Rankin-Cohen brackets for Calabi-Yau modular forms [1]

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Abstract

For any positive integer n, we consider a modular vector field R on a moduli space T of Calabi-Yau n-folds arising from the Dwork family enhanced with a certain basis of the n-th algebraic de Rham cohomology. The components of a particular solution of R, which are provided with definite weights, are called Calabi-Yau modular forms. Using R we introduce a derivation $\mathscr{D}$ and the Ramanujan-Serre type derivation ∂ on the space of Calabi-Yau modular forms. We show that they are degree 2 differential operators and there exists a proper subspace $\mathscr{M}^2$ of the space of Calabi-Yau modular forms which is closed under ∂. Employing the derivation $\mathscr{D}$, we define the Rankin-Cohen brackets for Calabi-Yau modular forms and prove that the subspace generated by the positive weight elements of $\mathscr{M}^2$ is closed under the Rankin-Cohen brackets.

1 Introduction

The proof of Fermat's last theorem led to the celebrated modularity theorem, which states that elliptic curves over the field of rational numbers $\mathbb{Q}$ are related with modular forms. Elliptic curves are 1-dimensional Calabi-Yau (CY for short) varieties, which makes it natural to ask whether a similar statement of modularity holds for higher dimensional CY varieties. This question persuaded mathematicians and theoretical physicists to the subject of *modularity of CY manifolds* which is one of the considerable present challenges of the modern algebraic number theory. Some relevant results can be found, for instance, in [NY13] and the references therein. Noriko Yui in [NY13] divides the modularity of CY varieties in arithmetic modularity and geometric modularity including (1) the modularity (automorphy) of Galois representations of CY varieties (or motives) defined over $\mathbb{Q}$ or number fields, (2) the modularity of solutions of Picard-Fuchs differential equations of families of CY varieties, and mirror maps (mirror moonshine), (3) the modularity of generating functions of invariants counting certain quantities on CY varieties, and (4) the modularity of moduli for families of CY varieties. But so far, in a general context, even there is no unified formulation or statement of the modularity of CY varieties. H. Movasati in [Mov16] says: "All the attempts to find an arithmetic modularity for mirror quintics have failed, and this might be an indication that maybe such varieties need a new kind of modular forms." In this way, he introduced CY modular forms which somehow can be considered as a modern generalization of the classical (quasi-)modular forms theory. The present work provides some evidences in favor of this generalization; namely, we introduce the space of CY modular forms $\mathscr{M}$ and furnish it with a Rankin-Cohen algebra structure. Then we find a proper subspace of $\mathscr{M}$ which is closed under the Rankin-Cohen brackets.

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This can be considered as a generalization of the work of Don Zagier [Zag94] for the space of classical (quasi-)modular forms.

In [MN16] we already offered other evidences in defense of the generalization of the (quasi-)modular forms theory using an algebraic method calling the *Gauss-Manin connection in disguise*, GMCD for short, which got started by H. Movasati in applying to elliptic curves [Mov12b] and then was used again by him in [Mov15] for the family of mirror quintic 3-folds, where he reencountered the so-called *Yukawa coupling* of Candelas et al. [COGP91]. More precisely, in [MN16] we introduced the *enhanced moduli space* $\mathsf{T} = \mathsf{T}_n$ of the pairs $(X, [\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha_{n+1}])$, where X is an n-dimensional CY variety arising from the so-called Dwork family and $\{\alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}$ refers to a basis of the n-th algebraic de Rham cohomology $H^n_{\rm dR}(X)$ which is compatible with the Hodge filtration of $H^n_{\rm dR}(X)$ (see (3.18)) and its intersection form matrix is constant (see (3.19)). We showed that there exist a unique vector field $\mathsf{R} = \mathsf{R}_n$, called *modular vector field*, and regular functions $\mathsf{Y}_i,\ 1 \le i \le n-2$, satisfying a certain equation involving the Gauss-Manin connection of the universal family of T (see Theorem 3.1 and also [Nik15, Theorem 1.1] in a more general context). Due to [Mov12b] we can say that the modular vector field R is a generalization of the vector field Ra introduced by S. Ramanujan in [Ram16] (see (2.1)). This is because R satisfies a similar equation to the one for Ra taking into account the Gauss-Manin connection of the universal family of a certain collection of elliptic curves (see (3.24) and (3.25)). It is known that the triple of Eisenstein series (E_2, E_4, E_6), where for $j = 1, 2, 3$:

$$(1.1)\quad E_{2j}(q) = 1 + b_j \sum_{k=1}^{\infty} \sigma_{2j-1}(k) q^k \text{ with } \sigma_i(k) = \sum_{d|k} d^i,\ (b_1, b_2, b_3) = (-24, 240, -504),$$

gives a solution of the Ramanujan vector field Ra. Note that E_2 is a quasi-modular form and $E_4,\ E_6$ are modular forms. Hence, for $n = 1$, where T is the enhanced moduli space of elliptic curves, it was expected that the components of a particular solution of the modular vector field R could be written in terms of (quasi-)modular forms. This fact was proved in [MN16] (see also [Ali17] for similar results); namely, if $n = 1, 2$, then we found the modular vector fields, respectively, as

$$(1.2)\qquad \mathsf{R}_1 : \begin{cases} \dot{t}_1 = -t_1 t_2 - 9(t_1^3 - t_3) \\ \dot{t}_2 = 81 t_1 (t_1^3 - t_3) - t_2^2 \\ \dot{t}_3 = -3 t_2 t_3 \end{cases}, \qquad \mathsf{R}_2 : \begin{cases} \dot{t}_1 = t_3 - t_1 t_2 \\ \dot{t}_2 = 2t_1^2 - \frac{1}{2} t_2^2 \\ \dot{t}_3 = -2 t_2 t_3 + 8 t_1^3 \\ \dot{t}_4 = -4 t_2 t_4 \end{cases},$$

where by $\dot{*}$ in R_1 we mean $\dot{*} = 3 \cdot q \cdot \frac{\partial *}{\partial q}$ and in R_2 we mean $\dot{*} = -\frac{1}{5} \cdot q \cdot \frac{\partial *}{\partial q}$, and furthermore in R_2 we have the polynomial equation $t_3^2 = 4(t_1^4 - t_4)$. For a complex number τ with ${\rm Im}\tau > 0$, if we set $q = e^{2\pi i \tau}$, then we got the following solutions of R_1 and R_2 respectively:

$$(1.3)\qquad \begin{cases} \mathsf{t}_1(q) = \frac{1}{3}(2\theta_3(q^2)\theta_3(q^6) \\ \qquad\qquad -\theta_3(-q^2)\theta_3(-q^6)), \\ \mathsf{t}_2(q) = \frac{1}{8}(E_2(q^2) - 9E_2(q^6)), \\ \mathsf{t}_3(q) = \frac{\eta^9(q^3)}{\eta^3(q)}, \end{cases}, \quad \begin{cases} \frac{10}{6}\mathsf{t}_1(\frac{q}{10}) = \frac{1}{24}(\theta_3^4(q^2) + \theta_2^4(q^2)), \\ \frac{10}{4}\mathsf{t}_2(\frac{q}{10}) = \frac{1}{24}(E_2(q^2) + 2E_2(q^4)), \\ 10^4 \mathsf{t}_4(\frac{q}{10}) = \eta^8(q)\eta^8(q^2), \end{cases}$$

in which η and θ_i's are the classical eta and theta series given as follows:

$$(1.4)\qquad \eta(q) = q^{\frac{1}{24}} \prod_{k=1}^{\infty}(1 - q^k),\ \ \theta_2(q) = \sum_{k=-\infty}^{\infty} q^{\frac{1}{2}(\frac{k+1}{2})^2},\ \ \theta_3(q) = 1 + 2\sum_{k=1}^{\infty} q^{\frac{1}{2}k^2}.$$

Besides that, these solutions satisfy some interesting enumerative properties. For example, in the solution of R_1 the coefficients of the q-expansion of t_1 yield the number of integer solutions of $x^2+3y^2=k$, and in the solution of R_2 the function $\mathsf{t}_1(q)$ is the generating function of the odd divisor function, i.e., $\frac{10}{6}\mathsf{t}_1(\frac{q}{10})=\sum_{k=0}^{\infty}\sigma^o(k)q^k$, where $\sigma^o(k)=\sum_{\substack{d|k \\ \text{d is odd}}} d$ (for more details and more properties see [MN16, §8]). In the case $n=3$, R_3 is explicitly computed in [Mov15] and it is verified that Y_1 is the Yukawa coupling introduced in [COGP91], which predicts the numbers of rational curves of various degrees on the general quintic three-folds. For $n=4$, we computed the modular vector field R_4 explicitly in [MN16] and we observed that $\mathsf{Y}_1^2=\mathsf{Y}_2^2$ is the same as 4-point function presented in [GMP95, Table 1, $d=4$]. In both cases $n=3,4$ we found the q-expansions of the components of a solution of R, in which all coefficients are integers, up to multiplying the solution components by a constant. Unlike the cases $n=1,2$, here we believe that it is not possible to write the solution components in terms of modular forms, since the coefficients of their q-expansions increase very rapidly. This leads us to think to another theory which generalizes the theory of modular forms. These components of a particular solution of R, which are called *CY modular forms*, are adequate candidates of the desired generalization. In the next paragraphs we give more evidences that convince us to continue with this generalization.

We know that the Ramanujan vector field Ra is deeply connected with the space of (quasi-)modular forms $\mathcal{M}_*$ ($\widetilde{\mathcal{M}}_*$) for $SL(2,\mathbb{Z})$, since $\mathcal{M}_*=\mathbb{C}[E_4,E_6]$ ($\widetilde{\mathcal{M}}_*=\mathbb{C}[E_2,E_4,E_6]$) and the triple (E_2,E_4,E_6) is a solution of Ra. Thus, our focus will be held on the properties of the Ramanujan vector field Ra. In particular, Ra along with the radial vector field $H=2t_1\frac{\partial}{\partial t_1}+4t_2\frac{\partial}{\partial t_2}+6t_3\frac{\partial}{\partial t_3}$ and the constant vector field $F=-12\frac{\partial}{\partial t_1}$ forms a copy of $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra. Our attention in [Nik20] was dedicated to this property, where we studied the AMSY-Lie algebra for the mirror CY varieties arising from the Dwork family. The AMSY-Lie algebra was discussed for the first time in [AMSY16] for non-rigid compact CY threefolds, and recently in [AV18] it is established for mirror elliptic $K3$ surfaces (note that the AMSY-Lie algebra is called Gauss-Manin Lie algebra by authors of [AV18]). In [Nik20] we introduced an algebraic group G acting from the right on T (see (3.36)) and described its Lie algebra $\mathrm{Lie}(\mathsf{G})$ (see (3.37)). We found the canonical basis of $\mathrm{Lie}(\mathsf{G})$ (see (3.38) and (3.39)) and observed that it is isomorphic to a Lie subalgebra of the Lie algebra $\mathfrak{X}(\mathsf{T})$ of the vector fields on T (see Theorem 3.2). We defined the AMSY-Lie algebra $\mathfrak{G}$ as the $\mathscr{O}_{\mathsf{T}}$-module generated by $\mathrm{Lie}(\mathsf{G})$ and the modular vector field R in $\mathfrak{X}(\mathsf{T})$. We stated the Lie structure of $\mathfrak{G}$ (see Theorem 3.3) and observed that $\dim(\mathfrak{G})=\dim(\mathsf{T})$. In this way, we could prove that the modular vector field R along with two other vector fields H and F generates a copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{G}\subset\mathfrak{X}(\mathsf{T})$ (see Theorem 3.4), which is the desired result (the notations H and F in the whole manuscript are used for the same vector fields given in Theorem 3.4).

It is well known that the derivation of a modular form is not necessarily a modular form. More precisely, for a positive integer r let $f\in\mathcal{M}_r$ be a modular form of weight r for $SL(2,\mathbb{Z})$. Then $f'\in\widetilde{\mathcal{M}}_{r+2}$ is a quasi-modular form of weight $r+2$. But the derivation f' can be corrected using the Ramanujan-Serre derivation $\partial f=f'-\frac{1}{12}rE_2f$ which yields $\partial f\in\mathcal{M}_{r+2}$. Besides this, for a given $f\in\mathcal{M}_r$, it is known that the polynomial relation $rff''-(r+1)f'^2$ gives another modular form of weight $2r+4$. R. Rankin in [Ran56] generalized the latter polynomial relation and described some necessary conditions under which a polynomial in a given modular form and its derivations is again a modular form. Then, in [Coh77] H. Cohen, for any non-negative integer k, defined a bilinear operator

$F_k(\cdot,\cdot)$ satisfying the imposed conditions by Rankin and proved that for all $f\in\mathcal{M}_r,\ g\in\mathcal{M}_s$ one gets $F_k(f,g)\in\mathcal{M}_{r+s+2k}$. For example, the last polynomial relation given above can be written as $rff''-(r+1)f'^2=\frac{1}{r+1}F_2(f,f)$. Later, Don Zagier in [Zag94] called these bilinear forms as *Rankin-Cohen brackets* an denoted by $[\cdot,\cdot]_k$ (see (2.6)). Furthermore, he developed the algebraic theory of *Rankin-Cohen algebras*, which are briefly described in Section 2.

In the present work we aim to endow the space of CY modular forms with a Rankin-Cohen algebra structure. We first need to assign the correct weights to the CY modular forms. In order to do this, we back again to the properties of the Ramanujan vector field Ra. We know that $\deg(E_2)=2,\ \deg(E_4)=4,\ \deg(E_6)=6$ and these integers appear as cofficients of the components of the vector field $H=2t_1\frac{\partial}{\partial t_1}+4t_2\frac{\partial}{\partial t_2}+6t_3\frac{\partial}{\partial t_3}$ which mentioned above. On the other hand, we observe that the vector field $\mathsf{H}\in\mathfrak{G}$ can be written in the form $\mathsf{H}=\sum_{j=1}^{\mathsf{d}}w_jt_j\frac{\partial}{\partial t_j}$, where $\mathsf{d}=\dim\mathsf{T}$, $(t_1,t_2,\ldots,t_{\mathsf{d}})$ is a chart of T and $w_j\in\mathbb{Z}^{\geq 0},\ j=1,2,\ldots,\mathsf{d}$ (see (3.48) and (3.50)). These facts lead us to define $\deg(t_j):=w_j,\ j=1,2,\ldots,\mathsf{d}$. Applying these weights, in Proposition 3.1 we show that for any positive integer n the modular vector field R is a quasi-homogeneous vector field of degree 2. We suppose that $\mathsf{t}_j,\ j=1,2,\ldots,\mathsf{d}$, is the component of a particular solution of R associated with the coordinate chart t_j carrying the same weight, i.e., $\deg(\mathsf{t}_j)=w_j$. An evidence of the truth of the attached weights are the solution components of R for $n=1,2$, given in (1.3), where the assigned weights of t_j's coincide with the real weights of the encountered (quasi-)modular forms. Hence, we define the space of CY modular forms as $\mathscr{M}:=\mathbb{C}[\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_3,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]$ and the subspace $\mathscr{M}^2$ of $\mathscr{M}$ as $\mathscr{M}^2:=\mathbb{C}[\mathsf{t}_1,\mathsf{t}_3,\mathsf{t}_4,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]$, where $\check{\mathsf{t}}$ is a product of a few number of t_j's (see (3.29)); indeed, $\mathscr{M}=\mathscr{M}^2[\mathsf{t}_2]$. From now on we call the elements of $\mathscr{M}^2$ the *2CY modular forms.* Note that in our approach t_2 plays the same role of the quasi-modular form E_2 in the theory of (quasi-)modular forms, which gives sense to the definition of $\mathscr{M}^2$ (recall that $\widetilde{\mathcal{M}}_*=\mathcal{M}_*[E_2]$).

