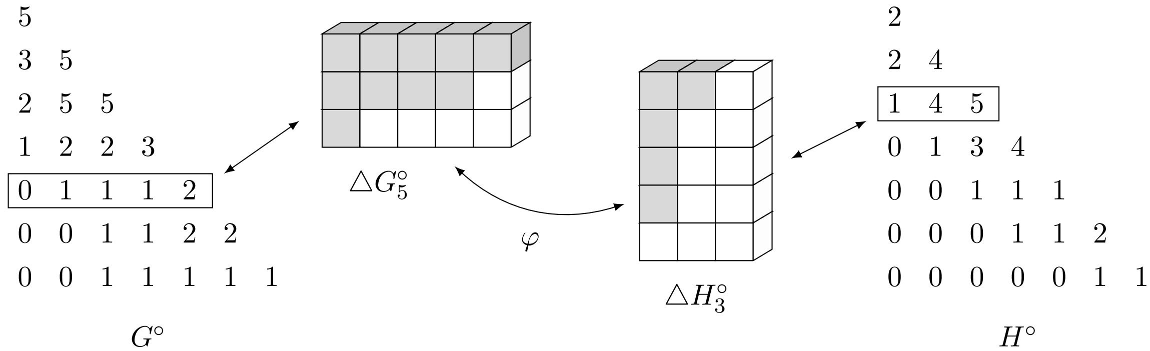


Figure 6.3: An ogog triangle G° from $\mathcal{G}_8^\circ$ and its image H° via the invertible mapping φ. Row 5 of triangle G° becomes pyramid wall $\triangle G_5^\circ$ which maps via φ to pyramid wall $\triangle H_3^\circ$ and then to row 3 of triangle H°.

7 Conclusion

Poset extensions of the de Finetti Lattice $F_{n,2}$ have interesting combinatorical connections. We have shown that $\mathcal{F}_{n,2}^{(1)}$ is enumerated by the ASM/TSSCPP sequence and that $\mathcal{F}_{n,2} = \mathcal{F}_{n,2}^{(2)}$ is enumerated by the strict-sense ballot numbers. We have also shown that there is a very natural involution on gog triangles that corresponds to reversing the rows of the associated alternating sign matrices. We conclude this work with some open research questions relating to both poset extentions and ASM/TSSCPP.

One natural continuation of this work is to consider the de Finetti extensions of $F_{n,3}$, namely $\mathcal{F}_{n,3}^{(1)}$, $\mathcal{F}_{n,3}^{(2)}$ and $\mathcal{F}_{n,3}^{(3)} = \mathcal{F}_{n,3}$. Are these families enunerated by known combinatorial sequences? If so, can we find a natural bijection to the appropriate combinatorial family? An understanding of these smaller families could provide valuable insight into the family $\mathcal{F}_n$ of de Finetti total orders. Any new perspective could have ramifications for comparative probability orders and completely separable preferences. One could further investigate the subfamily of de Finetti extensions $\mathcal{F}_{n,m}^{(k)}$ by defining a graph where we connect extensions via an appropriately atomic operation, such as transpositions [24] for $\mathcal{L}(B_n)$ or flips [22] for members of $\mathcal{F}_n$. Also, can the dimension [32] of a de Finetti extension of $F_{n,m}$ be achieved by restricting ourselves to de Finetti extensions?

This paper brings two novel families into the fold of ASM and TSSCPP combinatorial structures: the poset extensions $\mathcal{F}_{n,2}^{(1)}$ and the kagog triangles $\mathcal{K}_n$. Some recent efforts have focussed on statistic-preserving bijections between subfamilies of ASM and TSSCPP structures [31, 4]. Perhaps the properties $\mathcal{F}_{n,2}^{(1)}$ and $\mathcal{K}_n$ might reveal connections to help traverse the gap between ASM and TSSCPP. In particular, our two-color cube pyramid representation for triangular arrays revealed a natural bijection between magog triangles and kagog triangles, as well. as a nice involution on gog triangles. We are optimistic that this point of view could aid in the investigation of the other known triangular families.

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