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Solving Linear Recurrence Equations

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Abstract

The software is an implementation of the algorithms in [1], [2], and [3]. The main algorithm from [3] is implemented with additional base equations beyond what appear in [3] and is incorporated into [4]. Common to each algorithm is a transformation from a base equation to the input using transformations that preserve order and homogeneity (referred to as gt-transformations).

1 Introduction

Before calling any of these algorithms we first check currently available algorithms for a solution in the form of a first order right hand factor or some other general term solution. The software will then run through any or all of the following algorithms (all is the default setting). A help file is included with the implementation explaining the options.

2 'Find ${}_2F_1$'

Algorithm 'Find ${}_2F_1$' will find a gt-transformation to a recurrence relation satisfied by a hypergeometric series $u(n) = {}_2F_1\left(\begin{smallmatrix} a+n & b \\ & c \end{smallmatrix} \middle| z\right)$, if such a transformation exists. (${}_2F_1\left(\begin{smallmatrix} a+n & b \\ & c \end{smallmatrix} \middle| z\right)$ will be represented in our output by the equivalent notation: hypergeom$([a+n, b], [c], z)$.) As an example, consider sequence $A005572 = [1, 4, 17, 76, 354, 1704, 8421, \dots]$ from the OEIS ([5]) which represents the "Number of walks on cubic lattice starting and finishing on the xy plane and never going below it." $A005572$ has offset 0 (i.e. the first entry in the list is $A005572(0)$) and satisfies:

$$(12n + 12)A005572(n) + (-20 - 8n)A005572(n + 1) + (n + 4)A005572(n + 2) = 0.$$

The output from our program is:

$$A005572(n) = \frac{2^{n+1}\left(2\text{hypergeom}([1/2, n+2], [1], 2/3)\ -3\text{hypergeom}([1/2, n+1], [1], 2/3)\right)}{\sqrt{3}(n+2)},\ n \geqslant 0.$$

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3 'Find Liouvillian'

The algorithm 'Find Liouvillian' will find a gt-transformation to a recurrence relation of the form $u(n+2)+b(n)u(n)=0$ for some $b(n)\in\mathbb{C}(n)$, if such a transformation exists. 'Find Liouvillian' is not unique in terms of its purpose but, for second order recurrence relations, it is faster than prior algorithms (see [2] for details). As an example, for the recurrence relation satisfied by $A099364$ from the OEIS our program produces the solution:

$$A099364(n)=\left(\frac{1}{6}n+\frac{5}{6}\right)v(n)-\left(\frac{1}{12}n+\frac{1}{2}\right)v(n+1),$$
$$\text{where } v(n+2)-\frac{4(n+2)}{n+7}v(n)=0.$$

4 'Database Solver'

The algorithm 'Database Solver' takes advantage of a large database of sequences, the OEIS mentioned above, by using the recurrence relations that they satisfy as base equations. We have already searched the database and generated collections, with invariant properties that we compare, such that there exists a gt-transformation between any two members of a collection. If the input is a recurrence relation with initial conditions defining a sequence then the output will be a transformation from a representative sequence (if there is a match in our database). The output could be useful since the representatives of each collection are chosen, in large part, for how much information there is about that sequence (formulas, papers citing the sequence, ...).

As an example of this algorithm, suppose we were working on "Coefficients of series whose square is the weight enumerator of the [8,4,4] Hamming code" (sequence $A108095$ from the OEIS). This sequence, $[1,7,-24,168,-1464,14280,\dots]$ has offset 0 and satisfies:

$$(n-1)u(n)+(7+14n)u(n+1)+(n+2)u(n+2)=0.$$

We enter the recurrence equation satisfied by $A108095$ with the corresponding initial conditions and get the following output:

$$A108095(n)=7(-1)^n\left(\frac{(97n+49)A084768(n)}{7n(n-1)}-\frac{(n+1)A084768(n+1)}{n(n-1)}\right).$$

There is a little more information on $A084768$'s page then there is on the page for $A108095$ and we see that it is related to an interesting sequence: $A084768$ is defined as "the central coefficient of $(1+7x+12x^2)^n$" as well as "$P_n(7)$, where P_n is $n-$th Legendre polynomial." (The latter description can also be discovered with our next algorithm.)

5 'Special Functions'

By computing local data of a recurrence equation (see [3] for details) the program can find gt-transformations to many different known functions. Currently in our table of known functions are Bessel (first and second kind), Whittaker (W and M), and types of Jacobi, Legendre, Laguerre, and Gegenbauer functions. As an example, for the recurrence equation satisfied by the interesting sequence $A143415$ from the OEIS, the program will return the solution:

$$A143415(n)=\frac{1}{2}\sqrt{\frac{e}{\pi}}\left(\frac{\text{BesselK}(n-\frac{1}{2},\frac{1}{2})}{n(n+1)}+(2n-1)\frac{\text{BesselK}(n+\frac{1}{2},\frac{1}{2})}{n(n+1)}\right),\ n\geqslant 1.$$

6 Motivation

We provide an interesting problem as an example of the motivation for finding such transformations:

Let $u(0) := 0$, $u(1) := 1$, and

$$u(n+2) := \frac{(n+2)((9n+15)u(n)+(6n+28)u(n+1))}{(n+7)(3n+2)}.$$

Then $u(2) = 4$, $u(3) = 12$, $u(4) = 34$, etc.

Can we prove that $u(n)$ is an integer for every positive integer n? With no additional information, beyond the initial conditions and recurrence equation, it would be very difficult to prove that $u(n)$ is an integer sequence. It would help:

- to have some combinatorial description (e.g. $u(n)$ is the number of objects with a certain property)

or

- to find a relation between $u(n)$ and some other sequence for which a combinatorial description is known.

We input the recurrence equation and initial conditions into our software and determine that

$$u(n) = r_0(n)A001006(n) + r_1(n)A001006(n+1)$$

where

$$r_0(n) = \frac{3(n+1)(n+2)}{(n+4)(n+5)}, \quad r_1(n) = \frac{3(n^2+3n-2)}{(n+4)(n+5)}.$$

A check of the OEIS provides a description of $A001006$ that is clearly integer (and correctly modeled by the recurrence equation). We rewrite $u(n)$ as:

$$(n+6)A001006(n+3) - (7n+18)A001006(n+1) - (6n+6)A001006(n).$$

$u(n)$ is now seen to be the sum of products of integers and so $u(n)$ is an integer sequence.

References

[1] G. Levy. Solutions of second order recurrence relations. Ph.D. dissertation, Florida State University, 2010. Text and implementations available at: http://www.math.fsu.edu/∼hoeij/glevy/

[2] M. van Hoeij, G. Levy. Liouvillian Solutions of Irreducible Second Order Linear Difference Equations. ISSAC'2010 Proceedings, p. 297-302, 2010.

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[4] Implementation. http://www.math.fsu.edu/∼glevy/implementation/code.zip

[5] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. World-Wide Web URL www.research.att.com/∼njas/sequences/