The modular vector field R induces a degree 2 differential operator $\mathscr{R}$ on $\mathscr{M}$, but for odd integers n, except for $n=1$, the space of 2CY modular forms $\mathscr{M}^2$ is not closed under the Rankin-Cohen brackets defined by the derivation $\mathscr{R}$, which is our desired result. Hence, for odd integers n we need to correct two components of R and in general we obtain a new vector field D (see (3.59)), which coincides with R if n is an even integer or $n=1$. In Lemma 3.2 we prove the fundamental result of this paper which gives D is a quasi-homogeneous vector field of degree 2 in T, and also it implies that D along with H and the constant vector field $\frac{\partial}{\partial t_2}$ forms a copy of $\mathfrak{sl}_2(\mathbb{C})$ (see Corollary 3.1). Thus, D induces a degree 2 differential operator on $\mathscr{M}$ which is denoted by $\mathscr{D}$. It is not difficult to observe that the space of 2CY modular forms $\mathscr{M}^2$ is not closed under $\mathscr{D}$, but by excluding the terms which avoid the derivation of a given 2CY modular form under $\mathscr{D}$ to be again a 2CY modular form we can define the *Ramanujan-Serre type derivation* ∂ (see (4.5)). In Theorem 4.1 we state the first main result of this work which affirms that the Ramanujan-Serre type derivation ∂ is a degree 2 differential operator and $\mathscr{M}^2$ is closed under ∂. Employing the derivation $\mathscr{D}$ we define the Rankin-Cohen bracket of the CY modular forms in (4.27). Finally, in the second main result of the present paper, namely Theorem 4.2, we prove that the space of 2CY modular forms of positive weights is closed under the Rankin-Cohen brackets of the CY modular forms, and hence we provide this space with a Rankin-Cohen algebra structure. It is worth to mention that for different examples of 2CY modular forms of negative weights we used the computer, for $n=1,2,3,4$, and observed that their

Rankin-Cohen brackets is again 2CY modular forms. Thus, we conjecture that the whole space of the 2CY modular forms $\mathscr{M}^2$ is closed under the Rankin-Cohen brackets.

Remark 1.1. *The space of CY modular forms $\mathscr{M}$ can have different important subspaces which will be denoted using upper index in $\mathscr{M}$, namely $\mathscr{M}^j$, $j \geq 1$, where j increases in the order that the corresponding subspace appears in the literature. The first of such subspaces is the space of 1CY modular forms:*

$$\mathscr{M}^1 := \mathbb{C}[\mathsf{t}_1, \mathsf{t}_{n+2}, \frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2} - \mathsf{t}_1^{n+2})}],$$

which was studied in the case $n = 3$, see for instance [AMSY16], where it refers to the algebra of the so-called BCOV anomaly equation. In fact, $\mathscr{M}^1$ is associated with $\mathscr{O}_{\mathsf{S}}$, the $\mathbb{C}$-algebra of regular functions on the moduli space S given in (3.5)*. The second one is the space of 2CY modular forms $\mathscr{M}^2$ which is under our consideration in the present work. It would be interesting if one could find the interpretation of the moduli space associated with $\mathscr{M}^2$.*

Remark 1.2. *Analogous as [CS17, Proposition 5.3.27 and Corollary 5.3.29] we may apply the Rankin-Cohen brackets for CY modular forms to find a Chazy-type equation for the system of ODE's presented by the modular vector field R or the vector field D. This research is in progress and its eventual results will appear in the future works.*

This manuscript is organized as follows. In Section 2 we briefly review the relevant definitions and facts of [Zag94] which will be used in the rest of the text. Section 3 starts with a short summary of [MN16] and [Nik20] which constructs the foundation of the present research and also lets us to have a self contained manuscript. After that, we prove that the modular vector field R is a quasi-homogeneous vector field of degree 2, we define the vector field D and demonstrate the fundamental lemma. In Section 4 our main results are stated and proved. Namely, we define the concepts of: spaces of CY modular forms and 2CY modular forms, derivation $\mathscr{D}$, Ramanujan-Serre type derivation ∂ and Rankin-Cohen brackets of the CY modular forms. The main results are stated in Theorem 4.1 and Theorem 4.2. In different examples of this section, for $n = 1, 2, 3, 4$, the derivations $\mathscr{D}$, ∂ and Rankin-Cohen brackets of a few CY modular forms are explicitly calculated. Section 5 deals with the final remarks. In this section we state a conjecture which improves our results.

Acknowledgment. The initial inspiration of the present study came from a conversation between Hossein Movasati and Don Zagier. In fact, Hossein Movasati discussed his perspectives on the modular vector field R and the CY modular forms calculated in [Mov15] with D. Zagier, and received the recommendation that it would be very helpful if he could somehow define the Rankin-Cohen brackets in his context. This conversation was shared later with the author and others by H. Movasati. At that moment we did not succeed in solving the problem, because of the absence of some key points such as the correct weights of the CY modular forms and etc. After the work [Nik20], the author could find the missing points of the research and completed the present work. Because of this, the author would like to thank both of them, in particular he is very grateful to H. Movasati for his helpful discussions and comments, including his suggestions for unifying the notations.

2 Rankin-Cohen algebra

In this section we recall the important facts and terminologies of [Zag94] which are necessary to explain the motivations and the methods, and also help to understand better further discussions. We start with the initial steps which led to the construction of the Rankin-Cohen algebras, namely the theory of the (quasi-)modular forms. Let $\mathcal{M}_* = \bigoplus_{r\geq 0} \mathcal{M}_r$ be the graded algebra of modular forms, where $\mathcal{M}_r := \mathcal{M}_r(SL(2,\mathbb{Z}))$ is the space of modular forms of weight r for $SL(2,\mathbb{Z})$. We know that $\mathcal{M}_* = \mathbb{C}[E_4, E_6]$, i.e., $\mathcal{M}_*$ is generated by Eisenstein series E_4, E_6 given in (1.1). Note that E_4 and E_6 are modular forms of weight 4 and 6, respectively, while E_2 is a quasi-modular form of weight 2. If we denote the space of quasi-modular forms by $\widetilde{\mathcal{M}}_*$, then $\widetilde{\mathcal{M}}_* = \mathbb{C}[E_2, E_4, E_6]$. The triple (E_2, E_4, E_6) satisfies the system of ordinary differential equations

$$\text{(2.1)} \qquad \mathsf{Ra} : \begin{cases} t_1' = \frac{1}{12}(t_1^2 - t_2) \\ t_2' = \frac{1}{3}(t_1 t_2 - t_3) \\ t_3' = \frac{1}{2}(t_1 t_3 - t_2^2) \end{cases} , \qquad \textit{with } *' = q\frac{\partial *}{\partial q} = \frac{1}{2\pi i}\frac{d}{d\tau} \ \textit{ and } \ q = e^{2\pi i \tau} ,$$

which is known as the *Ramanujan relations between Eisenstein series*, and from now on we call it the *Ramanujan vector field*. The Ramanujan vector field $\mathsf{Ra} = t_1'\frac{\partial}{\partial t_1} + t_2'\frac{\partial}{\partial t_2} + t_3'\frac{\partial}{\partial t_3}$ together with two vector fields $H = 2t_1\frac{\partial}{\partial t_1} + 4t_2\frac{\partial}{\partial t_2} + 6t_3\frac{\partial}{\partial t_3}$ and $F = -12\frac{\partial}{\partial t_1}$ forms a copy of $\mathfrak{sl}_2(\mathbb{C})$; this follows from the fact that $[\mathsf{Ra}, F] = H$, $[H, \mathsf{Ra}] = 2\mathsf{Ra}$, $[H, \mathsf{F}] = -2F$, where $[\ ,\ ]$ refers to the Lie bracket of vector fields. We know that if $f \in \mathcal{M}_r$ is a modular form of weight r, then f' is not necessarily a modular form. If instead of the usual derivation, we use the so-called *Ramanujan-Serre derivation* ∂ given by

$$\text{(2.2)} \qquad \partial f = f' - \frac{1}{12}kE_2 f,$$

then ∂f is a modular form of weight $r+2$. After substituting (t_1, t_2, t_3) by (E_1, E_2, E_3) in the Ramanujan vector field (2.1) we get the following differential operator on $\widetilde{\mathcal{M}}_* = \mathbb{C}[E_1, E_2, E_3]$:

$$\text{(2.3)} \qquad \mathcal{D} : \widetilde{\mathcal{M}}_* \to \widetilde{\mathcal{M}}_{*+2}\,; \quad \mathcal{D}f = \frac{E_2^2 - E_4}{12}\frac{\partial f}{\partial E_2} + \frac{E_2E_4 - E_6}{3}\frac{\partial f}{\partial E_4} + \frac{E_2E_6 - E_4^2}{2}\frac{\partial f}{\partial E_6} ,$$

which is a degree 2 differential operator, i.e., for any $f \in \widetilde{\mathcal{M}}_r$, we get $\mathcal{D}f \in \widetilde{\mathcal{M}}_{r+2}$. Therefore, for any $f \in \mathcal{M}_r$ we have

$$\text{(2.4)} \qquad f' = \mathcal{D}f.$$

and since $\frac{\partial f}{\partial E_2} = 0$, we can express Ramanujan-Serre derivation (2.2) as follow:

$$\text{(2.5)} \qquad \partial f = -\frac{E_6}{3}\frac{\partial f}{\partial E_4} - \frac{E_4^2}{2}\frac{\partial f}{\partial E_6} ,$$

from which we get that Ramanujan-Serre derivation ∂f just excludes the terms including E_2 that prevent the derivation f' to be a modular form. Don Zagier [Zag94] in 1994, based on the works of Rankin [Ran56] and Cohen [Coh77], for any non-negative integer k introduced the k-th Rankin-Cohen bracket $[f,g]_k$ defined as follow:

$$\text{(2.6)} \qquad [f,g]_k := \sum_{i+j=k} (-1)^j \binom{k+r-1}{i}\binom{k+s-1}{j} f^{(j)} g^{(i)} , \quad f \in \mathcal{M}_r \textit{ and } g \in \mathcal{M}_s,$$

where $f^{(j)}$ and $g^{(j)}$ refer to the j-th derivation of f and g with respect to the derivation $*'$ given in (2.1). It was proven by Cohen that $[f,g]_k \in \mathcal{M}_{r+s+2k}$. Note that the 0-th bracket is considered as usual multiplication, i.e. $[f,g]_0 = fg$. We list some algebraic properties of the Rankin-Cohen bracket below, in which we assume $f \in \mathcal{M}_r,\ g \in \mathcal{M}_s$ and $h \in \mathcal{M}_l$:

$$[f,g]_k = (-1)^k [g,f]_k\,, \quad \forall\, k \geq 0\,, \tag{2.7}$$

$$[[f,g]_0,h]_0 = [f,[g,h]_0]_0\,, \tag{2.8}$$

$$[f,1]_0 = [1,f]_0 = f\,, \quad [f,1]_k = [1,f]_k = 0\,,\ \forall\, k > 0\,, \tag{2.9}$$

$$[[f,g]_1,h]_1 + [[g,h]_1,f]_1 + [[h,f]_1,g]_1 = 0\,, \tag{2.10}$$

$$[[f,g]_0,h]_1 + [[g,h]_0,f]_1 + [[h,f]_0,g]_1 = 0\,, \tag{2.11}$$

$$l[[f,g]_1,h]_0 + s[[g,h]_1,f]_0 + r[[h,f]_1,g]_0 = 0\,, \tag{2.12}$$

$$[[f,g]_0,h]_1 = [[g,h]_1,f]_0 - [[h,f]_1,g]_1\,, \tag{2.13}$$

$$(r+s+l)[[f,g]_1,h]_0 = r[[g,h]_0,f]_1 - s[[h,f]_0,g]_1\,, \tag{2.14}$$

$$\begin{aligned}(r+1)(s+1)[[f,g]_0,h]_2 &= -l(l+1)[[f,g]_2,h]_0 \\ &\quad + (r+1)(r+s+1)[[g,h]_2,f]_0 + (s+1)(r+s+1)[[h,f]_2,g]_0\end{aligned} \tag{2.15}$$

$$\begin{aligned}(r+s+l+1)(r+s+l+2)[[f,g]_2,h]_0 &= (r+1)(s+1)[[f,g]_0,h]_2 \\ &\quad - (r+1)(r+s+1)[[g,h]_0,f]_2 - (s+1)(r+s+1)[[h,f]_0,g]_2\end{aligned} \tag{2.16}$$

$$[[f,g]_1,h]_1 = [[g,h]_0,f]_2 - [[h,f]_0,g]_2 + [[g,h]_2,f]_0 - [[h,f]_2,g]_0\,. \tag{2.17}$$

Zagier defined a Rankin-Cohen algebra over a field k (of characteristic zero) as a graded k-vector space $M_* = \bigoplus_{r\geq 0} M_r$, with $M_0 = \mathsf{k}.1$ and $\dim_{\mathsf{k}} M_r$ finite for all r, together with bilinear operations $[\ ,\ ]_k : M_r \otimes M_s \to M_{r+s+2k},\ r,s,k \geq 0$, which satisfy (2.7)-(2.17) and all the other algebraic identities satisfied by the Rankin-Cohen brackets given in (2.6). A basic example of RC algebras can be constructed as follow, and for future uses we state it as a remark.

Remark 2.1. *Let M_* be a commutative and associative graded algebra with unit over the field k together with a derivation $D : M_* \to M_{*+2}$ of degree 2. Given $f \in M_r$ and $g \in M_s$, for any positive integer k define the Rankin-Cohen bracket $[f,g]_{D,k}$ as follow:*

$$[f,g]_{D,k} = \sum_{i+j=k} (-1)^j \binom{k+r-1}{i}\binom{k+s-1}{j} f^{(j)} g^{(i)} \in M_{r+s+2k}, \tag{2.18}$$

where $f^{(j)} = D^j f$ and $g^{(j)} = D^j g$ are the j-th derivation of f and g with respect to the derivation D. Then $(M_*, [\cdot,\cdot]_{D,*})$ is a Rankin-Cohen algebra which is called the standard Rankin-Cohen algebra.

For example $(\widetilde{\mathcal{M}}_*, [\cdot,\cdot]_{\mathcal{D},*})$ and $(\mathcal{M}_*, [\cdot,\cdot]_{\partial,*})$, where $\mathcal{D}$ and ∂ are respectively given in (2.3) and (2.5), are standard Rankin-Cohen algebras. On account of (2.4) we have $[\cdot,\cdot]_{\mathcal{D},k} = [\cdot,\cdot]_k,\ k \geq 0$, and hence $(\mathcal{M}_*, [\cdot,\cdot]_{\mathcal{D},*})$ is a Rankin-Cohen subalgebra of $(\widetilde{\mathcal{M}}_*, [\cdot,\cdot]_{\mathcal{D},*})$, although $\mathcal{M}$ is not closed under $\mathcal{D}$. Note that even though $(\mathcal{M}_*, [\cdot,\cdot]_{\partial,*})$ and $(\mathcal{M}_*, [\cdot,\cdot]_{\mathcal{D},*})$ are completely different, it is possible to reconstruct $(\mathcal{M}_*, [\cdot,\cdot]_{\mathcal{D},*})$ from $(\mathcal{M}_*, [\cdot,\cdot]_{\partial,*})$ by hiring (2.2). This fact, in a more general version, is given in the following proposition, and since a part of its proof will be needed, we summarize the proof and for more details the reader is referred to [Zag94, Proposition 1].

Proposition 2.1. *Let M_* be a commutative and associative graded k-algebra with $M_0 = \mathsf{k}\cdot 1$ together with a derivation $\partial : M_* \to M_{*+2}$ of degree 2, and let $\Lambda \in M_4$. For any $k \geq 0$ define brackets $[\cdot,\cdot]_{\partial,\Lambda,k}$ by*

$$[f,g]_{\partial,\Lambda,k} = \sum_{i+j=k} (-1)^j \binom{k+r-1}{i}\binom{k+s-1}{j} f_{(j)}g_{(i)} \in M_{r+s+2k}\ , \tag{2.19}$$

where $f \in M_r,\ g \in M_s$, and $f_{(j)} \in M_{r+2j},\ g_{(i)} \in M_{s+2i}$ are defined recursively as follows

$$f_{(j+1)} = \partial f_{(j)} + j(j+r-1)\Lambda f_{(j-1)},\ g_{(i+1)} = \partial g_{(i)} + i(i+s-1)\Lambda g_{(i-1)}, \tag{2.20}$$

with initial conditions $f_{(0)} = f,\ g_{(0)} = g$. Then $(M_*, [\cdot,\cdot]_{\partial,\Lambda,*})$ is a Rankin-Cohen algebra.

Sketch of proof. The only way is to embed $(M_*, [\cdot,\cdot]_{\partial,\Lambda,*})$ into a standard Rankin-Cohen algebra $(R_*, [\cdot,\cdot]_{D,*})$ for some larger R_* with derivation D. Indeed, it is taken $R_* = M[\lambda]_* := M_* \otimes_{\mathsf{k}} \mathsf{k}[\lambda]$, where λ has degree 2, and the derivation D is defined on the generators of R_* as follow

$$D(f) = \partial(f) + k\lambda f \in R_{k+2},\ \text{for any } f \in M_k,\ \text{and}\ \ D(\lambda) = \Lambda + \lambda^2 \in R_4, \tag{2.21}$$

which can be extended uniquely as a derivation on R_*. Then, for any $k \geq 0$ and any $f, g \in M_*$ we have:

$$[f,g]_{D,k} = [f,g]_{\partial,\Lambda,k} \text{ (see the proof of [Zag94, Proposition 1])}. \tag{2.22}$$

This completes the proof, since M_* is obviously closed under the brackets $[\cdot,\cdot]_{\partial,\Lambda,k}$. □

A Rankin-Cohen algebra $(M_*, [\cdot,\cdot]_*)$ is called *canonical* if its brackets are given as in Proposition 2.1 for some derivation $\partial : M_* \to M_{*+2}$ and some element $\Lambda \in M_4$, i.e., $[\cdot,\cdot]_k = [\cdot,\cdot]_{\partial,\Lambda,k}$. For example, $(\mathcal{M}_*, [\cdot,\cdot]_*)$ is a canonical Rankin-Cohen algebra with ∂ as Ramanujan-Serre derivation and $\Lambda = \frac{1}{12^2}E_4$.

3 GMCD attached to the Dwork family

In this section we first recall relevant facts and terminologies given in [MN16, Nik20] in subsections 3.1 and 3.2, and for more details one is referred to the same references. Then, we will observe some new important results which will be used in the subsequent section. In this manuscript for any positive integer n we fix the notation $m := \frac{n+1}{2}$ if n is odd, and $m := \frac{n}{2}$ if n is even.

3.1 Enhanced moduli space and modular vector field R

Let W_z, for $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, be an n-dimensional hypersurface in $\mathbb{P}^{n+1}$ given by the so-called Dwork family:

$$f_z(x_0, x_1, \ldots, x_{n+1}) := zx_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)x_0x_1x_2\cdots x_{n+1} = 0.$$

W_z represents a family of CY n-folds. The group $G := \{(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}) \mid \zeta_i^{n+2} = 1,\ \zeta_0\zeta_1\ldots\zeta_{n+1} = 1\}$, acts canonically on W_z as

$$(\zeta_0, \zeta_1, \ldots, \zeta_{n+1}).(x_0, x_1, \ldots, x_{n+1}) = (\zeta_0x_0, \zeta_1x_1, \ldots, \zeta_{n+1}x_{n+1}).$$

We obtain the variety $X = X_z,\ z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}$, by desingularization of the quotient space W_z/G (for more details see [MN16, §2]). From now on, we call $X = X_z$ the *mirror variety* which is also a CY n-fold. It is known that $\dim(H^n_{\rm dR}(X)) = n+1$ and all Hodge numbers $h^{ij},\ i+j=n$, of X are one.

We denote by S the moduli of the pairs (X, α_1), where X is an n-dimensional mirror variety and α_1 is a holomorphic n-form on X. We know that the family of mirror varieties X_z is a one parameter family and the n-form α_1 is unique, up to multiplication by a constant, therefore $\dim(\mathsf{S}) = 2$. Analogous to the construction of X_z, let $\mathsf{X}_{t_1,t_{n+2}}$, $(t_1, t_{n+2}) \in \mathbb{C}^2 \setminus \{(t_1^{n+2} - t_{n+2})t_{n+2} = 0\}$, be the mirror variety obtained by the quotient and desingularization of the CY n-folds given by

(3.1)
$$f_{t_1,t_{n+2}}(x_0, x_1, \dots, x_{n+1}) := t_{n+2}x_0^{n+2} + x_1^{n+2} + x_2^{n+2} + \cdots + x_{n+1}^{n+2} - (n+2)t_1x_0x_1x_2\cdots x_{n+1} = 0.$$

We fix two n-forms η and ω_1 in the families X_z and $\mathsf{X}_{t_1,t_{n+1}}$, respectively, such that in the affine space $\{x_0 = 1\}$ are given as follows:

$$\eta := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_z}\ , \quad \omega_1 := \frac{dx_1 \wedge dx_2 \wedge \ldots \wedge dx_{n+1}}{df_{t_1,t_{n+2}}}\ . \tag{3.2}$$

Any element of S is in the form $(X_z, a\eta)$ where a is a non-zero constant. The pair $(X_z, a\eta)$ can be identified by $(\mathsf{X}_{t_1,t_{n+2}}, \omega_1)$ as follows:

$$(X_z, a\eta) \mapsto (\mathsf{X}_{t_1,t_{n+2}}, \omega_1)\,, \qquad (t_1, t_{n+2}) = (a^{-1}, za^{-(n+2)})\,, \tag{3.3}$$

$$(\mathsf{X}_{t_1,t_{n+2}}, \omega_1) \mapsto (X_z, t_1^{-1}\eta)\,, \qquad z = \frac{t_{n+2}}{t_1^{n+2}}\,. \tag{3.4}$$

Hence, (t_1, t_{n+2}) construct a chart for S; in the other word

$$\mathsf{S} = {\rm Spec}(\mathbb{C}[t_1, t_{n+2}, \frac{1}{(t_1^{n+2} - t_{n+2})t_{n+2}}])\,, \tag{3.5}$$

and the morphism $\mathsf{X} \to \mathsf{S}$ is the universal family of (X, α_1). Let $\nabla : H^n_{\rm dR}(\mathsf{X}/\mathsf{S}) \to \Omega^1_{\mathsf{S}} \otimes_{\mathscr{O}_{\mathsf{S}}} H^n_{\rm dR}(\mathsf{X}/\mathsf{S})$ be the Gauss-Manin connection of the two parameter family of varieties X/S. We define the n-forms $\omega_i,\ i = 1, 2, \dots, n+1$, as follows

$$\omega_i := (\nabla_{\frac{\partial}{\partial t_1}})^{i-1}(\omega_1), \tag{3.6}$$

in which $\frac{\partial}{\partial t_1}$ is considered as a vector field on the moduli space S. Then $\omega := \{\omega_1, \omega_2, \dots, \omega_{n+1}\}$ forms a basis of $H^n_{\rm dR}(X)$ which is compatible with its Hodge filtration, i.e.,

$$\omega_i \in F^{n+1-i} \setminus F^{n+2-i}, i = 1, 2, \dots, n+1, \tag{3.7}$$

where F^i is the i-th piece of the Hodge filtration of $H^n_{\rm dR}(X)$. We can write the Gauss-Manin connection of X/S in the basis ω as follow

$$\nabla\omega = \mathsf{B}\omega\,, \text{ with } \omega = \begin{pmatrix} \omega_1 & \omega_2 & \ldots & \omega_{n+1} \end{pmatrix}^{tr}. \tag{3.8}$$

If we denote by $\mathsf{B}[i, j]$ the (i, j)-th entry of the Gauss-Manin connection matrix B, then we obtain:

$$\mathsf{B}[i,i]=-\frac{i}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1\leq i\leq n\,, \tag{3.9}$$

$$\mathsf{B}[i,i+1]=dt_1-\frac{t_1}{(n+2)t_{n+2}}dt_{n+2}\,,\ \ 1\leq i\leq n\,, \tag{3.10}$$

$$\mathsf{B}[n+1,j]=\frac{-S_2(n+2,j)t_1^j}{t_1^{n+2}-t_{n+2}}dt_1+\frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,,\ \ 1\leq j\leq n\,, \tag{3.11}$$

$$\mathsf{B}[n+1,n+1]=\frac{-S_2(n+2,n+1)t_1^{n+1}}{t_1^{n+2}-t_{n+2}}dt_1+\frac{\frac{n(n+1)}{2}t_1^{n+2}+(n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2}-t_{n+2})}dt_{n+2}\,, \tag{3.12}$$

where $S_2(r,s)$ is the Stirling number of the second kind defined by

$$S_2\left(r,s\right)\,:=\frac{1}{s!}\sum_{i=0}^{s}\left(-1\right)^i\left(\begin{array}{c}s\\ i\end{array}\right)\left(s-i\right)^r\,, \tag{3.13}$$

and the rest of the entries of B are zero. For any $\xi_1,\xi_2\in H^n_{\rm dR}(X)$, in the context of the de Rham cohomology, the *intersection form* of ξ_1 and ξ_2, denoted by $\langle\xi_1,\xi_2\rangle$, is given as

$$\langle\xi_1,\xi_2\rangle:=\frac{1}{(2\pi i)^n}\int_X\xi_1\wedge\xi_2\,,$$

which is a non-degenerate $(-1)^n$-symmetric form. We obtain

$$\langle\omega_i,\omega_j\rangle=0,\ \text{ if } i+j\leq n+1\,, \tag{3.14}$$

$$\langle\omega_1,\omega_{n+1}\rangle=(-(n+2))^n\frac{c_n}{t_1^{n+2}-t_{n+2}},\ \text{ where } c_n \text{ is a constant}\,, \tag{3.15}$$

$$\langle\omega_j,\omega_{n+2-j}\rangle=(-1)^{j-1}\langle\omega_1,\omega_{n+1}\rangle,\ \text{ for } j=1,2,\ldots,n+1\,. \tag{3.16}$$

On account of these relations, we can determine all the rest of $\langle\omega_i,\omega_j\rangle$'s in a unique way. If we set $\Omega=\Omega_n:=(\langle\omega_i,\omega_j\rangle)_{1\leq i,j\leq n+1}$ to be the intersection form matrix in the basis ω, then we have

$$d\Omega=\mathsf{B}\Omega+\Omega\mathsf{B}^{\mathsf{tr}}. \tag{3.17}$$

For any positive integer n by *enhanced moduli space* $\mathsf{T}=\mathsf{T}_n$ we mean the moduli of the pairs $(X,[\alpha_1,\cdots,\alpha_n,\alpha_{n+1}])$, where X is an n-dimensional mirror variety and $\{\alpha_1,\alpha_2,\ldots,\alpha_{n+1}\}$ constructs a basis of $H^n_{\rm dR}(X)$ satisfying the properties

$$\alpha_i\in F^{n+1-i}\setminus F^{n+2-i},\ \ i=1,\cdots,n,n+1, \tag{3.18}$$

and

$$[\langle\alpha_i,\alpha_j\rangle]_{1\leq i,j\leq n+1}=\Phi_n. \tag{3.19}$$

Here $\Phi=\Phi_n$ is the following constant $(n+1)\times(n+1)$ matrix:

$$\Phi_n:=\left(\begin{array}{cc}0_m & J_m\\ -J_m & 0_m\end{array}\right)\text{ if } n \text{ is odd, and } \Phi_n:=J_{n+1} \text{ if } n \text{ is even,} \tag{3.20}$$

where by $0_k, k \in \mathbb{N}$, we mean a $k \times k$ block of zeros, $J_1 = 1$ and

$$J_k := \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}, \text{ for } k > 1. \tag{3.21}$$

In [MN16] the universal family $\pi : \mathsf{X} \to \mathsf{T}$ together with the global sections $\alpha_i, \quad i = 1, \cdots, n+1$, of the relative algebraic de Rham cohomology $H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{T})$ was constructed, and in its main theorem we observed that:

Theorem 3.1. *There exist a unique vector field* $\mathsf{R} = \mathsf{R}_n \in \mathfrak{X}(\mathsf{T})$*, and unique regular functions* $\mathsf{Y}_i \in \mathscr{O}_\mathsf{T},\ 1 \le i \le n-2$*, such that:*

$$\nabla_{\mathsf{R}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \mathsf{Y}_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \mathsf{Y}_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}}_{\mathsf{Y}} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_n \\ \alpha_{n+1} \end{pmatrix}, \tag{3.22}$$

and $\mathsf{Y}\Phi + \Phi\mathsf{Y}^{\mathsf{tr}} = 0$*.*

Here $\mathscr{O}_\mathsf{T}$ refers to the $\mathbb{C}$-algebra of regular functions on T, and ∇_R stands for the algebraic Gauss-Manin connection

$$\nabla : H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}) \to \Omega^1_\mathsf{T} \otimes_{\mathscr{O}_\mathsf{T}} H^n_{\mathrm{dR}}(\mathsf{X}/\mathsf{T}),$$

composed with the vector field $\mathsf{R} \in \mathfrak{X}(\mathsf{T})$, in which Ω^1_T is hired for the $\mathscr{O}_\mathsf{T}$-module of differential 1-forms on T. We call R as *modular vector field* attached to Dwork family. Moreover, we found that:

$$\mathsf{d} = \mathsf{d}_n := \dim(\mathsf{T}) = \begin{cases} \frac{(n+1)(n+3)}{4} + 1, & \text{if } n \text{ is odd} \\ \frac{n(n+2)}{4} + 1, & \text{if } n \text{ is even} \end{cases}. \tag{3.23}$$

The above theorem is the key tool of GMCD. In the GMCD viewpoint, the vector field Ra given in (2.1), up to multiplying the coordinates by constants $(t_1, t_2, t_3) = (12t_1, 12t_2, \frac{12^3}{8}t_3)$, is the unique vector field that satisfies

$$\nabla_{\mathsf{Ra}}\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \alpha \ , \tag{3.24}$$

where $\alpha = (\ \alpha_1 \quad \alpha_2\ )^{\mathsf{tr}}$ and ∇ is the Gauss-Manin connection of the universal family of elliptic curves

$$y^2 = 4(x - t_1)^3 - t_2(x - t_1) - t_3\ , \quad \alpha_1 = [\frac{dx}{y}],\ \alpha_2 = [\frac{xdx}{y}], \text{ with } 27t_3^2 - t_2^3 \neq 0\ . \tag{3.25}$$

We can generalize the notion of Ramanujan-Serre derivation (2.5) and Rankin-Cohen bracket (2.6) for the modular vector fields $\mathsf{R} = \mathsf{R}_n$ using an analogous procedure explained for the Ramanujan vector field Ra, which will be treated in Section 4.

Next we are going to present a chart for the moduli space T. In order to do this, let $S = \begin{pmatrix} s_{ij} \end{pmatrix}_{1\leq i,j\leq n+1}$ be a lower triangular matrix, whose entries are indeterminates $s_{ij},\ \ i \geq j$ and $s_{11} = 1$. We define

$$\underbrace{\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_{n+1} \end{pmatrix}^{tr}}_{\alpha} = S \underbrace{\begin{pmatrix} \omega_1 & \omega_2 & \dots & \omega_{n+1} \end{pmatrix}^{tr}}_{\omega},$$

which implies that α forms a basis of $H^n_{\rm dR}(X)$ compatible with its Hodge filtration. We would like that $(X, [\alpha_1, \alpha_2, \dots, \alpha_{n+1}])$ be a member of T, hence it has to satisfy $\begin{pmatrix} \langle \alpha_i, \alpha_j \rangle \end{pmatrix}_{1\leq i,j\leq n+1} = \Phi$, from what we get the following equation

$$S\Omega S^{\mathsf{tr}} = \Phi. \tag{3.26}$$

Using this equation we can express $d_0 := \frac{(n+2)(n+1)}{2} - \mathsf{d} - 2$ numbers of parameters s_{ij}'s in terms of other $\mathsf{d} - 2$ parameters that we fix them as *independent parameters*. For simplicity we write the first class of parameters as $\check{t}_1, \check{t}_2, \cdots, \check{t}_{d_0}$ and the second class as $t_2, t_3, \dots, t_{n+1}, t_{n+3}, \dots, t_{\mathsf{d}}$. We put the independent parameters t_i inside S according to the following rule which is not canonical: t_i's are written in S from left to right and top to bottom in the entries (i, j) for $i + j < n + 2$ if n is even and $i + j \leq n + 2$ if n is odd. The position of $\check{t}_i$'s inside S can be chosen arbitrarily. For instance, for $n = 1, 2, 3, 4, 5$ we have:

$$\begin{pmatrix} 1 & 0 \\ t_2 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ t_2 & \check{t}_2 & 0 \\ \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 \\ t_4 & t_6 & \check{t}_2 & 0 \\ t_7 & \check{t}_4 & \check{t}_3 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 \\ t_4 & t_5 & \check{t}_3 & 0 & 0 \\ t_7 & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ t_2 & t_3 & 0 & 0 & 0 & 0 \\ t_4 & t_5 & t_6 & 0 & 0 & 0 \\ t_8 & t_9 & t_{10} & \check{t}_3 & 0 & 0 \\ t_{11} & t_{12} & \check{t}_7 & \check{t}_5 & \check{t}_2 & 0 \\ t_{13} & \check{t}_9 & \check{t}_8 & \check{t}_6 & \check{t}_4 & \check{t}_1 \end{pmatrix}.$$

Note that we have already used t_1, t_{n+2} as coordinate system of S. In particular we find:

$$s_{(n+2-i)(n+2-i)} = \frac{(-1)^{n+i+1}}{c_n(n+2)^n} \frac{t_1^{n+2} - t_{n+2}}{s_{ii}},\ \ 1 \leq i \leq m\,. \tag{3.27}$$

In this way, $\mathsf{t} := (t_1, t_2, \dots, t_d)$ forms a chart for the enhanced moduli space T, and in fact

$$\mathsf{T} = {\rm Spec}(\mathbb{C}[t_1, t_2, \dots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}])\,, \tag{3.28}$$

$$\mathscr{O}_{\mathsf{T}} = \mathbb{C}[t_1, t_2, \dots, t_{\mathsf{d}}, \frac{1}{t_{n+2}(t_{n+2} - t_1^{n+2})\check{t}}]\,. \tag{3.29}$$

Here, $\check{t}$ is the product of $m - 1$ independent parameters which are located in the main diagonal of S. From now on, we alternately use either s_{ij}'s, or t_i's and $\check{t}_j$'s to refer the entries of S. If we denote by A the Gauss-Manin connection matrix of the family X/T written in the basis α, i.e., $\nabla\alpha = \mathsf{A}\alpha$, then we calculate A as follow:

$$\mathsf{A} = (dS + S \cdot \mathsf{B})\ S^{-1}\,. \tag{3.30}$$

If for any vector field $\mathsf{E} \in \mathfrak{X}(\mathsf{T})$ we define the *Gauss-Manin connection matrix* attached to E as $(n + 1) \times (n + 1)$ matrix A_{E} given by:

$$\nabla_{\mathsf{E}}\alpha = \mathsf{A}_{\mathsf{E}}\alpha, \tag{3.31}$$

then from (3.30) we obtain:

$$\dot{S}_{\mathsf{E}} = \mathsf{A}_{\mathsf{E}} S - S\,\mathsf{B}(\mathsf{E})\,, \tag{3.32}$$

where $\dot{S}_{\mathsf{E}} = dS(\mathsf{E})$ and $\dot{x} := dx(\mathsf{E})$ is the derivation of the function x along the vector field E in T. Note that equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (3.32) give us respectively $\dot{t}_1$ and $\dot{t}_{n+2}$, and any $\dot{t}_i$, $1 \leq i \leq \mathsf{d}$, $i \neq 1, n+2$, corresponds to only one $\dot{s}_{jk}$, $1 \leq j,k \leq n+1$. In the following remarks we recall some useful results deduced from the proof of Theorem 3.1 in [MN16, §7].

Remark 3.1. *We obtain the functions* Y_i*'s given in* (3.22) *as follows: if n is odd, then*

$$\mathsf{Y}_i = -\mathsf{Y}_{n-(i+1)} = \frac{s_{22}\, s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, \frac{n-3}{2}\,, \tag{3.33}$$

$$\mathsf{Y}_{\frac{n-1}{2}} = (-1)^{\frac{3n+3}{2}} c_n (n+2)^n \frac{s_{22}\, s^2_{\frac{n+1}{2}\frac{n+1}{2}}}{t_1^{n+2} - t_{n+2}}\,, \tag{3.34}$$

and if n is even, then

$$\mathsf{Y}_i = -\mathsf{Y}_{n-(i+1)} = \frac{s_{22}\, s_{(i+1)(i+1)}}{s_{(i+2)(i+2)}}, \quad i = 1, 2, \ldots, \frac{n-2}{2}\,. \tag{3.35}$$

Remark 3.2. *Let* $\mathsf{E} \in \mathfrak{X}(\mathsf{T})$*. If* $\nabla_{\mathsf{E}}\alpha = 0$ *for any* $(X, [\alpha_1, \alpha_2, \ldots, \alpha_{n+1}]) \in \mathsf{T}$*, then* $\mathsf{E} = 0$*.*

3.2 AMSY-Lie algebra and $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra

In [Nik20] we observed that for any positive integer n the algebraic group:

$$\mathsf{G} = \mathsf{G}_n := \{\mathsf{g} \in \mathrm{GL}(n+1, \mathbb{C}) \mid \mathsf{g} \text{ is upper triangular and } \mathsf{g}^{\mathsf{tr}} \Phi \mathsf{g} = \Phi\}, \tag{3.36}$$

acts on the enhanced moduli space T from the right, and its Lie algebra:

$$\mathrm{Lie}(\mathsf{G}) = \{\mathfrak{g} \in \mathrm{Mat}(n+1, \mathbb{C}) \mid \mathfrak{g} \text{ is upper triangular and } \mathfrak{g}^{\mathsf{tr}} \Phi + \Phi \mathfrak{g} = 0\}\,, \tag{3.37}$$

is a $\mathsf{d} - 1$ dimensional Lie algebra with the canonical basis consisting of $\mathfrak{g}_{\mathfrak{a}\mathfrak{b}}$'s, $1 \leq \mathfrak{a} \leq m$, $\mathfrak{a} \leq \mathfrak{b} \leq 2m + 1 - \mathfrak{a}$, given as follows: if n is odd, then

$$\mathfrak{g}_{\mathfrak{a}\mathfrak{b}} = (g_{kl})_{(n+1)\times(n+1)}, \text{ where } \begin{cases} g_{\mathfrak{a}\mathfrak{b}} = 1, \; g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = -1, \text{when } \mathfrak{b} \leq m, \\ g_{\mathfrak{a}\mathfrak{b}} = g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = 1, \text{when } \mathfrak{b} \geq m+1, \\ \text{and the rest of the entries of } \mathfrak{g}_{\mathfrak{a}\mathfrak{b}} \text{are zero.} \end{cases} \tag{3.38}$$

and if n is even, then:

$$\mathfrak{g}_{\mathfrak{a}\mathfrak{b}} = (g_{kl})_{(n+1)\times(n+1)}, \text{ such that } \begin{cases} g_{\mathfrak{a}\mathfrak{b}} = 1, \; g_{(n+2-\mathfrak{b})(n+2-\mathfrak{a})} = -1, \\ \text{and the rest of the entries of } \mathfrak{g}_{\mathfrak{a}\mathfrak{b}} \text{ are zero.} \end{cases} \tag{3.39}$$

The following theorem was proved in [Nik20].

Theorem 3.2. *For any* $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$*, there exists a unique vector field* $\mathsf{R}_{\mathfrak{g}} \in \mathfrak{X}(\mathsf{T})$ *such that:*

$$\mathsf{A}_{\mathsf{R}_{\mathfrak{g}}} = \mathfrak{g}^{\mathsf{tr}}, \tag{3.40}$$

i.e., $\nabla_{\mathsf{R}_{\mathfrak{g}}}\alpha = \mathfrak{g}^{\mathsf{tr}}\alpha$*.*

This theorem yields that the Lie algebra generated by $\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}}$'s, $1 \leq \mathfrak{a} \leq m$, $\mathfrak{a} \leq \mathfrak{b} \leq 2m+1-\mathfrak{a}$, in $\mathfrak{X}(\mathsf{T})$ with the Lie bracket of the vector fields is isomorphic to $\mathrm{Lie}(\mathsf{G})$ with the Lie bracket of the matrices. Hence, we use $\mathrm{Lie}(\mathsf{G})$ alternately either as a Lie subalgebra of $\mathfrak{X}(\mathsf{T})$ or as a Lie subalgebra of $\mathrm{Mat}(n+1,\mathbb{C})$.

By *AMSY-Lie algebra* $\mathfrak{G}$ we mean the $\mathscr{O}_{\mathsf{T}}$-module generated by $\mathrm{Lie}(\mathsf{G})$ and the modular vector field R in $\mathfrak{X}(\mathsf{T})$. In what follows, δ_j^k denotes the Kronecker delta, $\varrho(n)=1$ if n is an odd integer, and $\varrho(n)=0$ if n is an even integer, Y_j's, $1\leq j\leq n-2$, are the functions given in Theorem 3.1, and besides them we let $\mathsf{Y}_0=-\mathsf{Y}_{n-1}:=1$. The following theorem determines the Lie bracket of $\mathfrak{G}$, which was demonstrated in [Nik20].

Theorem 3.3. *Followings hold:*

(3.41) $[\mathsf{R},\mathsf{R}_{\mathfrak{g}_{11}}]=\mathsf{R}$,

(3.42) $[\mathsf{R},\mathsf{R}_{\mathfrak{g}_{22}}]=-\mathsf{R}$,

(3.43) $[\mathsf{R},\mathsf{R}_{\mathfrak{g}_{\mathfrak{aa}}}]=0,\ 3\leq\mathfrak{a}\leq m\,,$

(3.44) $[\mathsf{R},\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}}]=\Psi_1^{\mathfrak{ab}}(\mathsf{Y})\,\mathsf{R}_{\mathfrak{g}_{(\mathfrak{a}+1)\mathfrak{b}}}+\Psi_2^{\mathfrak{ab}}(\mathsf{Y})\,\mathsf{R}_{\mathfrak{g}_{\mathfrak{a}(\mathfrak{b}-1)}},\ 1\leq\mathfrak{a}\leq m,\ \mathfrak{a}+1\leq\mathfrak{b}\leq 2m+1-\mathfrak{a}\,,$

where

$$\Psi_1^{\mathfrak{ab}}(\mathsf{Y}):=(1+\varrho(n)\delta_{\mathfrak{a}+\mathfrak{b}}^{2m}-\delta_{\mathfrak{a}+\mathfrak{b}}^{2m+1})\,\mathsf{Y}_{\mathfrak{a}-1}, \tag{3.45}$$

$$\Psi_2^{\mathfrak{ab}}(\mathsf{Y}):=(1-2\varrho(n)\delta_{\mathfrak{b}}^{m+1})\,\mathsf{Y}_{n+1-\mathfrak{b}}\,. \tag{3.46}$$

If $n=1,2$, then we see that $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. In general, for $n\geq 3$ we have a copy of $\mathfrak{sl}_2(\mathbb{C})$ as a Lie subalgebra of $\mathfrak{G}$ which contains the modular vector field R and we state it in the following theorem from Ref. [Nik20].

Theorem 3.4. *Let us define the vector fields* H *and* F *as follows:*

1. *if* $n=1$, *then* $\mathsf{H}:=-\mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F}:=\mathsf{R}_{\mathfrak{g}_{12}}$,
2. *if* $n=2$, *then* $\mathsf{H}:=-2\mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F}:=2\mathsf{R}_{\mathfrak{g}_{12}}$,
3. *if* $n\geq 3$, *then* $\mathsf{H}:=\mathsf{R}_{\mathfrak{g}_{22}}-\mathsf{R}_{\mathfrak{g}_{11}}$ *and* $\mathsf{F}:=\mathsf{R}_{\mathfrak{g}_{12}}$.

Then the Lie algebra generated by the vector fields $\mathsf{R},\mathsf{H},\mathsf{F}$ *in* $\mathfrak{G}\subset\mathfrak{X}(\mathsf{T})$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$*; indeed we get:*

$$[\mathsf{R},\mathsf{F}]=\mathsf{H}\ ,\quad [\mathsf{H},\mathsf{R}]=2\mathsf{R}\ ,\quad [\mathsf{H},\mathsf{F}]=-2\mathsf{F}\,.$$

According to Theorem 3.4, if $n=1,2$, then $\mathfrak{G}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ (see Example 4.1), and for $n\geq 3$ the Lie subalgebra of $\mathfrak{G}$ generated by R, $\mathsf{H}:=\mathsf{R}_{\mathfrak{g}_{22}}-\mathsf{R}_{\mathfrak{g}_{11}}$ and $\mathsf{F}:=\mathsf{R}_{\mathfrak{g}_{12}}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Employing the equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (3.32) for the vector fields $\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}}$'s we obtain the diagonal matrix $\mathsf{B}(\mathsf{R}_{\mathfrak{g}_{11}})=\mathrm{diag}(1,2,\ldots,n+1)$ and the null matrices $\mathsf{B}(\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}})=0$, for $1\leq\mathfrak{a}\leq m$, $\mathfrak{a}\leq\mathfrak{b}\leq 2m+1-\mathfrak{a}$, $\mathfrak{b}\neq 1$ (see [Nik20, § 4.4]). Due to these facts and again (3.32), we can find $\dot{S}_{\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}}}$'s, and consequently we obtain $\mathsf{R}_{\mathfrak{g}_{\mathfrak{ab}}}$'s. In particular, knowing that $\dot{S}_{\mathsf{H}}=\dot{S}_{\mathsf{R}_{\mathfrak{g}_{22}}}-\dot{S}_{\mathsf{R}_{\mathfrak{g}_{11}}}$, we get $dt_1(\mathsf{H})=t_1$, $dt_{n+2}(\mathsf{H})=(n+2)t_{n+2}$, and hence

$$\dot{S}_{\mathsf{H}}=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ 2s_{21} & 3s_{22} & 0 & 0 & \cdots & 0 & 0 & 0\\ s_{31} & 2s_{32} & 3s_{33} & 0 & \cdots & 0 & 0 & 0\\ s_{41} & 2s_{42} & 3s_{43} & 4s_{44} & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ s_{(n-1)1} & 2s_{(n-1)2} & 3s_{(n-1)3} & 4s_{(n-1)4} & \cdots & (n-1)s_{(n-1)(n-1)} & 0 & 0\\ 0 & s_{n2} & 2s_{n3} & 3s_{n4} & \cdots & (n-2)s_{n(n-1)} & (n-1)s_{nn} & 0\\ 2s_{(n+1)1} & 3s_{(n+1)2} & 4s_{(n+1)3} & 5s_{(n+1)4} & \cdots & ns_{(n+1)(n-1)} & (n+1)s_{(n+1)n} & (n+2)s_{(n+1)(n+1)} \end{pmatrix}. \tag{3.47}$$

Thus, for an even integer $n \geq 5$ we get:

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + \sum_{\substack{i=4\\ i\neq n+2}}^{\mathsf{d}-1} w_i t_i \frac{\partial}{\partial t_i} + (n+2)t_{n+2}\frac{\partial}{\partial t_{n+2}} + \frac{n+2}{2}t_{\mathsf{d}+1}\frac{\partial}{\partial t_{\mathsf{d}+1}}\,, \tag{3.48}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2}\,, \tag{3.49}$$

with $t_{\mathsf{d}+1}^2 = s^2_{\frac{n+2}{2}\frac{n+2}{2}} = \frac{(-1)^{\frac{n}{2}}}{c_n(n+2)^n}(t_1^{n+2} - t_{n+2})$ (see (3.27)), and for an odd integer $n \geq 5$ we obtain:

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + \sum_{\substack{i=4\\ i\neq n+2}}^{\mathsf{d}-3} w_i t_i \frac{\partial}{\partial t_i} + (n+2)t_{n+2}\frac{\partial}{\partial t_{n+2}} + t_{\mathsf{d}-1}\frac{\partial}{\partial t_{\mathsf{d}-1}} + 2t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}\,, \tag{3.50}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2} - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}\,. \tag{3.51}$$

In the both equations (3.48) and (3.50) we have $w_i = k$ if $t_i = s_{jk}$ for some $1 \leq j,k \leq n+1$, i.e., w_i is the number of the column of the entry t_i. Note that H and F have been computed explicitly for $n = 1,2,3,4$ in Example 4.1, which are similar to the H and F founded above for the cases $n \geq 5$. Hence, in general we can write H as:

$$\mathsf{H} = \sum_{i=1}^{\mathsf{d}} w_i t_i \frac{\partial}{\partial t_i}\,, \tag{3.52}$$

where w_i's are non-negative integers.

Remark 3.3. *1. If $n = 1$, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$.*

2. If $n = 2$, then $w_1 = 2$, $w_2 = 2$, $w_4 = 8$.

3. If $n = 3$, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_4 = 0$, $w_5 = 5$, $w_6 = 1$, $w_7 = 2$.

4. If $n \geq 4$ is an even integer, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_{n+2} = n+2$, $w_{\mathsf{d}} = 0$.

5. If $n \geq 5$ is an odd integer, then $w_1 = 1$, $w_2 = 2$, $w_3 = 3$, $w_{n+2} = n+2$, $w_{\mathsf{d}-2} = 0$, $w_{\mathsf{d}-1} = 1$, $w_{\mathsf{d}} = 2$.

3.3 R as a quasi-homogeneous vector field

Let us attach to any t_i in $\mathscr{O}_{\mathsf{T}}$ the weight $\deg(t_i) = w_i$, in which the non-negative integers w_i's are given in (3.52). Recall that a vector field $\mathsf{E} = \sum_{j=1}^{\mathsf{d}} \mathsf{E}^j \frac{\partial}{\partial t_j} \in \mathfrak{X}(\mathsf{T})$, with $\mathsf{E}^j \in \mathscr{O}_{\mathsf{T}}$, is said to be *quasi-homogeneous of degree d* if for any $1 \leq j \leq \mathsf{d}$ we have $\deg(\mathsf{E}^j) = w_j + d$. Hence, on account of (3.48), (3.49), (3.50), (3.51) and Remark 3.3 the vector fields H and F are quasi-homogeneous of degree 0 and -2, respectively. The vector field H is also known as the radial vector field. Moreover, in the following proposition we show that R is a quasi-homogeneous vector field as well.

Proposition 3.1. *The modular vector field R is a quasi-homogeneous vector field of degree 2 on T.*

Proof. Due to Example 4.1 the affirmation is valid for $n = 1, 2, 3, 4$. Hence we suppose that $n \geq 5$. First note that in the proof of Theorem 3.2 (see [Nik20, § 4.1]) it is verified that the equations $S\Omega S^{\mathsf{tr}} = \Phi$ and $\dot{S}_{\mathfrak{g}} = \mathsf{A}_{\mathfrak{g}} S - S\,\mathsf{B}(\mathfrak{g})$ are compatible for any $\mathfrak{g} \in \mathrm{Lie}(\mathsf{G})$. In particular, it holds for $\mathfrak{g} = \mathsf{H}$. This implies that the degree of any entry s_{jk} of S, $2 \leq j \leq n+1$, $1 \leq k \leq j$, is equal to the integer multiple of s_{jk} in the matrix $\dot{S}_{\mathsf{H}}$, which is stated in (3.47). If we set $\mathsf{R} = \sum_{i=1}^{\mathsf{d}} \dot{t}_i \frac{\partial}{\partial t_i}$, then $\dot{t}_i$'s follow from

$$\dot{S}_{\mathsf{R}} = \mathsf{Y} S - S\,\mathsf{B}(\mathsf{R})\,. \tag{3.53}$$

More precisely, from the equalities corresponding to $(1,1)$-th and $(1,2)$-th entries of (3.53) we obtain:

$$\dot{t}_1 = s_{22} - t_1 s_{12} \;\;\&\;\; \dot{t}_{n+2} = -(n+2) s_{21} t_{n+2}\,. \tag{3.54}$$

These equalities and (3.9)-(3.12) imply:

$$\left(-\frac{k}{(n+2)t_{n+2}} dt_{n+2}\right)(\mathsf{R}) = k s_{21},\;\; 1 \leq k \leq n\,,$$

$$\left(dt_1 - \frac{t_1}{(n+2)t_{n+2}} dt_{n+2}\right)(\mathsf{R}) = s_{22}\,,$$

$$\left(\frac{-S_2(n+2,j)t_1^j}{t_1^{n+2} - t_{n+2}} dt_1 + \frac{S_2(n+2,j)t_1^{j+1}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})} dt_{n+2}\right)(\mathsf{R}) = \frac{-S_2(n+2,j)t_1^j s_{22}}{t_1^{n+2} - t_{n+2}}\,,$$

$$\left(\frac{-S_2(n+2,n+1)t_1^{n+1}}{t_1^{n+2} - t_{n+2}} dt_1 + \frac{\frac{n(n+1)}{2} t_1^{n+2} + (n+1)t_{n+2}}{(n+2)t_{n+2}(t_1^{n+2} - t_{n+2})} dt_{n+2}\right)(\mathsf{R})$$
$$= (n+1)s_{21} - \frac{(n+1)(n+2)}{2} \frac{t_1^{n+1} s_{22}}{t_1^{n+2} - t_{n+2}}\,.$$

Note that in the above last equality we used the fact that $S_2(n+2, n+1) = \frac{(n+1)(n+2)}{2}$. Therefore:

$$\mathsf{B}(\mathsf{R}) = \begin{pmatrix} s_{21} & s_{22} & 0 & 0 & 0 \\ 0 & 2s_{21} & s_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n s_{21} & s_{22} \\ \frac{-S_2(n+2,1)t_1 s_{22}}{t_1^{n+2} - t_{n+2}} & \frac{-S_2(n+2,2)t_1^2 s_{22}}{t_1^{n+2} - t_{n+2}} & \cdots & \frac{-S_2(n+2,n)t_1^n s_{22}}{t_1^{n+2} - t_{n+2}} & (n+1)s_{21} - \frac{(n+1)(n+2)}{2} \frac{t_1^{n+1} s_{22}}{t_1^{n+2} - t_{n+2}} \end{pmatrix},$$

hence, $S\,\mathsf{B}(\mathsf{R})$ equals

$$\begin{pmatrix} s_{21} & s_{22} & 0 & 0 & \cdots & 0 & 0 \\ s_{21}s_{21} & s_{21}s_{22} + 2s_{22}s_{21} & s_{22}s_{22} & 0 & \cdots & 0 & 0 \\ s_{31}s_{21} & s_{31}s_{22} + 2s_{32}s_{21} & s_{32}s_{22} + 3s_{33}s_{21} & s_{33}s_{22} & \cdots & 0 & 0 \\ s_{41}s_{21} & s_{41}s_{22} + 2s_{42}s_{21} & s_{42}s_{22} + 3s_{43}s_{21} & s_{43}s_{22} + 4s_{44}s_{21} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n1}s_{21} & s_{n1}s_{22} + 2s_{n2}s_{21} & s_{n2}s_{22} + 3s_{n3}s_{21} & s_{n3}s_{22} + 4s_{n4}s_{21} & \cdots & s_{n(n-1)}s_{22} + n s_{nn}s_{21} & s_{nn}s_{22} \\ S\mathsf{B}(\mathsf{R})[n+1,1] & S\mathsf{B}(\mathsf{R})[n+1,2] & S\mathsf{B}(\mathsf{R})[n+1,3] & S\mathsf{B}(\mathsf{R})[n+1,4] & \cdots & S\mathsf{B}(\mathsf{R})[n+1,n] & S\mathsf{B}(\mathsf{R})[n+1,n+1] \end{pmatrix}, \tag{3.55}$$

in which:

$$S\mathsf{B}(\mathsf{R})[n+1,1] = s_{(n+1)1}s_{21} - \frac{S_2(n+2,1)t_1 s_{22} s_{(n+1)(n+1)}}{t_1^{n+2} - t_{n+2}}\,,$$

$$S\mathsf{B}(\mathsf{R})[n+1,j] = s_{(n+1)(j-1)}s_{22} + j s_{(n+1)j}s_{21} - \frac{S_2(n+2,j)t_1^j s_{22} s_{(n+1)(n+1)}}{t_1^{n+2} - t_{n+2}}\,,\quad 2 \leq j \leq n\,,$$

$$S\mathsf{B}(\mathsf{R})[n+1,n+1] = s_{(n+1)n}s_{22} + s_{(n+1)(n+1)}\left((n+1)s_{21} - \frac{(n+1)(n+2)}{2} \frac{t_1^{n+1} s_{22}}{t_1^{n+2} - t_{n+2}}\right).$$

Observe that

$$
\mathsf{Y}S = \begin{pmatrix}
s_{21} & s_{22} & 0 & 0 & \cdots & 0 & 0 \\
\mathsf{Y}_1 s_{31} & \mathsf{Y}_1 s_{32} & \mathsf{Y}_1 s_{33} & 0 & \cdots & 0 & 0 \\
\mathsf{Y}_2 s_{41} & \mathsf{Y}_2 s_{42} & \mathsf{Y}_2 s_{43} & \mathsf{Y}_2 s_{44} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\mathsf{Y}_{n-2} s_{n1} & \mathsf{Y}_{n-2} s_{n2} & \mathsf{Y}_{n-2} s_{n3} & \mathsf{Y}_{n-2} s_{n4} & \cdots & \mathsf{Y}_{n-2} s_{nn} & 0 \\
-s_{(n+1)1} & -s_{(n+1)2} & -s_{(n+1)3} & -s_{(n+1)4} & \cdots & -s_{(n+1)n} & -s_{(n+1)(n+1)} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}, \tag{3.56}
$$

and (3.33)-(3.35) imply that $\deg(\mathsf{Y}_1) = \deg(\mathsf{Y}_{n-2}) = 3$ and $\deg(\mathsf{Y}_j) = 2$, $2 \leq j \leq n-3$. If we denote the (i,j)-th entry of $\dot{S}_{\mathsf{R}}$ by $\dot{S}_{\mathsf{R}}[i,j]$, then (3.53), (3.55) and (3.56) yield $\deg(\dot{S}_{\mathsf{R}}[i,j]) = \deg(s_{ij}) + 2$, $2 \leq i \leq n+1$, $1 \leq j \leq i$, which complete the proof. □

Remark 3.4. *Using the matrix $\dot{S}_{\mathsf{R}} = \mathsf{Y}S - S\,\mathsf{B}(\mathsf{R})$ computed in the proof of the above proposition we can encounter the modular vector field R explicitly for any $n \geq 5$.*

The following lemma is useful for the future use.

Lemma 3.1. *If we write*

$$
\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(t_1, t_2, \ldots, t_{\mathsf{d}}) \frac{\partial}{\partial t_j}, \ \ \text{with} \ \ \mathsf{R}^j \in \mathscr{O}_{\mathsf{T}} \,,
$$

and define

$$
\Lambda(t_1, t_2, \ldots, t_{\mathsf{d}}) := \begin{cases} -\frac{1}{2}\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) - \frac{1}{4}t_2^2, & if \ n = 2; \\ \\ -\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) - t_2^2, & if \ n \neq 2, \end{cases} \tag{3.57}
$$

then $\deg(\Lambda) = 4$ *and* $\frac{\partial \Lambda}{\partial t_2} = 0$.

Proof. For $n = 1, 2, 3, 4$ the modular vector field R has been explicitly stated in Example 4.1 and one can easily check the truth of the statement. For $n \geq 5$ the component R^2 of the modular vector field R corresponds to the $(2,1)$-th entry of the matrix $\dot{S}_{\mathsf{R}} = \mathsf{Y}S - S\,\mathsf{B}(\mathsf{R})$ computed in the proof of Proposition 3.1 that yields:

$$
\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) = \mathsf{Y}_1 t_4 - t_2^2\,, \ \ \text{(note that } t_2 = s_{21} \text{ and } t_4 = s_{31}\text{)}.
$$

From (3.33) and (3.35) we get $\mathsf{Y}_1 = \frac{s_{22}^2}{s_{33}} = \frac{t_3^2}{t_6}$, which implies:

$$
\mathsf{R}^2(t_1, t_2, \ldots, t_{\mathsf{d}}) = \frac{t_3^2 t_4}{t_6} - t_2^2\,.
$$

Hence, for $n \geq 5$ we obtain $\Lambda = -\frac{t_3^2 t_4}{t_6}$ and the proof is complete. □

3.4 The fundamental lemma

Next we state the fundamental lemma of this work, which will be used to prove Theorem 4.1. First, we recall that if we have two vector fields $V = \sum_{j=1}^{\mathsf{d}} V^j \frac{\partial}{\partial t_j}$ and $W = \sum_{j=1}^{\mathsf{d}} W^j \frac{\partial}{\partial t_j}$, then

$$
[V, W] = VW - WV = \sum_{j=1}^{\mathsf{d}} \left(V(W^j) - W(V^j) \right) \frac{\partial}{\partial t_j}\,. \tag{3.58}
$$

Lemma 3.2. (Fundamental lemma) *For any positive integer n let:*

$$\mathsf{D} := \mathsf{R} + t_2\left([\mathsf{R}, (1+\delta_2^n)\frac{\partial}{\partial t_2}] - \mathsf{H}\right). \tag{3.59}$$

Then D *is a quasi-homogeneous vector field of degree 2 in the AMSY-Lie algebra* $\mathfrak{G}$ *that satisfies:*

$$[\mathsf{D}, \frac{\partial}{\partial t_2}] = \mathsf{H}. \tag{3.60}$$

Proof. If $n = 1, 2, 3, 4$, then R, F, H are given explicitly in Example 4.1, and one can easily find that the affirmations hold. For $n \geq 5$ we divide the proof in the following two cases:

Case 1. If $n \geq 5$ is even, then on account of (3.49) we have $\mathsf{F} = \frac{\partial}{\partial t_2}$. Hence, from Theorem 3.4, which gives $[\mathsf{R}, \mathsf{F}] = \mathsf{H}$, we get $\mathsf{D} = \mathsf{R}$ and due to Proposition 3.1 the proof is complete.

Case 2. Suppose that $n \geq 5$ is odd. Then by applying (3.32) to $\mathsf{R}_{\mathfrak{g}_{1n}}$ and $\mathsf{R}_{\mathfrak{g}_{1(n+1)}}$ we obtain $\mathsf{R}_{\mathfrak{g}_{1n}} = \frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2\frac{\partial}{\partial t_{\mathsf{d}}}$ and $\mathsf{R}_{\mathfrak{g}_{1(n+1)}} = \frac{\partial}{\partial t_{\mathsf{d}}}$. Therefore, by employing (3.44) given in Theorem 3.3 we find:

$$[\mathsf{R}, \frac{\partial}{\partial t_{\mathsf{d}}}] = [\mathsf{R}, \mathsf{R}_{\mathfrak{g}_{1(n+1)}}] = \mathsf{R}_{\mathfrak{g}_{1n}} = \frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2\frac{\partial}{\partial t_{\mathsf{d}}}. \tag{3.61}$$

If we write $\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j \frac{\partial}{\partial t_j}$, then Remark 3.4 yields $\mathsf{R}^{\mathsf{d}-2} = -t_{\mathsf{d}} - t_2 t_{\mathsf{d}-2}$, from which we get:

$$\begin{aligned}[\mathsf{R}, t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] &= \mathsf{R}(t_{\mathsf{d}-2})\frac{\partial}{\partial t_{\mathsf{d}}} + t_{\mathsf{d}-2}[\mathsf{R}, \frac{\partial}{\partial t_{\mathsf{d}}}] \\ &\overset{(3.61)}{=} \mathsf{R}^{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} + t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}} \\ &= t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} - t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}.\end{aligned} \tag{3.62}$$

Due to (3.51) we have $\frac{\partial}{\partial t_2} = \mathsf{F} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}$, hence

$$\begin{aligned}\mathsf{D} = \mathsf{R} + t_2\left([\mathsf{R}, \frac{\partial}{\partial t_2}] - \mathsf{H}\right) &= \mathsf{R} + t_2\left([\mathsf{R}, \mathsf{F} + t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] - \mathsf{H}\right) \\ &= \mathsf{R} + t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} - t_2 t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}.\end{aligned} \tag{3.63}$$

Note that in the last equality of the above equation we used (3.62) and the fact that $[\mathsf{R}, \mathsf{F}] = \mathsf{H}$. Thus,

$$\begin{aligned}[\mathsf{D}, \frac{\partial}{\partial t_2}] &= [\mathsf{R}, \frac{\partial}{\partial t_2}] + [t_2 t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}}, \frac{\partial}{\partial t_2}] - [t_2 t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}, \frac{\partial}{\partial t_2}] \\ &= [\mathsf{R}, \frac{\partial}{\partial t_2}] - \frac{\partial}{\partial t_2}(t_2 t_{\mathsf{d}-2})\frac{\partial}{\partial t_{\mathsf{d}-2}} + \frac{\partial}{\partial t_2}(t_2 t_{\mathsf{d}})\frac{\partial}{\partial t_{\mathsf{d}}} \\ &= [\mathsf{R}, \frac{\partial}{\partial t_2}] - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}} + t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}} \overset{(3.62)}{=} [\mathsf{R}, \frac{\partial}{\partial t_2}] - [\mathsf{R}, t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] \\ &= [\mathsf{R}, \frac{\partial}{\partial t_2} - t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}] \overset{(3.51)}{=} [\mathsf{R}, \mathsf{F}] = \mathsf{H}.\end{aligned}$$

We know that R is quasi-homogeneous of degree 2 and $\deg(t_2)=2$, hence (3.63) implies that D is quasi-homogeneous of degree 2. In order to get $\mathsf{D}\in\mathfrak{G}$, first observe that $\frac{\partial}{\partial t_{\mathsf{d}}}=\mathsf{R}_{\mathfrak{g}_{1(n+1)}}\in\mathfrak{G}$. Hence,

$$\frac{\partial}{\partial t_2}=\mathsf{F}+t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}}}\in\mathfrak{G}\,,$$

which yields $\mathsf{D}\in\mathfrak{G}$.

□

Corollary 3.1. *The Lie algebra generated by the vector fields* D, H *and* $\frac{\partial}{\partial t_2}$ *in the AMSY-Lie algebra* $\mathfrak{G}\subset\mathfrak{X}(\mathsf{T})$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C})$.

Proof. It suffices to show that $[\mathsf{D},\frac{\partial}{\partial t_2}]=\mathsf{H},\ [\mathsf{H},\mathsf{D}]=2\mathsf{D},\ [\mathsf{H},\frac{\partial}{\partial t_2}]=-2\frac{\partial}{\partial t_2}$. The truth of the first bracket is guaranteed by Lemma 3.2, and the last bracket follows from a simple computation after using (3.48) or (3.50) and (3.58). To demonstrate the second bracket $[\mathsf{H},\mathsf{D}]=2\mathsf{D}$, the same argument given in the proof of Lemma 3.2 works perfectly for the cases $n=1,2,3,4$ and even integers $n\geq 5$. For odd integers $n\geq 5$, we first use (3.63) to obtain:

$$[\mathsf{H},\mathsf{D}]=[\mathsf{H},\mathsf{R}]+[\mathsf{H},t_2t_{\mathsf{d}-2}\frac{\partial}{\partial t_{\mathsf{d}-2}}-t_2t_{\mathsf{d}}\frac{\partial}{\partial t_{\mathsf{d}}}].$$

Then the statement follows from the fact $[\mathsf{H},\mathsf{R}]=2\mathsf{R}$ given in Theorem 3.4 and using (3.58) for H stated in (3.50). □

4 Rankin-Cohen algebra for CY modular forms

Let us suppose that $\mathsf{t}_1,\mathsf{t}_2,\ldots,\mathsf{t}_{\mathsf{d}}$ present a solution of the modular vector field R, where each of which might have a q-expansion (this was proven for $n=1,2,3,4$ in [Mov15, MN16]). The reader should take care to differ the notations $\mathsf{t}_1,\mathsf{t}_2,\ldots,\mathsf{t}_{\mathsf{d}}$ used for a solutions of R from the notations $t_1,t_2,\ldots,t_{\mathsf{d}}$ which are used for the coordinate charts of T. Nevertheless, any solution component t_i is associated with the coordinate t_i. We define the *space of CY modular forms* $\mathscr{M}$ and the *space of 2CY modular forms* $\mathscr{M}^2$, respectively, as follows:

$$\mathscr{M}:=\mathbb{C}[\mathsf{t}_1,\mathsf{t}_2,\mathsf{t}_3,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]\,, \tag{4.1}$$

$$\mathscr{M}^2:=\mathbb{C}[\mathsf{t}_1,\mathsf{t}_3,\mathsf{t}_4,\ldots,\mathsf{t}_{\mathsf{d}},\frac{1}{\mathsf{t}_{n+2}(\mathsf{t}_{n+2}-\mathsf{t}_1^{n+2})\check{\mathsf{t}}}]\,, \tag{4.2}$$

in which $\check{\mathsf{t}}$ is associated with $\check{t}$ given in (3.28) or (3.29). Indeed, we have $\mathscr{M}=\mathscr{M}^2[\mathsf{t}_2]$ and in our generalization the CY modular form t_2 has the role of the quasi-modular form E_2 in the theory of (quasi-)modular forms. Remember that we call elements of $\mathscr{M}^2$ the *2CY modular forms*. Let us attach to any solution component t_i, $1\leq i\leq\mathsf{d}$, the weight $\deg(\mathsf{t}_i)=w_i$, in which the non-negative integers w_i's are given in (3.52). For any integer $r\in\mathbb{Z}$ we define $\mathscr{M}_r$ and $\mathscr{M}^2{}_r$ to be the $\mathbb{C}$-vector spaces generated by $\{\,f\in\mathscr{M}\mid\deg(f)=r\,\}$ and $\{\,f\in\mathscr{M}^2\mid\deg(f)=r\,\}$, respectively. Note that any constant in $\mathbb{C}$ is considered as a weight zero CY modular form and by convention we suppose that, for any $r\in\mathbb{Z}$, the element 0 is of weight r. Therefore, elements of $\mathscr{M}_r$ and $\mathscr{M}^2{}_r$ are CY modular forms and 2CY modular forms of weight r, respectively. In particular, t_2 is a CY modular form of weight 2, see Remark 3.3, and the other t_j's, $1 \le j \le \mathsf{d}$ and $j \neq 2$, are 2CY modular forms of weight w_j. In particular we have:

$$\mathscr{M} = \bigoplus_{r\in\mathbb{Z}} \mathscr{M}_r \quad \text{and} \quad \mathscr{M}^2 = \bigoplus_{r\in\mathbb{Z}} \mathscr{M}^2{}_r \,. \tag{4.3}$$

Thus, $\mathscr{M}$ and $\mathscr{M}^2$ are commutative and associative graded algebras on $\mathbb{C}$.

Notation 4.1. *From now on $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{F}$ refer to the differential operators on $\mathscr{M}$ obtained from the vector fields R, H and F, respectively, substituting the coordinate chart t_j, $1 \le j \le \mathsf{d}$, by the solution component t_j and $\frac{\partial}{\partial t_j}$ by the partial derivation $\frac{\partial}{\partial \mathsf{t}_j}$. For example, if $\mathsf{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(t_1,t_2,\ldots,t_d)\frac{\partial}{\partial t_j}$, with $\mathsf{R}^j(t_1,t_2,\ldots,t_d) \in \mathscr{O}_{\mathsf{T}}$, then $\mathscr{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j(\mathsf{t}_1,\mathsf{t}_2,\ldots,\mathsf{t}_d)\frac{\partial}{\partial \mathsf{t}_j}$. We consider the Lie bracket of the such obtained differential operators the same as the Lie bracket of the associated vector fields. Hence, due to Theorem 3.4 we get:*

$$[\mathscr{R},\mathscr{F}] = \mathscr{H} \ , \quad [\mathscr{H},\mathscr{R}] = 2\mathscr{R} \ , \quad [\mathscr{H},\mathscr{F}] = -2\mathscr{F} \,.$$

We recall that, for an integer d, a degree d differential operator D on $\mathscr{M}$, denoted by $D : \mathscr{M}_* \to \mathscr{M}_{*+d}$, is a differential operator that satisfies $D(\mathscr{M}_r) \subseteq \mathscr{M}_{r+d}$ for any positive integer r. Indeed, if we can write $D = \sum_{j=1}^{\mathsf{d}} D^j \frac{\partial}{\partial \mathsf{t}_j}$, with $D^j \in \mathscr{M}$, then D is of degree d provided $\deg(D^j) - w_j = d$ for any $1 \le j \le \mathsf{d}$. A degree d differential operator on $\mathscr{M}^2$ is defined analogously.

Definition 4.1. *We define the derivation $\mathscr{D}$ on $\mathscr{M}$ as the following differential operator:*

$$\mathscr{D} := \mathscr{R} + \mathsf{t}_2([\mathscr{R},(1+\delta_2^n)\frac{\partial}{\partial \mathsf{t}_2}] - \mathscr{H}) \,, \tag{4.4}$$

where δ_i^j refers to the Kronecker delta. Indeed, $\mathscr{D}$ is associated with the vector field D given in (3.59). By the Ramanujan-Serre type derivation *∂ on $\mathscr{M}$ we mean the differential operator which for any integer r and any $f \in \mathscr{M}_r$ satisfies:*

$$\partial f := \mathscr{D}f + (1 - \frac{1}{2}\delta_2^n) r \mathsf{t}_2 f \,. \tag{4.5}$$

We would like that the derivation $\mathscr{D}$ and the Ramanujan-Serre type derivation ∂ behave the same as the usual derivation (2.4) and the Ramanujan-Serre derivation (2.2) of the classical modular forms theory, respectively. In the following example we state the derivations $\mathscr{D}$ and ∂ explicitly for $n = 1,2,3,4$.

Example 4.1. *In [Nik20] we found $\mathsf{R},\mathsf{H},\mathsf{F}$ explicitly for $n = 1,2,3,4$. In these cases, we obtain the derivation $\mathscr{D}$ and the Ramanujan-Serre type derivation ∂ as follows:*

- $n = 1$.

$$\mathsf{R} = (-t_1t_2 - 9(t_1^3 - t_3))\frac{\partial}{\partial t_1} + (81t_1(t_1^3 - t_3) - t_2^2)\frac{\partial}{\partial t_2} + (-3t_2t_3)\frac{\partial}{\partial t_3} \,, \tag{4.6}$$

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3}, \tag{4.7}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2} \,. \tag{4.8}$$

By definition, the vector field (4.7) *implies* $\deg(\mathsf{t}_1)=1$, $\deg(\mathsf{t}_2)=2$ *and* $\deg(\mathsf{t}_3)=3$. *Since* $[\mathsf{R},\mathsf{F}]=\mathsf{H}$, *we observe that:*

$$\mathscr{D}=\mathscr{R}\,, \tag{4.9}$$

$$\partial=-9(\mathsf{t}_1^3-\mathsf{t}_3)\frac{\partial}{\partial \mathsf{t}_1}+(81\mathsf{t}_1(\mathsf{t}_1^3-\mathsf{t}_3)+\mathsf{t}_2^2)\frac{\partial}{\partial \mathsf{t}_2}\,. \tag{4.10}$$

If we let ∂ *acts just on* $\mathscr{M}^2$, *then we get:*

$$\partial=-9(\mathsf{t}_1^3-\mathsf{t}_3)\frac{\partial}{\partial \mathsf{t}_1}\,.$$

- $n=2$.

$$\mathsf{R}=(t_3-t_1t_2)\frac{\partial}{\partial t_1}+(2t_1^2-\frac{1}{2}t_2^2)\frac{\partial}{\partial t_2}+(-2t_2t_3+8t_1^3)\frac{\partial}{\partial t_3}+(-4t_2t_4)\frac{\partial}{\partial t_4}\,, \tag{4.11}$$

$$\mathsf{H}=2t_1\frac{\partial}{\partial t_1}+2t_2\frac{\partial}{\partial t_2}+4t_3\frac{\partial}{\partial t_3}+8t_4\frac{\partial}{\partial t_4}, \tag{4.12}$$

$$\mathsf{F}=2\frac{\partial}{\partial t_2}\,, \tag{4.13}$$

where the polynomial equation $t_3^2=4(t_1^4-t_4)$ *holds among* t_i*'s. From* (4.12) *we get* $\deg(\mathsf{t}_1)=2$, $\deg(\mathsf{t}_2)=2$, $\deg(\mathsf{t}_3)=4$ *and* $\deg(\mathsf{t}_4)=8$. *Hence, due to* (4.4) *and* (4.5) *we find:*

$$\mathscr{D}=\mathscr{R}\,, \tag{4.14}$$

$$\partial=\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1}+(2\mathsf{t}_1^2+\frac{1}{2}\mathsf{t}_2^2)\frac{\partial}{\partial \mathsf{t}_2}+8\mathsf{t}_1^3\frac{\partial}{\partial \mathsf{t}_3}\,. \tag{4.15}$$

In the case that ∂ *is considered on* $\mathscr{M}^2$ *we have:*

$$\partial=\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1}+8\mathsf{t}_1^3\frac{\partial}{\partial \mathsf{t}_3}\,.$$

- $n=3$.

$$\begin{aligned}\mathsf{R}=&(t_3-t_1t_2)\frac{\partial}{\partial t_1}+\frac{t_3^3t_4-5^4t_2^2(t_1^5-t_5)}{5^4(t_1^5-t_5)}\frac{\partial}{\partial t_2}\\&+\frac{t_3^3t_6-3\times 5^4t_2t_3(t_1^5-t_5)}{5^4(t_1^5-t_5)}\frac{\partial}{\partial t_3}+(-t_2t_4-t_7)\frac{\partial}{\partial t_4}\\&+(-5t_2t_5)\frac{\partial}{\partial t_5}+(5^5t_1^3-t_2t_6-2t_3t_4)\frac{\partial}{\partial t_6}+(-5^4t_1t_3-t_2t_7)\frac{\partial}{\partial t_7}\,,\end{aligned} \tag{4.16}$$

$$\mathsf{H}=t_1\frac{\partial}{\partial t_1}+2t_2\frac{\partial}{\partial t_2}+3t_3\frac{\partial}{\partial t_3}+5t_5\frac{\partial}{\partial t_5}+t_6\frac{\partial}{\partial t_6}+2t_7\frac{\partial}{\partial t_7}\,, \tag{4.17}$$

$$\mathsf{F}=\frac{\partial}{\partial t_2}-t_4\frac{\partial}{\partial t_7}\,. \tag{4.18}$$

We obtain $\deg(\mathsf{t}_1)=1,\ \deg(\mathsf{t}_2)=2,\ \deg(\mathsf{t}_3)=3,\ \deg(\mathsf{t}_4)=0,\ \deg(\mathsf{t}_5)=5,\ \deg(\mathsf{t}_6)=1,\ \deg(\mathsf{t}_7)=2$, *and we get* $\mathscr{D}:\mathscr{M}\to\mathscr{M}$ *as follow:*

$$\mathscr{D}=\mathscr{R}+\mathsf{t}_2\mathsf{t}_4\frac{\partial}{\partial \mathsf{t}_4}-\mathsf{t}_2\mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_7}\,. \tag{4.19}$$

If we define ∂ *on* $\mathscr{M}^2$, *then we find:*

$$\partial=\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1}+\frac{\mathsf{t}_3^3\mathsf{t}_6}{5^4(\mathsf{t}_1^5-\mathsf{t}_5)}\frac{\partial}{\partial \mathsf{t}_3}-\mathsf{t}_7\frac{\partial}{\partial \mathsf{t}_4}+(5^5\mathsf{t}_1^3-2\mathsf{t}_3\mathsf{t}_4)\frac{\partial}{\partial \mathsf{t}_6}-5^4\mathsf{t}_1\mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_7}\,. \tag{4.20}$$

- $n = 4$.

$$\mathsf{R} = (t_3 - t_1t_2)\frac{\partial}{\partial t_1} + \frac{6^{-2}t_3^2t_4t_8 - t_1^6t_2^2 + t_2^2t_6}{t_1^6 - t_6}\frac{\partial}{\partial t_2} + \frac{6^{-2}t_3^2t_5t_8 - 3t_1^6t_2t_3 + 3t_2t_3t_6}{t_1^6 - t_6}\frac{\partial}{\partial t_3} + \frac{-6^{-2}t_3^2t_7t_8 - t_1^6t_2t_4 + t_2t_4t_6}{t_1^6 - t_6}\frac{\partial}{\partial t_4} + \frac{6^{-2}t_3t_5^2t_8 - 4t_1^6t_2t_5 - 2t_1^6t_3t_4 + 5t_1^4t_3t_8 + 4t_2t_5t_6 + 2t_3t_4t_6}{2(t_1^6 - t_6)}\frac{\partial}{\partial t_5} + (-6t_2t_6)\frac{\partial}{\partial t_6} + \frac{6^{-2}t_4^2 - t_1^2}{2\times 6^{-2}}\frac{\partial}{\partial t_7} + \frac{-3t_1^6t_2t_8 + 3t_1^5t_3t_8 + 3t_2t_6t_8}{t_1^6 - t_6}\frac{\partial}{\partial t_8}\,, \tag{4.21}$$

$$\mathsf{H} = t_1\frac{\partial}{\partial t_1} + 2t_2\frac{\partial}{\partial t_2} + 3t_3\frac{\partial}{\partial t_3} + t_4\frac{\partial}{\partial t_4} + 2t_5\frac{\partial}{\partial t_5} + 6t_6\frac{\partial}{\partial t_6} + 3t_8\frac{\partial}{\partial t_8}\,, \tag{4.22}$$

$$\mathsf{F} = \frac{\partial}{\partial t_2}\,, \tag{4.23}$$

where the equation $t_8^2 = 36(t_1^6 - t_6)$ holds among t_i's. Analogous to the pervious cases we have $\deg(\mathsf{t}_1) = 1,\ \deg(\mathsf{t}_2) = 2,\ \deg(\mathsf{t}_3) = 3,\ \deg(\mathsf{t}_4) = 1,\ \deg(\mathsf{t}_5) = 2,\ \deg(\mathsf{t}_6) = 6,\ \deg(\mathsf{t}_7) = 0,\ \deg(\mathsf{t}_8) = 3$. *Due to* (4.4) *we find:*

$$\mathscr{D} = \mathscr{R} \tag{4.24}$$

and (4.5) *yields the Ramanujan-Serre type derivation on $\mathscr{M}^2$ as follow:*

$$\partial = \mathsf{t}_3\frac{\partial}{\partial \mathsf{t}_1} + \frac{6^{-2}\mathsf{t}_3^2\mathsf{t}_5t_8}{\mathsf{t}_1^6 - \mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_3} - \frac{6^{-2}\mathsf{t}_3^2\mathsf{t}_7\mathsf{t}_8}{\mathsf{t}_1^6 - \mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_4} + \frac{6^{-2}\mathsf{t}_3\mathsf{t}_5^2\mathsf{t}_8 - 2\mathsf{t}_1^6\mathsf{t}_3\mathsf{t}_4 + 5\mathsf{t}_1^4\mathsf{t}_3\mathsf{t}_8 + 2\mathsf{t}_3\mathsf{t}_4\mathsf{t}_6}{2(\mathsf{t}_1^6 - \mathsf{t}_6)}\frac{\partial}{\partial \mathsf{t}_5} + \frac{6^{-2}\mathsf{t}_4^2 - \mathsf{t}_1^2}{2\times 6^{-2}}\frac{\partial}{\partial \mathsf{t}_7} + \frac{3\mathsf{t}_1^5\mathsf{t}_3\mathsf{t}_8}{\mathsf{t}_1^6 - \mathsf{t}_6}\frac{\partial}{\partial \mathsf{t}_8}\,. \tag{4.25}$$

Remark 4.1. *1. If we look closely to all cases stated in Example 4.1 we find out that the derivation $\mathscr{D}$ and the Ramanujan-Serre type derivation ∂ are degree 2 differential operators. Besides these, the Ramanujan-Serre type derivation ∂ sends any element of $\mathscr{M}^2$ to another element of $\mathscr{M}^2$. More precisely, the same as what we mentioned for the Ramanujan-Serre derivation given in* (2.5)*, in all the above cases we observe that for any $f \in {\mathscr{M}^2}_r$ the term $(1 - \frac{1}{2}\delta_2^n)r\mathsf{t}_2 f$ in* (4.5) *kills all the terms including t_2 in $\mathscr{D}f$ which implies $\partial f \in {\mathscr{M}^2}_{r+2}$, and consequently $\mathscr{M}^2$ is closed under ∂. All these facts hold for any positive integer n which are stated in Theorem 4.1.*

2. *In Example 4.1 we stated the derivation $\mathscr{D}$ explicitly in the cases $\mathsf{n} = 1, 2, 3, 4$. For $\mathsf{n} \geq 5$, due to the proof of Lemma 3.2, we can state $\mathscr{D}$ explicitly as follows:*
   - *if $\mathsf{n} \geq 5$ is even, then $\mathscr{D} = \mathscr{R}$,*
   - *if $\mathsf{n} \geq 5$ is odd, then $\mathscr{D} = \mathscr{R} + \mathsf{t}_2\mathsf{t}_{\mathsf{d}-2}\frac{\partial}{\partial \mathsf{t}_{\mathsf{d}-2}} - \mathsf{t}_2\mathsf{t}_{\mathsf{d}}\frac{\partial}{\partial \mathsf{t}_{\mathsf{d}}}$.*

Theorem 4.1. *Followings hold.*

1. *The Rankin-Cohen derivation $\mathscr{D}$ is a degree 2 differential operator on $\mathscr{M}$, i.e.,*

$$\mathscr{D} : \mathscr{M}_* \to \mathscr{M}_{*+2}\,.$$

2. *The Ramanujan-Serre type derivation ∂ is a degree 2 differential operator on $\mathscr{M}^2$, i.e.,*

$$\partial : \mathscr{M}^2{}_* \to \mathscr{M}^2{}_{*+2} .$$

Proof. 1. Due to Lemma 3.2 the proof is straightforward, since the modular differential operator $\mathscr{D}$ is associated with the vector field D which is a quasi-homogeneous vector field of degree 2.

2. First note that on account of Remark 3.3 we always have $\deg(\mathsf{t}_2) = w_2 = 2$. Hence, from part 1 and (4.5) we deduce that ∂ is a degree 2 differential operator. To prove that for all $f \in \mathscr{M}^2$ we get $\partial f \in \mathscr{M}^2$, it is enough to observe that for all integers r:

$$\forall f \in \mathscr{M}^2{}_r : \ \partial f \in \mathscr{M}^2{}_{r+2} ,$$

which is equivalent to:

$$\begin{aligned}
\partial \mathsf{t}_j \in \mathscr{M}^2{}_{w_j+2} , \quad \forall j \neq 2, &\Leftrightarrow \frac{\partial}{\partial \mathsf{t}_2}(\partial \mathsf{t}_j) = 0 , \quad \forall j \neq 2, \\
&\Leftrightarrow \frac{\partial}{\partial \mathsf{t}_2}(\mathscr{D}\mathsf{t}_j + w_j \mathsf{t}_2 \mathsf{t}_j) = 0 , \quad \forall j \neq 2, \\
&\Leftrightarrow \frac{\partial}{\partial \mathsf{t}_2}(\mathscr{D}\mathsf{t}_j) = -w_j \mathsf{t}_j , \quad \forall j \neq 2, \\
&\Leftarrow \sum_{j=1}^{\mathsf{d}} \frac{\partial}{\partial \mathsf{t}_2}(\mathscr{D}\mathsf{t}_j)\frac{\partial}{\partial \mathsf{t}_j} = -\sum_{j=1}^{\mathsf{d}} w_j \mathsf{t}_j \frac{\partial}{\partial \mathsf{t}_j} = -\mathscr{H} , \\
&\Leftrightarrow \sum_{j=1}^{\mathsf{d}} \frac{\partial}{\partial \mathsf{t}_2}(\mathsf{D}^j)\frac{\partial}{\partial \mathsf{t}_j} = -\mathscr{H} , \quad \text{where } \mathscr{D} := \sum_{j=1}^{\mathsf{d}} \mathsf{D}^j \frac{\partial}{\partial \mathsf{t}_j} \\
&\Leftrightarrow [\frac{\partial}{\partial \mathsf{t}_2}, \mathscr{D}] = -\mathscr{H} , \\
&\Leftrightarrow [\mathscr{D}, \frac{\partial}{\partial \mathsf{t}_2}] = \mathscr{H} .
\end{aligned}$$

The last affirmation is valid due to Lemma 3.2, which completes the proof.

□

Next, to use Proposition 2.1, we need the CY modular forms of positive weights. Hence, we consider the spaces of CY modular forms $\mathscr{M}^{>0}$ and 2CY modular forms $\mathscr{M}^{2>0}$ of positive weights as follows:

$$\mathscr{M}^{>0} := \bigoplus_{r\geq 0} \mathscr{M}_r \quad , \quad \mathscr{M}^{2>0} := \bigoplus_{r\geq 0} \mathscr{M}^2{}_r , \tag{4.26}$$

in which we suppose that $\mathscr{M}_0 = \mathscr{M}^2{}_0 = \mathbb{C}$. Thus, the space of CY modular forms of positive weights $\mathscr{M}^{>0}$ is a commutative and associative graded algebra with unit over the field $\mathbb{C}$ together with the derivation $\mathscr{D} : \mathscr{M}^{>0}_* \to \mathscr{M}^{>0}_{*+2}$ of degree 2. Therefore, due to Remark 2.1, $(\mathscr{M}^{>0}, [\cdot,\cdot]_{\mathscr{D},*})$ is a standard Rankin-Cohen, and hence a Rankin-Cohen algebra. From now on, if no confusion arises, we denote the bracket $[\cdot,\cdot]_{\mathscr{D},*}$ simply by $[\cdot,\cdot]_*$ which is called the *Rankin-Cohen bracket for CY modular forms*, and for any non-negative integers k, r, s it is defined as

$$[f,g]_k := \sum_{i+j=k} (-1)^j \binom{k+r-1}{i} \binom{k+s-1}{j} f^{(j)} g^{(i)}, \ \forall f \in \mathscr{M}_r , \ \forall g \in \mathscr{M}_s , \tag{4.27}$$

where $f^{(j)} = \mathscr{D}^j f$ and $g^{(j)} = \mathscr{D}^j g$ refer to the j-th derivation of f and g under $\mathscr{D}$, respectively. It is evident that $[f,g]_k \in \mathscr{M}_{r+s+2k}$. In the next theorem we observe that the space of 2CY modular forms of positive weights $\mathscr{M}^{2>0}$ is closed under the Rankin-Cohen bracket for CY modular forms given in (4.27).

Theorem 4.2. *For all non-negative integers r, s, k and for any $f \in \mathscr{M}^2{}_r$, $g \in \mathscr{M}^2{}_s$ we have:*

$$[f,g]_k \in \mathscr{M}^2{}_{r+s+2k}\,.$$

Proof. The idea of the proof is to use Proposition 2.1 and its proof. To this end, first note that according to the part 2 of Theorem 4.1 the Ramanujan-Serre type derivation $\partial : \mathscr{M}^{2>0}_{\ *} \to \mathscr{M}^{2>0}_{\ *+2}$ is a degree 2 differential operator. If we set $\Lambda = \Lambda(\mathsf{t}_1, \mathsf{t}_2, \ldots, \mathsf{t}_\mathsf{d})$, where Λ is given in Lemma 3.1, then the same lemma yields $\Lambda \in \mathscr{M}^2{}_4$. Therefore, from Proposition 2.1 we get that $(\mathscr{M}^{2>0}, [\cdot,\cdot]_{\partial,\Lambda,*})$, where the k-th bracket $[\cdot,\cdot]_{\partial,\Lambda,k}\,,\ k \geq 0$, is given by (2.19), is a canonical Rankin-Cohen algebra. On the other hand, by letting $\lambda = (\frac{1}{2}\delta_2^n - 1)\mathsf{t}_2$, from (4.5) we obtain

$$\mathscr{D} f = \partial f + r\lambda f\,,\ \forall f \in \mathscr{M}^2{}_r\,. \tag{4.28}$$

Furthermore, if we write $\mathscr{D} = \sum_{j=1}^{\mathsf{d}} \mathsf{D}^j \frac{\partial}{\partial \mathsf{t}_j}$, with $\mathsf{D}^j \in \mathscr{M}$, then

$$\mathscr{D}(\lambda) = (\frac{1}{2}\delta_2^n - 1)\mathscr{D}(\mathsf{t}_2) = (\frac{1}{2}\delta_2^n - 1)\mathsf{D}^2\,. \tag{4.29}$$

Considering $\mathscr{R} = \sum_{j=1}^{\mathsf{d}} \mathsf{R}^j \frac{\partial}{\partial \mathsf{t}_j}$, with $\mathsf{R}^j \in \mathscr{M}$, the part 2 of Remark 4.1 yields $\mathsf{D}^2 = \mathsf{R}^2$. This fact along with (4.29) and (3.57) implies:

$$\mathscr{D}(\lambda) = \Lambda + \lambda^2\,. \tag{4.30}$$

The relations (4.28) and (4.30) show that (2.21) is satisfied. Hence, from the proof of Proposition 2.1 we obtain $[\cdot,\cdot]_{\partial,\Lambda,*} = [\cdot,\cdot]_*$ (note that $[\cdot,\cdot]_* = [\cdot,\cdot]_{\mathscr{D},*}$). Finally, since $\mathscr{M}^{2>0}$ is closed under $[\cdot,\cdot]_{\partial,\Lambda,*}$, we conclude that $\mathscr{M}^{2>0}$ is closed under $[\cdot,\cdot]_*$ and the proof is complete. □

In particular, Theorem 4.2 implies that $(\mathscr{M}^{2>0}, [\cdot,\cdot]_*)$ is a Rankin-Cohen subalgebra of $(\mathscr{M}^{>0}, [\cdot,\cdot]_*)$.

Corollary 4.1. *The Rankin-Cohen bracket for CY modular forms $[\cdot,\cdot]_*$ endows $\mathscr{M}^{2>0}$ with a Rankin-Cohen algebra structure.*

4.1 Examples of Rankin-Cohen brackets of CY modular forms

We know that the modular discriminant is given by $\Delta = \frac{1}{1728}(E_4^3 - E_6^2)$, which is related with the discriminant $t_2^3 - 27t_3^2$ of the family of elliptic curves stated in (3.25). One can easily compute (or find in [Zag94]) the following examples of Rankin-Cohen brackets (2.6) of modular forms:

$$\begin{aligned} &[E_4, E_6]_1 = -3456\Delta\,,\quad [E_4, E_6]_2 = 0\,,\quad [E_4, E_4]_2 = 4800\Delta,\\ &[E_6, E_6]_2 = -21168E_4\Delta\,,\quad [\Delta, \Delta]_2 = -13E_4\Delta^2\,. \end{aligned} \tag{4.31}$$

Note that for any (quasi-)modular form or any CY modular form f of non-negative weight r and any integer $k \geq 0$ it is evident by definition that :

$$[f, f]_{2k+1} = 0\,. \tag{4.32}$$

For any positive integer n, the discriminant of the Dwork family (3.1) is given by the polynomial $t_{n+2}(t_1^{n+2} - t_{n+2})$. Hence, in the rest of this section for any n we fix the notation $\Delta := \mathsf{t}_{n+2}(\mathsf{t}_1^{n+2} - \mathsf{t}_{n+2})$. Next, we compute a few examples of Rankin-Cohen brackets (4.27) of 2CY modular forms for $n = 1, 2, 3, 4$, which are motivated by examples given in (4.31).

- $n = 1$**.** In this case we found $\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3$ in the first list of (1.3) and we have $\Delta = \mathsf{t}_3(\mathsf{t}_1^3 - \mathsf{t}_3)$. The Rankin-Cohen brackets are calculated as follows:

$$\begin{aligned}&[\mathsf{t}_1, \mathsf{t}_3]_1 == 27\Delta\,, \quad [\mathsf{t}_1, \mathsf{t}_3]_2 = 729\mathsf{t}_1^2\Delta\,, \quad [\mathsf{t}_1, \mathsf{t}_1]_2 = 324\Delta,\\ &[\mathsf{t}_3, \mathsf{t}_3]_2 = -2916\mathsf{t}_1\Delta\,, \quad [\Delta, \Delta]_2 = -5103\mathsf{t}_1^4\Delta^2\,.\end{aligned} \tag{4.33}$$

  Before passing to the next case, we express the combinations of $\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_3$ which appeared in the right hand side of the above relations in terms of eta and theta functions that seem to us interesting. These relations are obtained thanks to [OEI64] and one can find out more about them by seeing the corresponding pages and references given there. By comparing the coefficients of t_1 with [OEI64, A004016] we find:

$$\mathsf{t}_1 = \frac{1}{3}(\theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3)), \tag{4.34}$$

  and for t_1^2 and t_1^4 the reader is referred to [OEI64, A008653] and [OEI64, A008655], respectively. After computing the q-expansion of Δ, from [OEI64, A007332] we get:

$$\Delta = \frac{1}{27}\eta^6(q)\eta^6(q^3)\,, \tag{4.35}$$

  and on account of [OEI64, A136747] we get:

$$\mathsf{t}_1^2\Delta = \frac{1}{243}\eta^6(q)\eta^4(q^3)\left(\eta^3(q) + 9\eta^3(q^9)\right)^2. \tag{4.36}$$

  The equations (4.34), (4.35) and (4.36) yield:

$$3\mathsf{t}_1 = \theta_3(q)\theta_3(q^3) + \theta_2(q)\theta_2(q^3) = \frac{\eta^3(q) + 9\eta^3(q^9)}{\eta(q^3)}. \tag{4.37}$$

- $n = 2$**.** Here $\mathsf{t}_1, \mathsf{t}_2, \mathsf{t}_4$ are stated in the second list of (1.3). We know that $\Delta = \mathsf{t}_4(\mathsf{t}_1^4 - \mathsf{t}_4)$, and we obtain:

$$\begin{aligned}&[\mathsf{t}_1, \mathsf{t}_4]_1 = -8\mathsf{t}_3\mathsf{t}_4\,, \quad [\mathsf{t}_1, \mathsf{t}_4]_2 = 192\mathsf{t}_1^3\mathsf{t}_4\,, \quad [\mathsf{t}_1, \mathsf{t}_1]_2 = 36\mathsf{t}_1^4 - 9\mathsf{t}_3^2 = 36\mathsf{t}_4,\\ &[\mathsf{t}_4, \mathsf{t}_4]_2 = -576\mathsf{t}_1^2\mathsf{t}_4^2\,, \quad [\Delta, \Delta]_2 = -1088\mathsf{t}_1^2\mathsf{t}_4(\mathsf{t}_1^4 + 8\mathsf{t}_4)\Delta\,.\end{aligned} \tag{4.38}$$

  Note that in the third bracket of (4.38) we used the fact that $\mathsf{t}_3^2 = 4(\mathsf{t}_1^4 - \mathsf{t}_4)$, which also implies:

$$[\mathsf{t}_1, \mathsf{t}_4]_1^2 = 64\mathsf{t}_3^2\mathsf{t}_4^2 = 256\mathsf{t}_4\Delta\,. \tag{4.39}$$

- $n=3$. In this case one can find the first 7 coefficients of the q-expansions of $\mathsf{t}_1,\mathsf{t}_2,\dots,\mathsf{t}_7$ in [Mov15]. We have $\Delta=\mathsf{t}_5(\mathsf{t}_1^5-\mathsf{t}_5)$, and we calculate the Rankin-Cohen brackets as follows:

$$\begin{aligned}(4.40)\qquad &[\mathsf{t}_1,\mathsf{t}_5]_1=-5\mathsf{t}_3\mathsf{t}_5\,,\quad [\mathsf{t}_1,\mathsf{t}_5]_2=\frac{-4\mathsf{t}_1\mathsf{t}_3^3\mathsf{t}_4\mathsf{t}_5+3\mathsf{t}_3^3\mathsf{t}_5\mathsf{t}_6}{125(\mathsf{t}_1^5-\mathsf{t}_5)}\,,\\ &[\mathsf{t}_1,\mathsf{t}_1]_2=\frac{-2500\mathsf{t}_3^2(\mathsf{t}_1^5-\mathsf{t}_5)-2\mathsf{t}_1\mathsf{t}_3^3(\mathsf{t}_1\mathsf{t}_4-\mathsf{t}_6)}{625(\mathsf{t}_1^5-\mathsf{t}_5)}\,,\quad [\mathsf{t}_5,\mathsf{t}_5]_2=\frac{-6\mathsf{t}_3^3\mathsf{t}_4\mathsf{t}_5^2}{25(\mathsf{t}_1^5-\mathsf{t}_5)}\,,\\ &[\Delta,\Delta]_2=\frac{\mathsf{t}_3^2\mathsf{t}_5^2}{25}(\mathsf{t}_1^3(-20625\mathsf{t}_1^5-55000\mathsf{t}_5+22\mathsf{t}_1\mathsf{t}_3\mathsf{t}_6)-44\mathsf{t}_3\mathsf{t}_4(\mathsf{t}_1^5-\mathsf{t}_5))\,.\end{aligned}$$

- $n=4$. Here, the first 7 coefficients of the q-expansions of $\mathsf{t}_1,\mathsf{t}_2,\dots,\mathsf{t}_7,\mathsf{t}_8$ are given in [MN16, Table 2]. We get $\Delta=\mathsf{t}_6(\mathsf{t}_1^6-\mathsf{t}_6)$ and hence:

$$\begin{aligned}(4.41)\qquad &[\mathsf{t}_1,\mathsf{t}_6]_1=-6\mathsf{t}_3\mathsf{t}_6\,,\quad [\mathsf{t}_1,\mathsf{t}_6]_2=\frac{-9\mathsf{t}_1\mathsf{t}_3^2\mathsf{t}_4\mathsf{t}_6\mathsf{t}_8+7\mathsf{t}_3^2\mathsf{t}_5\mathsf{t}_6\mathsf{t}_8}{12(\mathsf{t}_1^6-\mathsf{t}_6)}\,,\\ &[\mathsf{t}_1,\mathsf{t}_1]_2=\frac{-72\mathsf{t}_3^2(\mathsf{t}_1^6-\mathsf{t}_6)-\mathsf{t}_1\mathsf{t}_3^2\mathsf{t}_8(\mathsf{t}_1\mathsf{t}_4-\mathsf{t}_5)}{18(\mathsf{t}_1^6-\mathsf{t}_6)}\,,\quad [\mathsf{t}_6,\mathsf{t}_6]_2=\frac{-7\mathsf{t}_3^2\mathsf{t}_4\mathsf{t}_6^2\mathsf{t}_8}{\mathsf{t}_1^6-\mathsf{t}_6}\,,\\ &[\Delta,\Delta]_2=\mathsf{t}_3^2\mathsf{t}_6^2(\mathsf{t}_1^4(-1404\mathsf{t}_1^6-4680\mathsf{t}_6+26\mathsf{t}_1\mathsf{t}_5\mathsf{t}_8)-52\mathsf{t}_4\mathsf{t}_8(\mathsf{t}_1^6-\mathsf{t}_6))\,.\end{aligned}$$

The relations given in (3.54) yield $\mathscr{D}\mathsf{t}_1=\mathsf{t}_3-\mathsf{t}_1\mathsf{t}_2$ and $\mathscr{D}\mathsf{t}_{n+2}=-(n+2)\mathsf{t}_2\mathsf{t}_{n+2}$ for any integer $n\geq 3$, from which we conclude the following expected result (see (4.40) and (4.41)):

$$(4.42)\qquad [\mathsf{t}_1,\mathsf{t}_{n+2}]_1=-(n+2)\mathsf{t}_3\mathsf{t}_{n+2}\,,\quad \forall n\geq 3\,.$$

Another interesting point that we observe in the above examples is that in all the cases $n=1,2,3,4$ the bracket $[\Delta,\Delta]_2$ is expressed as a polynomial in terms of $\mathsf{t}_1,\mathsf{t}_2,\dots,\mathsf{t}_{\mathsf{d}}$, and we expect that this happens for higher dimensions as well.

It is also worth to point out that for any CY (quasi-)modular form f of weight r, the second Rankin-Cohen bracket $[f,f]_2$ provides a second order differential equation which is satisfied by f. More precisely, from (4.27) we obtain:

$$(4.43)\qquad [f,f]_2=6f\mathscr{D}^2f-9(\mathscr{D}f)^2,$$

which implies that f satisfies the second order ODE:

$$(4.44)\qquad 6y\mathscr{D}^2y-9(\mathscr{D}y)^2=[f,f]_2\,.$$

For example, if $n=1$, then from the third bracket of (4.33) we get that the function

$$\mathsf{t}_1=\frac{1}{3}(2\theta_3(q^2)\theta_3(q^6)-\theta_3(-q^2)\theta_3(-q^6))=\frac{1}{3}(\theta_3(q)\theta_3(q^3)+\theta_2(q)\theta_2(q^3))=\frac{\eta^3(q)+9\eta^3(q^9)}{3\eta(q^3)},$$

satisfies the following second order ODE:

$$(4.45)\qquad 2y\ddot{y}-3\dot{y}^2=4\eta^6(q)\eta^6(q^3)\,,$$

in which $\dot{y}=3q\frac{\partial y}{\partial q}=\frac{3}{2\pi i}\frac{dy}{d\tau}$.

5 Final remarks

One of weak points of Theorem 4.2 is that we are just considering the CY modular forms of positive weights. If we look closely to the definition of $\mathscr{M}$ and $\mathscr{M}^2$ given in (4.1) and (4.2), respectively, we observe that they contain non-constant elements of weight zero and elements of negative weights. For example for $n = 3$, the element $\mathsf{t}_4 \in \mathscr{M}^2$ is a non-constant element of weight zero and $\frac{1}{\mathsf{t}_5(\mathsf{t}_1^5-\mathsf{t}_5)} \in \mathscr{M}^2$ is an element of weight -10. Thus, in general it is not necessarily valid that $\mathscr{M}_0 = \mathscr{M}^2{}_0 = \mathbb{C}$; indeed, $\mathscr{M}_0$ and $\mathscr{M}^2{}_0$ are generated by $\mathbb{C} \cup \{ f \in \mathscr{M} \mid \deg(f) = 0 \}$ and $\mathbb{C} \cup \{ f \in \mathscr{M}^2 \mid \deg(f) = 0 \}$, respectively. We can consider the definition of the Rankin-Cohen bracket (4.27) for elements of negative weights as well, and one should note that if $k > 0$ is a positive integer, then for any $r \geq 0$ the binomial coefficient $\binom{-k}{r}$ is given as follow:

$$\binom{-k}{r} = (-1)^r \binom{k+r-1}{r}.$$

Thus, employing (4.27) we can endow $\mathscr{M}$ with a Rankin-Cohen algebra structure. Using the computer we observed that the Rankin-Cohen brackets of all examined 2CY modular forms of negative weights are again 2CY modular forms, in the cases $n = 1, 2, 3, 4$, but we could not prove theoretically the assertion that the space of 2CY modular forms $\mathscr{M}^2$ is closed under the Rankin-Cohen bracket (4.27). We believe to the truth of this assertion, but our main difficulty in carrying out its proof is the use of Proposition 2.1, where the weights of non-constant elements of the graded algebra are considered positive. This led us to the following conjecture.

Conjecture 1. *The proposition 2.1 holds if the graded algebra M_* also contains elements of negative weights or non-constant elements of weight zero, i.e., $M_* = \bigoplus_{k\in\mathbb{Z}} M_k$ and it is not necessary that $M_0 = \mathsf{k}.1$.*

In the above conjecture by constant elements we mean the elements of the field k. If we want to prove Conjecture 1 in an analogous way to the proof of D. Zagier given for [Zag94, Proposition 1], the unsolved part is the equality (2.22). Once we can prove the Conjecture 1, we can prove that the space of 2CY modular forms $\mathscr{M}^2$ is closed under the Rankin-Cohen brackets (4.27).

Another point which is worth to discuss is the modular vector field. As we observed in the part 2 of Remark 4.1, for even positive integers n the derivation $\mathscr{D}$ is associated with the modular vector field R, but for odd positive integers n, except for $n = 1$, $\mathscr{D}$ is not associated with R. The reason for which $\mathscr{D} \neq \mathscr{R}$, when $n \geq 3$ is an odd integer, is that if we use the differential operator $\mathscr{R}$ in the Rankin-Cohen bracket (4.27), then the space of CY modular forms $\mathscr{M}^2$ is not closed under the Rnakin-Cohen bracket. For example for $n = 3$ if we use the derivation $\mathscr{D}$, then

$$[\mathsf{t}_4, \mathsf{t}_5(\mathsf{t}_1^5 - \mathsf{t}_5)]_{\mathscr{D},1} = 10\mathsf{t}_5\mathsf{t}_7(\mathsf{t}_1^5 - \mathsf{t}_5) \in \mathscr{M}^2{}_{12},$$

but if we use the derivation $\mathscr{R}$, then

$$[\mathsf{t}_4, \mathsf{t}_5(\mathsf{t}_1^5 - \mathsf{t}_5)]_{\mathscr{R},1} = 10\mathsf{t}_5(\mathsf{t}_1^5 - \mathsf{t}_5)(\mathsf{t}_2\mathsf{t}_4 + \mathsf{t}_7) \notin \mathscr{M}^2{}_{12},$$

since in the right hand side of the above equality appears t_2 which is not a 2CY modular form. This fact leads us to think that we may change the definition of the modular vector field from being the unique vector field which satisfies the equation (3.22) to being the vector field that induces a Rankin-Cohen bracket under which the space of CY modular forms is closed. Hence, in this manuscript we may consider D as modular vector field which equals to R for even integers n and $n = 1$, and differs from R for odd integers $n \geq 3$. Furthermore, in Corollary 3.1 we observed that the Lie algebra generated by D, the radial vector field H and the constant vector field $\frac{\partial}{\partial t_2}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Since the vector field H stays unchanged, the weights w_j's remain the same. We should mention that one of the disadvantages of the vector field D in comparison with the vector field R is that the definition of D depends to the chosen chart $(t_1, t_2, \ldots, t_{\mathsf{d}})$ and, so far, we did not succeed to define it in a chart-independent way. Maybe studying the Gauss-Manin connection matrix of the vector field D be useful. Since the CY 3-folds are more important in the literature, we state the Gauss-Manin connection matrix of D for $n = 3$ here:

$$\mathsf{A}_{\mathsf{D}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 \\ t_2t_4 & 0 & 0 & -1 \\ -t_2(t_2t_4+t_7) & t_2t_4 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

in which $\mathsf{Y}_1 = \frac{t_3^3}{5^4(t_1^5-t_5)}$ is the Yukawa coupling. Note that, due to Theorem 3.1, the Gauss-Manin connection matrix of R is as follow:

$$\mathsf{A}_{\mathsf{R}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \mathsf{Y}_1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{5.2}$$

It would be very interesting, and maybe helpful, if one can find out the (physical) interpretation of the non-zero part of the lower triangle of the matrix A_{D} stated in (5.1).

References

[Ali17] Murad Alim. Algebraic structure of tt^* equations for Calabi-Yau sigma models. *Commun. Math. Phys.*, 353(3):963–1009, 2017.

[AMSY16] Murad Alim, Hossein Movasati, Emanuel Scheidegger, and Shing-Tung Yau. Gauss-Manin connection in disguise: Calabi-Yau threefolds. *Commun. Math. Phys.*, 344(3):889–914, 2016.

[AV18] Murad Alim and Martin Vorgin. Gauss-Manin Lie algebra of mirror elliptic K3 surfaces. *arXiv:1812.03185 [math.AG]*, 2018.

[COGP91] Philip Candelas, Xenia C. de la Ossa, Paul S. Green, and Linda Parkes. A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory. *Nuclear Phys. B*, 359(1):21–74, 1991.

[Coh77] H. Cohen. Sums involving the values at negative integers of L–functions of quadratic characters. *Math. Ann.*, 217:81–94, 1977.

[CS17] Henry Cohen and Fredrik Strömberg. Modular Forms: A Classical Approach. *American Mathematical Society, Providence, Rhode Island*, 2017.

[GMP95] Brian R. Greene, David R. Morrison, and M. Ronen Plesser. Mirror manifolds in higher dimension. *Comm. Math. Phys.*, 173:559–598, 1995.

[Mov12b] Hossein Movasati. Quasi modular forms attached to elliptic curves, I. *Annales Mathmatique Blaise Pascal*, 19:307–377, 2012.

[Mov15] Hossein Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. *Math. Zeit.*, 281, Issue 3, pp. 907-929(3):907–929, 2015.

[Mov16] Hossein Movasati. Gauss-Manin connection in disguise: Calabi-Yau modular forms. *International Press, Somerville, Massachusetts, U.S.A, and Higher Education Press, Beijing, China*, 2017.

[MN16] Hossein Movasati, and Younes Nikdelan. Gauss-Manin Connection in Disguise: Dwork-Family. *arXiv:1603.09411 [math.AG]*, 2016 (accepted by Journal of Differential Geometry).

[Nik15] Younes Nikdelan. Darboux–Halphen–Ramanujan vector field on a moduli of Calabi–Yau manifolds. *Qual. Theory Dyn. Syst.*, 14(1):71–100, 2015.

[Nik20] Younes Nikdelan. Modular vector fields attached to Dwork family: $\mathfrak{sl}_2(\mathbb{C})$ Lie algebra. *Moscow Math. J.*, 20(1):127–151, 2020.

[OEI64] The OEIS Foundation. The On-line Encyclopedia of Integer Sequences. `http://oeis.org/`, 1964.

[Ram16] S. Ramanujan. On certain arithmetical functions. *Trans. Cambridge Philos. Soc.*, 22:159–184, 1916.

[Ran56] R. A. Rankin. The construction of automorphic forms from the derivatives of a given form. *. Indian Math. Soc.*, 20:103–116, 1956.

[NY13] Noriko Yui. Modularity of CalabiYau Varieties: 2011 and Beyond. *In: Laza R., Schütt M., Yui N. (eds) Arithmetic and Geometry of K3 Surfaces and CalabiYau Threefolds. Fields Institute Communications*, vol 67. pp 101-139, Springer, New York, NY, 2013.

[Zag94] D. Zagier, Modular forms and differential operators. *Proceedings Mathematical Sciences*, 104(1):57–75, 1994.